Circular flows in mono-directed signed graphs

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January 19, 2024

Abstract

In this paper, the concept of circular r-flow in a mono-directed signed graph (G,σ) is introduced. That is a pair (D,f), where D is an orientation on G and $f : E(G) \to (-r,r)$ satisfies that $|f(e)| \in [1, r-1]$ for each positive edge e and $|f(e)| \in [0, \frac{r}{2}-1] \cup [\frac{r}{2}+1, r)$ for each negative edge e, and the total in-flow equals the total out-flow at each vertex. This is the dual notion of circular colorings of signed graphs and is distinct from the concept of circular flows in bi-directed graphs associated with signed graphs studied in the literature.

We first explore the connection between circular $\frac{2k}{k-1}$-flows and modulo k-orientations in signed graphs. Then we focus on the upper bounds for the circular flow indices of signed graphs in terms of the edge-connectivity, where the circular flow index of a signed graph is the minimum value r such that it admits a circular r-flow. We prove that every 3-edge-connected signed graph admits a circular 6-flow and every 4-edge-connected signed graph admits a circular 4-flow. More generally, for $k \geq 2$, we show that every $(3k-1)$-edge-connected signed graph admits a circular $\frac{2k}{k-1}$-flow, every $3k$-edge-connected signed graph has a circular r-flow with $r < \frac{2k}{k-1}$, and every $(3k+1)$-edge-connected signed graph admits a circular $\frac{4k+2}{2k-1}$-flow. Moreover, the $(6k-2)$-edge-connectivity condition is shown to be sufficient for a signed Eulerian graph to admit a circular $\frac{4k}{2k-1}$-flow, and applying this result to planar graphs, we conclude that every signed bipartite planar graph of negative girth at least $6k-2$ admits a homomorphism to the negative even cycles C_{-2k}.

1 Introduction

The concept of nowhere-zero integer flows in graphs, introduced by W.T. Tutte [28] in 1950s, is a central topic in graph theory. When restricted to planar graphs, this is the dual concept of proper vertex coloring. In 1988, A. Vince [30] introduced a refinement of proper vertex coloring of graphs, which is now called the *circular coloring*. The dual notion, *circular flow* in graphs, was introduced by L.A. Goddyn, M. Tarsi, and C.-Q. Zhang [7] in 1998.

A *signed graph*, denoted (G,σ) (or $\hat{G}$ if the signature is clear from the context), is a graph G together with a signature $\sigma : E(G) \to \{+,-\}$. Let $(G,+)$(or $(G,-)$) denote a signed graph where all the edges are positive (respectively, negative). In this work, graphs are allowed to have multi-edges, but no loop. The notion of the circular coloring of graphs has been recently extended to signed graphs in [22]. As the dual of the circular coloring of graphs, we define the concept of circular flow in mono-directed signed graphs as follows.

Definition 1.1. *Given a signed graph (G,σ) and a real number $r \geq 2$, a* circular r-flow *in (G,σ) is a pair (D,f) where D is an orientation on G and $f : E(G) \to (-r,r)$ satisfies the following conditions:*

- *For each positive edge e of (G,σ), $|f(e)| \in [1, r-1]$.*
- *For each negative edge e of (G,σ), $|f(e)| \in [0, \frac{r}{2}-1] \cup [\frac{r}{2}+1, r)$.*
- *For each vertex v of (G,σ), the total out-flow equals the total in-flow under the orientation D, i.e.,*
$$\partial_D f(v) := \sum_{e \in \overleftarrow{E_D}(v)} f(e) - \sum_{e \in \overrightarrow{E_D}(v)} f(e) = 0,$$
where $\overleftarrow{E_D}(v)$ and $\overrightarrow{E_D}(v)$ denote the sets of out-arcs and in-arcs at v respectively.

The circular flow index *of a signed graph (G,σ), denoted $\Phi_c(G,\sigma)$, is defined as*

$$\Phi_c(G,\sigma) = \inf\{r : (G,\sigma) \text{ admits a circular } r\text{-flow}\}.$$

To build the intuition for the conditions on $|f(e)|$, it is better to view the interval $[0,r)$ as a circle C^r with circumference r, points of the interval $[\frac{r}{2}+1, r) \cup [0, \frac{r}{2}-1]$ are antipodal to those points of $[1, r-1]$.

As in the case of graphs, given a flow (D,f) in (G,σ) and for an edge e of G, if D' is obtained from D by reversing the direction on e and f' is obtained from f by changing the value at e to $-f(e)$, then the pair (D',f') is also a flow in (G,σ). Thus, whenever needed, we may assume that $f(e) \geq 0$ for each edge e, and such a flow is called a *non-negative* circular flow.

It is easily observed that the third condition of Definition 1.1 extends to all edge-cuts: For each subset X of vertices,
$$\partial_D f(X) := \sum_{e \in \overleftarrow{E_D}(X)} f(e) - \sum_{e \in \overrightarrow{E_D}(X)} f(e) = 0. \tag{1}$$

The interval of allowed values for $|f(e)|$ is a closed interval of a circle of circumference r obtained from $[0,r]$ by identifying the two end points. It follows from the compactness of the circle that the infimum in the definition of the circular flow index can be replaced with the minimum, even for infinite graphs (see [1]). In finite signed graphs, which is the focus of this work, using the notion of tight cut, to be introduced in Section 2, we will see that in fact the circular flow index is always a rational number and that it can be computed for any given signed graph, though the problem of computing it is in the class of NP-hard problems.

It follows from the definition that viewing $(G,+)$ as a graph G, our study of the circular flow index of signed graphs includes that of graphs. Another notion of circular flows in signed graphs in which edges are bi-directed is studied in the literature (see for example [2] and [26]). That concept was motivated by the colorings of graphs embedded in non-orientable surfaces. However, there is no dual concept for circular flows in bi-directed graphs. The concept of circular flows of signed graphs studied in this paper is exactly the dual of circular colorings of signed graphs studied in [22].

In Definition 1.1, if the condition that "$\sum_{e \in \overleftarrow{E_D}(v)} f(e) - \sum_{e \in \overrightarrow{E_D}(v)} f(e) = 0$ for every vertex v" is replaced by the condition that "$\sum_{e \in C^F} f(e) - \sum_{e \in C^B} f(e) = 0$ for every cycle C" (where C^F and C^B denote the set of forward and backward arcs of C respectively), then f is called a *circular r-tension* in (G,σ). The *circular chromatic number* of (G,σ), denoted $\chi_c(G,\sigma)$, is the infimum of those r for which (G,σ) admits a circular r-coloring (equivalently, circular r-tension).

Given a signed plane graph (G,σ), the *dual signed graph* of (G,σ) is the signed plane graph (G^*,σ^*) defined as follows: G^* is the dual graph of the underlying graph G and $\sigma^*(e^*) = \sigma(e)$ for each edge $e^* \in E(G^*)$ where e^* is the dual edge of e. It is easy to see that when restricted to signed plane graphs, a circular r-flow in (G,σ) defined in this paper is equivalent to a circular r-tension of (G^*,σ^*), and hence $\Phi_c(G,\sigma)=\chi_c(G^*,\sigma^*)$.

A central topic in the study of circular flow in graphs is the relation between the edge-connectivity and the circular flow index. It is easily observed that, given a graph G, if a function f satisfies Equation (1), then for every bridge e of G we must have $f(e)=0$. Thus a signed graph with a positive bridge admits no circular flow. For a signed graph (G,σ) with no positive bridge, it is observed in Section 2, based on the 6-flow theorem of Seymour [27], that $\Phi_c(G,\sigma)\leq 12$. Note that if (D,f) is a circular r-flow in (G,σ) and $r'\geq r$, then $(D,\frac{r'}{r}f)$ is a circular r'-flow in (G,σ). One of the most influential problems in this area is Jaeger's circular flow conjecture [9]. Originally it asserted that for any positive integer k, every $4k$-edge-connected graph admits a circular $\frac{2k+1}{k}$-flow, observing that the $(4k-1)$-edge-connected graph K_{4k} does not admit such a flow. This exact assertion was disproved in [8] in 2018 for each integer $k\geq 3$. However, in support of the conjecture, many upper bounds for the circular flow indices of highly edge-connected graphs are proved. The best general results until now are that of [15] and [14]. In [15] it is proved that the circular flow index of a $6k$-edge-connected graph is at most $\frac{2k+1}{k}$. In [14] it is proved that the circular flow index of a $(6k+2)$-edge-connected graph is strictly less than $\frac{2k+1}{k}$ and that the circular flow index of a $(6k-2)$-edge-connected graph is at most $\frac{4k}{2k-1}$. For general values of k, none of these upper bounds are known to be tight and the search for the best upper bound continues.

The restriction of Jaeger's circular flow conjecture to planar graphs has attracted a considerable amount of attention and remains open. This restricted version, using the duality, is stated as follows: Every planar graph of girth at least $4k$ admits a circular $\frac{2k+1}{k}$-coloring. In [12, 33], supported by the folding lemma, the odd-girth version of this conjecture was introduced. It asserts that every planar graph of odd girth at least $4k+1$ has circular chromatic number at most $\frac{2k+1}{k}$. The case $k=1$ is the well-known Grötzsch theorem. For $k=2$, the best-known result of odd-girth 11 follows from the lower bound on the edge-density of C_5-critical graphs given in [5]. Similarly, that every planar graph of odd-girth at least 17 admits a circular $\frac{7}{3}$-coloring follows from [25]. Alternative proofs of these two results (for $k=2,3$) are given in [4]. For general values of k, the best results follow from the general results on flows mentioned above.

In this paper, we first present some basic properties of circular flows in mono-directed signed graphs, and explore relations between flows in graphs and flows in signed graphs. In Section 3, we consider the special case of circular $\frac{2k}{k-1}$-flows. After extending the notion of modulo k-orientations to signed graphs, we show a strong connection between the existence of a modulo k-orientation and the existence of a circular $\frac{2k}{k-1}$-flow. The main focus of this work, presented in Sections 4 and 5, is mostly on the relation between the edge-connectivity and the circular flow index of signed graphs. We observe that Tutte's 5-flow conjecture is equivalent to the statement that for any 2-edge-connected graph G and any signature σ, $\Phi_c(G,\sigma)\leq 10$, and Seymour's 6-flow theorem is equivalent to $\Phi_c(G,\sigma)\leq 12$. Our main results are as follows.

