Separating signatures in signed planar graphs

Reza Naserasr, Weiqiang Yu

Université Paris Cité, CNRS, IRIF, F-75006, Paris, France.
E-mail addresses: {reza, wyu}@irif.fr

Abstract

A signed graph (G, σ) is a graph together with an assignment σ of signs to the edges called signature. A switching at a vertex v is to reverse the sign of each edge incident to v. Two signatures σ_1 and σ_2 on G are equivalent if one can be obtained from the other by a sequence of switchings. The *packing number* of a signed graph (G, σ), denoted $\rho(G, \sigma)$, is defined to be the maximum number of signatures $\sigma_1, \sigma_2, \ldots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets of negative edges are pairwise disjoint. The question of determining the packing number in a class of signed graphs captures or relates to some of the most prominent studies in graph theory. For example the four-color theorem can be restated as: For every planar simple graph G we have $\rho(G, -) \geq 3$.

As a generalization of the packing number, instead of considering one signature and its equivalent signatures, we consider k signatures $\sigma_1, \sigma_2, \ldots, \sigma_k$ (not necessarily switching equivalent) and ask whether there exist signatures $\sigma'_1, \sigma'_2, \ldots, \sigma'_k$, where σ'_i is a switching of σ_i, such that the sets of negative edges $E^-_{\sigma'_i}$ are pairwise disjoint.

It is known that there exists a signed planar simple graph whose packing number is 1. Thus for a general planar graph separating two signatures is not always possible even if $\sigma_1 = \sigma_2$. In this work, we prove that given planar graph G with no 4-cycle and any two signatures σ and π on G, there are switchings σ' and π' of σ and π, respectively, such that $E^-_{\sigma'} \cap E^-_{\pi'} = \varnothing$. And as a corollary of 3-degeneracy, we could also separate two signatures on a planar graph with no triangle, or with no 5-cycle or with no 6-cycle. Moreover, we prove that one could separate three signatures on graphs of maximum average degree less than 3, in particular on planar graphs of girth at least 6.

Keywords: signed graph, packing and separating signatures.

1 Introduction

Graphs considered in this work are finite and simple. A graph G is called *planar* if it can be embedded in the plane such that any two edges intersect at most at their ends. A signed graph (G, σ) is a graph $G = (V, E)$ and a signature σ which is an assignment of signs, $+$ or $-$, to the edges. In the study of signed graphs, *switching* is a key concept: switching at a vertex v is to multiply the sign of each edge incident with v by $-$, and switching at a subset X of $V(G)$ is to multiply the signs of the edges in the edge cut $(X, V(G) - X)$ by $-$. Two signatures σ_1 and σ_2 on G are *switching equivalent*, denoted by $\sigma_1 \equiv \sigma_2$, if one can be obtained from the other by a sequence of switchings or, equivalently, by reversing the signs of the edges of an edge-cut. Given signed graphs (G, σ) and (H, π) a homomorphism of (G, σ) to (H, π) is a mapping φ of the vertices of G to the vertices of H, such that adjacencies and signs of closed walks are preserved. Given a signed graph (G, σ) and a subgraph H of G, (H, σ) is said to be a signed graph which keeps the sign of the edges as (G, σ). Moreover, if H is a spanning subgraph of G and we get (H, σ') from (H, σ) by switching at $X \subseteq V(H)$, then we say (G, σ') is obtained

from (G, σ) by switching at $X \subseteq V(G)$ since H is a spanning subgraph of G. One may easily observe that (H, σ') is also a signed graph that keeps the sign of the edges as (G, σ').
Let $E^-_\sigma(G)$ denote the set of negative edges of (G, σ). When G is clear from the context we would simply write E^-_σ. An *unbalanced* or *negative* cycle (*balanced* or *positive*) in signed graph is a cycle having an odd (even) number of negative edges. A theorem of Zaslavsky says that the set of unbalanced cycles (equivalently the set of balanced cycles) uniquely determines the equivalent class of signatures:

Theorem 1. *[7] Two signatures σ_1 and σ_2 on a graph G are switching equivalent if and only if they induce the same set of unbalanced cycles.*

The packing number of (G, σ), denoted $\rho(G, \sigma)$, is the maximum number of signatures $\sigma_1, \sigma_2, \ldots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets $E^-_{\sigma_i}$ are pairwise disjoint. Packing number captures and generalizes some of the most prominent results and conjectures in graph theory. For example the following conjecture strengthen the four-color theorem and relates to several related problems such as the edge-coloring conjecture of Seymour.

Conjecture 1. *Given a connected signed planar graph (G, σ) if there is no odd positive closed walk, then $\rho(G, \sigma) = g_-(G, \sigma)$.*

Here $g_-(G, \sigma)$ is the length of shortest negative closed walk of (G, σ) which is easily observed to be the same as the length of the shortest negative cycle of (G, σ). For further connections and more details on this part of the study we refer to [5].
In this work as a generalization of the packing number, instead of considering one signature and its equivalent signatures, we consider the following: given k signatures $\sigma_1, \sigma_2, \ldots, \sigma_k$ on a given graph G we say they are *separable* if there are signatures $\sigma'_1, \sigma'_2, \ldots, \sigma'_k$, where σ'_i is a switching of σ_i, such that the sets $E^-_{\sigma'_i}$ are pairwise disjoint. In particular, if we choose these k signatures to be σ, then being separable implies $\rho(G, \sigma) \geq k$. Given a graph G, if any set of k signatures on G are separable, then we say G has *k-separation property*.
The problem of packing number at least 2 is strongly connected to a notion of proper coloring of signed graphs first introduced by Zaslavsky in [8]. That is a coloring c of vertices of (G, σ) where colors are nonzero integers such that $c(x) \neq \sigma(xy)c(y)$. In a further study of this concept, Máčajová, Raspaud and Škoviera [3] conjectured that colors $\{\pm 1, \pm 2\}$ are enough for proper coloring of any signed planar simple graph. This conjecture was recently disproved by Kardoš and Narboni [2].
Connecting the two notions, it is shown in [4] and [5], a signed graph (G, σ) has packing number 2 if and only if $(G, -\sigma)$ admits a $\{\pm 1, \pm 2\}$-coloring, where $(G, -\sigma)$ is obtained from (G, σ) by turning the positive (resp. negative) edges to be negative (resp. positive).
This implies that there exists a signed planar simple graph whose packing number is 1, see [5] for more details.
In this work, we investigate sufficient conditions for a planar graph to have 2- or 3-separation property. We prove the followings.

