Homomorphism Bounded Classes of Graphs

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Abstract

A class $\mathcal{C}$ of graphs is said to be *H-bounded* if each graph in the class $\mathcal{C}$ admits a homomorphism to H. We give a general necessary and sufficient condition for the existence of bounds with special local properties. This gives a new proof of Häggkvist-Hell theorem [5] and implies several cases of the existence of triangle free bounds for planar graphs.

1 Introduction

In this paper we study mainly coloring of graphs in the setting of graph homomorphism. Recall that a homomorphism from G to H is any edge-preserving mapping $f : V(G) \longrightarrow V(H)$, (i.e. $xy \in E(G) \Longrightarrow f(x)f(y) \in E(H)$. The existence of a homomorphism from G to H is denoted by $G \longrightarrow H$. A homomorphism f from G to H is sometimes called an H-coloring of G. This notion captures the coloring problems by means of the following observation:

$$\chi(G) \leq k \quad \text{iff} \quad G \longrightarrow K_k.$$

It also allows us to treat many combinatorial problems in a more general setting, for example in the context of partial order, as the existence of a homomorphism defines a quasiorder $\leq$ which is called *coloring order*

$$G \leq H \text{ if and only if G} \rightarrow H.$$

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The following is the main concept of this paper: We say a class $\mathcal{C}$ of graphs is *bounded by* H if $G \leq H$ (or, equivalently, $G \longrightarrow H$) for any $G \in \mathcal{C}$. This simple order-theoretic concept may take the form of a profound problem when applied to a concrete class of graphs. For a complete introduction to graph homomorphisms we refer to [7].

Remark on terminology: In several earlier papers any graph H which bounds a class $\mathcal{C}$ is called universal. We believe that this is a bit confusing as the notion of universal graph (for a class $\mathcal{C}$) is usually reserved for those graphs H which belong to the class $\mathcal{C}$ (i.e. which is a greatest elements of the class $\mathcal{C}$). We believe that the present notations is more fitting in our context.

The characterization of boundedness (and the estimation of the size of H) is the basic problem of chromatic (extremal) theory. In particular, the 4CT asserts that the class $\mathcal{P}$ of all planar graphs is bounded by K_4. Another related classical result is the Grötzsch theorem [4]: The class of all triangle free planar graphs is K_3-bounded, see any graph-theory textbook, or [17] for a short proof. In our setting of the Grötzsch theorem a certain asymmetry of the statement becomes apparent: "triangle free bounded by triangle". This led to the following problems posed in [10], [11]:

Problem 1 *Is the class of K_4-free planar graphs bounded by a K_4-free graph?*

Problem 2 *Is the class of K_3-free planar graphs bounded by a K_3-free graph?*

One can ask whether the class of all planar graph is bounded by a K_5-free graph. Note that the answer to this last problem is positive by virtue of the 4CT [2], [15], but in view of the difficulty of the proofs we should perhaps ask for an independent proof (not relying on a computer).

The purpose of this paper is to give a necessary and sufficient condition for the existence of bounds of this type. This is related to the following concept which is of independent interest:
Given a graph property P we say that a proper coloring of a graph G is an (m, P)-coloring if the subgraph of G induced by any m or fewer color classes has property P.

Roughly speaking, we prove that, for certain properties P, a class $\mathcal{C}$ of graphs is H-bounded with H having a (m, P)-coloring iff all the graphs $G \in \mathcal{C}$ have an (m, P)-coloring by a fixed number of colors. This holds for properties P like $P(k)$, k-colorablity, or more generally, *locally F-colorable* graphs, see Theorem 7 below. Notice that a graph H is $(k, P(k-1))$-colorable iff it is K_k-free and that $(m, P(2))$-colorability is equivalent to high odd girth.