Theorem 1.2. *Let (G,σ) be a signed graph and k be an integer with $k\geq 2$.*

(1) If G is 3-edge-connected, then $\Phi_c(G,\sigma)\leq 6$.

(2) If G is 4-edge-connected, then $\Phi_c(G,\sigma)\leq 4$.

(3) If G is $(3k-1)$-edge-connected, then $\Phi_c(G,\sigma)\leq \frac{2k}{k-1}$.

(4) If G is $3k$-edge-connected, then $\Phi_c(G,\sigma)<\frac{2k}{k-1}$.

(5) If G is $(3k+1)$-edge-connected, then $\Phi_c(G,\sigma)\leq \frac{4k+2}{2k-1}$.

When restricted to the class of signed Eulerian graphs, we obtain a slightly better result that every $(6k-2)$-edge-connected signed Eulerian graph (G,σ) satisfies $\Phi_c(G,\sigma)\leq\frac{4k}{2k-1}$. Consequently, by the duality of circular flows and circular colorings of signed planar graphs, we obtain the following result on homomorphisms to negative even cycles.

Theorem 1.3. *Every signed bipartite planar graph of negative girth at least $6k-2$ admits a homomorphism to C_{-2k}.*

At the end of this section, we will introduce a few notations, for which we basically follow [31]. A *cycle* of a graph G is a 2-connected subgraph all whose vertices are of degree 2 (in the subgraph). A subgraph where the degree of each vertex is even is an *even-degree* subgraph, and a connected even-degree subgraph is an *Eulerian* subgraph. Given a proper and non-empty subset X of vertices, the *edge-cut* (or *cut* for short) consisting of edges with exactly one endpoint in X is denoted by (X, X^c). Moreover, for any edge-cut (X, X^c), we denote by $\overrightarrow{(X, X^c)}$ (or $\overleftarrow{(X, X^c)}$) the set of directed edges from X to X^c (or from X^c to X, respectively). A graph G or a signed graph (G, σ) is *k-edge-connected* if each edge-cut of G has at least k edges. Given a graph G, we denote by kG the graph obtained from G by replacing each edge with k parallel edges.

Given a signed graph $\hat{G} = (G, \sigma)$, the set of positive edges (respectively, negative edges) are denoted by $E^+_{\hat{G}}$ (respectively, $E^-_{\hat{G}}$). The *positive degree* of a vertex v is the number of positive edges incident to v and is denoted by $d^+(v)$. The *negative degree*, denoted $d^-(v)$, is defined analogously. Given an orientation D on a graph G, the *out-degree* of a vertex v is denoted by $\overleftarrow{d_D}(v)$ and the *in-degree* by $\overrightarrow{d_D}(v)$. Given an orientation D on a signed graph $\hat{G} = (G, \sigma)$, we have the following four types of degrees: *positive in-degree*, *positive out-degree*, *negative in-degree* and *negative out-degree* which are denoted, respectively, $\overrightarrow{d^+_{\hat{G}}}(v)$, $\overleftarrow{d^+_{\hat{G}}}(v)$, $\overrightarrow{d^-_{\hat{G}}}(v)$, and $\overleftarrow{d^-_{\hat{G}}}(v)$. In each of these notations, we may drop the subscript when it is clear from the context.

Given a signed graph (G, σ) and a vertex v, *switching* at v is to switch the signs of all the edges incident to v in (G, σ). We say two signed graphs are *switching equivalent* if one can be obtained from the other by switching at some vertices. The *sign* of a multiset F of edges in (G, σ) is $\prod_{e \in F} \sigma(e)$ where multiplicity counts. Given a signed graph $\hat{G} = (G, \sigma)$, the signed graph obtained from $\hat{G}$ by changing the signs of all the edges is denoted by $-\hat{G}$, that is to say, $-\hat{G} = (G, -\sigma)$. A *negative cycle* of length k, denoted C_{-k}, is the signed graph on the cycle of length k with only one negative edge, and it is switching equivalent to any signed graph on C_k with an odd number of negative edges. One may observe that for each even integer k, the signed cycle $-C_{-k}$ is switching equivalent to C_{-k}. But for each odd value k, $-C_{-k}$ is switching equivalent to C_k. The *negative girth* of a signed graph is the length of a smallest negative cycle of this signed graph.

2 Basic properties

In this section, we give some equivalent definitions and basic properties of circular r-flows in signed graphs. Most of these properties are direct extensions from graphs, and we shall omit some proofs.

2.1 Tight cut

In this subsection, using the notion of tight cuts (introduced below), we shall prove that the circular flow index of a signed graph is a rational number.

Definition 2.1. *Given a signed graph (G, σ) and a non-negative circular r-flow (D, f) in (G, σ), a positive edge e is said to be* tight *if either $f(e) = 1$ or $f(e) = r - 1$, and a negative edge e is said to be* tight *if either $f(e) = \frac{r}{2} - 1$ or $f(e) = \frac{r}{2} + 1$.*

A cut (X, X^c) *of* (G, σ) *is said to be* tight *with respect to* (D, f) *if*

$$f(e) = \begin{cases} 1, & \text{if } e \text{ is a positive edge in } \overleftarrow{(X, X^c)}, \\ r-1, & \text{if } e \text{ is a positive edge in } \overrightarrow{(X, X^c)}, \\ \frac{r}{2}+1, & \text{if } e \text{ is a negative edge in } \overleftarrow{(X, X^c)}, \\ \frac{r}{2}-1, & \text{if } e \text{ is a negative edge in } \overrightarrow{(X, X^c)}. \end{cases}$$

The concept of tight cuts with respect to the circular flow is the dual notion of the tight cycle with respect to the circular coloring. Similar to the case of circular coloring, this notion can be used to provide an equivalent definition of the circular flow index. This is of particular importance to compute the exact value of the circular flow index of a given signed graph.

Lemma 2.2. *Given a signed graph* (G, σ)*,* $\Phi_c(G, \sigma) = r$ *if and only if* (G, σ) *admits a circular* r*-flow and for any non-negative circular* r*-flow* (D, f)*,* (G, σ) *has a tight cut with respect to* (D, f)*.*

Proof. We first prove the "only if" part. Assume that $\Phi_c(G, \sigma) = r$ but (G, σ) has no tight cut with respect to some non-negative circular r-flows. We choose such a circular r-flow (D, f) with minimum number of tight edges. First we claim that there is no tight edge in (G, σ) with respect to (D, f). Otherwise, let uv be a tight edge. By symmetry, we may assume that $e = (u, v)$ is a positive arc with $f(e) = r - 1$. We say a vertex w is reachable from u if there is an oriented path P from u to w such that for each arc $e' = (x, y)$ on P, e' is either a positive (respectively, a negative) forward arc with $f(e) < r - 1$ (respectively, $f(e') < \frac{r}{2} - 1$), or a positive (respectively, a negative) backward arc with $f(e) > 1$ (respectively, $f(e') > \frac{r}{2} + 1$). As (G, σ) has no tight cut with respect to (D, f), v is reachable from u. Let P be the path from u to v defined as above. Then $B = P + (v, u)$ is an oriented cycle (where $e = (v, u)$ is viewed as a backward edge). Choose a sufficiently small $\epsilon > 0$, and let $f'(e') = f(e') + \epsilon$ for each forward arc e' of B and $f'(e') = f(e') - \epsilon$ for each backward arc e' of B, we obtain a circular r-flow f' in which e is no longer a tight edge and no new tight edge is created. This contradicts the choice of (D, f), and hence there is no tight edge in (G, σ) with respect to (D, f). Then for a sufficiently small $\epsilon > 0$, $(D, \frac{1}{1+\epsilon} f)$ is a circular $\frac{r}{1+\epsilon}$-flow in (G, σ), a contradiction.

For the "if" part, we need to show that if $\Phi_c(G, \sigma) < r$, then there is a circular r-flow in (G, σ) such that there is no tight cut. Assume $\Phi_c(G, \sigma) = r' < r$ and let (D, f') be a non-negative circular r'-flow in (G, σ). Let $f = \frac{r}{r'} f'$. Then (D, f) is a non-negative circular r-flow in (G, σ) and (G, σ) contains no tight edge with respect to (D, f). □

Assume that $\Phi_c(G, \sigma) = r$ and (D, f) is a circular r-flow in (G, σ). By Lemma 2.2, there exists a tight cut (X, X^c) with respect to (D, f). Assume that in (X, X^c), there are s_1 positive forward (i.e., from X to X^c) arcs e having $f(e) = r - 1$, s_2 positive backward arcs e having $f(e) = 1$, t_1 negative forward arcs e having $f(e) = \frac{r}{2} - 1$, and t_2 negative backward arcs e having $f(e) = \frac{r}{2} + 1$. Applying Equation (1) on the cut (X, X^c), we have that $s_1(r-1) + t_1(\frac{r}{2} - 1) = s_2 + t_2(\frac{r}{2} + 1)$. Thus, we can determine the circular flow index r of (G, σ):

$$r = \frac{2(s_1 + s_2 + t_1 + t_2)}{2s_1 + t_1 - t_2} = \frac{2|(X, X^c)|}{2s_1 + t_1 - t_2}. \tag{2}$$

The values of s_1, s_2, t_1 and t_2 are all bounded by the number of edges of G. Thus Formula (2) limits the possible choices of $\Phi_c(G, \sigma)$ to a rational number whose numerator and denominator each is bounded by $2|E(G)|$.

Theorem 2.3. *For every finite signed graph (G, σ) with no positive bridge,*

$$\Phi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-flow}, \ 1 \le 2q \le p \le 2|E(G)|\}.$$

In particular, $\Phi_c(G, \sigma)$ is a rational number.

