Walk-powers and homomorphism bound of planar signed graphs

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Abstract

As an extension of the Four-Color Theorem it is conjectured by the first author that every planar graph of odd-girth at least $2k+1$ admits a homomorphism to the projective cube of dimension $2k$, i.e., the Cayley graph $\mathcal{PC}(2k) = (\mathbb{Z}_2^{2k}, \{e_1, e_2, \cdots, e_{2k}, J\})$ where e_i's are the standard basis vectors of $\mathbb{Z}_2^d$ and J is the all 1 vector. Noting that $\mathcal{PC}(2k)$ itself is of odd-girth $2k+1$, in this work we show that if the conjecture is true, then $\mathcal{PC}(2k)$ is an optimal such graph both with respect to the number of vertices and the number of edges. The result is obtained using the notion of walk-power of graphs and their clique numbers.

An analogous result is proved for bipartite signed planar graphs of unbalanced-girth $2k$. The work is presented in the uniform framework of planar consistent signed graphs.

1 Introduction

The *projective cube* of dimension d, denoted $\mathcal{PC}(d)$, is defined to be the graph obtained from the hypercube of dimension $d+1$ by identifying antipodal vertices. It is easy to show that $\mathcal{PC}(d)$ is the Cayley graph $(\mathbb{Z}_2^d, \{e_1, e_2, \cdots, e_d, J\})$ where J is the all 1 vector.

It is conjectured by the first author that:

Conjecture 1. *[Na07] Every planar graph of odd-girth at least $2k+1$ admits a homomorphism to $\mathcal{PC}(2k)$.*

Recall that homomorphism of graphs is an edge-preserving mapping of vertices. Since $\mathcal{PC}(2)$ is isomorphic to K_4, the case $k=1$ of the conjecture is a restatement of the Four-Color Theorem and thus the conjecture is one of the venues to extend the most well known theorem of the theory of graphs.

The conjecture was motivated by a question of J. Nešetřil who asked in [Ne99]: is there a triangle-free graph to which every triangle-free planar graph admit a homomorphism?

This question was settled in a more general setting by P. Ossona de Mendez and J. Nešetřil:

Theorem 2. *[NO08] Given a set $\mathcal{H} = \{H_1, H_2, \cdots, H_k\}$ of connected graphs and a minor closed family $\mathcal{C}$ of graphs, there exist a graph $F(\mathcal{C}, \mathcal{H})$ such that (i) no H_i, $i = 1, 2, \cdots, k$, admits a homomorphism to $F(\mathcal{C}, \mathcal{H})$, (ii) any member of $\mathcal{C}$ which does not admit a homomorphism from any H_i admits a homomorphism to $F(\mathcal{C}, \mathcal{H})$.*

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An optimization question that follows is: what is the smallest possible order of $F(\mathcal{C}, \mathcal{H})$?

This question captures a large number of coloring problems in minor closed families. One of the most notable such questions is the Hadwiger conjecture which claims: every graph with no K_n minor is $(n-1)$-colorable. If we let $\mathcal{C}$ to be the class of graphs with no K_n-minor and $\mathcal{H} = \{K_n\}$, then the Hadwiger conjecture is equivalent to saying that the smallest possible order of $F(\mathcal{C}, \mathcal{H})$ is $n-1$.

When C is the class of planar graphs and $\mathcal{H} = \{C_{2k-1}\}$, since $\mathcal{PC}(2k)$ has odd-girth $2k+1$ (see [Na07]), Conjecture 1 suggests that projective cubes are examples of $F(\mathcal{C}, \mathcal{H})$. Furthermore, it is conjectured (and proved for special case) by the first author [Na13] that if the conjecture holds, then projective cubes give the smallest possible order and size of an $F(\mathcal{C}, \mathcal{H})$.

In this work we prove this latter claim. We also prove a similar result for the bipartite analogue of Conjecture 1, introduced by B. Guenin [Gu05]. Our work is presented in the general frame work of homomorphisms of (consistent) signed graphs, thus we shall introduce some notations, terminologies and background.

2 Notations and terminologies

For standard notations and terminologies of graph theory we follow [BM76]. We denote the clique number and minimum degree of a graph G by $\omega(G)$ and $\delta(G)$, respectively. The *length* of a cycle, a path or a walk is its number of edges. The *odd-girth* of a graph is the length of its shortest odd-cycle. A *thread* of a graph G is a path $P = uv_1v_2 \dots v_rw$, for some $r \geq 1$, where u, v have degree at least 3 and $v_1, v_2, \dots, v_r$ have degree exactly 2 in G.

2.1 Signed graphs

A *signed graph* is a graph where each edge is assigned one of the two signs *positive* $(+)$ and *negative* $(-)$. Assuming Σ is the set of negative edges we will use (G, Σ) to denote a signed graph. A notion of essential importance for signed graphs is the notion of *unbalanced cycle*, that is a cycle with an odd number of negative edges. A subgraph with no unbalanced cycle is called a *balanced* subgraph. In particular a *balanced cycle* is a cycle with even number of negative edges. The *unbalanced-girth* of a signed graph is the length of a shortest unbalanced cycle.

