Mapping planar graphs into projective cubes

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Abstract

Projective cubes are obtained by identifying antipodal vertices of hypercubes. We introduce a general problem of mapping planar graphs into projective cubes. This question, surprisingly, captures several well-known theorems and conjectures in the theory of planar graphs. As a special case we prove that the Clebsch graph, a triangle-free graph on 16 vertices, is the smallest triangle-free graph to which every triangle-free planar graph admits a homomorphism.

1. Introduction

We use the standard notation of graph theory. Graphs are simple and loopless. Given graphs G and H a mapping $\psi : V(G) \to V(H)$ is said to be a homomorphism of G to H if the image of every edge of G is an edge of H. A *core of a graph* G is the smallest subgraph of G to which G admits a homomorphism. The first theorem in graph homomorphisms is to show that core of a graph is well defined and that it is unique up to isomorphism. A *core* is a graph which is its own core.

For further notations and a comprehensive study of graph homomorphism we refer to [HN04]. Some recent or nonstandard notations, which we will use, are as follows: the class of all planar graphs of odd-girth at least $2k+1$ is denoted by $\mathcal{P}_{2k+1}$. Given a graph G and a positive integer k we define G^k to be a graph whose vertices are the same as $V(G)$ with two vertices being adjacent in G^k if there is a walk of length k joining x to y. If we take $A(G)$ to be the incident matrix of G, then G^k is the graph whose incident matrix is obtained from $A(G)^k$ by replacing every nonzero element with a 1 (we use the standard matrix product in defining $A(G)^k$).

It is an easy observation that if $\psi : V(G) \to V(H)$ is a homomorphism of G to H, then ψ is also a homomorphism of G^k to H^k. For even values of k we note that, unless G has no edge, G^k has a loop. The same is true if k is odd and at least as the odd-girth of G. Thus in both cases the problem of existence of a homomorphism from a graph H to G^k is solved by a trivial homomorphism. Therefore, from the

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homomorphism point of view, we are only interested in G^k for odd values of k that are smaller than the odd-girth of G.

For a class $\mathcal{G}$ of graphs we define: $\mathcal{G}^k = \{G^k | G \in \mathcal{G}\}$. Given a class $\mathcal{G}$ of graphs we say $\mathcal{G}$ is *b*ounded by a graph B if every member of $\mathcal{G}$ admits a homomorphism to B. It follows that if $\mathcal{G}$ is bounded by B, then $\mathcal{G}^k$ is bounded by B^k. Given two classes $\mathcal{G}_1, \mathcal{G}_2$ of graphs we say $\mathcal{G}_1$ is *b*ounded in $\mathcal{G}_2$ if there is a graph in $\mathcal{G}_2$ that bounds $\mathcal{G}_1$.

Given a finite set $\mathcal{H}$ of connected graphs we use $Forb_h(\mathcal{H})$ to denote the class of all graphs which do not admit a homomorphism from any member of $\mathcal{H}$. Similarly, given a set $\mathcal{M}$ of graphs we use $Forb_m(\mathcal{M})$ to denote the class of all graphs that have no member of $\mathcal{M}$ as a minor.

The question of bounding triangle-free planar graphs by a triangle-free graph was originally raised by J. Nešetřil [N99] as a strengthening of the celebrated theorem of Grötszch. It was shown in [MNN06] that such bound exists if and only if $\{\chi(G^3) | G \text{ is a triangle-free planar graph}\}$ is bounded. The question of Nešetřil was then studied in two different directions. On the one hand J. Nešetřil and P. Ossona de Mendez considered the general duality questions and, in a series of work, they proved the following theorem which, in the author's opinion, is the best approximation to the Hadwiger conjecture so far.

