The chromatic covering number of a graph

Reza Naserasr[a] *

Claude Tardif[b] †

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a: Department of Mathematics, Simon Fraser University, Burnaby BC, V5A 1S6, Canada. Current address: Department of Combinatorics and Optimization, University of Waterloo, Waterloo ON N2L 3G1 Canada. e-mail: naserasr@math.uwaterloo.ca
b: Department of Mathematics and Computer Science, Royal Military College of Canada, PO Box 17000, Station "Forces" Kingston ON K7K 7B4 Canada. e-mail: Claude.Tardif@rmc.ca

Abstract

Following [1], we investigate the problem of covering a graph G with induced subgraphs $G_1, \dots, G_k$ of possibly smaller chromatic number, but such that for every vertex u of G, the sum of reciprocals of the chromatic numbers of the G_i's containing u is at least 1. The existence of such "chromatic coverings" provides some bounds on the chromatic number of G.

1 Introduction

Let G be a graph and $G_1, \dots, G_k$ induced subgraphs of G. If for every vertex u of G we have $\sum\{\frac{1}{\chi(G_i)} : u \in V(G_i)\} \geq 1$, then $\{G_1, \dots G_k\}$ is called a *chromatic covering* of G. The *chromatic covering number* **cover-**$\chi(G)$ of G is the smallest value k such that G admits a chromatic covering with k subgraphs.

Grötzsch's graph G provides a good illustration of the dynamics of chromatic coverings. It is well known that this graph is 4-chromatic, yet it contains relatively large bipartite subgraphs. In particular the graph G_1 obtained from G by removing the vertices $0, 0', u$ is bipartite, as is the subgraph G_2 obtained from G by removing $1, 2, 4$. Noting that the subgraph G_3 of G induced by

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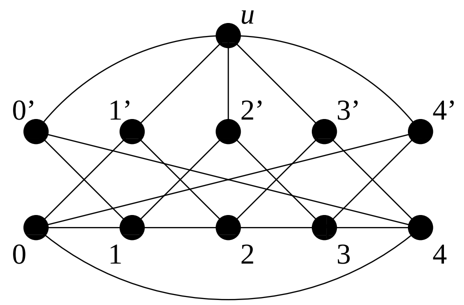

Figure 1: The Grötzsch graph

$\{0, 0', u, 1, 2, 4\}$ is again bipartite, we conclude that $\{G_1, G_2, G_3\}$ is a collection of bipartite induced subgraphs of G such that every vertex of G is in two of these subgraphs. Thus **cover-**$\chi(G) \leq 3$.

Amit, Linial and Matoušek [1] note that any graph parameter Ψ taking values in $[1, \infty)$ gives rise to a related "covering" parameter **cover-**Ψ, just like the chromatic number gives rise to the chromatic covering number discussed here. They use the "cover degeneracy plus one" **cover-**dng_1 to bound the chromatic number of random lifts of graphs; this parameter's relation to the chromatic number is also discussed in [3]. More generally, if $\Psi(G) = 1$ whenever G has no edges, then the inequality **cover-**$\Psi(G) \leq \chi(G)$ holds for all graphs. If $\Psi(G) \geq \chi(G)$ for every graph G, then the inequality **cover-**$\Psi(G) \geq \chi_f(G)$ holds for all graphs, where $\chi_f(G)$ is the fractional chromatic number of G. Therefore **cover-**χ is the smallest of a family of cover parameters sandwiched between $\chi_f(G)$ and $\chi(G)$; in particular the inequalities $\chi_f(G) \leq$ **cover-**$\chi(G) \leq$ **cover-**$\mathrm{dng}_1(G) \leq \chi(G)$ hold for any graph G.

It is well known that there exists no general upper bound for the chromatic number of a graph in terms of its fractional chromatic number. In contrast, such bounds can be found in terms of cover parameters. Amit, Linial and Matoušek [1] show that for any graph G we have $\chi(G) \leq 2\,(\textbf{cover-}\mathrm{dng}_1(G))^2$ and that a bound of the type $\chi(G) \leq O((\textbf{cover-}\chi(G))^2)$ is best possible. In this note we give a tight bound for $\chi(G)$ in terms of **cover-**$\chi(G)$.

Theorem 1 *For every graph G,* $\chi(G) \leq \left\lfloor \left(\frac{\textbf{cover-}\chi(G)+1}{2} \right)^2 \right\rfloor$.