Another notion of importance is the notion of *re-signing*; that is to switch the signs of all edges in an edge-cut of G. Given two signatures Σ_1 and Σ_2 of G, we say Σ_2 is *equivalent* to Σ_1 if it is obtained from a re-signing of Σ_1. In such a case we may also say (G, Σ_1) is equivalent to (G, Σ_2). It is easily observed that re-signing does not change balance of a cycle. A stronger relation is provided by the following theorem of Zaslavsky:

Theorem 3 (Zaslavsky [Za82])**.** *Two signed graphs (G, Σ_1) and (G, Σ_2) are equivalent if and only if they have the same set of balanced (equivalently, unbalanced) cycles.*

The set of all equivalent signed graphs will be denoted by $[G, \Sigma]$ where (G, Σ) is any member of the class and will be called *switch class.*

In this work we will only consider the subclasses of *consistent* signed graphs which itself consist of two subclasses. That is the class of signed graphs in which every balanced cycle is of even length and all unbalanced cycles are of a same parity. Thus there are two types of consistent signed graphs [NRS13]:

(i) when all unbalanced cycles are of odd length, in this case by Theorem 3 (G, Σ) is equivalent to $(G, E(G))$, such a signed graph will be referred to as an *anti-balanced signed graph.*

(ii) when all unbalanced cycles are of even length, this would be the case if and only if G is bipartite, such a signed graph thus will be referred to as a *signed bipartite graph*.

2.2 Homomorphisms and bounds

Given two switch classes $[G, \Sigma]$ and $[H, \Pi]$, a homomorphism of $[G, \Sigma]$ to $[H, \Pi]$ is a mapping ϕ of $V(G)$ to $V(H)$ such that for some representations (G, Σ') and (H, Π') of $[G, \Sigma]$ and $[H, \Pi]$ (respectively) the mapping ϕ preserves both adjacency and signs of edges (with respect to Σ' and Π'). While the existence of such a mapping depends on the choice of Σ', it is independent of the choice of Π'. We say there is a homomorphism of (G, Σ) to (H, Π) and write $(G, \Sigma) \to (H, \Pi)$ if there exist a homomorphism of $[G, \Sigma]$ to $[H, \Pi]$.

The length of shortest (even or odd) unbalanced cycle provides the first no homomorphism lemma:

Lemma 4. *[NRS13] Given two unbalanced cycles UC_r and UC_l we have $UC_r \to UC_l$ if and only if*

- $r \equiv l \pmod 2$,
- $r \geq l$.

We note that for consistent signed graphs, which are the main focus of this work, we only have one parity for unbalanced cycles.

Given a class $\mathcal{C}$ of signed graphs, a signed graph (H, Π) is said to *bound* $\mathcal{C}$ if for every member (G, Σ) of $\mathcal{C}$ we have $(G, \Sigma) \to (H, \Pi)$. It implies from Lemma 4 that in such a case (odd and even) unbalanced-girth of each member (G, Σ) of $\mathcal{C}$ is bounded below by (odd and even) unbalanced-girth of (H, Π).

2.3 Signed projective cubes

Recall that the *projective cube* of dimension d, denoted $\mathcal{PC}(d)$, is the Cayley graph $(\mathbb{Z}_2^d, \{e_1, e_2, \cdots, e_d, J\})$ where e_i's are the standard basis vectors of $\mathbb{Z}_2^d$ and J is the all 1 vector.

We define the *signed projective cube* of dimension d, denoted $\mathcal{SPC}(d)$, to be the signed graph obtained from $\mathcal{PC}(d)$ by assigning $+$ to each edge corresponding to an e_i vector and $-$ to each edge corresponding to the J vector. In other words $\mathcal{SPC}(d)$ is obtained from the hypercube of dimension d with all edges assigned positive sign and then adding a negative edge between each pair of antipodal vertices.

Projective cubes, also known as folded cubes, are well-studied graphs. We refer to [NRS13] and references there for some properties of signed projective cubes and for proofs of the following two theorems:

Theorem 5. *The signed projective cube of dimension d is a consistent signed graph and has unbalanced-girth $d + 1$.*

It follows that if a signed graph admits a homomorphism to a signed projective cube, it must be a consistent signed graph. The existence of such a homomorphism then becomes equivalent to a packing problem as the following theorem claims:

Theorem 6. *A consistent signed graph (G, Σ) admits a homomorphism to $\mathcal{SPC}(d)$ if and only if the edge set of G can be partitioned into $d + 1$ disjoint sets each of which induces a signature equivalent to Σ.*

The following conjecture, introduced in [Na07] and [Gu05] (see also [NRS12]) is the focus of this work:

Conjecture 7. *Given $d \geq 2$, every planar consistent signed graph of unbalanced-girth $d+1$ admits a homomorphism to $\mathcal{SPC}(d)$.*

Using the folding lemmas of [KZ00, NRS12] one can replace the condition of unbalanced-girth exactly $d+1$ with unbalanced-girth at least $d+1$ as long we remain in the same type of consistent planar signed graphs as the signed projective cube. Thus the conjecture is formed of two parts: for even values of d (by considering the signature in which all edges are negative) it claims that every planar graph of odd-girth at least $d+1$ admits a homomorphism to $\mathcal{PC}(d)$. For odd values of d it claims that every planar signed bipartite graph of unbalanced-girth at least $d+1$ admits a homomorphism to $\mathcal{SPC}(d)$. Since $\mathcal{PC}(2)$ is isomorphic to K_4, the very first case of this conjecture is the Four-Color Theorem.