Theorem 1.1. *[NN06] Given a finite set $\mathcal{M}$ of graphs and a finite set $\mathcal{H}$ of connected graphs, the class $Forb_h(\mathcal{H}) \cap Forb_m(\mathcal{M})$ of graphs is bounded in $Forb_h(\mathcal{H})$.*

As a corollary, by taking $\mathcal{M} = \{K_5, K_{3,3}\}$ and $\mathcal{H} = \{K_3\}$, one concludes that there is a triangle-free graph to which every triangle-free planar graph admits a homomorphism. Since the proof of the above theorem is constructive, one could build such a bound from the theorem. However, the graph obtained by this construction is too large.

On the other hand, trying to find the smallest triangle-free graph which answers the question of Nešetřil, we have arrived at several interesting questions and conjectures, some of which propose extensions of the four colour theorem. As a consequence we have shown in [N07] that the Clebsch graph, on 16 vertices, is a triangle-free graph which bounds the class of triangle-free planar graphs. In this work we introduce yet another question that captures several famous theorems and conjectures. Then we prove that the Clebsch graph is the smallest triangle-free graph, both in terms of number of vertices and number of edges, that bounds the class of triangle-free planar graph. To introduce our question we need an introduction to an interesting family of graphs which we call projective cubes (known by other names such as folded cube in the literature).

2. Projective cubes

Consider an $(n+1)$-dimensional hypercube which, using l_1-norm, is embedded as the unit sphere of dimension n in $\mathbb{R}^{n+1}$. To build a projective space of dimension n one would identify antipodal vertices of this sphere. In this projection the image of the hypercube graph $H(n+1)$ is a new graph which we call the *projective cube* of dimension n and denote by $PC(n)$. Thus $PC(n)$ is obtained from $H(n+1)$ by identifying vertices at maximum distance. There are several distinct ways of representing $PC(n)$, each useful for a different purpose.

In the binary representation of $H(n+1)$, vertices are the elements of $\mathbb{Z}_2^{n+1}$ with two vertices being adjacent if their hamming distance is 1. In other words $H(n+1)$ is a Cayley graph on $\mathbb{Z}_2^{n+1}$ with $x \sim y$ if $x - y \in \{e_1, e_2, \cdots e_{n+1}\}$ where $\{e_1, e_2, \cdots e_{n+1}\}$ is the standard basis of $\mathbb{Z}_2^{n+1}$. In this presentation of $H(n+1)$, given a vertex x, the vertex at maximum distance from x is $J + x$ where J is the all 1 vector. Thus vertices of $PC(n)$ are represented by pairs $\{x, x+J\}$ and in this presentation $(x, x+J) \sim (y, y+J)$ if either x is adjacent to y in $H(n+1)$ (i.e., $x - y \in \{e_1, e_2, \cdots e_{n+1}\}$) or x is adjacent to $y+J$ in $H(n+1)$, (i.e., $x - y + J \in \{e_1, e_2, \cdots e_{n+1}\}$). If we represent a vertex $\{x, x+J\}$ by either x or $x+J$, whichever whose last coordinate is 0, and then drop this last coordinate, we will have a presentation of $PC(n)$ using $\mathbb{Z}_2^n$ such that two vertices x and y are adjacent if $x - y \in \{e_1, e_2, \cdots e_n, J\}$ where $\{e_1, e_2, \cdots e_n\}$ is the standard basis for $\mathbb{Z}_2^n$ and J is the all 1 vector in $\mathbb{Z}_2^n$. Thus to build $PC(n)$ one could also take $H(n)$ and add a new edge between every pair of antipodal vertices.

Another presentation of $PC(n)$ is obtained by looking at $H(n+1)$ as a subset order. In this view, vertices of $H(n+1)$ are the 2^{n+1} subsets of an $(n+1)$-set with $A \sim B$ if the symmetric difference of A and B is of size 1 exactly. Thus the antipodal of a vertex A is A^c, the complement of A, and, therefore, vertices of $PC(n)$ are labeled by pairs of complementary sets in this notion. To simplify the notion we may choose to present each vertex $\{A, A^c\}$ with A or A^c whichever has fewer elements. When $n+1$ is even, we have to make a choice for sets of size $\frac{n+1}{2}$. However, when $n+1$ is odd, which is the primary focus of this work, this notation is well defined. In this notation, for $n = 2k$, two vertices A and B (subsets of an $(n+1)$-set) are adjacent if either $A \subset B$ and $|A| = |B| + 1$ or if $A \subset B^c$ and $|A| = |B^c| + 1$. Thus two vertices of the same cardinality cannot be adjacent unless they are of cardinality k, in which case they are adjacent if and only if they have no element in common. Therefore, the subgraph induced by the set of vertices of cardinality k is isomorphic to the Kneser graph $K(2k+1, k)$. These graphs are also known as odd graphs.