2.2 (p, q)-flow and modulo (p, q)-flow

For $r = \frac{p}{q}$, in defining a circular r-flow in a signed graph, by adjusting some small portion of flow values on certain cycles, it can be shown that we may restrict ourselves to the values of the form $\frac{i}{q}$, for $i \in \{0, 1, \dots, p-1\}$. Then multiplying all values by q, we may work with integer values. Hence, the following notion provides us an equivalent definition of circular $\frac{p}{q}$-flows.

Definition 2.4. *Given an even integer p and an integer q where $q \le \frac{p}{2}$, a* (p, q)-flow *in a signed graph (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfies the following conditions.*

- *For each positive edge e of (G, σ), $f(e) \in \{q, \dots, p-q\}$.*
- *For each negative edge e of (G, σ), $f(e) \in \{0, \dots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \dots, p-1\}$.*
- *For each vertex v of (G, σ), $\sum_{e \in \overleftarrow{E_D}(v)} f(e) = \sum_{e \in \overrightarrow{E_D}(v)} f(e)$.*

In the study of integer flow, Tutte [28] introduced the concept of modulo k-flow in a graph G as a pair (D, f), where D is an orientation of G and $f : E(D) \to \mathbb{Z}$ satisfies $\partial_D f(v) \equiv 0 \pmod{k}$, and proved the following lemma.

Lemma 2.5. [28] *If a graph admits a modulo k-flow (D, f), then it admits an integer k-flow (D, f') such that $f'(e) \equiv f(e) \pmod{k}$ for every edge e.*

Definition 2.6. *Given an even integer p and an integer q where $q \le \frac{p}{2}$, a* modulo (p, q)-flow *in (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}_p$ satisfies the following conditions.*

- *For each positive edge e of (G, σ), $f(e) \in \{q, \dots, p-q\}$.*
- *For each negative edge e of (G, σ), $f(e) \in \{0, \dots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \dots, p-1\}$.*
- *For each vertex v of (G, σ), $\sum_{e \in \overleftarrow{E_D}(v)} f(e) \equiv \sum_{e \in \overrightarrow{E_D}(v)} f(e) \pmod{p}$.*

The next result follows directly from Lemma 2.5.

Lemma 2.7. *A signed graph (G, σ) admits a (p, q)-flow if and only if it admits a modulo (p, q)-flow.*

2.3 Inversing operation

The circular chromatic number of a signed graph, as well as many other parameters of signed graphs, are invariant under the switching operation. The circular flow index, however, is invariant under the dual operation of switching.

Given a signed graph $\hat{G}$ and a cycle C of $\hat{G}$, *inversing* on C is to change the signs of all the edges of C. Two signed graphs (G, σ) and (G, σ') are said to be *inversing equivalent* if one can be obtained from the other by inversing on some cycles.

The combination of a sequence of switching at vertices is equivalent to switching at an edge-cut, i.e., changing the signs of all the edges in an edge-cut. Similarly, the combination of a sequence of inversing on cycles is to change the signs of all the edges in an even-degree subgraph. Switching at vertices does not change the sign of a cycle, and inversing on cycles does not change the sign of an edge-cut. It was proved by Zaslavsky [32] that (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of negative cycles. The dual statement follows from the fact that the symmetric difference of two inversing-equivalent signatures induces an even-degree subgraph:

Lemma 2.8. *Two signed graphs (G, σ) and (G, σ') are inversing equivalent if and only if they have the same set of negative edge-cuts.*

Note that for (G, σ) and (G, σ') to have the same set of negative cuts, it suffices that they have the same set of negative cuts among a basis of the bond space. In particular, it suffices to consider all but one of the cuts $(\{v\}, V(G) \setminus \{v\})$ from each connected component.

Lemma 2.9. *Let G be a connected graph and let T be a spanning tree of G. Given a signed graph (G, σ), there exists an inversing-equivalent signed graph (G, σ') of (G, σ) such that all the negative edges are in $E(T)$.*

Proof. For each edge $e \notin T$, let C_e be the unique cycle in the graph obtained from adding e to T. If e is a negative edge in (G, σ), then apply an inversing on C_e. After applying this process on all the edges of $E(G) \setminus E(T)$, we obtain a signed graph (G, σ') in which each edge in $E(G) \setminus E(T)$ is positive. □

One observes that if (D, f) is a modulo (p, q)-flow in (G, σ) and (G, σ') is obtained from (G, σ) by inversing on a cycle C, then (D, f') is a modulo (p, q)-flow in (G, σ') where $f'(e) = f(e) - \frac{p}{2}$ for each edge $e \in E(C)$ and $f'(e) = f(e)$ for all other edges. Thus the circular flow index is invariant under the inversing operation.

We note that the study of signed graphs with inversing operations is implicit in the study of T-joins of graphs. Given a graph G and a subset T of $V(G)$ where $|T|$ is even, a *T-join* of G is a subgraph of G where T is the set of the odd-degree vertices. Given a signed graph (G, σ), let T be the set of vertices v where the edge-cut $(\{v\}, V(G) \setminus \{v\})$ is negative. It is then easily observed that a subset E' of edges is the set of negative edges of a signature σ', inversing equivalent to σ, if and only if E' induces a T-join. For more on T-joins we refer to [3].

2.4 Relation with flows in graphs

Note that by definition $\Phi_c(G, -) = 2$ for any graph G. For signed graphs in general, it follows from the definition that a circular r-flow in G is also a circular $2r$-flow in (G, σ) for any signature σ. Thus we have the following proposition.

Proposition 2.10. *For any graph G without bridges and for any signature σ, $\Phi_c(G, \sigma) \leq 2\Phi_c(G)$.*

This upper bound is tight as below. A path of length k with all interior vertices having degree 2 is called a *k-thread*.

Lemma 2.11. *Given a graph G, let $T_2(G)$ be a signed graph obtained from G by replacing each edge with a negative 2-thread. Then $\Phi_c(T_2(G)) = 2\Phi_c(G)$.*

Proof. For one direction, assume that $\Phi_c(G) = \frac{p}{q}$ and (D, f) is a (p, q)-flow in G. We define an orientation D' of $T_2(G)$ based on D as follows: for each edge $uv \in G$ and its replacement thread uwv in $T_2(G)$, if $(u, v) \in D$, then $(u, w), (w, v) \in D'$. Moreover, we define $f'(uw) = f'(wv) = f(uv) \in$

$\{q, q+1, \ldots, p-q\}$. Then it is easy to verify that (D', f') is a $(2p, q)$-flow in $T_2(G)$ by definition. Thus $\Phi_c(T_2(G)) \le \frac{2p}{q} = 2\Phi_c(G)$.

For the other direction, given $\frac{p}{q} \ge 2$, assume that $\Phi(T_2(G)) = \frac{2p}{q}$ and $T_2(G)$ admits a $(2p, q)$-flow (D', f'). We may further choose the orientation D' such that for any 2-thread uwv, either $(u, w), (w, v) \in D'$ or $(v, w), (w, u) \in D'$. Hence, we also have $f'(uw) = f'(vw)$ for any 2-thread uwv. Now we define (D, f) in G as follows: we orient edge uv the same as 2-thread uwv in D' and let $f(uv) = f'(uw) = f'(vw) \in \{q, \ldots, 2p-q\} \cap (\{0, \ldots, p-q\} \cup \{p+q, \ldots, 2p-1\})$. Then we have $f(uv) \in \{q, \ldots, p-q\} \cup \{p+q, \ldots, 2p-q\}$ for any edge $uv \in E(G)$. By definition, (D, f) is a modulo (p, q)-flow in G. By Lemma 2.7, we have $\Phi_c(T_2(G)) \ge 2\Phi_c(G)$. This completes the proof. □

3 Modulo ℓ-orientation and homomorphism to cycles

In Jaeger's circular flow conjecture, the real numbers of the form $\frac{2k+1}{k}$ play a special role. A graph G is circular $\frac{2k+1}{k}$-colorable if and only if G admits a homomorphism to the odd cycle C_{2k+1}. While to determine whether a graph admits a homomorphism to an odd cycle is an interesting and difficult (NP-complete) problem [17], it is easy to decide whether a graph G admits a homomorphism to an even cycle. In contrast, to decide whether a signed graph $\hat{G}$ admits a homomorphism to a negative even cycle is difficult and related to many challenging conjectures [18, 20, 22]. The dual concept of admitting homomorphisms to cycles is "modulo ℓ-orientation". We propose the following generalization to signed graphs.

Definition 3.1. *A signed graph (G, σ) is said to be* modulo ℓ-orientable *if there exists an inversing-equivalent signature σ' and an orientation D on G such that, with respect to (G, σ'), for each vertex $v \in V(G)$,*

$$(\ell - 1)(\overleftarrow{d^+}(v) - \overrightarrow{d^+}(v)) = \overleftarrow{d^-}(v) - \overrightarrow{d^-}(v). \tag{3}$$

Such an orientation D is called a modulo ℓ-orientation *on (G, σ).*

Observation 3.2. *Assume (G, σ) admits a modulo ℓ-orientation.*

- *If ℓ is an odd number, then the left side of Equation (3), being multiplied by $\ell - 1$, is an even number. Thus in the subgraph induced by the set of negative edges with respect to σ', the difference of in-degree and out-degree is an even number. Hence, in this subgraph, the degree of each vertex is even. So we can apply some inversing on cycles to inverse all the edges into positive signs. So (G, σ) is inversing equivalent to $(G, +)$.*
- *If ℓ is an even number, then we conclude that at each vertex the difference of in-degree and out-degree for positive edges and negative edges, with respect to σ', is of the same parity. That means the total degree of each vertex is even. Hence, G is an even-degree graph.*

When restricted to graphs, the case of ℓ being even is easy and not interesting. However, in the framework of signed graphs, modulo ℓ-orientation for even ℓ is as interesting as for odd ℓ.