2.4 Walk-power

We will use two notions of graph powers, one for each type of consistent signed graphs. Since the homomorphism of anti-balanced signed graphs are reduced to graph homomorphism problems, we use the terminology of graphs for this case.

Given a graph G and a positive integer k, we define the k-th walk-power of G, denoted $G^{(k)}$, to be the graph whose vertex set is also $V(G)$ with two vertices x and y being adjacent if there is a walk of length k connecting x and y in G. Assuming G has at least one edge $G^{(k)}$ is loopless if and only if k is odd and G has odd-girth at least $k+2$. As an example we have:

Lemma 8. *We have $\mathcal{PC}_{2d}^{(2d-1)} \cong K_{2^{2d}}$.*

Proof. It follows from the fact that each pair of vertices of $\mathcal{PC}_{2d}$ belong to a cycle of length $2d+1$ (see for example [NRS12]). $\square$

A property of walk-power, which is important for our work, is that:

Lemma 9. *If ϕ is a homomorphism of a graph G to a graph H, then ϕ is also a homomorphism of $G^{(r)}$ to $H^{(r)}$ for any positive integer r.*

Note that for odd values of r if we consider a pair x, y of adjacent vertices in $G^{(r)}$ and identify them in G, then there will be a cycle of odd-length at most r in the resulting graph. This is a key tool for us and we define power of signed bipartite graph to have an analogous property. For this case we shall use the notion of unbalanced cycles instead of odd-cycles.

Given a signed bipartite graph (G, Σ) and an even integer $r \geq 2$ we define $(G, \Sigma)^{(r)}$ to be a graph (not signed) on vertex set $V(G)$ where vertices x and y are adjacent if the following two conditions satisfy:

- x and y are in a same part of bipartite graph G,
- if x and y are identified in (G, Σ), then there will be a (new) unbalanced cycle of (even) length at most r.

Note that second condition is equivalent to saying that there are x, y-paths P_1 and P_2 (connecting x and y), each of length at most r, such that one has an odd number of negative edges and the other has an even number of negative edges.

Using the fact that each pair of vertices from the same part in $\mathcal{SPC}_{2d+1}$ belong to an unbalanced cycle of length $2d+2$ we have the following bipartite analogue of Lemma 8.

Lemma 10. *We have* $\mathcal{SPC}^{(2d)}_{2d+1} \cong 2K_{2^{2d}}$.

Here $2K_{2^{2d}}$ means two disjoint copies of $K_{2^{2d}}$. The homomorphism property also holds the same:

Lemma 11. *Given a positive integer* r*, if* ϕ *is a homomorphism of a signed bipartite graph* (G, Σ) *to a singed bipartite graph* (H, Π)*, then* ϕ *is also a homomorphism of the graph* $(G, \Sigma)^{(2r)}$ *to the graph* $(H, \Pi)^{(2r)}$.

2.5 In this paper

We show that if Conjecture 7 holds, then the proposed projective cube is an optimal bound of the given unbalanced girth both in terms of number of vertices and number of edges. More precisely we prove the following.

Theorem 12. *If* (B, Ω) *is a consistent signed graph of unbalanced-girth* d *which bounds the class of consistent signed planar graphs of unbalanced-girth* d*, then* B *has at least* 2^{d-1} *vertices. Furthermore, if no subgraph of* (B, Ω) *bounds the same class, then minimum degree of* B *is at least* d*, and therefore,* B *has at least* $d2^{d-2}$ *edges.*

The first part of this theorem will follow from the following theorems (to be proved in the next two sections) and Lemmas 8, 9, 10 and 11.

Theorem 13. *There exists a planar graph* G *of odd-girth* $2k+1$ *with* $\omega(G^{(2k-1)}) \geq 2^{2k}$.

Theorem 14. *There exists a planar signed bipartite graph* (G, Σ) *of unbalanced-girth* $2k$ *for which there are two cliques of order* 2^{2k-2} *in* $(G, \Sigma)^{(2k-2)}$*, one for each part (induced by the bipartition) of* G.

Our proof of both theorems are constructive and we provide a concrete construction.

3 Bounding anti-balanced signed graphs

In this section we prove Theorem 13. Since this is for anti-balanced signed graphs, the homomorphism problem is equivalent to the homomorphisms of graphs. Thus, we will use the terminology of graphs rather than signed graph in this section.

As mentioned, our proof is constructive and we will build an example of a planar graph G of odd-girth $2k+1$ for which we have $\omega(G^{(2k)}) \geq 2^{2k}$. The construction is based on the following local construction.

Lemma 15. *Let* G *be the graph obtained by subdividing edges of* K_4 *such that in a planar embedding of* G *each of the four faces is a cycle of length* $2k+1$*. Then* $G^{(2k-1)}$ *is isomorphic to* K_{4k}.

Proof. Let a, b, c and d be the original vertices of the K_4 from which G is constructed. For $x, y \in \{a, b, c, d\}$ let P_{xy} be the subdivision of xy, and let t_{xy} be the length of this path. For an internal vertex v of P_{xy}, let P_{xv} (or P_{vx}) be the part of P_{xy} connecting v to x. Let t_{xv} be the length of P_{xv}. We have

$$\begin{aligned} t_{ab} + t_{bc} + t_{ca} &= t_{ab} + t_{bd} + t_{da} \\ &= t_{ac} + t_{cd} + t_{da} \\ &= t_{bc} + t_{cd} + t_{db} \\ &= 2k + 1. \end{aligned} \tag{1}$$

From Equation (1) we have

$$t_{xy} = t_{wz} \text{ for } \{x, y, w, z\} = \{a, b, c, d\}, \tag{2}$$

that is to say that if all four faces have the same length, then parallel edges of K_4 are subdivided the same number of times (the parity of the length of the faces is not important here and we will use this fact later to prove the analogous lemma, Lemma 17, for the bipartite case).