In presenting $PC(n)$ as a Cayley graph we may also use a difference set which would make symmetries of this graph more apparent. Let $C = \{(1100\ldots0), (0110\ldots0), \ldots (10\ldots01)\}$ be the set of elements in $\mathbb{Z}^{n+1}$ each having exactly two 1's which are also consecutive in a cyclic order. The Cayley graph $(\mathbb{Z}^{n+1}, C)$ has two isomorphic connected components each isomorphic to $PC(n)$. We leave the proof of the isomorphism to the reader, but we note that the two connected components are induced by the set of vertices with an even number of 1's and the set of vertices with an odd number of 1's. With this definition we see the following:

Lemma 2.1. *The graph $PC(n)$ is vertex transitive. It is, furthermore, distance transitive, i.e., for any two pairs $\{x, y\}$ and $\{u, v\}$ of vertices if $d(x, y) = d(u, v)$, then there is an automorphism σ of $PC(n)$ such that*

$\sigma(x) = u$ *and* $\sigma(y) = v$.

There are two basic families of automorphisms of $PC(n)$. Taking the Cayley presentation of $PC(n)$, for each $c \in \mathbb{Z}^n$ the permutation $\sigma(x) = c + x$ is an automorphism. Each permutation τ of the difference set also induces an automorphism. It is not hard to see that every other permutation is a composition of a σ and a τ.

Here we are primarily concerned with homomorphisms of planar graphs to projective cubes. Thus it is essential to understand basic homomorphism properties of the projective cubes themselves. We first note that $PC(n)$ for odd values of n is bipartite. Thus the ordinary homomorphism problem to this graph is easily solved. However $PC(n)$ for even values of n has a complicated structure from a homomorphism point of view. First we recall the following lemma from [N07]:

Lemma 2.2. *[N07] The odd-girth of* $PC(2k)$ *is* $2k + 1$.

Furthermore the following lemma shows that all the shortest odd cycles are isomorphic.

Lemma 2.3. *For every pair* C *and* C' *of* $2k + 1$*-cycles in* $PC(2k)$ *there is an automorphism that takes vertices of* C *into vertices of* C'.

The proof of this lemma is elementary and we leave it to the reader. We should note this statement would not hold for $2k$-cycles in $PC(2k-1)$. There are essentially two types of $2k$-cycles in $PC(2k-1)$. The ones that are k-edges coloured and the ones that are $2k$-edges coloured where edges are coloured with the corresponding elements of the difference set $\{e_1, e_2, \ldots e_{2k-1}, J\}$.

Lemma 2.4. *Each* $PC(2k)$ *is 4-chromatic. Furthermore we have* $PC(2k+2) \to PC(2k)$.

Proof. Let u be a vertex of $PC(2k+2)$ and let ψ_{2k} be the projection of u into its first $2k$ coordinates and let ψ_2 be the projection on the last two coordinates. Then define $\phi : V(PC(2k+2)) \to V(PC(2k))$ as follows:

$$\phi(u) = \begin{cases} \psi_{2k}(u) & \text{if } \psi_2(u) \in \{00, 11\} \\ \psi_{2k}(u) + J & \text{if } \psi_2(u) \in \{01, 10\} \end{cases}$$

We note that in the second case J is the all 1 vector in $\mathbb{Z}_2^{2k}$. It is easy to check that ϕ is a homomorphism. Thus, by transitivity, every $PC(2k)$ admits a homomorphism to $PC(2)$. Since this graph is isomorphic to K_4, we obtain a 4-colouring of $PC(2k)$. That $PC(2k)$ is not 3-colourable follows from the fact that it contains a generalized Mycielski graph as a subgraph; we refer to [P92] for details. $\square$

3. Bounding planar graphs by Projective Cubes

Our main purpose of this work is to introduce the following question which, surprisingly, captures many well-known theorems and conjectures and directs us toward new lines of research on the study of colouring planar graphs.

