Packing signatures in signed graphs

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Abstract

We define the *signature packing number* of a signed graph (G, σ), denoted $\rho(G, \sigma)$, to be the maximum number of signatures $\sigma_1, \sigma_2, \ldots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets $E^-_{\sigma_i}$, negative edges of (G, σ_i), are pairwise disjoint. In this work, first in connection to recent developments on the theory of homomorphisms of signed graphs we prove that for a signed graph (G, σ), $\rho(G, \sigma) \geq d + 1$ if and only if (G, σ) admits a homomorphism to $\mathcal{SPC}^o_d$, where $\mathcal{SPC}^o_d$ is obtained from $\mathcal{SPC}_d$ by adding a positive loop to every vertex. Noting that $\mathcal{SPC}_d$, *signed projective cube of dimension* d, is the signed (Cayley) graph built on $\mathbb{Z}_2^d$ where each pair of binary strings at hamming distance 1 are adjacent by a positive edge and those at hamming distance d are adjacent by a negative edge. In other words, $\mathcal{SPC}_d$ is built from the hypercube of dimension d by considering all its edges as positive edges and adding a negative edge for each pair of antipodal vertices.

In special cases we have: *I*. A simple graph G is 4-colorable if and only if $\rho(G, -) \geq 2$. *II*. A signed bipartite graph (G, σ) maps to SPC_3 if and only if $\rho(G, \sigma) \geq 3$ noting that SPC_3 is the same as $(K_{4,4}, M)$, that is a signed graph on $K_{4,4}$ where the set of negative edges forms a perfect matching.

On restriction to planar graphs, *I* is then a restatement of the 4-color theorem and *II* is implied by an unpublished work of B. Guenin. After further development of this theory of packing in signed graphs, we give an independent proof of *II* which works on the larger class of K_5-minor-free graphs. More precisely we prove that:

Theorem. *If G is a K_5-minor-free bipartite simple graph, then for any signature σ we have $\rho(G, \sigma) \geq 4$.*

The statement is shown to be strictly stronger than the four-color theorem and is proved assuming it. Furthermore, we show that *I* cannot be extended to the class of all signed planar simple graphs.

Further development, including algorithmic implications, are considered.
Keywords: signed graph, packing number, homomorphisms, signed projective cubes.

1 Introduction

Graphs in this work are allowed to have loops or multiedges. When loops are forbidden we will use the term loop-free and when both loops and multiedges are forbidden, we emphasize using the term "simple graph". Otherwise, we follow the standard notion of graph theory and refer to [2] for terminology.

A *signed graph* (G, σ) is a graph G equipped with a *signature* σ which assigns to each edge of G a sign (either $+$ or $-$). With $\{+, -\}$ viewed as a multiplicative group, the key concept that separates a signed graph from a 2-edge-colored graph is the notion of *switching* (also referred to as "resigning" by some authors). A switching at a vertex v, is to multiply the sign of all the edges incident to v by a $-$, noting that a loop on v is incident to it from both ends and, therefore, a switching at v does not change sign of a loop at v. To switch at each of the vertices of a subset X of vertices of G is to multiply the signs of all edges in the edge-cut $(X, V \setminus X)$ by $-$. A signed graph (G, σ') is said to be *switching-equivalent* (or simply equivalent) to (G, σ) if it is obtained by a switching on an edge-cut. It is straightforward to check that this is indeed an equivalence relation among all possible signatures.

Given a signed graph (G, σ), the set of negative edges is denoted by E_σ^-. It is easily observed that (G, σ) and (G, σ') are equivalent if and only if the symmetric difference $E_\sigma^- \triangle E_{\sigma'}^-$ is an edge cut of G. Given a labeled signature such as σ_i in (G, σ_i) and when there is no ambiguity, we may write E_i^- in place of $E_{\sigma_i}^-$. Given a signed graph (G, σ) with $E' = E_\sigma^-$, sometime we rather write (G, E') instead of (G, σ).

The sign of a structure in (G, σ) is the product of the signs of its edges, considering multiplicity. Structures of highest importance in this work are cycles and closed walks. Note that the sign of either of them is invariant under a switching operation. Moreover, it now is a classic result of Zaslavsky that the signs of cycles of G in (G, σ) uniquely determine the equivalence class to which σ belongs. More precisely:

Theorem 1. *[16] Given two signatures σ and σ' on a graph G, they are equivalent if and only if they have the same set of negative cycles.*

We note that in fact verifying the condition on a fundamental set of cycles suffices, and that a proof based on spanning tree leads to a simple algorithm as well (see [13] for more details).

A positive cycle is commonly referred to as *balanced* cycle and a negative cycle is commonly called *unbalanced*. Sign and parity of the length of closed walks partition them into four categories: positive and even, positive and odd, negative and even, negative and odd. Given a signed graph (G, σ), the length of a shortest closed walk in each of these categories will be denoted, respectively, by $g_{00}(G, \sigma)$, $g_{01}(G, \sigma)$, $g_{10}(G, \sigma)$, $g_{11}(G, \sigma)$ the logic being that the first index represents the parity of the number of negative edges and the second represents the parity of the total number of edges. Furthermore, the length of a shortest negative closed walk will be denoted by $g_-(G, \sigma)$ (i.e., $g_-(G, \sigma) = \min\{g_{10}(G, \sigma), g_{11}(G, \sigma)\}$). For each of these parameters, when there is no closed walk of the type that is considered, the corresponding parameter is set to be ∞.

As long as (G, σ) has at least one edge, $g_{00}(G, \sigma)$ is 2 as a traversing an edge in both direction is always a positive closed walk of length 2. It is not difficult to build an example of signed graph (G, σ) where the value of $g_{ij}(G, \sigma)$ for $ij \in \mathbb{Z}_2^2$, $ij \neq 00$ is obtained by a closed walk which is not a cycle. However, at least two of these values, if they are all bounded, will always be obtained by a proper cycle. More precisely, the two smallest of the values $\{g_{01}(G, \sigma), g_{01}(G, \sigma), g_{01}(G, \sigma)\}$ correspond to cycles because if a shortest closed walk of type ij, $ij \in \{01, 10, 11\}$, is not a cycle, it must be formed of merging of the two closed walks of types $\{01, 10, 11\} - ij$. Thus the only value of g_{ij} which is possibly not recognized by a cycle is the largest of the three values. This, in particular, implies that a shortest negative closed walk is always a cycle. Thus $g_-(G, \sigma)$ may also be defined as the length of a shortest negative cycle and referred to as the *negative girth* of (G, σ). We note that in these definitions a loop is considered as a cycle of length 1 and two parallel edges form a cycle of length 2.

Given a graph G, the signed graph $(G, -)$ (respectively, $(G, +)$) is the signed graph where all edges are negative (positive). For a positive integer l, C_{-l} is a negative cycle of length l together with any of its equivalent signatures.We may then denote a positive cycle of length l by C_{+l} or simply by C_l.