It is easy to check that G has $4k$ vertices. We will show that for every pair u, v of vertices of G there is a walk of length $2k - 1$ connecting u and v. If u and v are both vertices of a facial cycle of G, then we are done as each facial cycle is of length $2k + 1$. If there is no facial cycle of G containing both u and v, then they are internal vertices (after subdivision) of two distinct parallel edges of K_4, thus we may assume, without loss of generality, that u is a vertex of the path P_{ab} and v is a vertex of the path P_{cd}.

Note that by Equation (2) we have

$$\begin{aligned} t_{au} + t_{bu} &= t_{cv} + t_{dv} \\ &= t_{ab} = t_{cd}. \end{aligned} \tag{3}$$

If $t_{ab} = t_{cd}$ is even (odd respectively), then t_{au} and t_{bu} have the same parity (different parities respectively) and t_{cv} and t_{dv} have the same parity (different parities respectively). Moreover, since t_{cd} is even (odd respectively) and $t_{ac} + t_{cd} + t_{da} = 2k + 1$, t_{ac} and t_{ad} have different parities (same parity respectively).

Now one of the paths connecting u, v, say $P_{ua} \cup P_{ac} \cup P_{cv}$, is of length $t_{au} + t_{ac} + t_{cv}$, and another path, say $P_{ub} \cup P_{bd} \cup P_{dv}$, is of length $t_{bu} + t_{bd} + t_{dv}$. By (3) we have $(t_{bu} + t_{bd} + t_{dv}) + (t_{au} + t_{ac} + t_{cv}) = 2(t_{ab} + t_{bd})$, hence $t_{bu} + t_{bd} + t_{dv}$ and $t_{au} + t_{ac} + t_{cv}$ have the same parity. Furthermore, since $P_{ab} \cup P_{ad} \cup P_{bd}$ forms a facial cycle we have $t_{ab} + t_{ad} + t_{bd} = 2k + 1$, thus $2(t_{ab} + t_{bd}) = 4k + 2 - 2t_{ad} \leq 4k$.

Hence we have $min\{(t_{au} + t_{ac} + t_{cv}), (t_{bu} + t_{bd} + t_{dv})\} \leq 2k$. Similarly, we can show that $min\{(t_{au} + t_{ad} + t_{dv}), (t_{bu} + t_{bc} + t_{cv})\} \leq 2k$.

But note that $min\{(t_{au} + t_{ac} + t_{cv}), (t_{bu} + t_{bd} + t_{dv})\}$ and $min\{(t_{au} + t_{ad} + t_{dv}), (t_{bu} + t_{bc} + t_{cv})\}$ have different parities irrespective of the parity of $t_{ab} = t_{cd}$. Therefore, there is a walk of length $2k - 1$ from u to v. $\square$

The subdivided K_4 where two parallel edges are subdivided $2k - 1$ times will be the base of our construction. Next we will use two operations to enlarge this construction.

Operation copy threads: Let G be a graph and $\mathcal{P} = \{P_1, P_2, \cdots, P_k\}$ be a set of threads of G. For each thread $P_i = xv_1v_2 \cdots v_ry$ in P add a new thread $P'_i = xv'_1v'_2 \cdots v'_ry$ where all the internal vertices are new and distinct. Denote the new graph by $CT(G)$. Let W be a clique in $G^{(l)}$ on vertex set $V(W)$. Consider a set U of vertices of $CT(G)$ which consists of $V(W)$ and

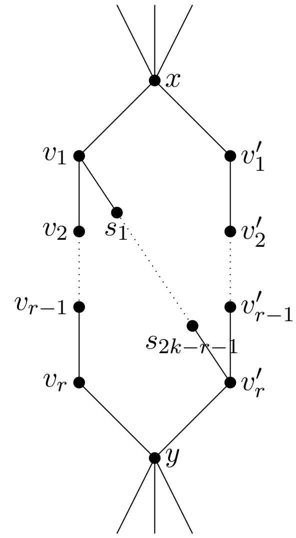

Figure 1: Copy and shortening of a thread.

copy vertices v' for each degree (in G) two vertex v of $V(W)$. It is now easy to check that the subgraph W' of $CT(G)^{(l)}$ induced by U is a complete graph minus a matching. To be precise the missing matching is between pairs v, v' where v is a degree two vertex of G and v' is its copy. Furthermore, it is easy to check that, if P_i belongs to a cycle of length x in G, then P_i' belongs to a cycle of length x in $CT(G)$.

Next we want to introduce an operation which will complete W' into a complete graph.