Theorem 2. *Given an integer i, $i \in \{3, 4, 5, 6\}$, any planar graph without a cycle of length i has 2-separation property.*

Theorem 3. *Every planar graph of girth at least 6 has 3-separation property.*

The last theorem is a corollary of a more general result on graphs of maximum average degree less than 3.
In the next section we prove Theorem 2. Proof of Theorem 3 is provided in Section 3. In the last section we have concluding remarks where we mention connection to homomorphisms.

2 Separating 2 signatures in subclasses of signed planar graphs

In the rest of this section G will be a minimum counterexample to Theorem 2. We will see soon that this minimum counterexample has to be 2-connected and be of minimum degree at least 4. Thus in developing the terminology that is followed we consider G to be 2-connected and of minimum degree at least 4.
The counterexample G will be regarded as a plane graph that is a graph together with a planar embedding. As we consider 2-connected graphs every face is bounded by a cycle of G. We use $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$ to denote its vertex set, edge set, face set, and minimum degree, respectively. A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. *k^+-vertex*, *k^--vertex*). Similarly, we define *k-face*, *k^+-face*, *k^--face* as well. We say that two faces (or cycles) are *adjacent* or *intersecting* if they share a common edge or a common vertex, respectively. Suppose that v is a k-vertex, and let $v_1, \ldots, v_k$ be the neighbours of v in the clockwise order. For $i = 1, \ldots, k$, $f_i(v)$ denotes the face incident with the vertex v with vv_i, vv_{i+1} (where the summation in the indices are taken modulo k) as boundary edges. As G is a plane graph of minimum degree at least 4, this is well defined.
For $a \in F(G)$, we write $a = [u_1u_2 \cdots u_l]$ if $u_1, u_2, \ldots, u_l$ are the incident vertices of a in a cyclic order of it. As G is 2-connected and minimum degree at least four, each edge $e = u_ju_{j+1}$ of a face a determines a face adjacent to a at e. This face will be denoted by $f_j(a)$, where $j = 1, \ldots, l$ and the summation in the indices are taken modulo l.
For two signatures σ and π on G, and for an edge $uv \in E(G)$, let $s_{\sigma\pi}(uv) = \{\sigma(uv)\pi(uv)\} \subseteq \{+,-\} \times \{+,-\}$. Observe that to separate σ and π is to find signatures σ', switching equivalent to σ, and π', switching equivalent to π, such that $s_{\sigma'\pi'}(uv) \neq --$ for every edge uv. For a vertex u define $S_{\sigma\pi}(u)$ as multiset $S_{\sigma\pi}(u) = [s_{\sigma\pi}(e) \mid e \text{ is incident with } u]$. Thus the order of $S_{\sigma\pi}(u)$ is the degree of u. Let $S^* = \{++, +-, -+\}$. We say a vertex v is *saturated* by σ and π if $S^* \subseteq S_{\sigma\pi}(v)$.
A path in G all whose vertices are of degree 4 in G is called a *light path*. Two paths are said to be *vertex disjoint* if their internal vertices are distinct. We say an m-face $a = [v_1v_2 \cdots v_m]$ is a *light face* if $d(v_i) = 4$ for all $i = 1, \ldots, m$. A 5-face with four vertices of degree 4 and one vertex of degree 5 is called a *weak 5-face*. A weak 5-face is said to be *super weak 5-face* if it is adjacent to at least four triangles. For $x \in V(G) \cup F(G)$, let $n_3(x)$ denote the number of triangles incident or adjacent to x and $n_w(x)$ be the number of incident or adjacent weak faces.
It is well-known that every planar graph is 5-degenerate and that every triangle-free planar graph is 3-degenerate. It is shown in [6] that every planar graph without a 5-cycle is 3-degenerate. Similarly it is shown in [1] that every planar graph without 6-cycles is 3-degenerate. In next section, we will see that in a minimum counterexample to Theorem 2, the minimum degree is at least 4, which cannot be the case for 3-degenerate graphs. This would imply the claim of the theorem for each of the conditions of being triangle-free, having no 5-cycle or having no 6-cycle. What remains to prove is that if G is a planar graph with no 4-cycle, then any two signatures on it can be separated.

2.1 Structural properties of a minimum counterexample

Recall that G is a minimum counterexample to our theorem. That is to say either G has no triangle, or no 4-cycle, or no 5-cycle or no 6-cycle and there are signatures σ and π on G such that no matter how we switch them there is an edge which is assigned a negative sign by each of the two signatures.
The first observation is that G is connected, as otherwise separating signatures on each connected component, which would be possible by minimality, would be also a separation of the two signatures on the whole graph. Almost the same argument implies the following stronger claim.

