Homomorphisms and Edge-colourings of Planar Graphs

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Abstract

We conjecture that every planar graph of odd-girth $2k+1$ admits a homomorphism to the Cayley graph $C(\mathbb{Z}_2^{2k+1}, S_{2k+1})$, with S_{2k+1} being the set of $(2k+1)$-vectors with exactly two consecutive 1's in a cyclic order. This is an strengthening of a conjecture of T. Marshall, J. Nešetřil and the author. Our main result is to show that this conjecture is equivalent to the corresponding case of a conjecture of P. Seymour, stating that every planar $(2k+1)$-graph is $(2k+1)$-edge-colourable.

Key words: Homomorphisms, Edge-coloring, Planarity, Clebsch graph.

1 Introduction

Let G and H be graphs. A homomorphism f of G to H is an edge preserving mapping of $V(G)$ to $V(H)$. The theory of graph homomorphisms can be viewed as a generalization of the theory of graph colourings, as a k-colouring of a graph G is exactly a homomorphism of G to the complete graph K_k. The existence of a homomorphism of G to H is normally denoted by $G \to H$. Homomorphism defines a quasiorder (a reflexive and transitive binary relation) on the set of graphs, by $G \preccurlyeq H$ if and only if $G \to H$. In this terminology, we say that H is a *bound* for a class $\mathcal{G}$ of graphs, if $G \preccurlyeq H$ for all $G \in \mathcal{G}$. For instance, we can now state Grötzsch's theorem (every triangle-free planar graph G is three-colourable) by saying that $H = K_3$ is a bound for the class of triangle-free planar graphs. For more on graph homomorphisms we refer to the recent book of Hell and Nešetřil [5].

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In a similar vein, the four-colour theorem (every planar graph is four-colourable) says that K_4 is a bound for the class of planar graphs. Since K_4 is a planar graph, there can be no smaller bound, i.e., no bound H with $H \preccurlyeq K_4$ and $K_4 \not\preccurlyeq H$. This led the authors of [2] to ask whether there exists a smaller bound, $H \preccurlyeq K_3, K_3 \not\preccurlyeq H$, for the class of triangle-free planar graphs. It is easy using graph products to reduce this question to the problem of constructing a triangle-free bound H for the class of triangle-free planar graphs. Such a bound is constructed in [10] using the methods of [7].

Let $\mathcal{P}_{2g+1}$ be the class of planar graphs of odd-girth at least $2g + 1$. It is conjectured in [7] that:

Conjecture 1 ([7]) *For any positive integer g, the class $\mathcal{P}_{2g+1}$ admits a bound B_{2g+1} of odd-girth $2g + 1$.*

This conjecture has been recently proved in [11]. See also [8] for a similar problem on planar graphs. Here, we introduce a strengthening of the conjecture by proposing a specific Cayley graph of odd girth $2k + 1$ to be the bound for $\mathcal{P}_{2k+1}$. Our main result, then, will be to connect this stronger conjecture to a conjecture of P. Seymour on edge-colouring of planar graphs.

The study of edge-colouring has a long history in graph theory, being closely linked to the four-colour problem. The edge-chromatic number of a graph is obviously at least Δ. By Vizing's well known theorem, the edge-chromatic number of a graph is at most $\Delta + \mu$, where μ is the maximum multiplicity of the edges of the graph. In the case of r-regular multigraphs, one natural obstacle to having an r-edge-colouring is to have a 'small odd cut': Let X, Y be a partition of the vertices of G and let $[X, Y]$ denote the set of all edges between X and Y. Then, $[X, Y]$ is said to be a cut of G. Moreover, $[X, Y]$ is an *odd cut* of G if $|X|$ or $|Y|$ is odd. The *size* of a cut $[X, Y]$ is $|[X, Y]|$.

If an r-regular graph G admits an r-edge-colouring, then every colour class is a perfect matching and, hence, meets every odd cut. Therefore, every odd cut must have size at least r. An r-regular graph with no odd cut of size less than r is called r-graph. In particular, an r-regular graph of edge-chromatic number r is an r-graph.

In 1878, Tait proved that the four-colour theorem is equivalent to the statement that every planar 3-graph is 3-edge-colourable. This theorem of Tait has played an important role in the development of graph theory. Not only has the study of edge-colouring been raised from the statement itself, but the proof of the equivalence was the first step in developing the theory of flows, as well. Another importance of this theorem was the flexibility of the equivalent form of the 4CC for possible generalizations. The following was a generalization conjectured by W. Tutte and proved in [12]:

Theorem 2 ([12]) *Every 3-graph with no Petersen minor is 3-edge-colourable.*

The other generalization, which is still mainly an open problem, was introduced by P. Seymour in [13].