Theorem 3.3. *A signed graph (G, σ) admits a modulo ℓ-orientation if and only if there is a partition $\{E_1, E_2, \ldots, E_\ell\}$ of $E(G)$ such that the following two conditions are satisfied:*

(i) Each E_i is the set of positive edges of a signature which is inversing equivalent to σ;

(ii) There is an orientation on G such that for each vertex v and any pair $i,j \in \{1,\dots,\ell\}$,

$$\overleftarrow{d_{E_i}}(v) - \overrightarrow{d_{E_i}}(v) = \overleftarrow{d_{E_j}}(v) - \overrightarrow{d_{E_j}}(v). \tag{4}$$

Proof. Suppose that $E(G)$ is partitioned into $E_1, E_2, \dots, E_\ell$, each E_i being the set of positive edges of the signature σ_i where each σ_i is inversing equivalent to σ, and D is an orientation satisfying Condition (ii) of the theorem. We then add up Equations (4) for the pairs $(1,j)$, $j = 2,3,\dots,\ell$, and we obtain Equation (3) for a modulo ℓ-orientation on (G,σ), with respect to the inversing-equivalent signed graph (G,σ_1).

Conversely, suppose that, with respect to an inversing-equivalent signature σ', the signed graph (G,σ') admits a modulo ℓ-orientation D. If at a vertex v of G there are both incoming and outgoing edges of the same sign with respect to the signature σ', then let uv and vw be such a pair of an incoming edge and an outgoing edge. We delete these two edges and add the oriented edge uw with the same sign as vw. We repeat this operation until there are no such pair of edges. Let (G_1,σ') be the resulting signed graph and let D' be the resulting orientation on (G_1,σ'). It follows from the construction that

1. $V(G_1) = V(G)$ (possibly G_1 contains isolated vertices),
2. $\forall v \in V(G)$, $\overleftarrow{d^+_G}(v) - \overrightarrow{d^+_G}(v) = \overleftarrow{d^+_{G_1}}(v) - \overrightarrow{d^+_{G_1}}(v)$ and $\overleftarrow{d^-_G}(v) - \overrightarrow{d^-_G}(v) = \overleftarrow{d^-_{G_1}}(v) - \overrightarrow{d^-_{G_1}}(v)$.

Thus D' is a modulo ℓ-orientation on (G_1,σ'), i.e., at each vertex v, $(\ell-1)d^+_{G_1}(v) = d^-_{G_1}(v)$. We split each vertex v into $d^-(v)$ copies $v_1, v_2, \dots, v_{d^-(v)}$ where each v_i takes one positive edge incident to v and $\ell-1$ negative edges incident to v. Let (G',σ'') be the resulting signed graph together with inherited orientation D'. Note that (G',σ'') (together with D') is an ℓ-regular graph oriented in a way that each vertex is either a source or a sink and where each vertex has exactly one positive edge incident to it. Thus first of all the underlying graph is a bipartite graph with source vertices forming one part and sink vertices forming the other part. Secondly, the set of positive edges, denoted by E'_1, form a perfect matching. We now consider the $(\ell-1)$-regular subgraph of G' obtained by removing edges of E'_1. By König's theorem, we can partition the edges of this $(\ell-1)$-regular bipartite graph into $\ell-1$ perfect matchings, denoted by $E'_2, E'_3, \dots, E'_\ell$. Note that there is a clear correspondence between the edges of G and G'. Moreover, with respect to this natural correspondence between the edges, the orientation on the edges will remain the same. So the corresponding edge sets $E_1, E_2, \dots, E_\ell$ in G form the partition as required. □

We note that for $\ell = 2k+1$, Definition 3.1 is equivalent to the classic definition of modulo $(2k+1)$-orientation of graphs in the literature, in which a modulo $(2k+1)$-orientation is defined as an orientation such that for each vertex the in-degree is congruent to the out-degree modulo $2k+1$. As observed before, when $\ell = 2k+1$, we only consider signed graphs $(G,+)$, and when $\ell = 2k$ is even, we only need to consider signed Eulerian graphs. Lemma 3.4 was proved in [9].

Lemma 3.4. [9] *A graph admits a circular $\frac{2k+1}{k}$-flow if and only if it admits a modulo $(2k+1)$-orientation.*

As an extension of Lemma 3.4 to the case that $\ell = 2k$ is even, we show below that a signed Eulerian graph admits a circular $\frac{4k}{2k-1}$-flow if and only if it admits a modulo $2k$-orientation. In fact, we obtain some more equivalent properties in the following lemma.

Lemma 3.5. *Let $\hat{G}$ be a signed Eulerian graph. Then the following statements are equivalent.*

(1) $\hat{G}$ admits a circular $\frac{4k}{2k-1}$-flow.

(2) $\hat{G}$ admits a modulo $4k$-flow (D, f) such that $f(e) \in \{2k-1, 2k+1\}$ for each positive edge e, and $f(e) \in \{-1, 1\}$ for each negative edge e.

(3) $\hat{G}$ admits an orientation D such that $\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) \equiv 2k \cdot d^+_{\hat{G}}(v) \pmod{4k}$ for each vertex $v \in V(G)$.

(4) $\hat{G}$ admits a modulo $2k$-orientation.

Proof. (1) ⇒ (2). Assume that $\hat{G}$ is a signed Eulerian graph and (D, φ) is a circular $\frac{4k}{2k-1}$-flow in $\hat{G}$. Then $\varphi(e) \in \{2k-1, 2k, 2k+1\}$ for each positive edge e and $\varphi(e) \in \{4k-1, 0, 1\}$ for each negative edge e. Since G is Eulerian, the set $E' = \{e \in E(G) : \varphi(e) \in \{0, 2k\}\}$ induces an even-degree subgraph. Partition E' into edge-disjoint cycles. For each cycle C in this partition, we modify φ on C as follows: If C is a directed cycle, then let $\varphi'(e) = \varphi(e) + 1$ for each edge $e \in C$. Otherwise, the cycle C is divided into segments $P_1, P_2, \ldots, P_{2t}$, where each P_i is a directed path and for each i, one of P_i or P_{i+1} is a forward directed path and the other is a backward directed path. Let $\varphi'(e) = \varphi(e) + 1$ for $e \in P_1, P_3, \ldots, P_{2t-1}$ and $\varphi'(e) = \varphi(e) - 1$ for $e \in P_2, P_4, \ldots, P_{2t}$. For edges $e \notin E'$, let $\varphi'(e) = \varphi(e)$. Then (D, φ') is a modulo $4k$-flow in $\hat{G}$ where $\varphi'(e) \in \{2k+1, 2k-1\}$ for each positive edge e and $\varphi'(e) \in \{-1, 1\}$ for each negative edge e.

(2) ⇒ (3). Assume that $\hat{G}$ admits a modulo $4k$-flow (D, f) such that $f(e) \in \{2k-1, 2k+1\}$ for each positive edge e, and $f(e) \in \{-1, 1\}$ for each negative edge e. We define a mapping $f' : E(G) \to \{0, 2k\}$ as follows: $f'(e) = 2k$ for each positive edge e, and $f'(e) = 0$ for each negative edge e. As $\overleftarrow{d^+_D}(v) - \overrightarrow{d^+_D}(v) \equiv d^+_{\hat{G}}(v) \pmod 2$, we have $\partial_D f'(v) \equiv 2k \cdot d^+_{\hat{G}}(v) \pmod{4k}$. Let $g = f + f'$. It is easy to observe that $g(e) \pmod{4k} \in \{-1, 1\}$ and as $\partial_D f(v) \equiv 0 \pmod{4k}$, we have $\partial_D g(v) \equiv 2k \cdot d^+_{\hat{G}}(v) \pmod{4k}$. We define a new orientation D' based on D and g as follows: an edge is oriented in D' the same as in D if $g(e) \equiv 1 \pmod{4k}$ and is orientated in D' opposite as in D if $g(e) \equiv -1 \pmod{4k}$. Such an orientation D' satisfies that $\overleftarrow{d_{D'}}(v) - \overrightarrow{d_{D'}}(v) \equiv 2k \cdot d^+_{\hat{G}}(v) \pmod{4k}$ for each vertex $v \in V(G)$.

(3) ⇒ (4). Let D be an orientation on $\hat{G} = (G, \sigma)$ such that $\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) \equiv 2k \cdot d^+_{\hat{G}}(v) \pmod{4k}$. We may regard D as a directed graph. Following the methods of the proof of Theorem 3.3, we build a new digraph as follows: First, we lift pair of arcs $(u, v), (v, w)$, and continue this process until we obtain a digraph D_1 where each vertex is either a source or a sink. We note that D_1 might not be unique and depends on how we apply the process. However, we have $\overleftarrow{d_{D_1}}(v) - \overrightarrow{d_{D_1}}(v) = \overleftarrow{d_D}(v) - \overrightarrow{d_D}(v)$ for any vertex $v \in V(D_1)$. Thus in the underlying graph of D_1, we conclude that the degree of each vertex is a multiple of $2k$. For the final step of building the new digraph, we split vertices of D_1 so to have a digraph D_2 whose underlying graph is a $2k$-regular bipartite graph. Being a bipartite regular graph, by König's theorem, we can partition the edges of this $2k$-regular bipartite graph into $2k$ perfect matchings. Let $E_1, E_2, \ldots, E_{2k}$ be such a partition. By identifying the arc set of D_1 with D_2, we note that in the process of constructing D_2 from D, each edge of D has a unique corresponding edge in D_2. More precisely, edges of D_2 correspond to a partition of edges of D into directed paths. This leads to an edge partition $E_1, E_2, \ldots, E_{2k}$ of $\hat{G}$. For each $i \in [2k]$, we define σ_i to be the signature on G where positive edges are those in E_i. By Theorem 3.3, it suffices to show that (i) σ_i is inversing equivalent to σ, and (ii) under the orientation D of $\hat{G}$, $\overleftarrow{d_{E_i}}(v) - \overrightarrow{d_{E_i}}(v) = \overleftarrow{d_{E_j}}(v) - \overrightarrow{d_{E_j}}(v)$.