For all n not congruent to 1 modulo 4, we exhibit a graph G such that **cover-**$\chi(G) = n$ and $\chi(G) = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor$ (see Section 3), proving that the bound is best possible. For $n = 4k+1$, our example satisfies **cover-**$\chi(G) = n$ and $\chi(G) \geq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor - \frac{n-1}{2}$, but it is reasonable to suspect that Theorem 1 is also best possible in this case.

2 Proof of Theorem 1

Let $\{G_1, \ldots G_k\}$ be a chromatic covering of G, where $k = \textbf{cover-}G$. Moreover, assume that $\chi(G_1) \le \chi(G_2) \le \ldots \le \chi(G_k)$. Then there exists a smallest index ℓ such that $\bigcup_{i=1}^{\ell} V(G_i) = V(G)$. Then considering a vertex $u \in V(G_\ell) \setminus (V(G_1) \cup \ldots \cup V(G_{\ell-1}))$, the covering condition reads

$$1 \le \sum_{u \in V(G_i)} \frac{1}{\chi(G_i)} \le \sum_{i \ge \ell} \frac{1}{\chi(G_i)} \le \frac{k - \ell + 1}{\chi(G_\ell)}; \tag{1}$$

therefore $\chi(G_\ell) \le k - \ell + 1$. On the other hand, since $V(G) = \bigcup_{i=1}^{\ell} V(G_i)$ we have

$$\chi(G) \le \sum_{i \le \ell} \chi(G_i) \le \sum_{i \le \ell} \chi(G_\ell) \le \ell(k - \ell + 1) \le \left\lfloor \left(\frac{k+1}{2} \right)^2 \right\rfloor. \tag{2}$$

□

In the next section we present a family of natural candidates to consider when trying to decide whether this bound is best possible.

3 Fractional multiples of graphs

Let n, r, s be integers such that $r \le s$. We define the graph $K_n^{r,s}$ as follows: The vertices of $K_n^{r,s}$ are the subsets $A = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ of $\{1, \ldots, s\} \times \{1, \ldots, n\}$ such that $i_1, \ldots, i_r$ are all distinct. Two of these subsets are joined by an edge in $K_n^{r,s}$ if they are disjoint. In [5], $K_n^{r,s}$ is called a *fractional multiple* of the complete graph K_n. It can also be represented as follows: The vertices of $K_n^{r,s}$ represent r-independent sets in a disjoint union of s copies of K_n, and two of these are joined by an edge in $K_n^{r,s}$ if they are disjoint.

Lemma 2 *A graph G admits a homomorphism to $K_n^{r,s}$ if and only if G can be covered by s n-colourable subgraphs $G_1, \ldots, G_s$ such that every vertex of G is in r of these subgraphs.*

Proof. Suppose that $\{G_1, \ldots, G_s\}$ is such a covering of G. For $i = 1, \ldots, n$, fix a n-colouring $f_i : G_i \to \{1, \ldots, n\}$ of G_i. We define a map $\phi : G \to K_n^{r,s}$ by

$$\phi(u) = \{(i, f_i(u)) : u \in V(G_i)\}.$$

Then for every edge $[u, v]$ of G, we have $f_i(u) \ne f_i(v)$ whenever $u, v \in G_i$. Therefore, $\phi(u)$ is disjoint from (that is, adjacent to) $\phi(v)$. This shows that ϕ is an edge preserving map, that is, a homomorphism.

Conversely, if $\phi : G \to K_n^{r,s}$ is a homomorphism, then for $i = 1, \ldots, s$, the graph G_i induced by

$$V(G_i) = \{u \in V(G) : \phi(u) \cap \{(i, 1), \ldots, (i, n)\} \ne \emptyset\}$$

is n-chromatic, and every vertex u of G belongs to r of these. □

Thus for $n \leq s$, $K_n^{n,s}$ admits a chromatic covering with s subgraphs hence **cover-**$\chi(K_n^{n,s}) \leq s$. Also, there is a natural $n(s-r+1)$-colouring of $K_n^{r,s}$, obtained by colouring each vertex with one element of its intersection with $\{1,\ldots,s-r+1\} \times \{1,\ldots,n\}$; hence $\chi(K_n^{r,s}) \leq n(s-r+1)$. We will show that equality holds when n is even, using the following result:

Lemma 3 (Schrijver [4]) *Let a, b be integers such that $b \geq 2a$. Let $H(a,b)$ be the graph whose vertices are the a-independent sets of the b-cycle, where two of these independent sets are joined by an edge if they are disjoint. Then $\chi(H(a,b)) = b - 2a + 2$.*

Corollary 4 *Let n, r, s be integers such that n is even and $r \leq s$. Then $\chi(K_n^{r,s}) = n(s-r+1)$.*

Proof. It suffices to show that $\chi(K_n^{r,s}) \geq n(s-r+1)$. We define a homomorphism ϕ from $H(a,b)$ to $K_n^{r,s}$, where $a = \frac{n}{2}(r-1)+1$ and $b = ns$. Suppose that the vertices of the b-cycle are labeled consecutively $(1,1),(1,2),\ldots,(1,n),(2,1),\ldots,(2,n),\ldots,(s,1),\ldots,(s,n)$. Then for any a-independent set I of the cycle, there exist at least r values $i_1,\ldots,i_r$ such that I intersects $\{(i_k,1),\ldots,(i_k,n)\}$ for $k = 1,\ldots,r$. We can then select j_k such that $(i_k,j_k) \in I$ for $k = 1,\ldots,r$, and put

$$\phi(I) = \{(i_1,j_1),\ldots,(i_r,j_r)\}.$$

This defines a homomorphism from $H(a,b)$ to $K_n^{r,s}$; therefore

$$\chi(K_n^{r,s}) \geq \chi(H(a,b)) = b - 2a + 2 = n(s-r+1).$$

□

For every $n \geq 1$, the graphs $K_n^{n,2n}$ and $K_{n+1}^{n+1,2n}$ both have a natural chromatic covering with $2n$ subgraphs and a natural $n(n+1)$-colouring. Since one of n and $n+1$ is even, Corollary 4 implies that one of these two graphs has chromatic number $n(n+1) = \left\lfloor \left(\frac{2n+1}{2}\right)^2 \right\rfloor$, whence by Theorem 1, its natural chromatic covering is also optimal. Also, for every $n \geq 1$, the graph $K_n^{n,2n-1}$ has a natural chromatic covering with $2n-1$ subgraphs and a natural n^2-colouring. When n is even, Corollary 4 implies that $\chi(K_n^{n,2n-1}) = n^2 = \left\lfloor \left(\frac{(2n-1)+1}{2}\right)^2 \right\rfloor$ whence by Theorem 1, its natural chromatic covering is again also optimal. When n is odd, we can at least say that $\chi(K_n^{n,2n-1}) \geq n^2 - n + 1$, since the vertex $\{(1,n),(2,n),\ldots,(n,n)\}$ is completely joined to $K_{n-1}^{n,2n-1}$ in $K_n^{n,2n-1}$. By Theorem 1 we then have **cover-**$\chi(K_n^{n,2n-1}) = 2n-1$. Summarizing, we have the following:

Corollary 5 *For every integer k not congruent to 1 modulo 4, there exists a graph G such that* **cover-**$\chi(G) = k$ *and* $\chi(G) = \left\lfloor \left(\frac{k+1}{2}\right)^2 \right\rfloor$. *When $k = 4\ell + 1$ there exist a graph G such that* **cover-**$\chi(G) = k$ *and* $\chi(G) \geq \left\lfloor \left(\frac{k+1}{2}\right)^2 \right\rfloor - \frac{k-1}{2}$.
□

4 Concluding comments

The graphs $K_1^{r,s}$ are the well-known Kneser graphs, and Lovász [2] has shown that when $s \geq 2r$, $\chi(K_1^{r,s}) = s - 2r + 2$. For larger odd values of n the hypothesis $\chi(K_n^{r,s}) = n(s - r + 1)$ seems reasonable, but it remains open for the moment. We note that it suffices to show that $\chi(K_3^{r,s}) = 3(s-r+1)$, since $K_{2n+3}^{r,s}$ contains a complete join of $K_{2n}^{r,s}$ and $K_3^{r,s}$.

We mention in closing that even though the graphs $K_2^{2,4}$ and $K_3^{3,4}$ each have a 4-chromatic covering, their disjoint union $K_2^{2,4} \cup K_3^{3,4}$ can be shown to have a chromatic covering number strictly greater than 4. Thus the disjoint union becomes a nontrivial operation when considering chromatic covering numbers.

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