Given $ij \in \mathbb{Z}_2^2$, $ij \neq 00$, the class $\mathcal{G}_{ij}$ of signed graphs is defined as follows:

$$\mathcal{G}_{ij} = \{(G, \sigma) \mid g_{i'j'}(G, \sigma) = \infty \text{ for } i'j' \in \mathbb{Z}_2^2 - 00, i'j' \neq ij\}.$$

In other words, given a signed graph $(G, \sigma) \in \mathcal{G}_{ij}$, every closed walk of (G, σ) is either a positive even closed walk or a closed walk whose parity of number of negative edges and the length are determined by i and j, respectively. Thus, based on Theorem 1 we have:

- $\mathcal{G}_{01}$ is the class of signed graphs (G, σ) which can be switched to $(G, +)$,
- $\mathcal{G}_{11}$ consists of signed graphs (G, σ) which can be switched to $(G, -)$,
- $\mathcal{G}_{10}$ is the class of all signed bipartite graphs.

Each of the first two items can be regarded as a natural embedding of graphs into the larger class of signed graphs. As $\rho(G, +) = \infty$, the preferred embedding of graphs into signed graphs in the study of packing signatures is $(G, -)$. This has extra advantage that works better with minor theory of signed graphs. The class $\mathcal{G}_{10}$ is also of importance for this study and is the main focus of this work.

2 Homomorphisms of signed graphs

Given signed graphs (G, σ) and (H, π), a *homomorphism* of (G, σ) to (H, π) is a mapping φ of the vertices and edges of G to the vertices and edges of H, respectively, such that adjacencies, incidences and signs of closed walks are preserved. Essentially, regarding Theorem 1, a homomorphism is expected to preserve the signs of cycles, however, the image of a cycle could be a closed walk rather than a cycle. One should note that replacing cycles with closed walks in Theorem 1 we still have the same conclusion.

When there exists a homomorphism (G, σ) to (H, π) we write $(G, \sigma) \to (H, \pi)$. A homomorphism of (G, σ) to (H, π) is said to be *edge-sign-preserving* if, furthermore, signs of the edges are preserved. When it is needed to distinguish the two notions, the former might be referred to as *switching homomorphism* because of the following connection:

Theorem 2. *[13] A signed graph (G, σ) admits a homomorphism to a signed graph (H, π) if for a signature σ' on G, equivalent to σ, the signed graph (G, σ') admits an edge-sign-preserving homomorphism to (H, π).*

The definition of homomorphism implies a basic no-homomorphism lemma:

Lemma 3. *If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for every $ij \in \mathbb{Z}_2^2$.*

It follows from the definitions and Theorem 2 that homomorphisms of signed graphs generalize the notion of chromatic number of graphs. More precisely, we have the following observation.

Observation 4. *Given a graph G, we have $\chi(G) \leq k$ if and only if $(G, -) \to (K_k, -)$.*

This restatement of k-coloring is also helpful to state the odd-Hadwiger conjecture of Gerards and Seymour (see for example [4]). Recall that minor of a signed graph (G, σ) is a signed graph obtained from (G, σ) by the following four operations: deleting vertices, deleting edges, contracting positive edges, and switching.

Conjecture 5 (Odd-Hadwiger)**.** *If $(G, -)$ has no $(K_{k+1}, -)$-minor, then $(G, -) \to (K_k, -)$.*

3 Packing number of signed graphs

Given a signed graph (G, σ), the *signature packing number*, or simply the packing number of (G, σ), denoted $\rho(G, \sigma)$, is the maximum number of signatures $\sigma_1, \sigma_2, \ldots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets E_i^- are pairwise disjoint. If (G, σ) is equivalent to $(G, +)$, then by taking the all positive signature any arbitrary number of times, the conditions are satisfied. Hence, in this case we may set $\rho(G, +) = \infty$. Note that this is a characterization of signed graphs with no negative cycle. For any other signed graph the packing number is a finite integer, which, moreover, admits the following basic upper bound.

Lemma 6. *Given a signed graph (G, σ), we have $\rho(G, \sigma) \leq g_-(G, \sigma)$.*

Proof. When (G, σ) has no negative cycle, then, by Theorem 1, it is equivalent to $(G, +)$ and in this case both $\rho(G, \sigma)$ and $g_-(G, \sigma)$ are set to be ∞. Otherwise, let C be a negative cycle of length $g_-(G, \sigma)$ and let $\sigma_1, \sigma_2, \ldots, \sigma_l$ be a packing of (G, σ). Then each σ_i must assign a negative sign to at least one distinct edge of C, thus proving that l cannot be more than the length of C. $\square$

The upper bound of this lemma in general can be far from equal. Indeed soon we will see how to find examples of signed graphs whose girth is as large as one wishes, but its packing number is 1. Furthermore, we will also observe that to decide if the equality holds in Lemma 6 for a general signed graph (G, σ) is an NP-complete problem. However, the study of sufficient conditions under which the equality in Lemma 6 holds captures a number of well studied theories in graph theory, with the 4-coloring problem and the Four-Color Theorem being among the most famous ones.

Given a signed graph (G, σ), we say it *packs* if $\rho(G, \sigma) = g_-(G, \sigma)$. Perhaps the most important signed graph that packs is $(K_4, -)$. In Figure 1 a 3-packing of $(K_4, -)$ is presented with indication of the switching that has resulted in each of the given signed graph. Observe that the negative edges of σ_1, σ_2 and σ_3 correspond to the (unique) proper 3-edge-coloring of K_4. This leads to further developments discussed in this work.

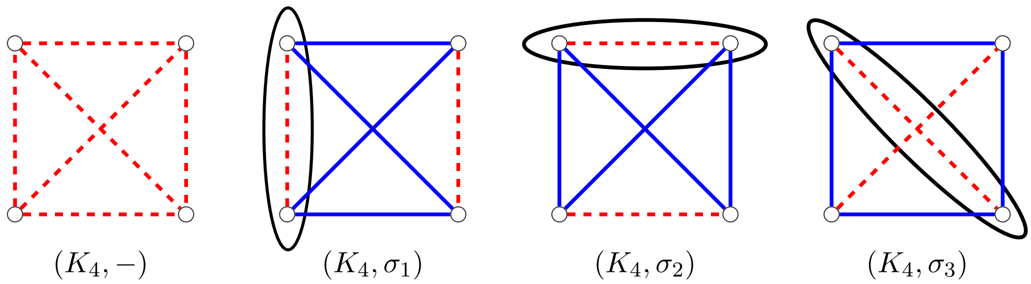

Figure 1: A 3-packing of $(K_4, -)$

On the other hand, as the smallest and perhaps simplest example of a simple signed graph whose packing number is 1 we have $(K_5, -)$. It can be easily checked that $\rho(K_5, -) = 1$ and thus $\rho(K_k, -) = 1$ for every $k \geq 5$. A strong conjecture, (see Section 9 for a precise statement) is that under certain restriction this signed graph is also a minimal signed graph with respect to taking minor and having packing number 1.