To formulate the next result we introduce the following:

For a finite set $\mathcal{F}$ of graphs we denote by $Forb_h(\mathcal{F})$ the class of all graphs G satisfying $F \not\longrightarrow G$ for any $F \in \mathcal{F}$.
The main result of [5] and [3] can be reformulated as follows:

Theorem 3 *For any $d \geq 1$ and any finite class of graphs $\mathcal{F}$, the class of connected graphs in $Forb_h(\mathcal{F})$ with maximum degree d and chromatic number k is bounded by a k-chromatic graph in $Forb_h(F)$.*

Below (in Section 3) we give a proof of this result as a consequence of our main result (see Theorem 7 and Proposition 3). One can see that most of our results are partial results toward solving the following:

Problem 4 *Is it true that for any finite set of connected graphs $\mathcal{F}$ any set of planar graphs in $Forb_h(\mathcal{F})$ is bounded in $Forb_h(\mathcal{F})$?*

The paper is organized as follows: In section 2 we reduce the boundedness of classes of graphs to the existence of (m, P)-colorings while in Section 3 we prove the existence of (m, P)-colorings for bounded degree graphs. In Section 4 we present results on triangle free and large odd girth planar graphs. Section 5 contains some further remarks and open problems.

2 Construction of bounds for bounded classes

For a graph property P we introduced the notion of an (m, P)-coloring of a graph G (recall: the subgraph induced by the union of any m or fewer color classes should have property P). We also denote by $\chi_{m,P}(G)$ the minimal number of classes in a (m, P)-coloring of G (provided that it exists).

The following definition is a key construction, compare [1, 14, 16] for constructions of similar flavour:

Definition Let m and n be positive integers and U a graph, then let $\Pi = \Pi(n, m, U)$ be the graph whose vertex set is the set of ordered pairs (i, ϕ), where $1 \leq i \leq n$ and ϕ is a function from the m-sets of $\{1, 2, \ldots n\}$ which contain i, to $V(U)$ and whose edge set is the set $((i, \phi), (j, \psi))$ for which $i \neq j$ and $\phi(S)$ is adjacent to $\psi(S)$ for all m-sets S of $\{1, 2, \ldots n\}$ which contain both i and j.
The graph $\Pi(n, m, U)$ is n-partite and has order $n \times |V(U)|^{\binom{n-1}{m-1}}$.

Proposition 5 *Let P_U be the property of U-colorability. If $\chi_{m,P_U}(G) \leq n$ then there is a homomorphism from G to $\Pi(n, m, U)$.*

Proof. Let $c : V(G) \to [n]$ be an (m, P_U)-coloring of G, then for each $S \subseteq [n]$ of cardinality m the vertices colored from S induce a subgraph G_S of G with property P_U and so there is a homomorphism $\rho_S : G_S \to U$. Now define $f : V(G) \to V(\Pi(n, m, U))$ by $f(v) = (c(v), \phi)$, where ϕ is defined by $\phi(S) = \rho_S(v)$. We must show that f is a homomorphism. If u and v are adjacent vertices in G then set $f(u) = (i, \phi)$ and $f(v) = (j, \psi)$. Since c is a proper coloring $i \neq j$ and, if $\{i, j\} \subseteq S$ then $\phi(S) = \rho_S(u)$ which is adjacent to $\rho_S(v) = \psi(S)$ so f is a homomorphism. $\square$

For special type of properties we prove the converse of this result.
Definition Let G and U be graphs. The graph G is said to be *m-locally U-colorable* if every subgraph of G induced by m or fewer vertices admits a homomorphism into U.
Examples
i. m-locally K_2-colorable graphs are graphs with odd girth at least $m+1$.
ii. k-locally K_{k-1}-colorable graphs are just graphs not containing K_k.