Operation shorten threads: Let G be a graph. Consider a collection $\mathcal{P}$ of threads of G and let $CT(G)$ be the graph obtained after the operation copy thread with respect to $\mathcal{P}$. For $P \in \mathcal{P}$ let P' be its copy in $CT(G)$. Suppose P is of length $r+1$ ($1 \leq r \leq 2k-2$) with x and y its end vertices and with $v_1, v_2 \cdots v_r$ as its internal vertices. Let $v_1', v_2' \cdots v_r'$ be the internal vertices of P'. Add a new path to $CT(G)$ of length $2k-r$ which connects v_1 and v_r' (all internal vertices are new and distinct). The new graph obtained after repeating the process for all paths in $\mathcal{P}$ will be denoted by $ST(G)$. Note that the operation $ST(G)$ creates two shortened threads; $v_1v_2 \cdots v_ry$ a shortening of P and $xv_1'v_2' \cdots v_r'$ a shortening of P'. The length of each of the shortened threads is one less that the length of P.

See Figure 3 for presentation of the two operations. Note that if we start with a planar graph after the two operations mentioned above, what we get are also planar graphs. The next lemma is the key property of this operation.

Lemma 16. *Let G be of odd-girth $2k+1$ and let $\mathcal{P}$ be a collection of threads of G, each of them contained in a cycle of length $2k+1$. Let $ST(G)$ be the graph obtained after operations copy threads and shorten threads with respect to $\mathcal{P}$. Then $ST(G)$ is also of odd-girth $2k+1$.*

Proof. Note that a new cycle C' in $CT(G)$ must contain at least one copy thread P' of a thread P of G in it. If C' contains both P and P', then C' is formed of the union of the two and is of even length. Otherwise, by replacing each such copy thread P' by the corresponding original thread P we obtain a cycle C of G from C' such that the length of C is the same as the length of C'. As G has odd-girth at least $2k+1$, the cycle C' must either be of even length or have odd length at least $2k+1$. Thus, $CT(G)$ is also of odd-girth $2k+1$.

In $ST(G)$ the only other kinds of odd cycles with length less than $2k+1$ could be the ones created by the newly added threads. Let $P = v_1v_2\cdots v_r$ be a thread of G and $P' = v'_1v'_2\cdots v'_r$ be its copy thread. Furthermore, let $S = v_1s_1s_2\cdots s_{2k-r-1}v'_r$ be the newly added thread. Two cycles created by this are the cycles $v_1s_1s_2\cdots s_{2k-r-1}v'_ryv_rv_{r-1}\cdots v_2$ and $v_1s_1s_2\cdots s_{2k-r-1}$ $v'_rv'_{r-1}\cdots v'_1x$ (see Figure 3 for pictorial assistance). Each of these cycles are of length $2k+1$.

Clearly, if there is any other odd cycle with length less than $2k+1$ containing the thread S, there is one of the type: $C^* = xv_1s_1s_2\cdots s_{2k-r-1}v'_ryw_1w_2\cdots w_l$ where w_i's are not vertices of P or P'. Assume that C^* is an cycle of length at most $2k-1$ such that it contains minimum number of new threads (the ones added due to operation $ST(\cdot)$).

Since each thread of $\mathcal{P}$ is contained in a cycle of length $2k+1$, there exists a cycle $xv_1v_2\cdots v_ry$ $u_1\cdots u_{2k-r-1}$ in G of length $2k+1$ containing P. If we replace $v_1s_1s_2\cdots s_{2k-r-1}v'_r$ of C^* by $u_1\cdots u_{2k-r-1}$, then we get an odd closed walk of length less than $2k-1$ in $ST(G)$ that must contain an odd cycle of length less than $2k-1$ with less number of new threads than in C^*, a contradiction. □

Now we are ready to prove Theorem 13.

Proof of Theorem 13. Let G_0 be the graph obtained from K_4 by subdividing two parallel edges $2k-2$ times each. Note that the resultant is a graph of odd-girth $2k+1$ in which each face is of length $2k+1$. Thus by Lemma 15 the $(2k-1)$-th walk power of G_0 is a clique of order $4k$. Let $\mathcal{P}_0$ be the two threads of G_0 (each of length $2k-1$), note that each of them belongs to a cycle of length $2k+1$.

Starting form G_0 and $\mathcal{P}_0$ we will build a graph inductively in $2k-1$ steps as follows: given G_i and $\mathcal{P}_i$ we define G_{i+1} to be $ST(G_i)$ with respect to $\mathcal{P}_i$. We then define $\mathcal{P}_{i+1}$ to be collections of shortened threads and their copies. Thus $\mathcal{P}_{i+1}$ has twice as many elements as $\mathcal{P}_i$. Also, note that each thread in $\mathcal{P}_{i+1}$ has $2k-i$ vertices. Thus,

$$\begin{aligned}
\omega(G_{2k-2}^{(2k-2)}) &\geq 4k + \sum_{j=1}^{2k-2} 2^j(2k-j-1) \\
&= 4k + (2k-1)\sum_{j=1}^{2k-2} 2^j - 2\sum_{j=1}^{2k-2} j2^{j-1} \\
&= 4k + [(2k-1)(2^{2k-1}-2)] - \\
&\quad 2[(1-2^{2k-1}) - (-1)(2k-1)2^{2k-2}] \\
&= 4k + (k2^{2k} - 4k - 2^{2k-1} + 2) - \\
&\quad (2 - 2^{2k} + k2^{2k} - 2^{2k-1}) \\
&= 2^{2k}.
\end{aligned}$$

Note that G_{2k-2} is also a planar graph of odd-girth $2k+1$ by Lemma 16. □

In Figure 2 we present our construction for the case of $k = 2$. The result is a graph on 26 vertices. Note that the construction of [Na13] has more than 600 vertices.