Problem 3.1. Given integers $l \geq k \geq 1$, what is the smallest subgraph of $PC(2k)$ to which every planar graph of odd-girth $2l+1$ admits a homomorphism?

Intuitively speaking one could see a planar graph of a given odd-girth as an approximation or partition of the plane with certain properties. The question is what is the smallest section of the projective cube that this approximation or partition can be encoded to?

For a few choices of k and l we know the answer or we have a conjectured answer:

Case $k = l = 1$. In this case $PC(2)$, i.e., K_4 itself is a planar core and thus no proper subgraph could be an answer. By the virtue of the four colour theorem we know that $PC(2)$ is an answer.

Case $k = 1$**,** $l \geq 2$. Since $\mathcal{P}_{2l+1}$ contains graphs that are not bipartite, the smallest answer could be K_3. This is indeed the answer by Grötzsch's Theorem.

Case $k = l$. In this case we have introduced the following conjecture:

Conjecture 3.2. The class $\mathcal{P}_{2k+1}$ of planar graphs of odd-girth $2k+1$ is bounded by $PC(2k)$, the projective cube of dimension $2k$. Furthermore, $PC(2k)$ is the smallest graph of odd-girth $2k+1$ which bounds $\mathcal{P}_{2k+1}$.

We note that in this conjecture we claim $PC(2k)$ is generally the smallest graph of odd-girth $2k+1$ which bounds $\mathcal{P}_{2k+1}$. However it follows from [N03] that if B is a minimal graph of odd-girth $2k+1$ bounding $\mathcal{P}_{2k+1}$, then the minimum degree of B is at least $2k+1$. Thus if the first part of Conjecture 3.2 (that $PC(2k)$ bounds $\mathcal{P}_{2k+1}$) is true, then $PC(2k)$ itself is the only subgraph that will work. This first part of the conjecture is equivalent to a conjecture of P. Seymour in the theory of edge-colouring of planar graphs, see [N07]. Thus the answer to this question in this particular case would solve the problem of finding the edge chromatic number of planar graphs. On the other hand this special case is also related to other conjectures such as the cycling conjecture on clutters, see [G05].

As mentioned the case $k = 1$ of the previous conjecture is equivalent to the four colour theorem. The first part of the case $k = 2$ of this conjecture follows from the four colour theorem as shown in [N07]. In the next section we prove the second part of the conjecture for this case. More precisely we prove that if a triangle-free graph bounds $\mathcal{P}_5$, then it has at least 16 vertices.

Case $l = k+1$. For this case we introduce the following conjecture:

Conjecture 3.3. For $l = k+1$ the smallest subgraph of $PC(2k)$ to which every planar graph of odd-girth $2l+1$ admits a homomorphism is the Kneser graph $K(2k+1, k)$.

This conjecture is related to the study of the fractional chromatic number of planar graphs of a given odd-girth. We have conjectured ([N09]) that the fractional chromatic number of planar graphs of odd-girth $2l+1$ is bounded by $2 + \frac{1}{l-1}$ and we have shown that this is the best possible, if possible at all.

The case $k = 1$ of Conjecture 3.3 is obtained by Grötzsch's Theorem. For $k = 2$ the best known result is that of Dvořák, Škrekovski and Valla, [DSV08]. They prove that the Petersen graph (i.e., $K(5,2)$) bounds $\mathcal{P}_9$, while our conjecture claims that the Petersen graph bounds the larger family $\mathcal{P}_7$.