Proposition 6 *The graph $\Pi(n, m, U)$ is m-locally U-colorable.*

Proof. Let Π_1 be the subgraph of $\Pi(n, m, U)$ induced by vertices $\{(i_k, \phi_k)\}_{k=1}^{m}$. Let $S' = \{i_1, i_2, \dots i_m\}$ and let S be any m-subset containing S'. Then the mapping $(i_k, \phi_k) \to \phi_k(S)$ is a homomorphism. □

Together these propositions give

Theorem 7 *A class $\mathcal{C}$ of graphs is bounded by an m-locally U-colorable graph if and only if $\{\chi_{m,P_U}(G)|\ G \in \mathcal{C}\}$ is bounded where P_U stands for U-colorability.*

Proof. Propositions 5 and 6 give the "if" part of the theorem. For the converse, suppose that $\mathcal{C}$ is bounded by an m-locally U-colorable graph H. If $G \in \mathcal{C}$ then color it by a homomorphism $\phi : G \longrightarrow H$. If G' is a subgraph of G which takes at most m colors, then so does its image $\phi(G')$ which thus maps to U. Hence $\chi_{m,P_U}(G) \leq |H|$ for all $G \in \mathcal{C}$. □

We will be mainly concerned with the following two instances of this theorem (propositions 8 and 9).

Proposition 8 *For each n and each class $\mathcal{C}$ of graphs the following statements are equivalent, where $P(n-1)$ is the property of $(n-1)$-colorability.*

(a) $\mathcal{C}$ is bounded by a K_n-free graph.

(b) $\{\chi_{n,P(n-1)}(G)|\ G \in \mathcal{C}\}$ is bounded.

(c) $\{\chi_{n+1,P(n-1)}(G)|\ G \in \mathcal{C}\}$ is bounded.

Proof. Observe that K_n-free is equivalent to both n-locally K_{n-1}-colorable and $(n+1)$-locally K_{n-1}-colorable (since the smallest n-chromatic graph not containing K_n has $n+2$ vertices) and apply theorem 7. □

Our second particular choice is the property $B = P(2)$ of being bipartite.

Proposition 9 *For each odd n and each class $\mathcal{C}$ of graphs the following statements are equivalent where B is the property of being bipartite.*

(a) $\mathcal{C}$ is bounded by a graph of odd girth n.

(b) $\{\chi_{n-2,B}(G)|\ G \in \mathcal{C}\}$ is bounded.

(c) $\{\chi_{n-1,B}(G)|\ G \in \mathcal{C}\}$ is bounded.

Proof. Having odd girth n is equivalent to both $(n-1)$-locally K_2-colorability and $(n-2)$-locally K_2-colorability. Now apply theorem 7. □

3 Proof of Theorem 3

Let $\mathcal{F}$ be a finite set of connected graphs. Put $m = \max\{|V(F)|\ F \in \mathcal{F}\}$ and let U be the disjoint union of all (non-isomorphic) graphs in $Forb_h(\mathcal{F})$ which have at most m vertices. Then $Forb_h(\mathcal{F})$ is precisely the class of all m-locally U-colorable graphs. (This is easy to see: any $G \in Forb_h(\mathcal{F})$ is clearly m-locally U-colorable. If G is m-locally U-colorable then the existence of a homomorphism $F \to G$ for an $F \in \mathcal{F}$ would violate the U-colorability of the image of F in G.)

Let $\mathcal{C}_d$ be the subclass of $Forb_h(\mathcal{F})$ of all graphs G with maximal degree $\Delta(G) \leq d$. We prove that for any $G \in \mathcal{C}_d$ holds $\chi_{m,P_U}(G) \leq d^{2m+1}$. This will finish the proof as, by Theorem 7, the class $\mathcal{C}_d$ is then bounded by a graph H in $Forb_h(\mathcal{F})$. Moreover if all graphs in $\mathcal{C}_d$ are k-colorable then $\mathcal{C}_d$ is bounded by the k-colorable graph $H \times K_k \in Forb_h(\mathcal{F})$.

Given $G \in \mathcal{C}_d$ consider the graph $G^{(m)}$ on the same set of vertices where two distinct vertices are joined by an edge iff they are joined in G by a path of length $\leq m$. As $\Delta(G^{(m)}) < d^{m+1}$ there exists a d^{m+1}-coloring of G so that any two distinct vertices of G with their distance $\leq m$ are colored differently.