Proof of Theorem 12 (for even values of d). Let $G = G_{2k-2}$ be the graph built in the previous proof. Since G is of odd-girth $2k+1$, by the assumption, it maps to B. Since B is also of odd-girth $2k+1$, both $B^{(2k-1)}$ and $G^{(2k-1)}$ are simple graphs and $G^{(2k-1)}$ admits a homomorphism to $B^{(2k-1)}$. Hence $K_{2^{2k}} \subset B^{(2k-1)}$ which, in particular, implies $|V(B)| \geq 2^{2k}$.

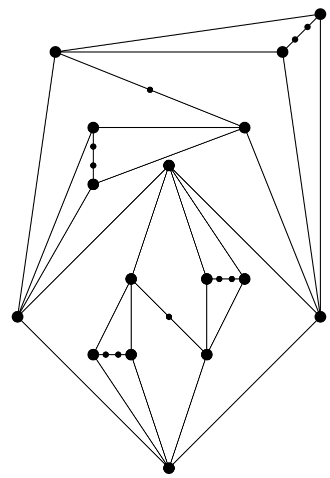

Figure 2: Example of a planar graph G of odd-girth five such that $G^{(4)}$ has a clique of order 2^4. The big vertices corresponds to the clique of order 2^4 in $G^{(4)}$.

To prove the lower bound on the minimum degree, we first introduce the following graph: let $P = x_1, x_2, \cdots, x_{2k+1}$ be a path of length $2k$. Now subdivide each edge $x_i x_{i+1}$ of P by replacing it with the path $x_i y_1^i y_2^i \cdots y_{2k-2}^i x_{i+1}$. Note that now x_i is at distance $2k-1$ from x_{i+1}. Then we obtain a new graph P' by adding some shortcut edges $x_1 y_1^2$, $y_1^2 y_2^3$, $y_2^3 y_3^4, \cdots, y_{2k-2}^{2k} x_{2k+1}$ so that the shortest odd walk between each x_i and x_j becomes of length $2k-1$. Now, given a vertex u, the graph P_u is the graph formed from a disjoint copy of P' by adding the edges ux_i for all $i \in \{1, 2, \cdots, 2k+1\}$. Note that the graph P_u is of odd-girth $2k+1$ and that in $P_u^{(2k-1)}$ vertices of P (i.e., x_i's) induce a $(2k+1)$-clique.

Now since B is minimal, there exists a planar graph G_B of odd-girth $2k+1$ whose mappings to B are always onto. Let G_B^* be a new graph obtained from G_B by adding a P_u for each vertex u of G_B. This new graph is also of odd-girth $2k+1$, thus, by the choice of B, it maps to B. Let ϕ be such a mapping of G_B^* to B. This mapping induces a mapping of G_B to B. Thus each vertex v of B is the image of a vertex u of G_B by the choice of G_B. But in the mapping G_B^* to B, all x_i's of P_u must map to distinct vertices all of which are neighbours of $\phi(u) = v$. $\square$

Note that since $\mathcal{PC}(2k)$ is a $(2k+1)$-regular graph on 2^{2k} vertices, it is an optimal homomorphism bound if Conjecture 7 holds.

4 Optimal homomorphism bound for planar signed bipartite graphs

The development of the notion of homomorphisms for signed graphs has began very recently and, therefore, it is not yet known if an analogue of Theorem 2 would hold for the class of signed bipartite graphs. While we believe that it would be the case, here we prove that $\mathcal{SPC}(d)$ is the optimal homomorphism bound for the signed bipartite case of Conjecture 7 if the conjecture holds.

Note that if both graphs are of unbalanced-girth at least $r+2$, then $(G, \Sigma)^r$ and $(H, \Pi)^r$ are both loopless, and, therefore, the existence of a homomorphism $\phi : (G, \Sigma) \to (H, \Pi)$ would imply $\omega((G, \Sigma)^r) \leq \omega((H, \Pi)^r)$. Furthermore, assuming that G and H are both connected, since ϕ is also a homomorphism of G to H, it would preserve bipartition. Thus in what follows we will built a signed bipartite planar graph (G, Σ) of unbalanced-girth $2k$ such that each part of G contains a clique of size 2^{k-2} in $(G, \Sigma)^{2k-2}$.

To this end we start with the following lemma which is the signed bipartite analogue of Lemma 15.

Lemma 17. *Let (G, Σ) be a planar signed graph which is obtained by assigning a signature to a subdivision of K_4 in such a way that each of the four facial cycles is an unbalanced cycle of length $2k$. Then $(G, \Sigma)^{(2k-2)}$ is isomorphic to two disjoint copies of K_{2k-1} induced by the two parts of G.*

Proof. We consider a fixed signature Σ of (G, Σ). We will use the same notations (P_{xy}, t_{xy}, etc.) as in Lemma 15. Thus as proved in that lemma, parallel edges of K_4 are subdivided the same number of times. Furthermore, repeating the same argument modulo 2, we can conclude that the number of negative edges in P_{xy} and the number of negative edges in P_{wz} have the same parity for all $\{x, y, w, z\} = \{a, b, c, d\}$.