The next lemmas are among earliest observation in the study of packing number of signed graphs.

Lemma 7. *Given a graph G which is not bipartite, the packing number of the signed graph $(G, -)$ is an odd number.*

Proof. Since G is not bipartite it has an odd cycle which is a negative cycle in $(G, -)$. Thus $\rho(G, -)$ is a finite number. Let $\sigma_1, \sigma_2, \ldots, \sigma_{2l}$ be a packing of even order, that is to say the sets E_i^- are pairwise disjoint. Let $E_{2l+1} = E(G) - E_1^- \cup E_2^- \cdots \cup E_{2l}^-$. Then it is straightforward to verify that each odd cycle of G intersects E_{2l+1} in an odd number of edges and each even cycle intersects it in an even number of edges. Thus, by Theorem 1, the assignment σ_{2l+1} which assigns a negative sign to the edges in E_{2l+1} and positive sign to all other edges produces a signed graph (G, σ_{2l+1}) equivalent to $(G, -)$. Therefore, the packing number of $(G, -)$ can never be an even number. $\square$

The proof of the next lemma is quite similar to the proof of the previous lemma and we leave it as an exercise.

Lemma 8. *The packing number of any signed bipartite graph is an even number.*

The notion of packing signatures of a signed graph is developed from a discussion between the first author and T. Zaslavsky. A parallel and somewhat similar study is then carried on by N. Lacasse, a Ph.D. student of Zaslavsky. His results are presented in [8] where the notion of *negation set* is employed to refer to the set of negative edges in a signed graph. However, the parameter packing signature number is equal to packing negative cycles cover which has been first considered in [3]. This formulation together with the main contribution of [3] to this subject is mentioned in the following subsection.

3.1 Packing negative cycle covers

We point out here that the signature packing number is the same as negative cycle cover packing number of a signed graph. Based on this equivalence we have the following theorem of Gan and Johnson which can be regarded as the first result on this subject.

Given a signed graph (G, σ), a set CC of the edges of G is said to be *negative cycle cover* of (G, σ), or simply a cycle cover of (G, σ) if it contains at least one edge from each negative cycle of (G, σ). A collection $CC_1, CC_2, \ldots, CC_i$ of cycle covers of (G, σ) is said to be a *cycle cover packing* if no pair of them have a common element. The maximum number of cycle covers in a cycle cover packing is said to be *cycle cover packing number* of (G, σ). It turns out that the cycle cover packing number of any signed graph is equal to the signature packing number of it. This claim is immediately followed by employing a notion of minimality and a correspondence between minimal elements of the two notions.

Given a signed graph (G, σ), a signature σ' obtained from a switching of σ is said to be *minimal* if for no other switching σ'' we have $E^-_{\sigma''} \subsetneq E^-_{\sigma'}$. Similarly, a cycle cover CC of (G, σ) is said to be minimal if no proper subset of it is a negative cycle cover of (G, σ). It is immediate that every equivalent signature of (G, σ) contains a minimal signature and that every cycle cover of it contains a minimal cycle cover. Thus, in each of the definition of the packing numbers if we restrict ourselves to the minimal elements of the corresponding set, we have the same result. That the signature packing number and the negative cycle packing number of a signed graph (G, σ) are equal then follows from the following lemma first proved in [7] (see Theorem 7 of this reference).

Lemma 9. *Given a signed graph (G, σ), every minimal cycle cover is a minimal signature and vice versa: every minimal signature is a minimal cycle cover.*

Restated in our language of packing signatures, one of the results of [3] is to show that K_3^2, that is the signed graph of Figure 2, is a minor minimal signed graph which does not pack. It is easily observed that $\rho(K_3^2) = 1$ while $g_-(K_3^2) = 2$. On the other hand:

Theorem 10. *[3] If a signed graph (G, σ) has no K_3^2-minor, then it packs, i.e., $\rho(G, \sigma) = g_-(G, \sigma)$.*

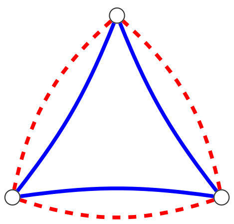

Figure 2: $\rho(K_3^2) = 1$

This result would also follow from the structural result of Gerarad's from Chapter 3 of [5], where he provide a decomposition theorem for the class of signed graphs with no K_3^2-minor.

4 Signed Projective Cubes

The *signed projective cube* of dimension d, denoted $\mathcal{SPC}_d$, is a signed graph on $\mathbb{Z}_2^d$ as the vertex set where two vertices are adjacent by a positive edge if they are at hamming distance 1 and by a negative edge if they are at hamming distance d. That is to say $\mathcal{SPC}_d$ is built from the hypercube of dimension d by taking all the edges to be positive and adding a negative edge between each pair of antipodal vertices. For the sake of completeness we also define $\mathcal{SPC}_0$ to be the signed graph on one vertex with a negative loop. The first few signed projective cubes are depicted in Figure 3. For equivalent definitions of $\mathcal{SPC}_d$ and for a proof of the following lemma we refer to [12] and [13].

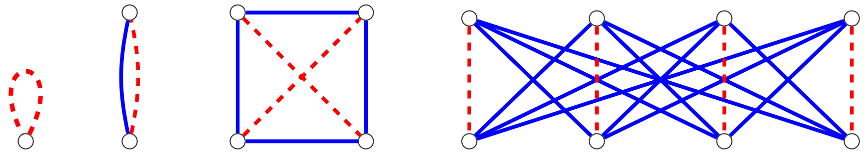

Figure 3: $\mathcal{SPC}_d$ for $d \in \{0, 1, 2, 3\}$

Lemma 11. *For odd values of d, $\mathcal{SPC}_d \in \mathcal{G}_{10}$ and for even values of d, $\mathcal{SPC}_d \in \mathcal{G}_{11}$. Moreover $g_-(\mathcal{SPC}_d) = d + 1$.*

Given the signed graph $\mathcal{SPC}_d$, one may label its positive edges by the coordinate that is the witness of the hamming distance 1 between its two ends and label negative edges by J. It is easily observed that this labeling is a proper edge-coloring of the underlying graph $\mathcal{PC}_d$. Furthermore, in this edge-coloring each pair of colors induces an edge cut of $\mathcal{PC}_d$. Thus the signed graph $(\mathcal{PC}_d, \pi_i)$, where π_i assigns a negative sign to the edges labeled i for $i \leq d$ and to the edge labeled J for $i = d+1$, is switching equivalent to $\mathcal{SPC}_d$. As no pair of these $d + 1$ signatures share a common negative edge, and together with $g_-(\mathcal{SPC}_d) = d + 1$ we have:

Lemma 12. *Given a nonnegative integer d, the signed graph $\mathcal{SPC}_d$ packs. More precisely $\rho(\mathcal{SPC}_d) = g_-(\mathcal{SPC}_d) = d + 1$.*

Observe that in the above example of $(d + 1)$-packing of $\mathcal{SPC}_d$ we not only find examples of signatures without sharing a negative edge, but also partition the set of edges of $\mathcal{PC}_d$ into sets of negative edges of the signatures. It is shown in [12] that the problem of decomposing edges of a signed graph into $d + 1$ sets, each corresponding to the negative edges of an equivalent signature, is equivalent to a homomorphism problem where the signed graph $\mathcal{SPC}_d$ plays the role of universal target. More precisely, we have the following theorem.