We prove that this is an (m, P_U)-coloring: Let G' be a subgraph of G induced by any m classes and let G'' be one of its components. Every pair of vertices in G'' is joined by a path in G''. If any of these paths is of length at least m then its vertices take at least $m+1$ colors (on the first $m+1$ vertices of the path). This is impossible so every two distinct vertices in G'' are joined by a path of length at most $m-1$, and so take distinct colors. Therefore G'' has at most m vertices and thus, by the definition of U, is U-colorable, whence so is G'.

4 Bounds with given odd girth

The problem of bounds for the set of planar graphs with a given odd girth has been studied in various papers. The Grötszch theorem is of this type. We use $\mathcal{P}_k$ to denote the set of all planar graphs with odd girth at least k (note that $\mathcal{P}_{2k} = \mathcal{P}_{2k+1}$). Grötzsch's theorem then simply states that C_3 is a bound for $\mathcal{P}_4$. Another result of this type is Zhu's recent proof that C_{2k+1} is a bound for $\mathcal{P}_{8k-3}$, and it is conjectured that C_{2k+1} even bounds $\mathcal{P}_{4k+1}$ [19].
In this section we will study the following problem:

Problem 10 *For any given $k \geq 1$ does there exists a bound H for the subclass $\mathcal{P}_{2k+1}$ of planar graphs with $oddgirth(H) = 2k+1$?*

By Proposition 9 this is equivalent to asking whether $\{\chi_{2k-1,B}(G)|\ G \in \mathcal{P}_{2k+1}\}$ (or, equivalently, $\{\chi_{2k,B}(G)|\ G \in \mathcal{P}_{2k+1}\}$) is bounded.
In this setting Problem 1 reads as follows:

Conjecture 11 *$\mathcal{P}_4$ is bounded by a triangle free graph.*

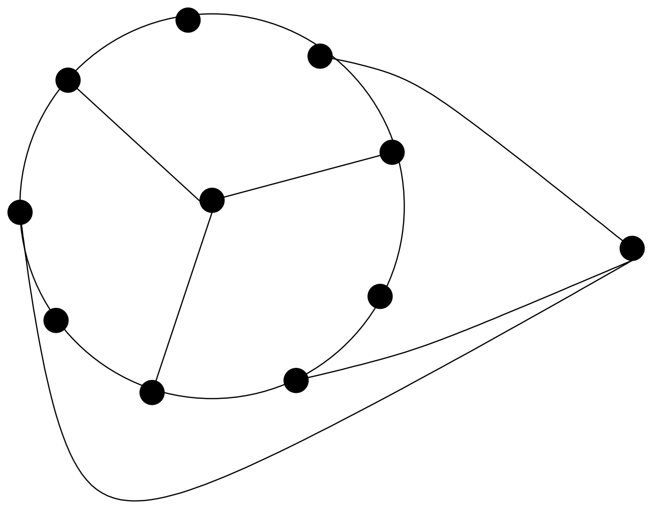

Figure 1: H

By applying Proposition 9 this is equivalent to:

Conjecture 12 *There exists a number l for which every $G \in \mathcal{P}_4$ has an l-coloring such that every odd cycle takes at least 4 different colors.*

Examples like the graph of Fig. 1 shows that a triangle-free bound for $\mathcal{P}_4$, if it exists, can not be very small. In fact any such a graph has to contain graph H of Fig. 1 as subgraph (as any 2 non-adjacent vertices of this graph are joined by a path of length 3). It is also not hard to see that any triangle-free bound for $\mathcal{P}_4$ is nonplanar:

Theorem 13 *$\mathcal{P}_4$ is not bounded by a triangle-free planar graph H.*

Proof. For a contradiction, suppose that H is a planar triangle-free bound and assume H has minimum number of vertices. It follows that H is a core (i.e. it does not have a proper retract). We show that H has minimum degree at least four. Since H is also triangle-free, this contradicts Euler's formula.