Let u and v be two vertices from the same part of G (thus any path connecting u and v has even length). We would like to prove that they are adjacent in $(G, \Sigma)^{(2k-2)}$. If they both belong to a facial cycle, then the two paths connecting these two vertices in that (unbalanced) cycle satisfy the conditions and we are done. Hence, assume without loss of generality that $u \in P_{ab}$ and $v \in P_{cd}$.

Removing the edges of the parallel paths P_{ad} and P_{bc} will result in a cycle of length $4k - 2t_{ad}$ containing u, v. This implies:

$$\begin{aligned}
(t_{ua} + t_{ac} + t_{cv}) + (t_{ub} + t_{bd} + t_{dv}) &\leq 4k - 2, \\
\text{and thus } \min\{(t_{ua} + t_{ac} + t_{cv}), (t_{ub} + t_{bd} + t_{dv})\} &\leq 2k - 2. \qquad (4)
\end{aligned}$$

Similarly by removing P_{ac} and P_{bd} we get

$$\min\{(t_{ua} + t_{ad} + t_{dv}), (t_{ub} + t_{bc} + t_{cv})\} \leq 2k - 2. \qquad (5)$$

It remains to show that the two paths of Equations (4) and (5) have different numbers of negative edges modulo 2. To see this note that the union of any of the two paths from (4) with a path from (5) covers a facial cycle exactly once and one of P_{ab} or P_{cd} twice. Since each facial cycle is unbalanced, our claim is proved. $\square$

Next we will use two operations, similar to the ones done the previous section, to enlarge this construction.

Operation copy threads: Let (G, Σ) be a signed bipartite graph and $\mathcal{P} = \{P_1, P_2, \cdots, P_k\}$ be a set of threads of (G, Σ). For each thread $P_i = xv_1v_2 \cdots v_ry$ in P add a new thread $P'_i = xv'_1v'_2 \cdots v'_ry$ where all the internal vertices are new and distinct. Assign signs to the new edges in such a way that the edges xv'_1 and v_ry have the same sign, the edges v'_ry and xv_1 have the same sign and the edges $v'_iv'_{i+1}$ and $v_{r-i+1}v_{r-i}$ have the same sign. Denote the new signed graph by $(CT(G), CT(\Sigma))$.

Let W be a clique in $(G, \Sigma)^{(l)}$ on vertex set $V(W)$. Consider a set U of vertices of $(CT(G), CT(\Sigma))$ which consists of $V(W)$ and copy vertices v' for each degree (in G) two vertex v of $V(W)$. It is now easy to check that the subgraph W' of $(CT(G), CT(\Sigma))^{(l)}$ induced by U is a complete graph minus a matching. To be precise the missing matching is between pairs v, v' where v is a degree two vertex of (G, Σ) and v' is its copy.

Next we want to introduce an operation which will complete W' into a complete graph.

Operation shorten threads: Let (G, Σ) be a signed bipartite graph. Consider a collection $\mathcal{P}$ of threads of (G, Σ) and let $(CT(G), CT(\Sigma))$ be the graph obtained after the operation copy thread with respect to $\mathcal{P}$. For $P \in \mathcal{P}$ let P' be its copy in $(CT(G), CT(\Sigma))$. Suppose P is of length $r+1$ with x and y its end vertices and with $v_1, v_2 \cdots, v_r$ as its internal vertices. Let $v_1', v_2' \cdots v_r'$ be the internal vertices of P'. Add a new path N of length $2k-r$ to $(CT(G), CT(\Sigma))$ which connects v_1 and v_r' (all internal vertices are new and distinct). Moreover, we assign signs to the edges of the new path N in such a way that the cycles induced by $V(N) \cup (V(P) \setminus \{x\})$ and $V(N) \cup (V(P') \setminus \{y\})$ are both unbalanced. The new graph obtained after repeating the process for all paths in $\mathcal{P}$ will be denoted by $(ST(G), ST(\Sigma))$. Note that the operation $(ST(G), ST(\Sigma))$ creates two shortened threads; $v_1 v_2 \cdots v_r y$ a shortening of P and $x v_1' v_2' \cdots v_r'$ a shortening of P'. The length of each of the shortened threads is one less that the length of P.

The next lemma is the key property of this operation.

Lemma 18. *Let (G, Σ) be a signed bipartite graph of unbalanced-girth $2k$ and let $\mathcal{P}$ be a collection of threads of (G, Σ), each of them contained in an unbalanced cycle of length $2k+1$. Let $(ST(G), ST(\Sigma))$ be the signed graph obtained after operations copy threads and shorten threads with respect to $\mathcal{P}$. Then $(ST(G), ST(\Sigma))$ is also a signed bipartite graph of unbalanced-girth $2k$.*

The proof of this lemma is analogous to the proof of Lemma 16. We are now ready to prove Theorem 14.

Proof of Theorem 14. Let (G_0, Σ_0) be the signed graph obtained from K_4 by subdividing two parallel edges and by assigning a signature in such a way that each of the four facial cycles is an unbalanced cycle of length $2k$. Note that the resultant is a planar bipartite signed graph of unbalanced-girth $2k$ in which each face is of length $2k$. Thus by Lemma 17 the $(2k-2)$-th walk power of (G_0, Σ_0) is disjoint union of two cliques, each of order $2k-1$. Let $\mathcal{P}_0$ be the two threads of (G_0, Σ_0) (each of length $2k-2$).