Lemma 4. *The minimum counterexample G is 2-vertex-connected.*

Proof. Suppose to the contrary that v is a cut vertex of G. Let $G = G_1 \cup G_2$ such that v is the unique common vertex of G_1 and G_2, and there does not exist any edges between $V(G_1) - v$ and $V(G_2) - v$. Given two signatures σ and π on G, we consider subgraphs (G_1, σ), (G_1, π), (G_2, σ), and (G_2, π). By the assumption of the minimality of G, there are switchings σ_1 and π_1 on G_1 (resp. σ_2 and π_2 on G_2) of σ and π, respectively, such that they have no common negative edge.
In particular, in G_1 (resp. G_2), in order to get the switchings σ_1 and π_1 (resp. σ_2 and π_2) of σ and π, we could choose to switch at a subset V_1 (resp. V_2) of $V(G_1)$ (resp. $V(G_2)$) which does not contain v. Thus in G, if we switch at subset $V_1 \cup V_2$ which does not contain v as well, we find switchings σ' and π' of σ and π, such that σ' and π' have no common negative edge. This shows that a minimal counterexample cannot have a vertex cut of one vertex. □

Lemma 5. *Given an edge $uv \in E(G)$ let $G' = G - uv$ and assume σ' and π' are switchings of σ and π, respectively, such that (G', σ') and (G', π') are separated. Then both u and v are saturated by σ' and π' in G'.*

Proof. Towards a contradiction and without loss of generality, assume $S^* \subsetneq S_{\sigma'\pi'}(u)$. Since σ' and π' have no common negative edge as signatures on $G - uv$, and G is counterexample, considering the extension of these signatures to G we have $s_{\sigma'\pi'}(uv) = --$. Assume $\alpha\beta \notin S_{\sigma'\pi'}(u)$, $\alpha\beta \in S^*$. If $\alpha = +$, switch σ' at u; if $\beta = +$, switch π' at u. After this operation, we have signatures σ'' and π'' both on G which agree with σ' and π' (respectively) on every edge that is not incident to u. Thus, by the choice of σ' and π', no edge which is not incident to u is negative in both. But, furthermore, based on our switchings $\{--\} \notin S(u)$ and thus σ'' and π'' are switchings of σ and π that are separated, a contradiction. □

Corollary 6. *The minimum degree of G is at least* 4.

Thus as mentioned above, the case when G has no triangle or no 5-cycle or no 6-cycle is settled because any such a planar graph must be 3-degenerate.

Lemma 7. *Let P be a light path of G, $e \in P$. Assume σ_e and π_e are switchings of σ and π, respectively, such that (G, σ_e) and (G, π_e) have only e as their common negative edge. Then given an edge e' of P, by switching σ_e on a set X of vertices of P and switching π_e on a set Y of the vertices of P, for some choices of X and Y, we have signatures $\sigma_{e'}$ and $\pi_{e'}$ where e' is the only common negative edge of $(G, \sigma_{e'})$ and $(G, \pi_{e'})$.*

Proof. Suppose $P = v_1 v_2 \cdots v_k$ and $e = \{v_i v_{i+1}\}$, where $i \in \{1, 2, \dots k-1\}$. By our assumption $s_{\sigma_e \pi_e}(v_i v_{i+1}) = \{--\}$. By Lemma 5, $S_{\sigma_e \pi_e}(v_i) = S_{\sigma_e \pi_e}(v_{i+1}) = S^*$. With the same idea as in the proof of Lemma 5, and assuming $i \geq 2$, we may apply switchings at the vertex v_i so that $v_{i-1} v_i$ is the only common negative edge of the resulting two signatures. Similarly, assuming $i \leq k-2$ we may apply switchings at the vertex v_{i+1} so that $v_{i+1} v_{i+2}$ is the only common negative edge of the resulting two signatures. Continuing this process, and noting that each time switchings are only done on one of v_j's, $j = 2, \dots k-1$, we have the desired claim. □

Lemma 8. *There is no pair of vertices connected by three vertex disjoint light paths.*

Proof. Assume to the contrary that P_1, P_2, P_3 are three vertex disjoint light uv-paths and label them as follows: $P_1 = u x_1 \cdots x_i v$, $P_2 = u y_1 \cdots y_j v$, and $P_3 = u z_1 \cdots z_k v$, where $i, j, k \geq 0$, noting that $k = 0$ means $P_3 = uv$ and that, since G is a simple graph, only one of these values can be 0. Thus, without loss of generality, we may assume $i \geq j \geq 1$ and $k \geq 0$. Since G has no 4-cycle, we also conclude that $i \geq 2$. Moreover, we may choose P_1, P_2, P_3 to be shortest subject to being internally vertex disjoint. This implies, in particular, that for any pair of non-consecutive vertices on a path P_i $(i = 1, 2, 3)$, they are not adjacent in G. Recalling that all vertices of a light path are of degree 4 in G, let t, w be the neighbours of u, v which are not on any of P_1, P_2, or P_3, respectively. Let $G' = G - u x_1$. By the minimality of G, assume σ' and π' are switchings of σ and π, respectively, such that (G', σ') and (G', π') are separated. Thus when σ' and π' are viewed as signatures on G we have $s_{\sigma'\pi'}(ux_1) = \{--\}$ and both u and x_1 are saturated. Noting that k is allowed to be 0, we consider two cases depending on this.

First consider the case $k \geq 1$, as depicted in Figure 1. We may apply Lemma 7 to switch only at the internal vertices of P_1 to obtain signatures σ'' and π'' such that $x_i v$ is the only edge with $s_{\sigma''\pi''}(x_i v) = --$. Therefore, considering signatures σ'' and π'', and by Lemma 5, the vertex v must be saturated. Recall that in the process of getting σ'' and π'' from σ' and π' we are considering only switchings at the internal vertices of P_1. Furthermore, since P_i's chosen to be shortest, no internal vertex of P_i is adjacent to v. That means, in particular, that the signs of the three edges $y_j v$, $z_k v$, wv each incident to v remain untouched when switching σ' to σ'' and π' to π''. We conclude that

$$\{s_{\sigma'\pi'}(y_j v), s_{\sigma'\pi'}(z_k v), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}. \tag{1}$$

Next, restarting from signatures σ' and π' and applying Lemma 7 to the path $x_1 u y_1 \cdots y_j v$ (that is the path obtained from P_2 by adding the edge $x_1 u$ at the start), and as before, we conclude that

$$\{s_{\sigma'\pi'}(x_i v), s_{\sigma'\pi'}(z_k v), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}. \tag{2}$$

In this argument that $k \geq 1$ helps us to confirm that the signs of the three edges incident to v other than $y_j v$ remain the same.

Equations 1 and 2 imply that $s_{\sigma'\pi'}(x_i v) = s_{\sigma'\pi'}(y_j v)$.