Theorem 13. *[12] Given a nonnegative integer d, the edge set of a signed graph (G, σ) can be decomposed into $d+1$ sets $E_1, E_2, \ldots, E_{d+1}$, with each E_i being the set of negative edges of a switching equivalent signed graph (G, σ_i), if and only if $(G, \sigma) \to \mathcal{SPC}_d$.*

Here using a modification on a signed projective cube we introduce a variant of this theorem which captures packing problems of signed graphs where the edge set is not necessarily decomposed, but rather a number of disjoint subsets are selected.

Definition 14. *We define $\mathcal{SPC}_d^o$ to be the signed graph obtained from the signed projective cube of dimension d by adding a positive loop to each of its vertices.*

The first few examples of $\mathcal{SPC}_d^o$ are given in Figure 4.

Theorem 15. *Given a nonnegative integer d, for a signed graph (G, σ), we have $\rho(G, \sigma) \geq d + 1$ if and only if $(G, \sigma) \to \mathcal{SPC}_d^o$.*

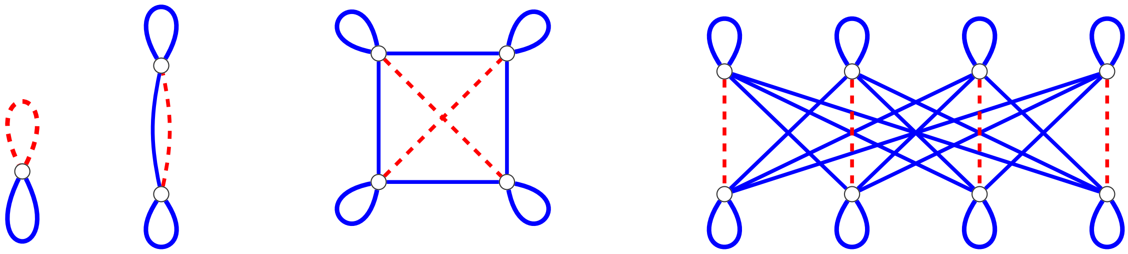

Figure 4: $\mathcal{SPC}_d^o$ for $d \in \{0, 1, 2, 3\}$

Proof. Let (G, σ) be a signed graph. First suppose $(G, \sigma) \to \mathcal{SPC}_d^o$. Following the discussion on equivalent signatures of $\mathcal{SPC}_d$, we denote by $(\mathcal{SPC}_d^o, \pi_i)$ the signed graph on $\mathcal{SPC}_d^o$ where for $i = 1, 2, \ldots, d$, the edges labeled i are the negative edges and for $i = d+1$ the edges labeled J are the negative edges. Then for each i, $i = 1, 2, \ldots, d+1$, the set of edges of G mapped to the negative edges of $(\mathcal{SPC}_d^o, \pi_i)$ forms the set of negative edges of a signature σ_i of G which is equivalent to σ. As these sets are disjoint, we have $\rho(G, \sigma) \geq d+1$.

For the inverse assume that $\rho(G, \sigma) \geq d+1$. Thus there are at least $d+1$ signatures $\sigma_1, \sigma_2, \ldots, \sigma_{d+1}$ such that each σ_i is switching equivalent to σ and the sets E_i^- are pairwise disjoint. Let $E' = E(G) - \bigcup_{i=1}^{d+1} E_i^-$. Let G' be the graph obtained by contracting all edges in E'. Let σ_i' be the signature on G' induced by the signature σ_i on G; that is to say the set of edges assigned a negative sign by σ_i' is the set E_i^-.

We claim that each pair of σ_i' and σ_j' are switching equivalent signatures on G'. This follows from discussion in Section 4 of [16], and can verified directly as well. Thus the edges of G' are decomposed into $d+1$ disjoint parts as the negative edges of the signatures σ_i', and, therefore, by Theorem 13, (G', σ_1') admits a homomorphism to $\mathcal{SPC}_d$. This then easily extends to a homomorphism of (G, σ_1) to $\mathcal{SPC}_d^o$ by noting that the edges in E' are positive in (G, σ_1) and are mapped to the positive loops. □

Following the proof technique of Lemmas 7 and 8 we have the following lemma which connects Theorem 13 and Theorem 15.

Lemma 16. *A signed graph (G, σ) belongs to $\mathcal{G}_{10} \cup \mathcal{G}_{11}$ if and only if its edge set can be partitioned into sets $E_1, E_2, \ldots, E_l$, for some integer l, each of which is the set of negative edges of a signature σ_i equivalent to σ.*

Theorem 17. *Given a signed graph (G, σ) of packing number $d+1$, we have $(G, \sigma) \in \mathcal{G}_{10} \cup \mathcal{G}_{11}$ if and only if $(G, \sigma) \to \mathcal{SPC}_d$.*

5 4-coloring of graphs and Packing signed graphs

Since $\mathcal{SPC}_2$ is switching equivalent to $(K_4, -)$, and considering the fact that for a non-bipartite graph G the packing number of $(G, -)$ is always an odd number we have the following.

Theorem 18. *A graph G is 4-colorable if and only if $\rho(G, -) \geq 2$.*

Proof. If G is bipartite, then $\rho(G, -) = \infty$ and G is 4-colorable, in which case there is nothing left to prove. Thus we assume G is not bipartite.

A graph G is 4-colorable if and only if it admits a homomorphism to K_4. By Theorem 2, that is to say: A graph G is 4-colorable if and only if the signed graph $(G, -)$ admits a homomorphism to $(K_4, -)$. Since $(K_4, -)$ is switching equivalent to $\mathcal{SPC}_2$, we have: a graph G is 4-colorable if and only if $(G, -)$ maps to

$\mathcal{SPC}_2$. As $(G, -) \in \mathcal{G}_{11}$, by Theorem 17 and Theorem 13, G is 4-colorable if and only it has packing number at least 3. Finally, since $(G, -) \in \mathcal{G}_{11}$, and by Lemma 7, a graph G is 4-colorable if and only if $(G, -)$ has a packing of order 2. □

Using the four-color theorem, or rather a strengthening of it on the class of K_5-minor-free graphs, and by Lemma 7, we have the following corollary.

Corollary 19. *Given a K_5-minor-free graph G with no loop, we have $\rho(G, -) \geq 3$.*

Given a graph G, a signed bipartite graph $S(G)$ is defined as follows: vertices of $S(G)$ consist of vertices of G as one part of $S(G)$ and for each edge uv two vertices labeled x_{uv}, y_{uv} on the other part of $S(G)$. For each edge uv of G then we build a 4-cycle $ux_{uv}vy_{uv}$. The signature of $S(G)$ is an assignment π_0 which assigns a negative sign to exactly one edge of each 4-cycle of the constructed bipartite graph. We note that the choice of π_0 is arbitrary and that different choices are not necessarily switching equivalent but they result in (switching) isomorphic graphs. This construction was first introduced in [11]. The following theorem is implied using a result of [11] and Theorem 17.