Starting from (G_0, Σ_0) and $\mathcal{P}_0$ we will build a graph inductively in $2k-1$ steps as follows: given (G_i, Σ_i) and $\mathcal{P}_i$ we define (G_{i+1}, Σ_{i+1}) to be $(ST(G_i), ST(\Sigma_i))$ with respect to $\mathcal{P}_i$. We then define $\mathcal{P}_{i+1}$ to be collections of shortened threads and their copies. Thus $\mathcal{P}_{i+1}$ has twice as many elements as $\mathcal{P}_i$. Also, note that each thread in $\mathcal{P}_{i+1}$ has $2k-2-i$ vertices. Thus the number of vertices in each of the two maximum cliques of $(G_{2k-2}, \Sigma_{2k-2})^{(2k-2)}$ is equal to

$$\begin{aligned}
&\omega((G_{2k-2}, \Sigma_{2k-2})^{(2k-2)}) = \\
&\qquad = 2k-1+\sum_{j=1}^{2k-2} 2^{j-1}(2k-j-2) = 2k-1+(k-1)\sum_{j=1}^{2k-2} 2^j - \sum_{j=1}^{2k-2} j2^{j-1} \\
&\qquad = 2k-1+[(k-1)(2^{2k-1}-2)] - [(1-2^{2k-1}) - (-1)(2k-1)2^{2k-2}] \\
&\qquad = 2k-1+[k2^{2k-1}-2k-2^{2k-1}+2] - [1-2^{2k-1}+k2^{2k-1}-2^{2k-2}] \\
&\qquad = 2^{2k-2}.
\end{aligned}$$

Note that $(G_{2k-2}, \Sigma_{2k-2})$ is also a planar bipartite signed graph of unbalanced-girth $2k$ by Lemma 18. $\square$

Now we are ready to conclude the proof of Theorem 12.

Proof of Theorem 12 (for odd values of d). The proof is similar to the proof for even values of d. The only thing we need to do is to provide a gadget graph similar to P_u in this case also. We will use the graph $(G_{2k-2}, \Sigma_{2k-2})$ as the gadget graph P_u where the role of u is played by one of the original vertices of the K_4 from which the graph was built.

More formally, let x be one of the original vertices of the K_4 from which the signed graph $(G_{2k-2}, \Sigma_{2k-2})$ was built in the previous proof. Note that x has exactly $2k$ neighbors in $(G_{2k-2}, \Sigma_{2k-2})$, each of which is part of a clique in $(G_{2k-2}, \Sigma_{2k-2})^{(2k-2)}$.

Now since B is minimal, there exists a planar bipartite signed graph (G_B, Σ_B) of unbalanced-girth $2k$ whose mappings to B are always onto. Let (G^*_B, Σ^*_B) be a new graph obtained from (G_B, Σ_B) by gluing a copy of $(G_{2k-2}, \Sigma_{2k-2})$ to each vertex u of (G_B, Σ_B) by identifying the vertex x of $(G_{2k-2}, \Sigma_{2k-2})$ with the vertex u of (G_B, Σ_B). This new graph (G^*_B, Σ^*_B), clearly, is a planar bipartite signed graph of unbalanced-girth $2k$. The rest of the proof is similar to the proof for even values of d. □

5 Concluding remarks

P. Seymour has conjectured in [Se75] that the edge chromatic number of a planar multi-graph is equal to its fractional edge chromatic number. It turns out that the restriction of this conjecture for k-regular multigraphs can be proved if and only if Conjecture 7 is proved for $d = k + 1$ [Na13, NRS12]. This special case of Seymour's conjecture is proved for $k \leq 8$ in a series of works using induction and the Four-Color Theorem in [Gu12] ($k = 4, 5$), [DKK] ($k = 6$), [Ed11] ($k = 7$) and [CES12] ($k = 8$). Thus Conjecture 7 is verified for $d \leq 7$. Hence we have the following corollary.

Theorem 19. *For $d \leq 7$ the signed graph $\mathcal{SPC}(d)$ is the smallest consistent graph (both in terms of number of vertices and edges) of unbalanced-girth $d + 1$ which is a homomorphism bound for all consistent planar signed graphs of unbalanced-girth at least $d + 1$.*

B. Guenin has proposed a strengthening of Conjecture 7 by replacing the condition of planarity by the condition of having no $(K_5, E(K_5))$-minor [Gu05].

For further generalization one can consider the following general question:

Problem 20. *Given d and r, $d \geq r$ and $d \equiv r \pmod 2$, what is the optimal homomorphism bound having unbalanced girth r for all consistent signed graphs of unbalanced-girth d with no $(K_n, E(K_n))$-minor?*

We do not know yet whether such a homomorphism bound exists in general. For $n = 3$, consistent signed graphs with no $(K_n, E(K_n))$-minor are bipartite graphs with all edges positive, and, therefore, have K_2 as their homomorphism bound. For $n = 5$ if the input and target graphs are both of unbalanced-girth $d+1$, then our work and Geunin's extension of Conjecture 7 propose projective cubes as the optimal solutions. For $d = r = 3$, the answer would be K_{n-1} if Odd Hadwiger Conjecture is true. For the case $n = 4$ some partial answers are given by L. Beadou, F. Foucaud and first author. For all other cases there is not even a conjecture yet.

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