Similarly, considering paths P_1 and $x_1 u z_1 \cdots z_k v$ we conclude that $s_{\sigma'\pi'}(x_i v) = s_{\sigma'\pi'}(z_k v)$. However, this leads to contradiction with either of the identities 1 and 2. This concludes the statement for the case that $k \geq 1$.

Now assume $k = 0$, that is to say uv is an edge of G, this case is depicted in Figure 1. First suppose that, except for the edge uv, no vertex of P_1 is connected to a vertex of P_2. Our first claim in this case is that $s_{\sigma'\pi'}(uy_1) = s_{\sigma'\pi'}(y_1 y_2) = \cdots = s_{\sigma'\pi'}(y_j v)$. That is because by applying Lemma 7 and Lemma 5 to the path $x_1 u y_1 y_2 \cdots y_j v$ we get that $S_{\sigma'\pi'}(y_l) - s_{\sigma'\pi'}(y_l y_{l-1}) = S^*$ and by applying the same lemma to the path $u x_1 x_2 \cdots x_i v y_j y_{j-1} \cdots y_1$ we get that $S_{\sigma'\pi'}(y_l) - s_{\sigma'\pi'}(y_l y_{l+1}) = S^*$.

Next we claim that $s_{\sigma'\pi'}(x_i v) = s_{\sigma'\pi'}(uv)$. That is for similar reasons as the previous claim and by considering the two paths P_1 and $x_1 uv$. Furthermore, applying Lemma 5 to signature σ'' and π'' which have only $x_i v$ as common negative edge, and are obtained from switching of σ' and π' (respectively) on internal vertices of P_1, we conclude that:

$$\{s_{\sigma'\pi'}(uv), s_{\sigma'\pi'}(y_j v), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}. \tag{3}$$

Recall that u is saturated by σ' and π' where ux_1 is negative in both signatures. This means

$$\{s_{\sigma'\pi'}(ut), s_{\sigma'\pi'}(uy_1), s_{\sigma'\pi'}(uv)\} = \{++, +-, -+\}. \tag{4}$$

Comparing identities 3 and 4 we have: $s_{\sigma'\pi'}(ut) = s_{\sigma'\pi'}(vw)$.

Observe that when applying Lemma 7 to get uy_1 as the only common negative edge, we apply switchings at u in one or both of the signatures. Assuming the new signatures are σ'' and π'' one observes that $s_{\sigma''\pi''}(ux_1) = s_{\sigma'\pi'}(uy_1)$ and thus $s_{\sigma''\pi''}(uv) = s_{\sigma'\pi'}(ut)$. Therefore, $s_{\sigma''\pi''}(uv) = s_{\sigma'\pi'}(vw)$.

If we now apply Lemma 7 to σ'' and π'' on the path P_2 so to have $y_j v$ as the only common negative edge, as we will not change signs of the other three edges incident with v we will end up with a vertex v which is not saturated, contradicting Lemma 5.

For the final case, suppose beside uv, there exists another edge connecting a vertex of P_1 to a vertex of P_2. Let $x_p y_q$ be such an edge. Since $i \geq 2$, and by exchanging the roles of u and v, if needed, we may assume that $p \leq i - 1$. In this case, as before we apply Lemma 7 to the following three paths: P_1, $x_1 uv$, and $u x_1 \cdots x_p y_q \cdots y_j v$. From the first we conclude that $\{s_{\sigma'\pi'}(uv), s_{\sigma'\pi'}(y_j v), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}$.

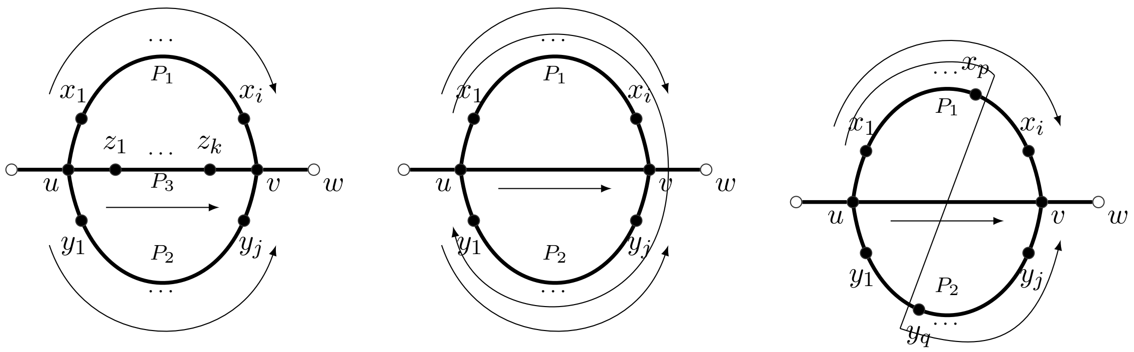

Figure 1: 3 disjoint light paths between u and v.

From the second we conclude that $\{s_{\sigma'\pi'}(x_iv), s_{\sigma'\pi'}(y_jv), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}$. And the last one implies $\{s_{\sigma'\pi'}(uv), s_{\sigma'\pi'}(x_iv), s_{\sigma'\pi'}(wv)\} = \{++, +-, -+\}$. Comparing the first two we conclude that $s_{\sigma'\pi'}(uv) = s_{\sigma'\pi'}(x_iv)$, then first with second $s_{\sigma'\pi'}(uv) = s_{\sigma'\pi'}(y_jv)$ which contradicts, say, the third identity. $\square$

Corollary 9. *There are no adjacent light faces in G.*

We may now apply discharging technique to conclude our claim.

2.2 Discharging for planar graphs without 4-cycles

In this section, we apply discharging technique to complete the proof of Theorem 2 for the case of C_4-free planar graphs.

We define a weight function ω on the vertices and faces of G by letting $\omega(v) = d(v) - 4$ for each $v \in V(G)$ and $\omega(f) = d(f) - 4$ for $f \in F(G)$. It follows from Euler's formula and the relation $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E(G)|$ that the total sum of weights of the vertices and faces satisfies the following

$$\sum_{v\in V(G)} (d(v) - 4) + \sum_{f\in F(G)} (d(f) - 4) = -8.$$

Next we design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. The total sum of weights is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, we will show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This contradiction implies that no such counterexample exists.