Let $V^o(\hat{G}) = \{v \in V(\hat{G}) \mid d^+_{\hat{G}}(v) \text{ is odd}\}$. By Lemma 2.8 and noting that $\hat{G}$ is Eulerian, to show that two signatures σ and σ' are inversing equivalent, it suffices to prove that $V^o(G, \sigma) = V^o(G, \sigma')$. Note that for each vertex w in $V^o(G, \sigma)$, we have $\overleftarrow{d_D}(w) - \overrightarrow{d_D}(w) \equiv 2k \pmod{4k}$. Hence, it has been split into an odd number of vertices in D_2, and each of which is exactly in one edge of E_i', $i \in [2k]$. As each edge of E_i' is represented by an edge-disjoint path in E_i and $\hat{G}$ is Eulerian, in (G, σ_i), also there is an odd number of positive edges incident to w. Hence $V^o(G, \sigma) = V^o(G, \sigma_i)$ for each $i \in [2k]$. This completes the proof of (i). Since in D_1 each vertex is either a sink or source, each of them contains the same number of edges from each of E_i. The edge pair $(u, v), (v, w)$ which has been lifted contributes 0 to the term "$\overleftarrow{d_{E_i}}(v) - \overrightarrow{d_{E_i}}(v)$" as they are in the same class E_i' in D_1. So the value $\overleftarrow{d_{E_i'}}(v) - \overrightarrow{d_{E_i'}}(v)$ is the number of edges of D_1 in E_i. Hence, under the orientation D of $\hat{G}$, $\overleftarrow{d_{E_i}}(v) - \overrightarrow{d_{E_i}}(v) = \overleftarrow{d_{E_j}}(v) - \overrightarrow{d_{E_j}}(v)$.

(4) $\Rightarrow$ (1). Assume that, with respect to an inversing-equivalent signature, $\hat{G}$ admits an orientation D such that for each vertex v, $(2k-1)(\overleftarrow{d^+}(v) - \overrightarrow{d^+}(v)) = \overleftarrow{d^-}(v) - \overrightarrow{d^-}(v)$. Let f be defined as follows: $f(e) = 2k-1$ if e is a positive edge and $f(e) = -1$ if e is a negative edge. Based on (D, f), for each vertex v, we have

$$\partial_D f(v) = (2k-1) \cdot (\overleftarrow{d_D^+}(v) - \overrightarrow{d_D^+}(v)) + (-1) \cdot (\overleftarrow{d_D^-}(v) - \overrightarrow{d_D^-}(v)) = 0.$$

Such a pair (D, f) is a modulo $(4k, 2k-1)$-flow of $\hat{G}$ and thus $\hat{G}$ admits a circular $\frac{4k}{2k-1}$-flow. □

Observe that a signed Eulerian planar graph (G, σ) admits a modulo $2k$-orientation if and only if its dual signed graph (G^*, σ^*) admits a homomorphism to C_{-2k}. However, a translation of this theorem to a homomorphism theorem (stated below) holds for all signed graphs. Considering the similarity of the proofs, we omit its proof.

Theorem 3.6. *A signed graph (G, σ) admits a homomorphism to $-C_{-k}$ if and only if there is a partition $E_1, E_2, \ldots, E_k$ of edges of G such that the following two conditions hold:*

(i) Each E_i is the set of positive edges of a signature which is switching equivalent to σ.

(ii) There is an orientation D on G with the following property: For each cycle C of G, there is a constant w_C such that the difference between the number of the forward and the number of the backward edges of C that are in E_i, for any $i \in [k]$, is w_C.

4 Edge connectivity and circular flow index

In this section, we focus on the circular flow indices of signed graphs in terms of edge-connectivities.

Seymour's 6-flow theorem states that every 2-edge-connected graph has a circular 6-flow. By Proposition 2.10 and Lemma 2.11, Seymour's 6-flow theorem can be restated in signed graphs as below.

Theorem 4.1. [Seymour's 6-flow theorem restated] *Every 2-edge-connected signed graph admits a circular 12-flow.*

Similarly, Tutte's 5-flow conjecture in graphs can be, equivalently, stated as:

Conjecture 4.2. [Tutte's 5-flow conjecture restated] *Every 2-edge-connected signed graph admits a circular 10-flow.*

For graphs of higher connectivity, stronger results can be obtained using the notion of group connectivity, introduced by Jaeger, Linial, Payan, and Tarsi [10].

Definition 4.3. *Given a graph G, a mapping $\beta : V(G) \to \mathbb{Z}_k$ is called a* $\mathbb{Z}_k$-boundary *of G if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{k}$.*

A graph G is said to be $\mathbb{Z}_k$-connected *if for every $\mathbb{Z}_k$-boundary β, there is an orientation D on G and a mapping $f : E(G) \to \mathbb{Z}_k$ such that for each vertex $v \in V(G)$, $\partial_D f(v) \equiv \beta(v) \pmod{k}$.*

Proposition 4.4 and Theorem 4.5 below were proved in [10].

Proposition 4.4. [10] *Let G be a connected graph with an arbitrary orientation D. The following statements are equivalent:*

- *G is $\mathbb{Z}_k$-connected.*
- *For any function $g : E(G) \to \mathbb{Z}_k$, there exists a modulo k-flow (D, f) such that for each edge e of G, $f(e) \not\equiv g(e) \pmod{k}$.*

Theorem 4.5. [10] *Every 3-edge-connected graph is $\mathbb{Z}_6$-connected, and every graph with two edge-disjoint spanning trees is $\mathbb{Z}_4$-connected. In particular, every 4-edge-connected graph is $\mathbb{Z}_4$-connected.*

Now we use Theorem 4.5 to derive results concerning circular flow in signed graphs.

Theorem 4.6. *Every 3-edge-connected signed graph admits a circular 6-flow, and every 4-edge-connected signed graph admits a circular 4-flow.*

Proof. Assume (G, σ) is a 3-edge-connected signed graph. Let $g : E(G) \to \mathbb{Z}_6$ to be defined as

$$g(e) = \begin{cases} 0, & \text{if } e \text{ is a positive edge,} \\ 3, & \text{if } e \text{ is a negative edge.} \end{cases}$$

By Proposition 4.4 and Theorem 4.5, G admits a modulo 6-flow (D, f) such that $f(e) \not\equiv g(e) \pmod{6}$ for each edge e. Then (D, f) is a modulo 6-flow in (G, σ).

The other half of the theorem is proved in the same way. □

Theorem 4.7. *For any signed graph (G, σ) that contains 3 edge-disjoint spanning trees, we have $\Phi_c(G, \sigma) < 4$. In particular, for every 6-edge-connected signed graph (G, σ), we have $\Phi_c(G, \sigma) < 4$.*

Proof. Let T_1, T_2 and T_3 be three edge-disjoint spanning trees of the underlying graph G. By Lemma 2.9, we consider the inversing equivalent signed graph (G, σ') where all the negative edges are in T_1. By Lemma 2.2, it suffices to construct a circular 4-flow (D, f) of (G, σ') that has no tight cut.

Let D be an orientation on G and let $g : E(G) \to \mathbb{Z}_4$ be defined as

$$g(e) = \begin{cases} 0, & \text{for each positive edge } e \in E(T_1) \cup E(T_2), \\ 2, & \text{otherwise.} \end{cases}$$

Let $\beta^* : V(G) \to \mathbb{Z}_4$ be the map satisfying that $\beta^*(v) \equiv \partial_D g(v) \pmod{4}$. Then β^* is a $\mathbb{Z}_4$-boundary. Let $H = T_1 \cup T_2$. By Theorem 4.5, H is $\mathbb{Z}_4$-connected, and hence there exists a mapping $f^* : E(H) \to \mathbb{Z}_4 \setminus \{0\}$ such that $\partial_D f^*(v) \equiv \beta^*(v) \pmod{4}$ for each vertex $v \in V(H)$. Extend f^* to the whole graph G by letting $f^*(e) = 0$ for $e \in E(G \setminus H)$. Let $f = f^* - g$. Then $\partial_D f(v) \equiv 0 \pmod{4}$ for every vertex $v \in V(G)$ and (D, f) is a modulo 4-flow in (G, σ'), since $f(e) \neq 2$ for each negative edge $e \in T_1$ and $f(e) \neq 0$ for each positive edge e. As $f(e) = 2$ for each edge $e \in T_3$ and any cut contains an edge in T_3, (G, σ') contains no tight cut with respect to (D, f).

By Nash-Williams-Tutte Theorem [23, 29], $2k$-edge-connected graphs have k edge-disjoint spanning trees. Thus $\Phi_c(G, \sigma) < 4$ for every 6-edge-connected signed graph (G, σ). □

It was proved in [15] that 6-edge-connected graph is $\mathbb{Z}_3$-connected. However, the argument in the proof of Theorem 4.6 cannot be applied to show that every 6-edge-connected signed graph admits a circular 3-flow. It is because our definition of circular (modulo) 3-flow in signed graphs is quite different from the notion of $\mathbb{Z}_3$-connected graphs.

Before moving to the higher edge-connectivity, we need the following definitions and results concerning orientations with boundaries from [14].

Definition 4.8. *(1) Given a graph G, a* parity-compliant $2k$-boundary *(or $2k$-*pc-boundary*) of G is a mapping $\beta : V(G) \to \{0, \pm1, \dots, \pm k\}$ satisfying the following two conditions:*

- $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2k}$, *and*
- *for every vertex $v \in V(G)$, $\beta(v) \equiv d(v) \pmod 2$.*

(2) Given a $2k$-pc-boundary β, an orientation D on G is called a $(\mathbb{Z}_{2k}, \beta)$-orientation *if for every vertex $v \in V(G)$,*

$$\overleftarrow{d_D}(v) - \overrightarrow{d_D}(v) \equiv \beta(v) \pmod{2k}.$$

Given a $2k$-pc-boundary β, for a subset $A \subset V(G)$, we define $\beta(A) \in \{0, \pm1, \dots, \pm k\}$ by $\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod{2k}$.