Theorem 20. *Given a simple graph G, we have $\rho(G, -) \geq 3$ if and only if $\rho(S(G)) \geq 4$ (for any choice of π_0).*

Thus to prove that $\rho(S(G)) \geq 4$ is the same as proving that G is four-colorable. Noting that for every planar graph G, the associated signed graph $S(G)$ is a signed bipartite planar graph, to claim that every signed planar simple bipartite graph has packing number at least 4 is stronger than the four-color theorem. This is proved to be the case and is discussed in more details in the next section.

6 Packing signed planar graphs

An example of a signed planar simple graph which does not map to $\mathcal{SPC}_1^o$ is given in [10]. Combined with Theorem 15 we have the following.

Proposition 21. *There exists a signed planar simple graph (G, σ) satisfying $\rho(G, \sigma) = 1$.*

Thus in Corollary 19, the assumption on the signature, i.e., that $(G, \sigma) \in \mathcal{G}_{11}$, is essential. However, with this kind of restriction a generalization of the 4CT can be proposed as follows.

Conjecture 22. *Every signed planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ packs.*

That is to say: given a signed planar graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, the packing number of (G, σ) is equal to the negative girth of (G, σ). We note that a signed connected graph is in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ if it has no positive odd closed walk, i.e. $g_{01}(G, \sigma) = \infty$.

From the discussion of Section 4 it follows that Conjecture 22 is equivalent to:

Conjecture 23. *Given a signed planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$, if $g_-(G, \sigma) = d + 1$, then $(G, \sigma) \to \mathcal{SPC}_d$.*

This conjecture, which is partly proposed in [9] and partly in [6], is shown [9] and [12] to be equivalent to the following conjecture of P. Seymour.

Conjecture 24. *Given a k-regular planar graph, it is k-edge-colorable if for each set X of odd number of vertices the edge cut $(X, V \setminus X)$ is of size at least k.*

It is easily observed that the connectivity condition in this conjecture is necessary. The conjecture is a generalization of Tait's reformulation of the 4CT. Thus the case $k = 3$ is implied by the 4CT. The cases $k = 4, 5$ were settled by B. Guenin, in 2002 using the notion of packing T-joins but it remains unpublished. The claimed proof is based on induction on k, thus the 4CT (the case $k = 3$) is assumed. The result is extended by several authors for $k = 6, 7, 8$. Our result in this work, based on the notion of packing, implies a proof of the case $k = 4$. Our proof has some similar elements to that of Guenin. There are advantages in our approach, a notable one being that: since faces are not needed, our result works for any minor closed family of 4-colorable graphs. The largest of those is the class of K_5-minor-free graphs, but taking some smaller class one may get a proof without using the 4CT. More precisely we prove that:

Theorem 25. *Any signed bipartite simple K_5-minor-free graph has a packing number at least* 4*.*

To prove Theorem 25 we establish a number of lemmas that could be of use for the general case of Conjecture 22. These are collected in the next section.

7 Packing and minors

The advantage of Conjecture 22 is that induction on the negative girth looks possible and indeed we will prove the case of negative girth being 4 use negative girth 3 (which is equivalent to the 4CT). This is based on the following easy lemma. We recall that for a subset E_1 of the edges of a graph G, the graph obtained from contracting all edges in G is denoted by G/E_1.

Lemma 26. *Let (G, σ_1) and (G, σ'_1) be two switching equivalent signed graphs with no common negative edge. Then $\rho(G, \sigma_1) \geq \rho(G/E_1, E'_1) + 1$, where E_1 and E'_1 are the sets of the negative edges of (G, σ_1) and (G, σ'_1), respectively.*

Proof. Let $\sigma_2, \sigma_3, \ldots, \sigma_{k+1}$ be k signatures on G/E_1 such that each is equivalent to $(G/E_1, E'_1)$ and that no pair of them have a common negative edge. Let $E_2, E_3, \ldots, E_{k+1}$ be the set of negative edges in $(G/E_1, \sigma_2)$, $(G/E_1, \sigma_3), \ldots, (G/E_1, \sigma_{k+1})$, respectively. Then it is quite straightforward to check that (G, E_1), (G, E_2), $(G, E_3), \ldots, (G, E_{k+1})$ is a packing of (G, σ_1). □

In applying this lemma one should note that if (G, σ_1) is in $\mathcal{G}_{11}$, then $(G/E_1, E'_1)$ is in $\mathcal{G}_{10}$ and that conversely, if $(G, \sigma_1) \in \mathcal{G}_{10}$, then $(G/E_1, E'_1) \in \mathcal{G}_{11}$. Thus if we are attempting to prove that for a minor closed family $\mathcal{C}$ of graphs, every signed graph (G, σ), $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$ and $G \in \mathcal{C}$, packs, then in an approach which is based on induction on the negative girth of (G, σ), assuming the claim holds as long as $g_-(G, \sigma) \leq k$, and given a signed graph (G, σ) in the class satisfying $g_-(G, \sigma) = k + 1$, it would be enough to find signatures σ_1 and σ'_1, each equivalent to σ and such that $g_-(G/E_1, \sigma'_1) \geq k$.

When (G, σ) is in $\mathcal{G}_{10}$, finding σ'_1 or rather E'_1 is quite simple, it would be enough to set $E'_1 : E \setminus E_1$. Thus in this case the main task in hand would be to find an appropriate σ_1. When (G, σ) is in $\mathcal{G}_{11}$, then we must provide both σ_1 and σ'_1 when applying this technique. However, in this case finding σ' can also be done with a condition on σ_1: let (G, σ_1) be a switching of $(G, -)$ with the property that every negative cycle of $(G, -)$, that is every odd cycle of G, has at least one (therefore, at least 2) positive edges. Thus in the minor (G/E_1) of G every negative closed walk of G has an image which is a nontrivial closed walk of G/E_1. The set of all these closed walks have a θ-property: that if we take three $x - y$ walks P_1, P_2 and P_3, then of the three closed walks P_1P_2, P_1P_3 and P_2P_3 either none or exactly two of them are in the set. Then it follows from Theorem 10 of [13] that this set of closed walks is the set of negative closed walks of a signature on G/E_1. Taking E'_1 as the set of negative edges of such a signature then works.

Thus based on this discussion, Conjecture 22 is equivalent to the following conjecture:

Conjecture 27. *Given a signed planar graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, there is an equivalent signature σ_1 such that every negative cycle of (G, σ) has at least $g_-(G, \sigma) - 1$ positive edges.*

Theorem 28. *Conjecture 22 and Conjecture 27 are equivalent.*

Proof. That Conjecture 22 implies Conjecture 27 is straightforward: if $\sigma_1, \sigma_2, \ldots, \sigma_k$ is a packing of (G, σ), then any of σ_i's satisfies the condition of Conjecture 27: every negative cycle of (G, σ_i) has at least one negative edge in each of (G, σ_j), $j \neq i$, all of which are positive in (G, σ_i).