Conjecture 3 ([14,13]) *Every planar r-graph is r-edge-colourable.*

It is important to admit multiple edges for r-graphs in Conjecture 3. Otherwise, there is no planar r-graph for $r \geq 6$. On the other hand, multiple edges are irrelevant for questions related to homomorphisms and vertex colourings.

A classical result of Kotzig implies that $r = 4$ of Conjecture 3 is stronger than the four-colour theorem. This case, together with the case $r = 5$, has been proved by B. Guenin (using the four-colour theorem) in [4].

A planar graph with a specific planar drawing is called a *plane* graph. We will need the following lemma of W. Klostermeyer and C. Q. Zhang, known as *Folding* lemma:

Lemma 4 ([6]) *Let G be a plane graph with odd-girth $2g+1$. If $C = v_0v_1 \cdots v_{r-1}v_0$ is a facial cycle of G with $r \neq 2g+1$, then there is an $i \in \{0, 1, \cdots, r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (mod r) is still of odd-girth $2g+1$.*

Corollary 5 *Given a planar graph G of odd girth greater than or equal to $2k+1$, there is a plane graph G' of odd girth $2k+1$ with every face of G' being a $(2k+1)$-cycle and for which $G \to G'$.*

Proof. If G has more than one component, then we choose a vertex from each component and identify them all. Now, the corollary is implied by a repeated application of Lemma 4 and by removing parallel edges. $\square$

2 Some Cayley graphs

Let Γ be an additive group and S a subset of Γ closed under taking inverses. Then, the Cayley graph $C(\Gamma, S)$ is defined to have the vertex set Γ, with two vertices x and y being adjacent if and only if $x - y \in S$.

Let $k \geq 1$ and $\Gamma_k = \mathbb{Z}_2^k$ be the k-dimensional group over $\mathbb{Z}_2$. Let $S_k = \{s_i | i = 1, 2, \cdots k\}$ be the set of k-vectors with exactly two consecutive 1's in a cyclic order. The Cayley graph $C(\Gamma_k, S_k)$ has two isomorphic connected components. The set of vectors with an even number of 1's induces one component and the set of vectors with an odd number of 1's induces the other component. Let H_k be one component of this graph.

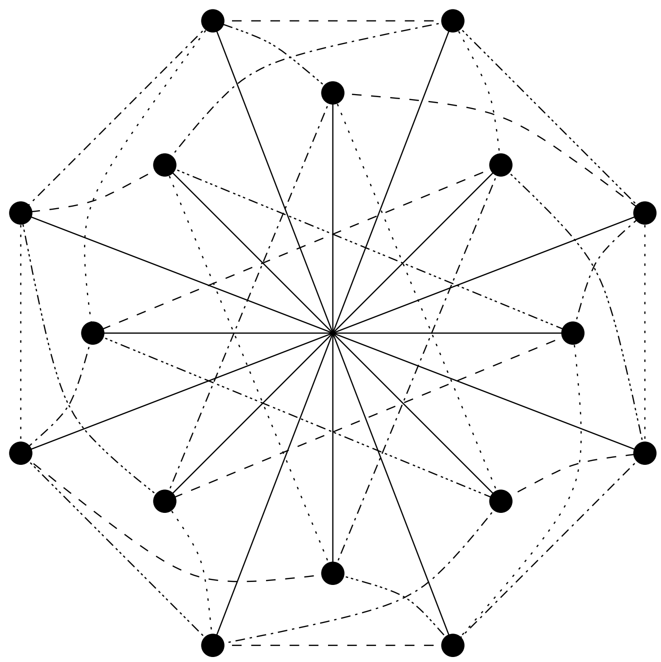

Fig. 1. H_5 (the Clebsch graph) with a canonical edge-colouring.

The graphs H_1, H_2, H_3 and H_4 are, respectively, isomorphic to K_1, K_2, K_4 and $K_{4,4}$. In general, H_k is a bipartite graph for all even values of k. The graph H_5 is well known independently in two different areas. In Ramsey theory, it is called the Greenwood-Gleason graph, see [1]. It is mainly called the Clebsch graph as it is the intersection graph of the straight lines in some algebraic surface. It is also one of the few known triangle-free strongly regular graphs, see [3]. In the next section, we will show that it is also a bound for $\mathcal{P}_5$ and that this statement is a direct generalization of the four-colour theorem.

Lemma 6 *The graph H_{2k+1} has the following properties:*

(a) It is $(2k+1)$-regular.
(b) It has edge-chromatic number equal to $2k+1$.
(c) It is of odd-girth $2k+1$.

Proof. Statement (a) is obvious. For (b), it is enough to give a $(2k+1)$-edge-colouring. This is easy because every $s_i \in S_k$ induces a perfect matching, it matches a vertex x to a unique vertex $x+s_i$. The edge-colouring obtained this way will be called the *canonical* edge-colouring of H_{2k+1}.