Definition 4.9. *Given a (partial) orientation D of a graph G and an arc $e = (x, y)$ in D, let D_e be the (partial) orientation obtained from D by flipping the arc e, that is to remove (x, y) and to add (y, x). Similarly, given a $2k$-pc-boundary β and already oriented edge (x, y) we define β_e as follows:*

$$\beta_e(x) = \beta(x) - 2, \quad \beta_e(y) = \beta(y) + 2, \quad \text{and for every other vertex } v,\ \beta_e(v) = \beta(v).$$

It is easy to see that β_e is also a $2k$-pc-boundary of G. Here the calculations are done in $\mathbb{Z}_{2k}$ whose presentation of elements will be understood from the context. Normally, as in the previous definition, we will present them by $\beta : V(G) \to \{0, \pm1, \dots, \pm k\}$ where $\pm k$ present the same element but it is more suitable to allow both presentations. With these definitions, we have the following immediate observation.

Observation 4.10. *Given a signed graph (G, σ) and a $2k$-pc-boundary function β, an orientation D on G is a $(\mathbb{Z}_{2k}, \beta)$-orientation if and only if D_e is a $(\mathbb{Z}_{2k}, \beta_e)$-orientation.*

The following theorem of [15, 14] is one of the key elements of our proofs. It basically claims that subject to some connectivity condition, some partial orientations satisfying basic necessary conditions can be extended to a $(\mathbb{Z}_{2k}, \beta)$-orientation of the full graph.

Theorem 4.11. [15, 14] *Let G be a graph and let β be a $2k$-pc-boundary of G, where $k \geq 3$. Let z be a vertex of $V(G)$ such that $d(z) \leq 2k - 2 + |\beta(z)|$. Assume that D_z is an orientation on $E(z)$ (edges incident to z) which achieves boundary $\beta(z)$ at z. Let $V_0 = \{v \in V(G) \setminus \{z\} \mid \beta(v) = 0\}$. If $V_0 \neq \emptyset$, then let v_0 be a vertex of V_0 with the smallest degree. Assume that $d(A) \geq 2k - 2 + |\beta(A)|$ for any $A \subset V(G) \setminus \{z\}$ with $A \neq \{v_0\}$ and $|V(G) \setminus A| > 1$. Then the partial orientation D_z can be extended to a $(\mathbb{Z}_{2k}, \beta)$-orientation on the entire graph G.*

Theorem 4.12. [14] *Let G be a $(3k - 3)$-edge-connected graph, where $k \geq 3$. Then for any $2k$-pc-boundary β of G, G admits a $(\mathbb{Z}_{2k}, \beta)$-orientation.*

Recall that $d^+_{\hat{G}}(v)$ (or simply $d^+(v)$ when the graph is clear from the context) denotes the number of positive edges incident to v in the signed graph $\hat{G}$.

Theorem 4.13. *Given positive integers p and q satisfying that $p \geq q$, a signed graph $\hat{G}$ admits a $(2p, q)$-flow if and only if the graph $(2p-2q)G$ admits a $(\mathbb{Z}_{4p}, \beta)$-orientation with $\beta(v) \equiv 2p \cdot d^+_{\hat{G}}(v) \pmod{4p}$ for each vertex $v \in V(G)$.*

Proof. Assume that D is a $(\mathbb{Z}_{4p}, \beta)$-orientation on $(2p-2q)G$ where $\beta(v) \equiv 2p \cdot d^+_{\hat{G}}(v) \pmod{4p}$ for each vertex $v \in V(G)$. This in particular means that β is a $4p$-pc-boundary. Observe that if we define $h(e) = 1$ for each edge e of $(2p-2q)G$, then we have $\partial_D h(v) \equiv \beta(v) \pmod{4p}$.

Let D' be an orientation on G. For each $e \in E(G)$, let $[e]$ denote the set of the corresponding $2p-2q$ parallel edges in $(2p-2q)G$. Let $I_e : [e] \to \{\pm 1\}$ be defined as follows: $I_e(e') = 1$ if $e' \in [e]$ in D is oriented as the same as e in D' and $I_e(e') = -1$ otherwise. Then we define a mapping $f_I : E(G) \to \mathbb{Z}_{4p}$ as follows:

$$f_I(e) = \sum_{e_i \in [e]} I_e(e_i).$$

Note that for each edge $e \in E(G)$, $f_I(e) \in \{-(2p-2q), \ldots, -2, 0, 2, \ldots, 2p-2q\}$, i.e., $|f_I(e)|$ is even and it satisfies that $|f_I(e)| \leq 2p - 2q$, and (D', f_I) in G satisfies that for each vertex $v \in V(G)$, $\partial_{D'} f_I(v) \equiv \beta(v) \pmod{4p}$. Next, we define another mapping $g : E(G) \to \mathbb{Z}_{4p}$ as follows: $g(e) = 0$ if e is a negative edge and $g(e) = 2p$ if e is a positive edge. Then, for each $v \in V(G)$, we have $\partial_{D'} g(v) \equiv 2p \cdot d^+(v) \pmod{4p}$.

Let $f = f_I + g$. Then $f : E(G) \to \mathbb{Z}_{4p}$ satisfies the following conditions: for each positive edge e, $f(e) = f_I(e) + 2p \in \{2q, 2q+2, \ldots, 4p-2q\}$ and for each negative edge e, $f(e) = f_I(e) \in \{-(2p - 2q), \ldots, -2, 0, 2, \ldots, 2p-2q\}$. Furthermore, considering the orientation D' on G,

$$\partial_{D'} f(v) = \partial_{D'} f_I(v) + \partial_{D'} g(v) \equiv \beta(v) + 2p \cdot d^+(v) \equiv 0 \pmod{4p}.$$

Hence, (D', f) is a modulo $(4p, 2q)$-flow in $\hat{G}$. By Theorem 2.7, $\hat{G}$ admits a $(4p, 2q)$-flow and hence a $(2p, q)$-flow.

By reversing the process above, one can build a $(\mathbb{Z}_{4p}, \beta)$-orientation on the graph $(2p-2q)G$ with $\beta(v) \equiv 2p \cdot d^+_{\hat{G}}(v) \pmod{4p}$ from a given $(2p, q)$-flow. □

We are now ready to give our main results about the upper bounds on the circular flow indices of signed graphs based on the edge connectivity of the underlying graphs. To prove the theorem, rather than study the circular flow in signed graphs directly, we apply Theorem 4.13 to study an orientation property of αG for some choice of α.

Theorem 4.14. *Let k be an integer with $k \geq 2$. Given a graph G and a signature σ on G, the following claims hold.*

(1) If G is $(3k-1)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{2k}{k-1}$.
(2) If G is $3k$-edge-connected, then $\Phi_c(G, \sigma) < \frac{2k}{k-1}$.
(3) If G is $(3k+1)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{4k+2}{2k-1}$.

Proof. Let (G, σ) be a signed graph and let D be an orientation on G. We define a mapping $\beta_\ell : V(G) \to \{0, 2\ell\}$ satisfying the following:

$$\beta_\ell(v) \equiv 2\ell(\overleftarrow{d^+}(v) - \overrightarrow{d^+}(v)) \pmod{4\ell}.$$

Note that $\sum_{v\in V(G)} \beta_{\ell}(v) \equiv 0 \pmod{4\ell}$. Moreover, $\beta_{\ell}(v) = 2\ell$ if $d^+(v)$ is odd and $\beta_{\ell}(v) = 0$ otherwise. Thus $\beta_{\ell}(v) \equiv 2\ell \cdot d^+(v) \pmod{4\ell}$.

(1). To prove that (G, σ) admits a $(2k, k-1)$-flow, by Theorem 4.13, it would be enough to show that $2G$ admits a $(\mathbb{Z}_{4k}, \beta_k)$-orientation. To this end, we must first verify that β_k is a $4k$-pc-boundary of $2G$. That is because $\beta_k(v)$ is an even value for each vertex v of G and in $2G$ every vertex is of even degree. To get the required orientation on $2G$ we apply Theorem 4.12, noting that $2G$ is a $(6k-2)$-edge-connected graph.

The proof of (3) is quite similar and we provide this proof before proving (2).

(3). To prove that (G, σ) admits a $(4k+2, 2k-1)$-flow, by Theorem 4.13, it would be enough to show that $4G$ admits a $(\mathbb{Z}_{8k+4}, \beta_{2k+1})$-orientation. The fact that β_{2k+1} is a $(8k+4)$-pc-boundary of $4G$ is implied similar to the previous case. To get the required orientation on $4G$ once again we apply Theorem 4.12, noting that $4G$ is a $(12k+4)$-edge-connected graph.

(2). For this claim, we aim to prove that there exists a sufficiently large $s = s(G)$ such that (G, σ) admits a $(2ks-2, ks-s)$-flow. Our claim then follows by observing that $\frac{2ks-2}{(k-1)s} < \frac{2k}{k-1}$. In order to get a $(2ks-2, ks-s)$-flow in (G, σ), using Theorem 4.13, it would be sufficient to find a $(\mathbb{Z}_{4ks-4}, \beta_{ks-1})$-orientation on $(2s-2)G$. One may easily check that β_{ks-1} is a $(4ks-4)$-pc-boundary of $(2s-2)G$.

To this end we first build a graph H by adding a vertex z to the graph $(2s-2)G$ and connecting it to each vertex of G with $6k-8$ parallel edges. Observe that $d_H(z) = (6k-8)|V(G)|$.

Next we extend β_{ks-1} to z by defining $\beta_{ks-1}(z) = 0$, but with slight abuse of notation we use the same name β_{ks-1}. By the construction of H (from $(2s-2)G$)), the degree of each vertex in H is even. It is then easily verified that the extended β_{ks-1} is a $(4ks-4)$-pc-boundary of H.