Suppose Conjecture 27 holds. Let (G, σ) be a counterexample to Conjecture 22 of minimum possible negative girth, say k. By the statement of Conjecture 27 there is a switching equivalent signature σ_1 where each negative cycle has at least $k - 1$ positive edges. Considering the signed graph $(G/E_1, \sigma'_1)$, where σ'_1 is a signature equivalent to σ but disjoint from it, the negative girth is $k - 1$. By our choice of (G, σ), which has minimal negative girth among all counterexamples, $(G/E_1, \sigma'_1)$ packs. Thus there are signatures $\sigma_2, \sigma_3, \ldots \sigma_k$ where no pair of them have a common negative edge. Together with E_1, then they correspond to signatures $\sigma_1, \sigma_2, \ldots, \sigma_k$ proving that (G, σ) packs. $\square$

Following this formulation, given a signed graph (G, σ) of negative girth k, a negative cycle whose number of positive edges is (strictly) less than $k - 1$ will be referred to as *super negative* cycle. Thus Conjecture 27, and, therefore, Conjecture 22, are to say that any planar signed graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$ can be switched so that it has no super negative cycle.

There are a couple of important remarks to make here: first is that we did not really use the assumption of planarity here, rather we used the fact that we are working with a minor closed family of graphs, or even more precisely, we want the minor G/E_1 to be in our family. The second remark is that if we restrict both conjectures on subclass of signed graphs of negative girth at most k, then these restricted versions are still equivalent.

Following these observation, we would like to work with a minor closed family $\mathcal{C}$ of graphs such that any signed graph (G, σ) with $G \in \mathcal{C}$ and $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$ packs. If we take all signed graphs $(G, -)$ in this family, where G is a simple graph, then the fact that $(G, -)$ packs implies, in particular, that G is 4-colorable. Thus, in particular, K_5 is not in $\mathcal{C}$ and as $\mathcal{C}$ is a minor closed family, we are working with a subclass of K_5-minor-free graphs. One may assume that $\mathcal{C}$ is indeed the class of K_5-minor-free graphs, but there is advantage in this general statement which will be pointed out in Section 9.

Before continuing, we state a couple of facts on K_5-minor-free graphs.

The first is the following classic theorem of Wagner on characterization of K_5-minor-free graphs. Here W is the graph of Figure 5.

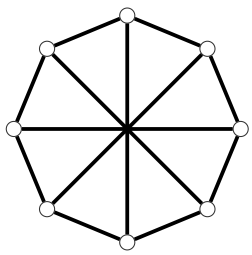

Figure 5: Wagner graph

Theorem 29. *(Wagner) Every edge-maximal graph with no K_5-minor can be obtained by means of 3-sum and 2-sum, starting from planar triangulations and copies of W.*

A 3-sum of two graphs G and H is to identify the vertices of one triangle of G with the vertices of a triangle of H. Similarly, their 2-sum is to identify the vertices of an edge from G with the vertices of an edge from H.

A first and classic corollary of this decomposition theorem is that, the four-color theorem can be extended to the class of K_5-minor-free graphs, this is a classic application of this decomposition theorem. A second corollary is to extend the application of the Euler formula to bound the number of edges of a triangle-free members of the class, we give a proof of this folklore fact for the sake of completeness.

Proposition 30. *If G is a K_5-minor-free graph of girth at least 4, then $|E(G)| \leq 2|V(G)| - 4$.*

Proof. First we build a graph G' from G by adding edges to make it edge-maximal while it remains K_5-minor-free. Obviously, G is a spanning subgraph of G'. Then by Theorem 29, G' is obtained from 3-sum or 2-sum of planar triangulations and copies of W. Suppose G' is obtained by clique-sums of $G'_1, G'_2, \dots, G'_n$. Without loss of generality, let G''_i be the clique-sums of $G'_1, \dots, G'_i$. Let G_i be the subgraph of G''_i contained in G, let H_i be the subgraph of G'_i contained in G. Then $G = G_n$, and it suffices to prove that $|E(G_n)| \leq 2|V(G_n)| - 4$.

We first claim that the inequality holds for each H_i. That is because each H_i is either planar and triangle-free, in which case $|E(G_1)| \leq 2|V(G_1)| - 4$ by application of the Euler formula, or it is a spanning subgraph of W, and the inequality holds for W itself. Thus in particular $G_1 = H_1$ satisfies the conditions. We complete the proof by induction on i, showing that each G_i satisfies the bound. That is because if G''_i is obtained from 3-sum of G''_{i-1} and G'_i, then G_i is formed from G_{i-1} and H_i by identifying three vertices and at most two edges. Since they both satisfy the inequality, G_i also satisfies it. If G''_i is obtained from 2-sum of G''_{i-1} and G'_i, then G_i is formed from G_{i-1} and H_i by identifying two vertices and at most one edge, and similarly, G_i also satisfies the inequality. □

We are now ready to state and prove the following.

Theorem 31. *Let $\mathcal{C}$ be a minor closed family of graphs whose members are 4-colorable. Then for any bipartite simple graph G in $\mathcal{C}$ and for any signature σ we have $\rho(G, \sigma) \geq 4$.*

Proof. Assume that (G, σ) is a minimal counterexample to the theorem. That is to say that G is a simple bipartite graph in $\mathcal{C}$ with a signature σ such that $\rho(G, \sigma) = 2$ and that for any edge e of G, the signed bipartite graph $(G - e, \sigma)$ has packing number at least 4.

Here the signature in $(G - e, \sigma)$ is the restriction of the signature of (G, σ), thus, with a minor abuse of notation, we use σ to denote both. Furthermore, if $(G - e, \sigma')$ is obtained from $(G - e, \sigma)$ by switching at a subset X of vertices, then we may use (G, σ') to denote the signature which is obtained from (G, σ) by switching at the same vertex set X, in this case $(G - e, \sigma')$ will be induced signed subgraph of (G, σ').

With this notation and with the assumption on the minimality of (G, σ), we conclude that for each edge e of G, there are four signatures σ_1, σ_2, σ_3 and σ_4 such that any pair of them either have no common negative edge, or e is their only common negative edge. We recall, by Theorem 28, (G, σ) is also a minimal counterexample to Conjecture 27. As we are considering negative girth to be 4, given a signature, a super negative cycle is a negative cycle with only one positive edge. If for any signature equivalent to σ, in particular for one of the signatures σ_i, $i = 1, 2, 3, 4$, the signed graph (G, σ_i) has no super negative cycle then we are done. On the other hand $(G - e, \sigma_i)$ has no super negative cycle for $i = 1, 2, 3, 4$ because each negative cycle has at least one negative edge in each σ_i which is a positive edge in the other three signatures. Thus each (G, σ_i), $i = 1, 2, 3, 4$, must have a super negative cycle which contains e.