Case $l \geq 2k$. The smallest subgraph of the projective cube $PC(2k)$ which is not bipartite is C_{2k+1}. It is a classical result that when l is much larger than k, then C_{2k+1} is the answer to Problem 3.1. Zhang in [Zh02] conjectured that C_{2k+1} is the answer to this problem as soon as $l \geq 2k$. This is related to the theory of flows and is a strengthening of a dual statement of a conjecture of Jaeger on the existence of a certain kind of flows (Jaeger's conjecture is for general graphs but to use the duality one must restrict it to planar graphs only). To our knowledge the best known result in this direction is that of X. Zhu who proved in [Zh02] that for $l \geq 4k - 2$, C_{2k+1} bounds $\mathcal{P}_{2l+1}$. For $k = 2$ and $l \geq 5$, M. DeVos and A. Deckelbaum claim to have a proof that C_5 is the answer.

For any other values of k and l, the problem, to our knowledge, is a new problem whose study may lead to new concepts in the theory of homomorphisms of planar graphs.

4. Bounding $\mathcal{P}_5$

We have shown previously that $PC(4)$, also known as the Clebsch graph (or the Greenwood-Gleason graph in some other literature), bounds $\mathcal{P}_5$. Two different representations of $PC(4)$ are shown in Figure 1. We have proved in [N03] that $\chi(\mathcal{P}_5^3) \geq 15$. Here we show that any triangle-free bound for $\mathcal{P}_5$ has at least 16 vertices. Since a minimal triangle-free bound for P_5 has minimum degree at least 5 (see Fact 2 in the proof of the following theorem) it turns out that $PC(4)$ is the smallest triangle-free bound for $\mathcal{P}_5$ both in terms of number of vertices and number of edges.

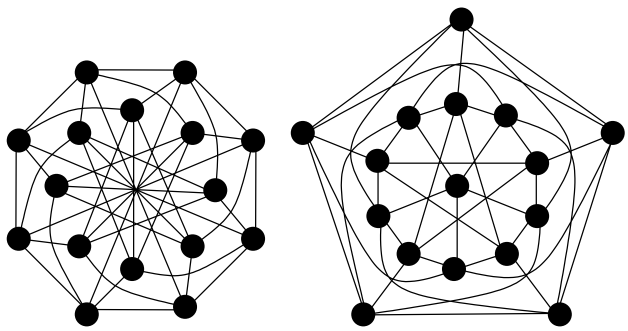

Figure 1: Two different presentations of the Clebsch graph

Theorem 4.1. *Every triangle-free bound for $\mathcal{P}_5$ has at least 16 vertices.*

Proof. Let B be a minimal triangle-free bound for $\mathcal{P}_5$, i.e., every triangle-free planar graph admits a homomorphism to B but for every proper subgraph B' of B, with fewer number of vertices, there is a triangle-free planar graph $G_{B'}$ that does not map to B'. Taking the union of all $G_{B'}$ we find a triangle-free planar graph G_B that maps to B by the choice of B but any such mapping must be onto. We note that one could take G_B to be connected by identifying some vertices from disconnected components.

Since B is triangle-free, B^3 is a loopless graph and since B bounds $\mathcal{P}_5$, B^3 bounds $\mathcal{P}_5^3$ but we have already proved in [N03] that $\chi(\mathcal{P}_5^3) \geq 15$. Thus to prove the theorem we must show that B cannot be on 15 vertices. Towards a contradiction, from here on we assume B is a triangle-free bound of $\mathcal{P}_5$ on 15 vertices. We collect some facts about B and we put them together to finally get a contradiction.

Fact 1. *B is a core.*

This is immediate from the fact that B is a minimal bound.

The next fact was implicitly proved in [N03], but we include a proof for the sake of completeness.

Fact 2. *$\delta(B) \geq 5$.*

We use the graph of Figure 2 as a gadget to prove this fact. We first note that there is a walk of length 3 between every pair of the five emphasized vertices of the graph of Figure 2.

