The Petersen graph is not 3-edge-colorable - a new proof

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Abstract

We give a new proof that the Petersen graph is not 3-edge-colorable.

J. Petersen introduced the most well known graph, the Petersen graph, as an example of a cubic bridgeless graph that is not Tait colorable, i.e. it is not 3-edge-colorable. It is easy to see the equivalence between the following statements, but most proofs for each of them use a case by case argument [1].

Theorem 1 *For the Petersen graph P the following are equivalent:*

(1) P is not 3-edge-colorable;

(2) P is not Hamiltonian;

(3) P does not admit a nowhere-zero 4-flow.

Here we give a new proof of:

Proposition 2 *The Petersen graph is not 3-edge-colorable.*

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Proof. The Petersen graph is usually drawn as an outer 5-cycle, an inner 5-cycle where edges join vertices that are cyclically two apart, and a matching joining corresponding vertices on the two cycles drawn as depicted in Figure 1. Assuming a proper 3-edge-coloring, we obtain a contradiction by showing that each of the three colors must be used twice on the inner cycle, which has only five edges.

Since the outer cycle is of odd length, each of the three colors appears on it. Let uv be an edge on the outer cycle with color a. In a proper 3-edge-coloring of a 3-regular graph, each color must appear at each vertex. Since a cannot appear on ux or vy, where x and y are the neighbors of u and v on the inner cycle, and xy is not an edge, color a appears on distinct edges of the inner cycle at x and y. □

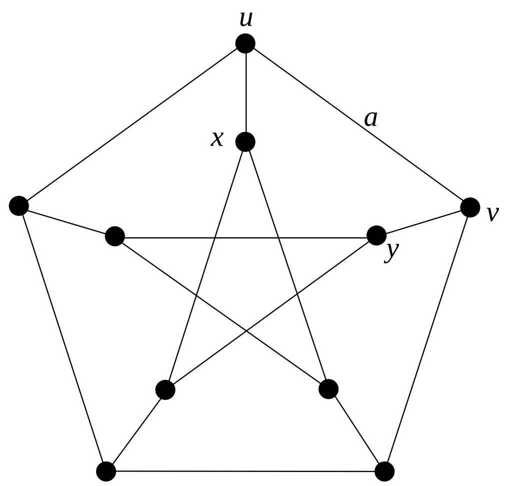

Figure 1: The Petersen graph

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References

[1] D. A. Holton and J. Sheehan, *The Petersen Graph*, Cambridge University Press, Cambridge (1993).