One easily observes that replacing a signature σ_i, $i = 1, 2, 3, 4$, with a minimal signature contained in σ_i may only decrease the number of super negative cycles. Thus we may assume each σ_i is a minimal signature. This in particular implies that:

$$\text{not all edges incident to the same vertex are negative in a given } \sigma_i. \tag{1}$$

Let $e = uv$ be an edge where $d(u) = 2$. Let σ_1, σ_2, σ_3, σ_4 be a 4-packing of $(G - e, \sigma)$ consisting of four minimal signatures. We claim that, for each signature σ_i, $i = 1, 2, 3, 4$, at least one super negative cycle C_i in (G, σ_i) has the following property:

P1. Except possibly the two vertices of the only positive edge of C_i, every (other) vertex of C_i has a degree at least 4 in G.

Since σ_i's are assumed to be minimal, and by (1), in none of (G, σ_i) the two edges incident to u are negative. They cannot be both positive either, as otherwise (G, σ_i) has no super negative cycle and we are done. If necessary by switching at u we may assume $e = uv$ is the negative edge in each of (G, σ_i) and that the other edge incident to u, say uw, is positive in all of them. We now consider a super negative cycle C_i of (G, σ_i). Observe that, as this cycle must contain e, and since u is a vertex of degree 2, it must also contain uw, and thus uw is its only positive edge. Let x be a vertex on C_i which is of degree 2 or 3 in G and $x \notin \{u, w\}$. Then x is not of degree 2 because of (1), thus $d(x) = 3$. Let xy be the edge incident to x which is not on C_i. Observe that, again by (1), xy is a positive edge of (G, σ_i). Moreover, as $x \notin \{u, w\}$, xy is distinct from uw. Thus no super negative cycle of (G, σ_i) contains the edge xy. Let σ'_i be the signature on G obtained from a switching at the vertex x. Observe the following 3 facts: 1. C_i is not a super negative cycle in (G, σ'_i), 2. Because $x \notin \{u, w\}$, the number of positive edges incident to u is not decreased but it may have gone up if $x = v$. 3. If C'_i is a super negative cycle of (G, σ'), then C'_i is also a super negative cycle of (G, σ_i) and moreover, signs of each edge of C'_i are the same in both (G, σ_i) and (G, σ'_i). Thus if a super negative cycle of (G, σ'_i) satisfies the conditions of **P1** then we are done, otherwise we repeat the process. As we are working with a finite graph, and the number of super negative cycles is finite, at the end either we find a super negative cycle that satisfies the conditions of **P1**, or we obtain a signature with no super negative cycle in which case we can find a packing of four signatures and we are done.

In conclusion, we have a 4-packing σ_1, σ_2, σ_3, σ_4 of $(G - e, \sigma)$ with the property that each (G, σ_i), $i = 1, 2, 3, 4$, contains a super negative cycle C_i in which uw is the only positive edge, and, except for u and w, every other vertex on C_i is of degree at least 4 in G. Let x_i be the neighbor of v on C_i distinct from u. Observe that as G is bipartite, x_i is also distinct from w. We observe, furthermore, that any pair of the signatures σ_i and σ_j, $i, j \in \{1, 2, 3, 4\}$, can only have $e = uv$ as the common negative edge. We conclude that v has (at least) four neighbors each of which is of degree 4.

This argument can be repeated exchanging the roles of v and w, thus we conclude that:

Claim 1. *For each vertex u of degree 2, each of its neighbors v and w has four neighbors each of degree at least 4.*

Next we aim to prove a similar claim for the neighborhood of a 3-vertex. Proofs are quite similar, but we need to take care of further details.

Let u be a vertex of degree 3 and let v, w and t be its three neighbors. Consider $e = uv$ and let σ_1, σ_2, σ_3, σ_4 be a 4-packing of $(G - e, \sigma)$ consisting of four minimal signatures. We first observe that in each of (G, σ_i), $i = 1, 2, 3, 4$, not all three edges uv, uw, ut are of the same sign. That is because three of them being negative would contradict our choice of σ_i's being minimal and three of them being positive will leave no room for a super negative cycle in (G, σ_i) containing uv, noting that there is also no super negative cycle in $(G - e, \sigma_i)$ by our choice of σ_1, σ_2, σ_3, σ_4. If for any of σ_i the signed graph (G, σ_i) contains two negative edges incident to u, then we will switch at u to get a signature σ'_i.

So altogether we will work with signatures σ'_1, σ'_2, σ'_3, σ'_4 such that in each signed graph (G, σ'_i), the signature σ'_i assigns one negative and two positive signs to the edges uv, uw, ut and σ'_i is either the same as σ_i, or is obtained from σ_i by switching at u. Observe that, by the choice of σ_i, $i = 1, 2, 3, 4$, any pair of signatures among σ'_i's have at most one common negative edge, and if so, that edge is one of uv, uw, ut. We may further modify σ'_i's to have them as minimal signatures. One may remind the reader again that replacing a

σ_i' with potentially minimal subset would not create a new intersection among σ_i''s and that the only affect such a replacement may have on super negative cycles is to kill off some.

We claim again that, for each signature σ_i', $i = 1, 2, 3, 4$, at least one super negative cycle C_i in (G, σ_i') has the following property: every vertex of C_i not incident with the positive edge of C_i has degree at least 4 in G.

To prove the claim we first note that C_i is also a super negative cycle of (G, σ_i). That is because first of all edges not incident to u that are negative in (G, σ_i') are also negative in (G, σ_i). Secondly, since the only positive edge of C_i is incident to u, each edge of C_i which is not incident to u is negative in both (G, σ_i') and, therefore, in (G, σ_i). Thirdly, since C_i is a negative cycle of (G, σ), and as G is bipartite, in both (G, σ_i) and (G, σ_i') one of the two edges incident with u is positive and the other is negative.

We conclude two facts from this: 1. that every super negative cycle of (G, σ_i') must contain the edge uv, and, therefore, 2. the positive edge of every super negative cycle of (G, σ_i') is incident to u. We note that this is not necessarily true for (G, σ_i).

We now consider a shortest super negative cycle C_i of (G, σ_i') and assume that it contains a vertex x not incident to the positive edge of C_i and that $d(x) \leq 3$. Once again by the fact that σ_i' is a minimal signature, we conclude that x must be of degree exactly 3 and that the edge xy which is the edge incident with x but not in C_i must be positive. We claim that $y \neq u$. Otherwise, since C_i must contain u as well, xy is a chord of C_i. Then xy creates two cycles with C_i and the part that does not contain the positive edge of C_i is a super negative cycle of (G, σ_i') but it is shorter than C_i, contradicting the choice of C_i.