Let v be vertex of degree 4 whose neighbours in clockwise orientation are v_1, v_2, v_3, and v_4. Let f_1, f_2, f_3, and f_4 be the face containing v_1vv_2, v_2vv_3, v_3vv_4, and v_4vv_1 respectively. If $d(v_3) = d(v_4) = 4$, $d(v_1) = d(v_2) \geq 5$, $d(f_2) = d(f_4) = 3$, $d(f_3) = 5$, and $d(f_1) \geq 5$, then we say f_3 is a *receiver* of f_1.

For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the amount of weights transferred from x to y.

Our first discharging rule is as follows:

$R1$: Each 5^+-face sends $\frac{1}{3}$ to each adjacent 3-face and $\frac{2}{15}$ to each of its receiver.

Let v be a 5-vertex with $f_1, f_2, \dots, f_5$ being the faces incident to v. Assume f_1 and f_3 are triangles and, furthermore, that f_4 is a super weak 5-face. Then it is easily observed that f_5 is not a super weak 5-face.

The next two discharging rules are as follows:

$R2$: If $d(v) = 5$, $n_3(v) = 1$, say $d(f_1) = 3$, then let $\tau(v \to f_2) = \tau(v \to f_5) = \frac{1}{3}$.

$R3$: If $d(v) = 5$ and $n_3(v) = 2$, say $d(f_1) = d(f_3) = 3$, then $\tau(v \to f_2) = \frac{2}{3}$. Furthermore, if there exists one super weak 5-face f', $f' \neq f_2$, then $\tau(v \to f') = \frac{1}{3}$, otherwise $\tau(v \to f_4) = \tau(v \to f_5) = \frac{1}{6}$.

The remaining two rules are about 6^+-vertices.

$R4$: If $d(v) \geq 6$ and f is a face incident to v and adjacent to one triangle also incident to v, then $\tau(v \to f) = \frac{1}{3}$.

$R5$: If $d(v) \geq 6$ and f is a face incident to v and adjacent to two triangles each incident to v, then $\tau(v \to f) = \frac{2}{3}$.

In the following, we will show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.
First we consider vertices, let $v \in V(G)$. By Corollary 6, $d(v) \geq 4$. Note that no 4-vertex participates in discharging argument, so $\omega^*(v) = \omega(v) = d(v) - 4 = 0$ for any 4-vertex v. Next we consider 5-vertices. Let v be any such a vertex, then $\omega(v) = 1$. By the fact that G contains no 4-cycle we have $0 \leq n_3(v) \leq 2$. If $n_3(v) = 0$, then the charge of v is not changed, i.e., $\omega^*(v) = \omega(v) = 1$. If $n_3(v) = 1$, the charge of v is changed (only) by the $R2$, and in this case $\omega^*(v) = \omega(v) - 2 \times \frac{1}{3} = \frac{1}{3}$. If $n_3(v) = 2$, then $R3$ is the only rule that changes the charge of v and under this rule at most a charge of 1 is given from v to its incident face. Thus $\omega^*(v) \geq 0$.
It remains to consider 6^+-vertices. Let v be such a vertex. $d(v) \geq 6$. For $i = 1, 2$, let $m_i(v)$ denote the number of incident faces adjacent to i triangles each incident to v. Observe that, by definition, $m_1(v) + 2m_2(v) \leq 2n_3(v) \leq d(v)$ (the latter inequality because of being C_4-free). In applying $R3$ the vertex v loses a charge of $\frac{m_1(v)+2m_2(v)}{3}$. Thus $\omega^*(v) = d(v) - 4 - \frac{m_1(v)+2m_2(v)}{3}$. Therefore, $\omega^*(v) \geq d(v) - 4 - \frac{d(v)}{3}$. As $d(v) \geq 6$ we have $\omega^*(v) \geq 0$.

Now we consider faces, let $f \in F(G)$. First assume $d(f) = 3$, in other words f is a triangle. Recall that original charge $\omega(f) = -1$. Since G has no C_4, each of the faces adjacent to f is of size at least 5. Then by rule $R1$, each of them sends a charge of $\frac{1}{3}$ to f and thus $\omega^*(f) = 3 - 4 + 3 \times \frac{1}{3} = 0$.
Next we consider 5-faces, let $f = [v_1 \cdots v_5]$ be such a face. For the original charge of f we have $\omega(f) = 5 - 4 = 1$. If f is adjacent to at most two triangles, then f gives a charge of $\frac{1}{3}$ to each of the triangles it is adjacent to and it has at most one receiver, so can only lose a charge of $2 \times \frac{1}{3} + \frac{2}{15} = \frac{4}{5}$, thus the final charge is at least $\frac{1}{5}$.
Suppose f is adjacent to precisely 3 triangles. If f has no receiver, then it only loses charge by $R1$ and by this rule loses exactly a charge of $3 \times \frac{1}{3} = 1$, hence $\omega^*(f) = 0$. If f has exactly one receiver, let v_2 be the common vertex of f and its receiver. Then, by the definition of a receiver, v_1, v_3 each has degree at least 5. We now consider the position of the third triangle adjacent to f. If it is one of f_3 or f_5, say f_3, then by $R3$ or $R5$, depending on if $d(v_3) = 5$ or $d(v_3) \geq 6$, the vertex v_3 gives a charge of $\frac{2}{3}$ to f, concluding that $\omega^*(f) \geq \frac{8}{15}$. Otherwise f_4 is the third triangle adjacent to f. In such a case the two faces f_3 and f_5 are 5^+-faces. We claim that neither is a super weak 5-face. By contradiction, suppose f_3 is a super weak 5-face. Then, it must be adjacent to at least four triangles. As f is not a triangle, all the other faces adjacent to f_3 are triangles. This implies that vertices v_3 and v_4 are each of degree at least 5, but this contradicts the second condition of being a super negative 5-face which is to have four vertices of degree 4. If f_3 (or f_5) is a 6^+-face, then, by $R4$, it gives a charge of $\frac{1}{3}$ to f, raising $\omega^*(f)$ to at least $\frac{1}{5}$. If they are both 5-faces, then, by $R3$, each of v_1 and v_3 gives a charge of $\frac{1}{6}$ to f. The final charge of f in this is at least $1 + \frac{1}{6} - 3 \times \frac{1}{3} - \frac{2}{15} = \frac{1}{30}$.
Suppose f is adjacent to 4 triangles and by symmetry assume f_1, f_2, f_3, and f_4 are the triangles. The receiver face implies that there are at most two receivers for f, and moreover if there is at least one, then one of v_2, v_3