For (c), first note that H_{2k+1} is not a bipartite graph (for $k \geq 1$). For example, the set C of vertices defined by $C = \{v_i \,|\, v_i = s_1 + s_2 + \cdots s_i,\ i = 1, \cdots 2k+1\}$ induces an odd cycle of length $2k+1$. To show that H_{2k+1} does not contain any smaller odd cycle, consider the canonical edge-colouring of H_{2k+1} and let C be any cycle in the graph. Note that the sum of the colours of the edges of C is zero in $\mathbb{Z}_2^{2k+1}$, because it is $2\sum_{x \in C} x = 0$.

Now, if C is an odd cycle, then one of the colours, say s_i, appears an odd number of times. In order for s_i to vanish in the sum, both s_{i+1} and s_{i-1} have to appear an odd number of times. By repeating this argument, we conclude that all s_j's $j = 1, 2, \cdots 2k+1$ must appear on C an odd number of times. In particular, $|C| \geq 2k+1$. $\square$

The above proof also implies that:

Corollary 7 *In the canonical $(2k+1)$-edge-colouring of H_{2k+1}, every $(2k+1)$-cycle takes $2k+1$ different colours.*

We would like to remark that for any k-subset S of $\mathbb{Z}_2^{2k+1}$, if $\sum_{x \in S} x = 0$ but no other subset of S sums up to zero, then the Cayley graph $C(\mathbb{Z}_2^{2k+1}, S)$ will have two isomorphic copies of H_k as its components. It is also worth noting that H_k can be obtained from the hypercube of order $k-1$ by adding an edge to the vertices at maximum distance (i.e., distance $k-1$). H_5, together with a canonical 5-edge-colouring, is depicted in Figure 1. To obtain a labeling of vertices, one may pick an arbitrary vertex to be 0 and then add the colour of edges in a path from 0 to any selected vertex to get the label of that vertex.

3 The four-colour theorem and homomorphisms

The following is a strengthening of Conjecture 1:

Conjecture 8 *The class $\mathcal{P}_{2k+1}$ is bounded by H_{2k+1}.*

Notice that for $k = 1$ this is exactly the four-colour theorem. Our next theorem shows that this conjecture is equivalent to the corresponding case of Conjecture 3.

Theorem 9 *The class $\mathcal{P}_{2k+1}$ is bounded by H_{2k+1} if and only if every planar $(2k+1)$-graph is $(2k+1)$-edge-colourable.*

For a given abelian group Γ and a subset S of Γ, (Γ, S)-flow on a graph G is a flow f on G for which $f(e) \in S$ for every edge e of G. It is a well known fact that a 3-regular graph is 3-edge-colourable if and only if it admits a nowhere zero 4-flow. The following theorem is a generalization of this fact to $(2k+1)$-regular graphs and will help us in proving Theorem 9.

Theorem 10 *Given a $(2k+1)$-regular graph G, it is $(2k+1)$-edge-colourable if and only if it admits a $(\mathbb{Z}_2^{2k+1}, S_{2k+1})$-flow.*

Proof. One direction of the theorem is easy to observe. If G admits a $(\mathbb{Z}_2^{2k+1}, S_{2k+1})$-flow, then every element of S_{2k+1} must be assigned to exactly one edge incident to a given vertex. This gives a $(2k+1)$-edge-colouring. For the other side, assume G is $(2k+1)$-edge-colourable. Let $E_1, E_2, \cdots E_{2k+1}$ be the colour classes. Then, for each i, the subgraph induced by $E_i \cup E_{i+1}$ (indices are being taken modulo $2k+1$) is a union of cycles and, therefore, φ_i defined by

$$\varphi_i(e) = \begin{cases} 1 & \text{if } \ e \in E_i \cup E_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$

is a $\mathbb{Z}_2$-flow on G.

Now φ, defined by $\varphi(e) = (\varphi_1(e), \varphi_2(e), \cdots \varphi_{2k+1}(e))$, is a $(\mathbb{Z}_2^{2k+1}, S_{2k+1})$-flow on G. $\square$

Proof of Theorem 9. Let $\mathcal{P}'_{2k+1}$ be the class of plane graphs such that for each graph in $\mathcal{P}'_{2k+1}$ the odd girth is $2k+1$ and, moreover, every facial cycle is of length $2k+1$. By corollary 5, a graph B bounds $\mathcal{P}_{2k+1}$ if and only if it bounds $\mathcal{P}'_{2k+1}$.