Let B be the graph obtained by joining two 5-cycles with an edge. This graph contains a set of four vertices $\{a, b, c, d\}$, each of which is joined to the others by a path of length three and no two of which are adjacent. Let H_1 be the disjoint union of H and one copy, say B_v, of B for each vertex v of H. Let $\{a_v, b_v, c_v, d_v\}$ be the four vertices in B_v corresponding to $\{a, b, c, d\}$ in B. Now form a new graph H' by joining each vertex v of H in H_1 to the vertices a_v, b_v, c_v and d_v.

Since H' is still in $\mathcal{P}_4$, there is homomorphism from H' to H. As H is a core, the restriction of this homomorphism to H is an automorphism of H. For each $v \in V(H)$, the images of the vertices in $\{a_v, b_v, c_v, d_v\}$ are distinct (otherwise H would contain a triangle) so each vertex of H has degree at least four, as required. $\square$

The following related result puts our results in yet another context.

Proposition 14 *The class of planar graphs with girth at least $2k$ is bounded by a graph H with oddgirth$(H) = k + 1$.*

Proof. Let P have girth at least $2k$, then the dual $P^\star$ of P has edge connectivity at least $2k$ and so, by the theorem of Tutte and Nash-Williams, [9, 18], has k edge disjoint spanning trees $T_1, T_2, \ldots, T_k$. Let D_i be the union of T_i and T_{i+1} (taking subscripts modulo k).

For each $i > 0$ and each edge $e \in E(T_i)$ we define a Z_2-flow $f_{i,e}$ by taking a cycle which contains the edge e and otherwise lies entirely in T_{i+1} and letting $f_{i,e}$ take the value 1 on the directed edges of this cycle and zero elsewhere. Let ψ_i be the sum of the $f_{i,e}$ taken over all edges in T_i. Thus ψ_i takes the value 1 on all edges of T_i and zero on all edges not in D_i. Symmetrically we construct a Z_2-flow ψ_i' which is 1 on T_{i+1} and zero off D_i. Let ϕ_i be the cartesian product of these two flows so that ϕ_i is a flow taking values in the Klein 4-group V, which is nonzero exactly for edges in D_i. Define T_0 to be the set of edges that do not lie in any T_i $(i \geq 1)$. A similar construction gives a Z_2-flow ϕ_0 which is nonzero on all edges in T_0 (and generally on some other edges too).

The cartesian product of the ϕ_i $(0 \leq i \leq k)$ is thus a nonzero $Z_2 \times V^k$-flow ϕ on $P^\star$. An oriented edge e is in T_i $(i \geq 1)$ if and only if $\phi(e)$ vanishes on all of the last k coordinates except the ith and the $(i-1)$th and is in T_0 if and only if $\phi(e)$ is non- vanishing on the first coordinate. Thus we have:

Every edge e of $P^\star$ lies in a unique T_i $(i \geq 0)$ and this T_i is determined by the value of $\phi(e)$.

The flow ϕ induces a corresponding $Z_2 \times V^k$-coloring $\tilde{\phi}$ on P. In view of Proposition 9, it suffices to prove that $\tilde{\phi}$ gives each odd cycle in P at least k colors. In fact we prove this for every cycle.

Let C be a cycle in P which takes m vertex colors under $\tilde{\phi}$. Define an edge-colored graph H whose vertex set is the set of colors taken by vertices in C and join vertices c and c' by an edge colored i if there are adjacent vertices v and v' in C, colored c and c' respectively, which are joined by an edge whose dual is in T_i. This edge coloring is well defined because, as observed above, the value of $c - c'$ determines i.