or v_4 has to be of degree at least 5. If one of v_2, v_3 or v_4 is of degree at least 5, either by $R3$ or by $R5$ it will give a charge of $\frac{2}{3}$ to f and thus the final charge of f would be at least $1 + \frac{2}{3} - 4 \times \frac{1}{3} - 2 \times \frac{2}{15} = \frac{1}{15}$. Let now assume v_2, v_3, and v_4 are all of degree 4. In such case, if v_5 and v_1 each has degree at least 5 or one of them has degree at least 6, then f_5 cannot be a weak face and either by applying $R3$ to both v_1 and v_5 or applying $R4$ to the one which is a 6^+-vertex, a total charge of at least $\frac{1}{3}$ is given to f and thus the final charge of f is non-negative. If one of v_1 and v_5 is degree 4 and the other, say v_5 is of degree 5, then f is a super weak 5-face and thus by $R3$ the vertex v_5 will give a charge of $\frac{1}{3}$ to f, resulting a final charge of f to be positive. If all vertices $v_1, \dots, v_5$ are of degree 4, i.e., f is a light face, then since there is no adjacent light faces (Corollary 9) for each of the triangles $f_1, \dots, f_4$ the vertex of f_i which is not on f is a 5^+-vertex. Then f is a receiver for the face adjacent to f_1 and f_2 and for the face adjacent to f_2 and f_3 and also for the face adjacent to f_3 and f_4. It, therefore, receives a charge of $\frac{2}{15}$ from each of these 3 for a final charge of $\omega^*(f) \geq 1 + 3 \times \frac{2}{15} - 4 \times \frac{1}{3} = \frac{1}{15}$.
Finally we consider the case where all faces adjacent to f are triangles. Recall that f has at most two receivers. So it loses at most $5 \times \frac{1}{3} + 2 \times \frac{2}{15}$. If two of v_i's are 5^+-vertices, then either by $R3$ or by $R5$ they each gives a charge of $\frac{2}{3}$ to f and the final charge of f is positive. If only one of v_i's, say v_1, is a 5^+-vertex, then f has no receiver and only loses a charge of $5 \times \frac{1}{3}$ but gains $\frac{2}{3}$ from v_1 and again the final charge would be non-negative. If none of v_i's is a 5^+-vertex, i.e., f is a light face, by Corollary 9, for each of the triangles $f_1, \dots, f_5$ the vertex of f_i which is not on f is a 5^+-vertex. Thus f is a receiver of five faces determined by consecutive triangles around it. Hence by $R1$, it receives $5 \times \frac{2}{15}$ from each of these five faces, to have a final charge of 0. This conclude all the cases for a 5-face.
Next assume that $f = [v_1 \cdots v_6]$ is a 6-face. Then $\omega(f) = 2$. If f is adjacent to at most 5 triangles, then it has at most two receivers, and hence it loses at most $5 \times \frac{1}{3} + 2 \times \frac{2}{15}$ (all in $R1$) hence $\omega^*(f) \geq \frac{1}{15}$. If all the six faces adjacent to f are triangles, we consider two possibilities depending on the degrees of $v_1, \dots, v_6$. If at least one of them is a 5^+-vertex, then either by $R3$ or by $R5$ it gives a charge of $\frac{2}{3}$ to f. As f can have at most three receivers, the final charge of f remains non-negative. If all vertices on f are of degree 4, then f has no receiver and the final charge of f is 0.
Finally we consider 7^+-faces. Recall that faces only lose charge by $R1$. There are at most $d(f)$ triangles adjacent to f, and it can have at most $\lceil \frac{d(f)}{2} \rceil$ receivers. Thus for the final charge of f we have $\omega^*(f) = d(f) - 4 - \frac{1}{3}d(f) - \frac{1}{2} \times \frac{2}{15}d(f) = \frac{3}{5}d(f) - 4 \geq \frac{1}{5}$. This completes the proof.

3 Separating 3 signatures in signed planar graphs of girth 6

In this section we provide a maximum average degree condition which is sufficient for any three signatures on a graph to be separated. Theorem 3 will be immediate consequence then.

Theorem 10. *Every simple graph of maximum average degree less than 3 has a 3-separation property.*

Proof. Let G be a minimum counterexample. That, in particular means there are three signatures σ_1, σ_2, and σ_3 on G that are not separable but for any edge e, the restrictions of the three signatures on $G - e$ are separable. After proving a few claims, and using discharging technique then we will show that G itself must have average degree at least 3 contradicting our hypothesis on the maximum average degree of G.
For three signatures σ_1, σ_2 and σ_3 on G, and for an edge $uv \in E(G)$, let $s_{\sigma_1\sigma_2\sigma_3}(uv) = \{\sigma_1(uv)\sigma_2(uv)\sigma_3(uv)\} \subseteq \{+,-\} \times \{+,-\} \times \{+,-\}$. For a vertex u define a multiset $S_{\sigma_1\sigma_2\sigma_3}(u) = [s_{\sigma_1\sigma_2\sigma_3}(e) | e \in E_u]$, where E_u is the set of edges incident to u. We may use $s(uv)$ and $S(u)$ when the signatures are clear from the context. Let $S^* = \{+++, -++, +-+, ++-\}$.
The first observation, whose verification we leave to the reader, is that G is 2-connected. Thus in particular the minimum degree is at least 2. To achieve our goal then we have three claims about the neighbourhood of vertices of degree 2.