Next we shall apply Theorem 4.11 to obtain a $(\mathbb{Z}_{4ks-4}, \beta_{ks-1})$-orientation on H. To that end, we first consider the partial orientation D_z at the vertex z defined as follows: For each vertex v of G, orient half of the edges connected to z toward v and the other half away from v. If we choose s large enough, then we have $d(z) \leq (4ks-4) - 2 + |\beta_{ks-1}(z)|$. Here the choice of s depends on the order of G. For each subset A of $V(G)$ with $|V(G) \setminus A| > 1$, since $(2s-2)G$ is $(6ks-6k)$-edge-connected, we have at least $6ks-6k$ edges connecting A to $V(G) \setminus A$. Note that, since $z \notin A$, there are $(6k-8)|A|$ edges connecting z to A. Thus $d_H(A) \geq 6ks - 6k + (6k-8)|A| \geq 6ks - 8$, the inequality being the consequence of the fact that $|A| \geq 1$ and $k \geq 2$. Therefore, noting that $|\beta_{ks-1}(A)| \leq 2ks-2$, we have that $d_H(A) \geq 6ks - 8 \geq (4ks-4) - 2 + |\beta_{ks-1}(A)|$.

As the conditions of Theorem 4.11 are satisfied for H with z being the special vertex, we have an extension of D_z to a $(\mathbb{Z}_{4ks-4}, \beta_{ks-1})$-orientation D on H. We claim that the restriction of D to $(2s-2)G$ is also a $(\mathbb{Z}_{4ks-4}, \beta_{ks-1})$-orientation on it. This is the case because, for each vertex v, the number of edges oriented to z from v and oriented to v from z are chosen to be the same. We can then apply Theorem 4.13 to get a $(2ks-2, ks-s)$-flow in (G, σ). □

One of the key points of the proof in the previous theorem is to consider $2G$ or $4G$ so that the β function we consider is a parity-compliant boundary. If the graph itself had no odd-degree vertex, then we can directly work with G. In the case that G is $(6k-2)$-edge-connected, this leads to a slight improvement on the bound for the flow index as follows.

Theorem 4.15. *For any signed Eulerian graph (G, σ), if G is $(6k-2)$-edge-connected, then $\Phi_c(G, \sigma) \leq \frac{4k}{2k-1}$.*

Proof. Applying Theorem 4.12 to G we get a $(\mathbb{Z}_{4k}, \beta_k)$-orientation on G, where $\beta_k \equiv 2k \cdot d^+_{\hat{G}}(v) \pmod{4k}$. The claim then is concluded by the equivalence of part (1) and part (3) in Lemma 3.5. □

5 Application to planar graphs

As mentioned in the introduction, the circular flow index of a signed plane graph is equal to the circular chromatic number of its dual. Thus we have the following corollary of Theorem 4.14.

Corollary 5.1. *Let k be an integer with $k \geq 2$. Given a planar graph G and a signature σ on G, the following claims hold.*

(1) If G is of girth at least $3k-1$, then $\chi_c(G, \sigma) \leq \frac{2k}{k-1}$.

(2) If G is of girth at least $3k$, then $\chi_c(G, \sigma) < \frac{2k}{k-1}$.

(3) If G is of girth at least $3k+1$, then $\chi_c(G, \sigma) \leq \frac{4k+2}{2k-1}$.

For the dual of Theorem 4.15, we will present a stronger result by replacing the girth condition with the negative girth condition. To this end, we first present two lemmas.

Lemma 5.2. *Given a positive integer k, a graph G, and a vertex z of it, assume that the cut $(\{z\}, V(G) \setminus \{z\})$ is of size at most $6k-2$, but every other cut (X, X^c) is of size at least $6k-2$. Then given any $4k$-pc-boundary β of G and any orientation D_z of the edges incident to z satisfying that $\overleftarrow{d_{D_z}}(z) - \overrightarrow{d_{D_z}}(z) \equiv \beta(z) \pmod{4k}$, D_z can be extended to a $(\mathbb{Z}_{4k}, \beta)$-orientation on G.*

Proof. Assume that β and D_z are given as in the lemma. Given an orientation D, let $-D$ be the orientation obtained from D by flipping every arc. Then D is a $(\mathbb{Z}_{4k}, \beta)$-orientation on G if and only if $-D$ is a $(\mathbb{Z}_{4k}, -\beta)$-orientation on G. Hence, we may assume that $\beta(z) \in \{0, 1, \ldots, 2k\}$.

Our goal is to apply Theorem 4.11. Observing that since $\beta(A)$ is assumed to be in $\{0, \pm 1, \ldots, \pm 2k\}$, we have $|\beta(A)| \leq 2k$ for every $A \subset V(G)$. Thus the condition $d(A) \geq 4k - 2 + |\beta(A)|$ holds for every choice of A except $A = \{z\}$, for which the conditions are not required. We only need to consider the condition on the vertex z. If $d(z) \leq 4k - 2 + \beta(z)$, then we can directly apply Theorem 4.11. So we assume $d(z) - 4k + 2 - \beta(z) > 0$. Combining the fact that $d(z) \leq 6k - 2$ and $\beta(z) > 0$, we have $4k - 2 < d(z) \leq 6k - 2$. We aim to modify both D_z and β following the operation defined in Definition 4.9 so that by Observation 4.10 we can apply Theorem 4.11 to the new partial orientation and the new boundary function. What remains to do is to modify D_z and β to D^*_z and β^* respectively such that $\beta^*(z) = d(z) - 4k + 2$ and D^* achieves β^* at z.

Since $\overleftarrow{d_{D_z}}(z) - \overrightarrow{d_{D_z}}(z) \equiv \beta(z) \pmod{4k}$, $\overleftarrow{d_{D_z}}(z) + \overrightarrow{d_{D_z}}(z) = d(z) \leq 6k - 2$ and $0 \leq \beta(z) \leq 2k$, we have the following three possibilities: $\overleftarrow{d_{D_z}}(z) - \overrightarrow{d_{D_z}}(z) \in \{\beta(z), \beta(z) + 4k, \beta(z) - 4k\}$.

(1) $\overleftarrow{d_{D_z}}(z) - \overrightarrow{d_{D_z}}(z) = \beta(z)$.

In this case, we have $\overleftarrow{d_{D_z}}(z) = \frac{d(z)+\beta(z)}{2}$ and $\overrightarrow{d_{D_z}}(z) = \frac{d(z)-\beta(z)}{2}$. We need to flip $\frac{d(z)-\beta(z)}{2} - (2k-1)$ many in-arcs at z in D_z to out-arcs. Note that $0 \leq \frac{d(z)-\beta(z)}{2} - (2k-1) \leq \overrightarrow{d_{D_z}}(z)$. Let D^*_z be the partial orientation obtained from D_z by flipping $\frac{d(z)-\beta(z)}{2} - (2k-1)$ in-arcs of z and let β^* be the corresponding boundary function obtained as defined in Observation 4.10. We have $\beta^*(z) = \beta(z) + 2 \times (\frac{d(z)-\beta(z)}{2} - (2k-1)) = d(z) - 4k + 2$. Thus we may apply Theorem 4.11 to extend the partial orientation D^*_z to a $(\mathbb{Z}_{4k}, \beta^*)$-orientation D^*.

(2) $\overleftarrow{d_{D_z}}(z) - \overrightarrow{d_{D_z}}(z) = \beta(z) + 4k$.

In this case, we have $\overleftarrow{d_{D_z}}(z) = \frac{d(z)+\beta(z)}{2} + 2k$ and $\overrightarrow{d_{D_z}}(z) = \frac{d(z)-\beta(z)}{2} - 2k$. We need to flip $(4k-1) - \frac{d(z)-\beta(z)}{2}$ many in-arcs at z in D_z to out-arcs. Note that $0 \leq (4k-1) - \frac{d(z)-\beta(z)}{2} \leq \overrightarrow{d_{D_z}}(z)$. Let D_z^* be the partial orientation from D_z by flipping $(4k-1) - \frac{d(z)-\beta(z)}{2}$ in-arcs at z, and let β^* be the $2k$-pc-boundary obtained from β following Observation 4.10. Thus $\beta^*(z) = d(z) - 4k + 2$. Theorem 4.11 can then be applied to extend D_z^* to an orientation D^* which is a $(\mathbb{Z}_{4k}, \beta^*)$-orientation on G.

(3) $\overleftarrow{d_{D_z}}(z) - \overrightarrow{d_{D_z}}(z) = \beta(z) - 4k$.

In this case, we have $\overleftarrow{d_{D_z}}(z) = \frac{d(z)+\beta(z)}{2} - 2k$ and $\overrightarrow{d_{D_z}}(z) = \frac{d(z)-\beta(z)}{2} + 2k$. Recall that $d(z) - \beta(z) \geq 2k$, thus $0 \leq \frac{d(z)-\beta(z)}{2} + 1 \leq \overrightarrow{d_{D_z}}(z)$. We may flip a set of $\frac{d(z)-\beta(z)}{2} + 1$ in-arcs at z. After so many flips, and following Observation 4.10, we have $\beta^*(z) \equiv d(z) + 2 \pmod{4k}$. Since $4k-2 < d(z) \leq 6k-2$, we have $\beta^*(z) = d(z) - 4k + 2$. Thus, as before, we may apply Theorem 4.11 on D_z^* and β^* to get the orientation D^*.

This completes the proof. □

Lemma 5.3. [Bipartite folding lemma][19] *Let $\hat{G}$ be a signed bipartite plane graph and let $2k$ be the length of its shortest negative cycle. Assume that C is a facial cycle that is not a negative $2k$-cycle. Then there are vertices v_{i-1}, v_i, v_{i+1}, consecutive in the cyclic order of the boundary of C, such that identifying v_{i-1} and v_{i+1}, after a possible switching at one of the two vertices, the resulting signed graph remains a signed bipartite plane graph whose shortest negative cycle is still of length $2k$.*

By applying this lemma repeatedly, one gets a homomorphic image of $\hat{G}$ which is also a signed bipartite plane graph in which every facial cycle is a negative cycle of length exactly $2k$. Based on this fact and Lemma 5.2, we are ready to prove the following.