Since each of the trees T_i $(i \geq 1)$ meets each cutset in $P^\star$, H has at least one edge of each color $i \geq 1$.
Choose one edge of each color $i \geq 1$ in H, and suppose $|V(H)| \leq k$. In this case a subset of these edges forms a cycle in H with vertices, in order, $c_1, c_2, \ldots c_s$. Since

$$\sum (c_{j+1} - c_j) = 0,$$

(taking subscripts modulo s) and each difference $c_{j+1} - c_j$ is nonzero on a different adjacent pair of the last k coordinates, we must have $s \geq k$. Thus, in any case,

$|V(H)| \geq k$ and, since $m = |V(H)|$, the proof is completed. □

5 Remarks and open problems

1. If m is odd and B is the property of being bipartite, then the (m, B)-colorings, considered in the previous section, are only possible on graphs with odd girth at least $m + 2$. On the other hand nothing is said about the number of colors taken by an even cycle. We now consider colorings which are (m, B)-colorings when the odd girth is at least $m + 2$ but are defined for all graphs and for which there is a specified minimum number of colors on even as well as odd cycles.

Conjecture 15 *For each n there exists a constant C_n for which every planar graph admits an coloring by at most C_n colors, for which every cycle of length $\geq k$ takes at least* $\min\{\lceil (k/2) \rceil + 1, n\}$ *distinct colors.*

The proof of Proposition 14 shows that this result holds for planar graphs of girth at least $2n$.

Remark For each k and C there is a planar graph G such that, for every C-coloring of G there is a cycle of length k which is colored by at most $\lceil (k/2) \rceil + 1$ distinct colors. If k is even, say $k = 2m$, then we can let G be a graph with many independent paths of length m with common end points. By the pigeonhole principle, if there are enough paths, some two must take the same colors and hence the circuit they form takes at most $k/2 + 1$ colors. If $k = 2m + 1$, then we modify the preceding graph G by joining two adjacent interior points of each path by a path of length two. This graph then has the property that, for every C-coloring, there is a cycle of length $k+1$ which is colored by at most $\lceil ((k+1)/2) \rceil + 1$ distinct colors. Hence the numbers in the conjecture above cannot be increased.

2. For a graph G, we denote by G^k the graph with the same set of vertices where the edges correspond to pairs of distinct vertices joined by a trail of length k. We confine our attention to odd k. Note that G^k is loopless iff G has odd girth at least $k+2$.

It is an easy observation that if k is odd and a class $\mathcal{C}$ is bounded by a graph in H with odd girth at least $k + 2$ then the graphs in $\mathcal{C}$ also have odd girth at least $k+2$ and H^k bounds the graphs G^k whenever $G \in \mathcal{C}$. Thus there is a bound (namely $|V(H)|$) for $\chi(G^k)$ when $G \in \mathcal{C}$.

We do not know whether the converse to this statement is true. This can be formulated as follows:

Conjecture 16 *If k is odd, the graphs in $\mathcal{C}$ have odd girth at least $\geq k + 2$ and $\{\chi(G^k) | G \in \mathcal{C}\}$ is bounded, then $\mathcal{C}$ is bounded by a graph with odd girth at least $k + 2$.*

In particular the following is a weakening of Conjecture 11

Conjecture 17 $\{\chi(G^3) | G \in \mathcal{P}_4\}$ *is bounded*

We do not know any example of a graph $G \in P_4$ for which $\chi(G^3) > 11$. The graph of the Fig. 1 shows that 11 can be attained.

Added in proof: There has been some progress on the problem studied in this paper. Problem 10 has been answered affirmatively in [13]. The same authors have constructed the first triangle free bound for P_4 in [12]. A smaller bound together with a connection between Problem 10 and a conjecture of P. Seymour (on edge coloring of planar graphs) can be found in [8]. Conjecture 16 was also recently proved by Claude Tardif, personal communication. The result of [8] shows $\chi(G^3) \leq 16$ for every $G \in P_4$ and the same author has shown that there are graphs in P_4 with $\chi(G^3) = 15$. Whether the bound of 16 can be achieved remains an open problem.

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