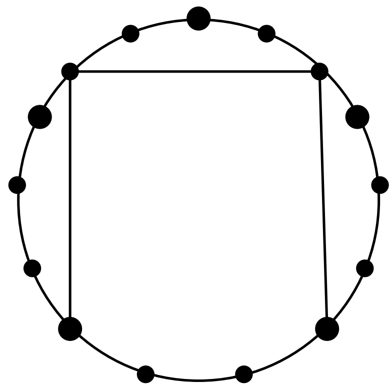

Figure 2: A triangle-free outer planar graph

Let G be a graph obtained from G_B by adding a copy H_v of the graph of Figure 2 for each vertex v of G_B and joining v to five emphasized vertices of H_v. We note that G is also in P_5 and admits a homomorphism ϕ to B by the choice of B. The restriction of ϕ on G_B is an onto homomorphism of G_B to B. Thus each vertex x of B is an image of a vertex v of G_B in G. But then the five emphasized vertices of H_v in G must map to five distinct neighbours of x in B, proving that $d_B(x) \geq 5$.

Fact 3. *For each vertex x of B, the non-neighbours of x induce a subgraph which contains a matching of size at least 3.*

To show this we use the graph of Figure 3. We use H to denote the graph of Figure 3 till the end of proof of this fact. We note that $H \in \mathcal{P}_5$ and that $H^3 \cong K_{11}$. Thus any homomorphic image of H in B is

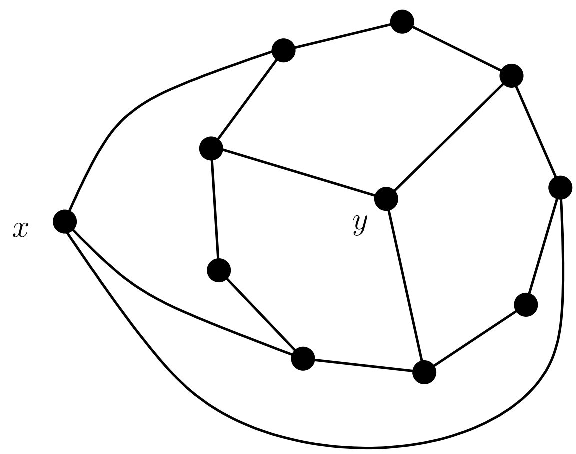

Figure 3: An 11 clique in $\mathcal{P}_5^3$

isomorphic to H. We further note that beside xy any other edge of $\overline{H}$ incident to x or y if added to H would create a triangle. Finally if we delete x, y and the neighbours of x from H, the remaining graph, call it H_x, is a matching of size 3. Now as in the previous case we build a new graph G from G_B by adding a disjoint copy H_v of H for each vertex v of G_B and then identifying the copy of x in H_v with v. Since $G \in \mathcal{P}_5$, there is a homomorphism φ of G to B which must be onto on the subgraph G_B of G, by the choice of G_B. For an arbitrary vertex u of B let v be a vertex in G_B such that $\varphi(v) = u$. The subgraph isomorphic to H_x in H_v is a matching of size 3 in the subgraph induced by the non-neighbours of u in B.

Fact 4. $\Delta(B) = 6$.

It follows from Fact 3 that each vertex of B has degree at most 8. Let u be a vertex of maximum degree in B and suppose $d_B(u) = i$, $7 \leq i \leq 8$. Let X be the set of neighbours of u and Y be the set of non-neighbours of u. Then $|Y| = 14 - i$ and, again by Fact 3, the graph induced by Y has a 3-matching. Since B is triangle-free and has minimum degree at least 5, any vertex in X must be joined to at least 4 vertices in Y. But it cannot be adjacent to more than 3 vertices of the matching induced by Y, thus Y has at least 7 vertices. However, if it has exactly 7 vertices, then the only vertex not in the induced 3-matching, say u', must be connected to all the vertices of X. Now the mapping of $u \to u'$ (every other vertex is fixed) induces a homomorphism of B to a proper subgraph of itself contradicting the Fact 1. On the other hand by Fact 2 every vertex of B has minimum degree at least 5. Since there are no 5-regular graph on 15 vertices (implied by the first theorem of graph theory), we have $\Delta(B) = 6$.