Theorem 5.4. *Every signed bipartite planar graph of negative girth at least $6k-2$ admits a circular $\frac{4k}{2k-1}$-coloring.*

Proof. Assume to the contrary that (G, σ) is a minimum counterexample with respect to $|E(G)| + |V(G)|$. By Lemma 5.3, we may assume that (G, σ) is a signed bipartite plane graph of negative girth $6k-2$ in which each facial cycle is a negative $(6k-2)$-cycle and (G, σ) admits no circular $\frac{4k}{2k-1}$-coloring. Let $\hat{G}^* = (G^*, \sigma^*)$ be the dual signed plane graph of (G, σ). Hence, the signed graph $\hat{G}^*$ is Eulerian, $(6k-2)$-regular and moreover, each of its negative cut has size at least $6k-2$. If G^* is $(6k-2)$-edge-connected, then we are done by Theorem 4.15. Thus we may assume that $\hat{G}^*$ has a positive even cut of size strictly less than $6k-2$. Let (X, X^c) be such a cut with X being inclusion-wise minimal among all the possibilities. That is to say, for every proper subset Y of X we have $|(Y, Y^c)| \geq 6k-2$.

Let $\hat{H}$ denote the signed subgraph of $\hat{G}^*$ induced by X. Observing that $|X| \geq 2$ and the subgraph $\hat{H}$ is connected, we consider $\hat{G}^*/\hat{H}$ where all the edges of $\hat{H}$ are contracted but the remaining edges get their signs from σ^*. We claim that $\hat{G}^*/\hat{H}$ admits a circular $\frac{4k}{2k-1}$-flow. Otherwise, its dual signed graph, which is a proper subgraph of (G, σ) (because $\hat{H}$ is connected), admits no circular $\frac{4k}{2k-1}$-coloring, contradicting to the minimality of (G, σ). By the equivalence of (1) and (3) in Lemma 3.5, we know that $\hat{G}^*/\hat{H}$ admits a $(\mathbb{Z}_{4k}, \beta')$-orientation with $\beta'(v) \equiv 2k \cdot d^+(v) \pmod{4k}$ for $v \in V(\hat{G}^*/\hat{H})$. Let D' be such a $(\mathbb{Z}_{4k}, \beta)$-orientation on $\hat{G}^*/\hat{H}$.

Next we build a signed graph $\hat{G}_1$ from $\hat{G}^*$ by identifying all vertices in X^c to a vertex z, deleting resulting loops, but keeping all parallel edges. Note that $d_{G_1}(z) = |(X, X^c)| < 6k-2$ but for any other vertex subset $S \subset V(G_1)$, $|(S, S^c)| \geq 6k-2$. Let D'_z be the orientation on the edges incident to z (i.e., $E(X, X^c)$ in $\hat{G}^*$) induced by D' and let β'' be a $4k$-pc-boundary of $\hat{G}_1$ satisfying that $\beta''(v) \equiv 2k \cdot d^+(v)$ (mod $4k$) for $v \in V(\hat{G}_1)$. Note that

$$\beta''(z) \equiv 2k \cdot d^+(z) \equiv \overrightarrow{d_{D'_z}}(z) - \overleftarrow{d_{D'_z}}(z) \equiv \overleftarrow{d_{D'_z}}(z) - \overrightarrow{d_{D'_z}}(z) \pmod{4k}$$

and thus we know that D'_z achieves β'' at z. We may now apply Lemma 5.2 to extend the partial orientation D'_z and obtain a $(\mathbb{Z}_{4k}, \beta')$-orientation D'' on $\hat{G}_1$.

Combining D' and D'', and also combining β' (restricted to X^c) and β'' (restricted to X), we get a $4k$-pc-boundary β of $\hat{G}^*$ with $\beta(v) \equiv 2k \cdot d^+(v)$ (mod $4k$) and the $(\mathbb{Z}_{4k}, \beta)$-orientation D on $\hat{G}^*$. Using Lemma 3.5 once again, we conclude that $\hat{G}^*$ admits a circular $\frac{4k}{2k-1}$-flow. Equivalently, as its dual, (G, σ) must admit a circular $\frac{4k}{2k-1}$-coloring, contradicting the fact that this was a (minimum) counterexample to our claim. □

It is shown in [21] that the signed cycle C_{-2k} is the bipartite circulant $(4k, 2k-1)$-clique, that is to say, a signed bipartite graph (G, σ) satisfies $\chi_c(G, \sigma) \leq \frac{4k}{2k-1}$ if and only if (G, σ) admits a homomorphism to C_{-2k}. Thus we have the following corollary.

Corollary 5.5. *Every signed bipartite planar graph of negative girth at least* $6k-2$ *admits a homomorphism to* C_{-2k}.

6 Conclusion and Questions

Recall a restatement of Tutte's 5-flow conjecture is that every 2-edge-connected signed graph admits a circular 10-flow. It has been proved in [24] that for any rational number r between 2 and 5, there exists a graph G whose circular flow index is r. Thus by Lemma 2.11 and considering the signed graph $T_2(G)$, we have the following.

Proposition 6.1. *For any rational number* $r \in [2, 10]$*, there exists a* 2*-edge-connected signed graph whose circular flow index is* r.

For 3-edge-connected signed graphs, on the one hand, we have a 6-flow theorem (Theorem 4.6), and on the other hand, we do not know any example whose circular flow index is larger than 5.

Problem 6.2. *Is it true that every* 3*-edge-connected signed graph admits a circular* 5*-flow?*

The technique of [27] can be adapted to reduce the above problem to 3-edge-connected signed cubic graphs.

Tutte's 4-flow conjecture asserts that every 2-edge-connected Petersen-minor-free graph admits a circular 4-flow. Similar to the restatement of Tutte's 5-flow conjecture, by applying Proposition 2.10 and Lemma 2.11, this conjecture can be reformulated as follows.

Conjecture 6.3. [Tutte's 4-flow conjecture restated] *Every* 2*-edge-connected signed Petersen-minor-free graph admits a circular* 8*-flow.*

If the conjecture is true, then a potential strengthening would be to use the notion of minors of signed graphs and consider 2-edge-connected signed graphs with no $(P, -)$-minor where P is the Petersen graph.

Note that the Petersen graph has circular flow index 5. Furthermore, an infinite family of snarks with circular flow index 5 are built in [16]. For each such snark G, by Lemma 2.11, $T_2(G)$ is a 2-edge-connected signed graph whose circular flow index is 10. Considering the same T_2-operation, and in combination with the result of [6], it follows that to decide if an input 2-edge-connected signed graph has circular flow index strictly smaller than 10 is an NP-hard problem.

In Theorem 4.6, we have seen that every 4-edge-connected signed graph admits a circular 4-flow. This upper bound is tight because a sequence of signed bipartite plane simple graphs of girth 4 with circular chromatic numbers approaching 4 is given in [22]. Their duals provide a sequence of 4-edge-connected signed Eulerian plane graphs with circular flow indices approaching 4. The following however remains open:

Problem 6.4. *Is there a 4-edge-connected signed graph (G, σ) satisfying that $\Phi_c(G, \sigma) = 4$?*

A result of [11] can be restated as: a signed graph verifying Problem 6.4 positively cannot be planar and Eulerian. Moreover, Theorem 4.7 implies that such an example cannot be 6-edge-connected either.

Tutte's 3-flow conjecture states that every 4-edge-connected graph admits a nowhere-zero 3-flow. Later, M. Kochol proved in [13] that Tutte's 3-flow conjecture is equivalent to the seemingly weaker statement that every 5-edge-connected graph admits a nowhere-zero 3-flow. Motivated by Kochol's result, we may propose the next conjecture which implies Tutte's 3-flow conjecture if it is true.

Conjecture 6.5. *Every 5-edge-connected signed graph admits a circular 3-flow.*

In conclusion, generalizing the notion of circular flow indices of graphs to signed graphs, we have provided upper bounds for the circular flow indices of several classes of signed graphs, in particular for the class of k-edge-connected signed graphs. The best-known results are summarized in Table 1. Our bounds are not shown to be tight and in fact, in most cases, they are not even expected to be tight. However, the problem of finding tight bounds captures some of the most challenging open problems in graph theory.

Edge connectivity	Conjectures	Known bounds
2	$\Phi_c \leq 10$ (Conj. 4.2)	$\Phi_c \leq 12$ (Thm. 4.1)
3	$*$	$\Phi_c \leq 6$ (Thm. 4.6)
4	$*$	$\Phi_c \leq 4$ (Thm. 4.6)
5	$\Phi_c \leq 3$ (Conj. 6.5)	—
6		$\Phi_c < 4$ (Thm. 4.7)
$\cdots$	$\cdots$	$\cdots$
$3k-1$	$*$	$\Phi_c \leq \frac{2k}{k-1}$ (Thm. 4.14)
$3k$	$*$	$\Phi_c < \frac{2k}{k-1}$ (Thm. 4.14)
$3k+1$	$*$	$\Phi_c \leq \frac{4k+2}{2k-1}$ (Thm. 4.14)
$6k-2$+Eulerian	$*$	$\Phi_c \leq \frac{4k}{2k-1}$ (Thm. 4.15)

Table 1: Edge Connectivity and Circular Flow Indices of Signed Graphs

Acknowledgment. This work is partially supported by the following grants: National Key Research and Development Program of China (No. 2022YFA1006400), National Natural Science Foundation of China (Nos. 12222108, 12131013), Natural Science Foundation of Tianjin (No. 22JCYBJC01520), and the Fundamental Research Funds for the Central Universities, Nankai University; ANR

(France) project HOSIGRA (ANR-17-CE40-0022); National Natural Science Foundation of China (Nos. 12301444) and the Fundamental Research Funds for the Central Universities, Nankai University, and NSERC-Canada grant number R611450; National Natural Science Foundation of China grant NSFC 11971438 and U20A2068 and by Zhejiang Natural Science Foundation grant ZJNSF LD19A010001.

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