It is also important to note that a planar graph G is in $\mathcal{P}'_{2k+1}$ if and only if the dual G^* is a planar $(2k+1)$-graph. To see this, first note that for a planar graph being $(2k+1)$-regular is equivalent to having every facial cycle of its dual to be of size $2k+1$. Secondly, by a simple counting argument, a cut of a $(2k+1)$-regular graph is an odd cut if and only if it has an odd number of edges. Therefore, the condition of no small odd cut for a $(2k+1)$-regular planar graph G^* is equivalent to the condition of no small odd cycle for the dual G.

Our next important observation is about the dual of the edge-colouring of a planar $(2k+1)$-graph. It is not hard to see that a planar $(2k+1)$-graph, G^*, admits a proper $(2k+1)$-edge-colouring if and only if its dual, G, admits an edge-colouring (most possibly an improper one) in which every facial cycle takes all the $2k+1$ different colours.

With these observations, one direction of the theorem is easy to prove. Suppose $\mathcal{P}_{2k+1}$ is bounded by H_{2k+1}. Let G^* be a planar $(2k+1)$-graph. Then the dual G^* of G is in $\mathcal{P}'_{2k+1}$ and, therefore, it admits a homomorphism to H_{2k+1}. This homomorphism induces a $(2k+1)$-edge-colouring c on G using the canonical $(2k+1)$-edge-colouring of H_{2k+1}. Since every $(2k+1)$-cycle of G and, therefore, every facial cycle of G must map to a $(2k+1)$-cycle of H_{2k+1}, the edge-colouring c has the property that every facial cycle of G takes all the $2k+1$ different colours. This colouring, therefore, induces a proper $(2k+1)$-edge-colouring on G^* (note that we are using Corollary 7 here).

For the other direction, suppose every planar $(2k+1)$-graph is $(2k+1)$-edge-colourable. It is enough to prove that every member of $\mathcal{P}'_{2k+1}$ admits a homomorphism to H_{2k+1}. Let G be a graph in $\mathcal{P}'_{2k+1}$. Then, the dual G^* of G is a planar $(2k+1)$-graph. Hence, by the assumption, G^* admits a $(2k+1)$-edge-colouring and, therefore, by Theorem 10, it admits a $(\mathbb{Z}_2^{2k+1}, S_{2k+1})$-flow φ^*.

Now, for every edge e of G define $\varphi(e) = \varphi^*(e^*)$, where e^* is the corresponding edge of e in the dual G^* of G. The homomorphism $\overline{\varphi}$ of G to H_{2k+1} can be defined as follows: For an arbitrary but fixed vertex x of G, let $\overline{\varphi}(x) = (0, 0, \cdots 0)$. For any other vertex y of G, choose an arbitrary xy-path P and let $\overline{\varphi}(y) = \sum_{e_i \in P} \varphi(e_i)$. The fact that $\overline{\varphi}$ is well defined is an easy classical result in the theory of flows. $\overline{\varphi}$ is normally called the tension of φ. It is then straightforward to see that $\overline{\varphi}$ is a homomorphism of G to H_{2k+1}. $\square$

The following theorem is now a consequence of Theorem 9 and the recent proof of Guenin for Conjecture 3 in the case of $r = 5$.

Theorem 11 *The class of triangle-free planar graphs, $\mathcal{P}_5$, is bounded by H_5.*

4 Remarks and open problems

For a given positive integer k we define

$$G^{2k-1} = (V(G), \{xy \mid \text{there is a } (2k+1) - \text{walk in } G \text{ joining } x \text{ and } y\}).$$

Note that G^{2k-1} is loopless if and only if G is of odd girth at least $2k+1$. Moreover, any homomorphism of G to H is also a homomorphism of G^{2k-1} to H^{2k-1}. The following is now a relaxation of Conjecture 8:

Conjecture 12 *For every $G \in \mathcal{P}_{2k+1}$ we have $\chi(G^{2k-1}) \leq 2^{2k}$.*

The weaker claim that $\{\chi(G^{2k-1}) | G \in \mathcal{P}_{2k+1}\}$ is bounded is equivalent to Conjecture 1 and, therefore, is proved in [11].

It is also of an independent interest to question the tightness of the bound of Conjecture 12. More precisely, we would like to ask:

Problem 13 *Is there a graph G in $\mathcal{P}_{2k+1}$ with $\chi(G^{2k-1}) = 2^{2k}$?*

A relatively simpler problem to ask is whether every bound B of odd-girth $2k+1$ for $\mathcal{P}_{2k+1}$ has at least 2^{2k} vertices. It is shown by the author that such a bound with a minimal number of vertices must have a minimum degree of at least $2k+1$.

When $k = 2$, the upper bound of 16 in Conjecture 12 is implied by Theorem 11. A triangle-free graph with $\chi(G^3) \geq 15$ is constructed by the author, see [9]. We do not know whether the right bound is 15 or 16.

We finally would like to ask whether the equivalence of Theorem 9 can be extended beyond planar graphs. In particular, is there any dual for Theorem 2?

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