Fact 5. *The subgraph of B induced by the set of non-neighbours of a vertex x contains a 5-cycle.*

We use the graph of Figure 4 to prove this fact. Till the end of proving this fact H will denote the graph of Figure 4. Since $H \in \mathcal{P}_5$ we have $H \to B$. Let ψ be any mapping of H to B. The central vertex x of this

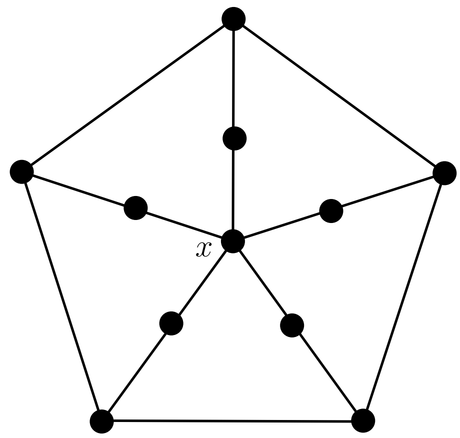

Figure 4: Forcing a 5-cycle in the non-neighbours

graph is in a 5-cycle with every other vertex, thus its image in B is distinct from the image of all the other vertices. For the same reason the five vertices of the outer face are also mapped to five distinct vertices of B under ψ and the image is also an induced 5-cycle. Let y be a vertex of the outer face. Since the image of y is distinct from that of x, the 2-path connecting them must also map to a 2-path. Thus if $\psi(y) \sim \psi(x)$, then we have a triangle in G, a contradiction. We now build a graph G from G_B by adding a disjoint copy H_v of H and identifying the vertex x in this copy with v. The proof of this fact now follows from considering a mapping of G to B; the rest of the proof is similar to the proof of the previous facts and we leave it as an exercise.

Let u be a vertex of degree 6 in B. Let $N(u)$ be its six neighbours and $\overline{N}(u)$ be its eight non-neighbours. By Fact 5, the subgraph induced by $\overline{N}(u)$ contains a 5-cycle. We denote this cycle by C_u and we use x, y, z to denote the three vertices of $\overline{N}(u)$ that are not in C_u.

Fact 6. *x, y, z induce an independent set.*

By contradiction suppose xy is an edge of B. By Fact 2 and since B is triangle-free, each vertex v in $N(u)$ must be joined to at least four vertices in $\overline{N}(u)$. Since v can only be joined to two vertices of C_u, it must be joined to at least two of x, y and z. But it cannot be joined to both x and y, thus it is joined to z. This means every neighbour of u is a neighbour of z thus the mapping that maps u to z and fixes all other vertices is a homomorphism of B to a proper subgraph of itself, contradicting Fact 1.

Fact 7. *Each of x, y, z is adjacent to exactly 4 vertices of $N(u)$.*

To prove this we first show that x cannot be adjacent to five or six vertices in $N(u)$. By contradiction suppose x is adjacent to five vertices of $N(u)$ and let v be the only vertex in $N(u)$ which is not adjacent to x. We note that v must be adjacent to both y and z. Since mapping x to u (while fixing all the other vertices) is not a homomorphism x must be adjacent to another vertex, and since x, y, z induce an independent set, x must be adjacent to a vertex of C_u, we call it a. Thus of the vertices in $N(u)$, v is the only vertex that could be adjacent to a. Hence to complete the degree of a to a minimum of 5 it must be adjacent to at least one of y or z and, therefore, a is not adjacent to v either. Since a is not adjacent to any neighbour of u, by adding the edge au to B we get a new triangle-free graph B' on fifteen vertices which contains B and, therefore, bounds $\mathcal{P}_5$. But B' has a vertex of degree 7. A contradiction can now be obtained as in Fact 4 (we note that we did not use edge minimality in the proof of Fact 4).