Claim 1. Both neighbours of a 2-vertex v in G have degree at least 4.

Proof of the claim. Let $N(v) = \{v_1, v_2\}$ and assume to the contrary, that $d(v_2) \leq 3$. Let $G' = G - vv_2$. By the minimality of G, assume σ'_1, σ'_2, and σ'_3 are switchings of σ_1, σ_2, and σ_3, respectively, such that (G', σ'_1), (G', σ'_2), and (G', σ'_3) are separated. Since $d_{G'}(v) = 1$, and by a switching at v in any signature that needs, we may assume $s_{\sigma'_1\sigma'_2\sigma'_3}(vv_1) = \{+ + +\}$. When σ'_1, σ'_2, and σ'_3 are viewed as signatures on G, vv_2 is the only edge not satisfying the condition which means at least two of the signatures must assign negative to vv_2. If one of them, say σ'_3 assigns positive to vv_2, then by switching v at the signature σ'_2 (or σ'_1) we have separation. Thus we may assume $s_{\sigma'_1\sigma'_2\sigma'_3}(vv_2) = \{- - -\}$.

At this point, it suffices to find one or two signatures, σ'_i and σ'_j, such that if σ'_i or both σ'_i and σ'_j are switched at v_2, then vv_2 is still the only edge not satisfying our condition. If we manage to find σ'_i, or σ'_i and σ'_j, then we may also switch signature σ'_l, $l \notin \{i, j\}$, at v. After these switchings, vv_1 will be negative at one signature only, and vv_2 with be either positive in all or negative in just one signature, and thus we have three separated signatures. To choose σ'_i and possibly σ'_j among σ'_1, σ'_2, and σ'_3 we consider the two edges, e_1 and e_2 incident to v_2 but different from vv_2. If they are both negative in a signature, we choose that one to be σ'_i and no need for a second. If each of the edges is assigned only positive sign by each of the signatures, then σ'_i can be any of the three signatures and again no need for a second choice. Otherwise, we note that at most two of the signatures can assign different signs to e_1 and e_2. If only one, then we choose that signature to be σ'_i and if two we take them both to be σ'_i and σ'_j. $\diamond$

Claim 2. A 4-vertex v can have at most two 2-neighbours.

Proof of the claim. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. Toward a contradiction assume that $d(v_i) = 2$ for $i = 1, 2, 3$. For each i, $i = 1, 2, 3$, let the other neighbour of v_i be v'_i. Let $G' = G - vv_1$. By the minimality of G, we have signatures σ'_1, σ'_2, and σ'_3 as switchings of σ_1, σ_2, and σ_3, respectively, such that (G', σ'_1), (G', σ'_2), and (G', σ'_3) are separated. In what follows, we consider signatures σ'_1, σ'_2, and σ'_3 on G. Again since $d_{G'}(v_1) = 1$, without loss of generality, we may assume $s(v_1v'_1) = \{+ + +\}$. The same argument as in the previous case then implies that $s(vv_1) = \{- - -\}$.

If $S(v) \cap S^* \leq 2$, then we continue the same argument as in the previous case, where v is a neighbour of the 2-vertex v_1 and to our purpose it is of degree $|S(v) \cap S^*| + 1$. So we assume $S(v) \cap S^* = 3$. We observe that by switching at v_2, in the signatures that are needed, we may exchange $s(vv_2)$ and $s(v_2v'_2)$. If after such switchings the previous condition holds, we are done. If not, either $s(vv_2) = s(v_2v'_2)$ in which case by switchings at v_2 we may conclude that $s(vv_2) = s(vv_3)$ and then we are done as before, or $s(v_2v'_2)$ is distinct from each of $s(vv_2)$, $s(vv_3)$, and $s(vv_4)$. Repeating the same argument we conclude that $s(v_3v'_3) = s(v_2v'_2)$. We may now do enough switchings at v_2 and v_3 so that $s(vv_2) = s(vv_3)$. Then the process can be completed as before. $\diamond$

Claim 3. A 5-vertex v can have at most four 2-neighbours.
Proof of the claim. Let $N(v) = \{v_1, \ldots, v_5\}$. Assume to the contrary that $d(v_i) = 2$ for $i = 1, \ldots, 5$. We name the other neighbour of v_i as v'_i. Let $G' = G - vv_1$. By the minimality of G, assume σ'_1, σ'_2, and σ'_3 are switchings of σ_1, σ_2, and σ_3, respectively, such that (G', σ'_1), (G', σ'_2), and (G', σ'_3) are separated. As in the previous two cases, we may assume $s(v_1v'_1) = \{+ + +\}$ and $s(vv_1) = \{- - -\}$. Furthermore, by switching at v_i's, if necessary, we can assume that none of $s(vv_i)$, $i = 2, \ldots, 5$, is $\{+ + +\}$. Thus for some i and j, $2 \leq i < j \leq 5$, we have $s(vv_i) = s(vv_j)$. At this point we note that in the proof of Claim 2 we never applied a switching at v_4. Thus we may now continue the same proof as in the Claim 2 by treating v_i and v_j as v_4 and not switching at these two vertices. $\diamond$

Finally to complete the proof we show that the three forbidden configurations of Claims 1, 2, and 3 imply an average degree of at least 3.

We first define ω on the vertices of G by letting $\omega(v) = d(v)$ for each $v \in V(G)$. The single discharging rule is as follows.

R' : Each 4^+-vertex sends $\frac{1}{2}$ to each 2-neighbour.