Now since each vertex in $\{x, y, z\}$ is adjacent to at most four vertices in $N(u)$ (twelve edges at most) and since each vertex in $N(u)$ is adjacent to at least two vertices in $\{x, y, z\}$ (twelve edges at least), it must be the case that each vertex in $\{x, y, z\}$ is adjacent to exactly four vertices in $N(u)$ and each vertex in $N(u)$ is adjacent to exactly two vertices in $\{x, y, z\}$. Thus the graph of Figure 4 presents a partial structure of B.

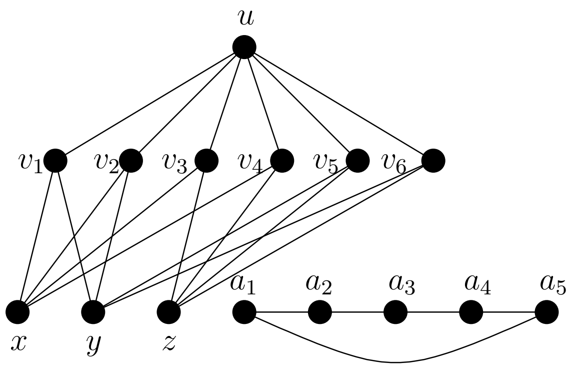

Figure 5: Forced structure of a bound on 15 vertices

Now to get the final contradiction we assume x is adjacent to a vertex a_1 of C_u (else mapping x to u and fixing other vertices is a homomorphism, contradicting Fact 1) . Then each v_i ($i = 1, 2, 3, 4$) is either adjacent to both a_2, a_4 or to both a_3, a_5. Furthermore v_1 and v_2 must be adjacent to different pairs of vertices in C_u (else they have identical neighbours and one could identify them). Similarly v_3 and v_4 are adjacent to different pairs of vertices in C_u. Now y and z cannot be joined to any vertex of C_u but a_1. Thus a_1 is not adjacent to any vertex in $N(u)$. We now get a new triangle-free bound for $\mathcal{P}_5$ by adding the edge ua_1. This new graph has a vertex of degree 7 which contradicts Fact 7. □

5. Concluding Remarks

1. Though $PC(k)$ is a bipartite graph and the ordinary homomorphism problems into these graphs is trivial, the story changes when we consider them as signed graphs. In this regard an analogous of Conjecture 3.2 for bipartite signed planar graphs is introduced in [G05]. We hope to write more on this subject in the future.

2. Conjecture 3.2 can be viewed as an optimal answer (in terms of order of the bounds) for Theorem 1.1. In this regard it is interesting to know that even formulating a conjecture in finding the optimal bound for $Forb_h(\{C_{2k+1}\}) \cap Forb_m(K_4)$ has proved to be challenging. For $k = 1$ and for $k = 2$, the triangle and the Wagner graph are the answers respectively. For $k = 3$, an optimal answer on 14 vertices is found recently in a joint work with F. Foucaud.

3. Given a class $\mathcal{G}$ odd-girth at least $2l + 1$ and $l >$ we define $\chi(\mathcal{G}^{2k+1}) = \max\{\chi(G^{2k+1}) | G \in \mathcal{G}\}$. Another problem related to Problem 3.1 is to ask: what is the exact value of $\chi(\mathcal{P}_{2l+1}^{2k+1})$ for $l > k$? The number of vertices of the subgraph of the projective cube found by the answer to Problem 3.1 is an upper bound on this question. But we do not know if this gives the exact value of $\chi(\mathcal{P}_{2k+1}^{2l+1})$. In particular while we proved in this paper that the smallest triangle-free graph bounding the class of all triangle-free graphs has 16 vertices, we do not know if $\chi(\mathcal{P}_5^3)$ =15 or 16.

4. Finally we would like to point out that our proof is constructive, i.e., if we follow the steps of the proof, starting with a triangle-free planar graph G with $\chi(G^3) \geq 15$, which is constructed in [N09], we build a triangle-free planar graph whose triangle-free images each have at least 16 vertices. Smaller examples of this type may help studying the questions we asked in this paper.

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