Let $\omega^*(v)$ be the charge of v after applying the rule. Let $v \in V(G)$. As observed before, $d(v) \geq 2$. If $d(v) = 2$, then by Claim 1, v is adjacent to two vertices of degree at least 4. Thus, $\omega^*(v) = 2 + 2 \times \frac{1}{2} = 3$ by (R'). The discharging rule does not change $\omega(v)$ if $d(v) = 3$. If $d(v) = 4$, then by Claim 2, v has at most two 2-neighbours, thus $\omega^*(v) \geq 4 - 2 \times \frac{1}{2} = 3$. When $d(v) = 5$, by Claim 3, v has at most four 2-neighbours, thus $\omega^*(v) \geq 5 - 4 \times \frac{1}{2} = 3$. Finally if $d(v) \geq 6$, then $\omega^*(v) \geq d(v) - \frac{d(v)}{2} = \frac{d(v)}{2} \geq 3$. □

4 Conclusion

The problem of packing signatures in signed graphs relates to some of the most prominent problems in graph theory such as the four-color theorem and edge-coloring problems as shown in [5]. In particular it is shown that a signed graph (G, σ) admits a $(k+1)$-packing of signatures if and only if it admits a homomorphism to the graph obtained from the projective cube of dimension k by adding a positive loop to each vertex. We recall that a homomorphism of a signed graph (G, σ) to (H, π) is a mapping of vertices and edges of G to the vertices and edges of H, respectively, such that adjacencies and incidences, as well as signs of closed walks are preserved. Projective cube of dimension k is the signed graph with $\mathbb{Z}_2^k$ as the vertex set where pairs of vertices at hamming distance 1 are adjacent by a positive edge and pairs of vertices at hamming distance k are adjacent by a negative edge.
The first four projective cubes with positive loops added to the vertices are presented in Figure 2. The claim that every antibalanced signed planar simple graph $(G, -)$ maps to SPC_2^o is equivalent to the four-color theorem and the claim that every signed bipartite planar simple graph maps to SPC_3^o is stronger than the four-color theorem and is proved using it.

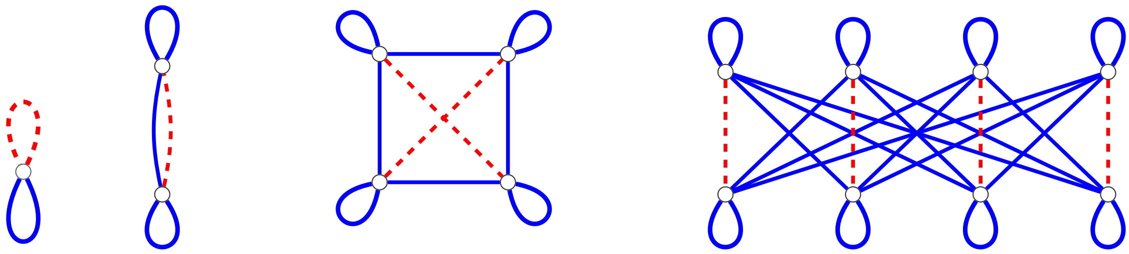

Figure 2: $\mathcal{SPC}_d^o$ for $d \in \{0, 1, 2, 3\}$

The question of separating a given set of k signatures captures the k-packing of signature problem because one can simply take k identical signatures. The question then can be translated back to a homomorphism problem as follows.
A multi-signed graph, denoted $(G, \sigma_1, \sigma_2, \ldots, \sigma_l)$, is a graph G together with l signatures. A multi-signed graph $(G, \sigma_1, \sigma_2, \ldots, \sigma_l)$ is said to admit a homomorphism to a multi-signed graph $(H, \pi_1, \pi_2, \ldots, \pi_l)$ if there is a mapping f of vertices and edges of G to vertices and edges of H, respectively, which is a homomorphism of (G, σ_i) to (H, π_i) for every i, $i = 1, 2, \ldots, l$. That is to say incidences and adjacencies are preserved, and the sign of any closed walk in (G, σ_i) is the same as the sign of its image in (H, π_i).

Given an integer l, let L_l be the multi-signed graph on a single vertex with $l+1$ loops $e_0, e_1, \dots e_l$ where e_0 is assigned a positive sign by each of the signatures and e_i is assigned a negative sign by σ_i and positive sign by all other signatures. The cases $l = 1, 2, 3$ are presented in Figures 3, 4, and 5. It is then immediate to restate the separating problem we have studied here as a homomorphism problem.

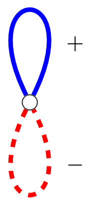

Figure 3: (L_1, σ)

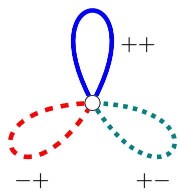

Figure 4: $(L_2, \sigma_1, \sigma_2)$

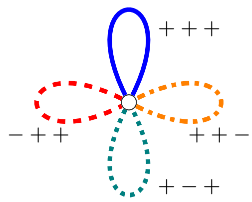

Figure 5: $(L_3, \sigma_1, \sigma_2, \sigma_3)$

Theorem 11. *A multi-signed graph $(G, \sigma_1, \sigma_2, \dots, \sigma_l)$ admits a separation if and only it admits a homomorphism to L_l.*

Acknowledgement. This work is supported by the ANR (France) project HOSIGRA (ANR-17-CE40-0022). And we also gratefully acknowledge financial support from China Scholarship Council.

References

[1] G. Fijavž, M. Juvan, B. Mohar, R. Škrekovski, Planar graphs without cycles of specific lengths. European J. Combin. 23 (2002) 377–388.

[2] F. Kardoš, J. Narboni, On the 4-color theorem for signed graphs. European J. Combin. 91 (2021) 103215.

[3] E. Máčajová, A. Raspaud and M. Škoviera, The chromatic number of a signed graph. Electron. J. Combin. 23 (2016) Paper 1.14.

[4] R. Naserasr, É. Sopena and T. Zaslavsky, Homomorphisms of signed graphs: An update. European J. Combin. 91 (2021) 103222.

[5] R. Naserasr, W. Yu, Packing signatures in signed graphs. 2023. To appear in SIAM J. Discrete Math.(Online version: https://hal.science/hal-03960731)

[6] W. Wang, L. Ko-Wei, Choosability and edge choosability of planar graphs without five cycles. Appl. Math. Lett. 15 (2002) 561–565.

[7] T. Zaslavsky, Signed graphs. Discrete Appl. Math. 4 (1982) 47–74.

[8] T. Zaslavsky, Signed graph coloring. Discrete Math. 39 (1982) 215–228.