That $y \neq u$ implies that no super negative cycle of (G, σ_i') contains xy. Let σ_i'' be the signature obtained from a switching of (G, σ_i') at x. What we have observed is that: 1. C_i, which was a super negative cycle of (G, σ_i'), is not a super negative cycle in (G, σ_i''), and 2. for every super negative cycle C of (G, σ_i'') each edge of C has the same sign in (G, σ_i'') and (G, σ_i'). We observe that σ_i'' is not necessarily minimal, however, replacing it with a minimal signature can only kill off some super negative cycles without any change on the signs of edges of the remaining one. Thus the remaining super negative cycles are the super negative cycles of (G, σ_i') without any change to the signs of their edges. We continue this process, if we end up with a signature where there is no super negative cycles, then we have found a 4-packing of (G, σ). Else we must end up with a super negative cycle C_i' where each vertex not incident with the positive edge of C_i' is of degree at least 4 in G. Since we have retained the sign of super negative cycles during the process, C_i' is also super negative cycle of (G, σ_i') with the property that each vertex not incident with the positive edge of C_i' is of degree at least 4 in G. We recall that each super negative cycle of (G, σ_i') must contain the edge $e = uv$ and that its only positive edge must be incident to u. Thus if vz_i, $z_i \neq u$, is an edge of C_i', then z_i is of degree at least 4 in G. As this must be true for every i, $i = 1, 2, 3, 4$, we have proved the following claim.

Claim 2. *If v is a vertex of degree 3 in G, then its nieghbors x, y and z each has at least four neighbors of degree at least 4.*

We may now employ the discharging technique to obtain a contradiction.

Discharging procedure

The initial charge of each vertex v is defined as: $\omega(v) = d(v)$. As G is K_5-minor-free and bipartite (thus triangle-free), by Proposition 30, we have $\displaystyle\sum_{v \in V(G)} \omega(v) \leq 2|V(G)| - 8$. However, the following discharging rule will redistribute charges such that each vertex has a charge of at least 4, contradicting this formula.

(R1) Each vertex of degree 2 or 3 receives a charge of 1 from each of its neighbors.

Our two claims imply that for vertex v of degree 2 or 3 all neighbors are of degree at least 5, and thus while v gets a charge of 1 from each of its neighbors, it looses no charge, and thus has a final charge of at least 4. On the other hand a neighbor of such a vertex v has at least four vertices each of which is of degree at least 4, thus its charge will never go below 4. $\square$

Corollary 32. *Any signed bipartite K_5-minor-free graph admits a homomorphism to SPC_3.*

8 Algorithmic conclusion

We recall that the proof of the four-color theorem provided in [15] leads to a quadratic time algorithm for 4-coloring of planar graphs. More precisely, that is an algorithm A which takes as an input a simple planar graph G and gives as an output a proper 4-coloring of G, in time $O(|V(G)|^2)$. Using the Wagner decomposition theorem, this works on the class of K_5-minor-free graphs as well. This is equivalent to giving a 3-packing of the signed graph $(G, -)$ as we discussed before. We may then use this algorithm to give an algorithm BP which takes as an input a signed bipartite planar simple graph (G, σ) and gives, as an output, a 4-packing of (G, σ) in time $O(|V(G)|^3)$. This follows easily from our proof: Since G is planar, bipartite and simple, it has at most $2n - 4$ edges. We may simply assume G is 2-connected as one may combine solutions on distinct 2-connected blocks. Our discharging proof implies that G has either a vertex v of degree 2 where at least one of the neighbors, say x, has at most 3 neighbors of degree at least 4, or it has a vertex u of degree 3 each of whose neighbors have at most 3 neighbors of degree 4 or more.

Having found such a vertex v or u, that can be done in a linear time, we remove from (G, σ) an edge e incident to v or u. Assume a solution $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ is provided for $(G - e, \sigma)$. By the proof given in the previous section we know one of the four signed graphs $(G, \sigma_1), (G, \sigma_2), (G, \sigma_3), (G, \sigma_4)$ has no super negative cycle. This can be verified by checking for a loop in the graphs G/E_1, G/E_2, G/E_3 and G/E_4, noting that contracting these edges and looking for a loop can all be done in linear time. Suppose G/E_4 has no loop. Then we apply algorithm A on the graph G/E_4 to get signatures σ'_1, σ'_2, σ'_3. These three signatures together with σ_4 form a 4-packing of (G, σ).

To find a solution for $(G - e, \sigma)$, which we had assumed in the argument above, one may repeat the same process. Assuming G is on n vertices, since G has at most $2n - 8$ edges, the algorithm A might be recalled at most $2n - 8$ times. As algorithm A runs in time $O(n^2)$, the running time of the full algorithm is $O(n^3)$.

Mapping a signed bipartite graph (G, σ) to SPC_3, given a 3-packing $\sigma_1, \sigma_2, \sigma_3$, can be done in linear time: label negative edges in (G, σ_1) by 001, those in (G, σ_2) by 010, ones in (G, σ_3) by 100 and then label the remaining edges 111 noting that they form the negative edges of an equivalent signature. Observe that sum of the labels of the edges in each cycle is 000. Now for each connected component of (G, σ) take an arbitrary vertex, say x and map it to the vertex 000 of SPC_3. Then for a vertex y in the same component as x, take an xy path P and map y to sum of labels of edges of the path P. It can be readily verified that this is a mapping of (G, σ) to SPC_3.

We note that the algorithm works the same for signed bipartite K_5-minor-free graphs. However, the planar case has the following application on the dual.

Corollary 33. *Given a 4-regular planar multigraph G where each set X of odd number of vertices is connected to $V \setminus X$ by at least 4 edges, we have $\chi'(G) = 4$. Moreover, a 4-edge-coloring can be found in time $O(|F(G)|^3)$, where $|F(G)|$ is the number of faces of G.*

9 Concluding remarks

We introduced the notion of packing signatures in signed graph and we established connections with a number of problems such as 4-coloring of graphs, edge-coloring of planar graphs, etc.

We proved that given a minor closed family $\mathcal{C}$ of 4-colorable graphs, for any bipartite simple graph in $\mathcal{C}$ and any signature σ on it, the packing number of (G, σ) is at least 4. The largest family to which this result may apply is the class of K_5-minor-free graphs where 4-colorability of a general member is established by the four-color theorem. However, if we take smaller classes where 4-coloring can be verified without the use of the four-color theorem, then the result on the packing number will also be independent of the four-color theorem. An interesting case to mention is the following.

Theorem 34. *Given a signed bipartite simple graph (G, σ) where G has treewidth at most 3, we have $\rho(G, \sigma) \geq 4$.*

Corollary 35. *Every signed bipartite simple graph of treewidth at most 3 admits a homomorphism to SPC_3.*

The class of graphs of treewidth at most 3 is a minor closed family of graphs that is a subclass of K_5-minor-free graphs. More precisely, as proved in [1], it consists of graphs which do not have any of the four graphs of Figure 6 as a minor. That loop-free members of this class are 4-colorable follows from the fact that edge-maximal elements are 3-trees. Thus Theorem 34 is proved without using the four-color theorem.

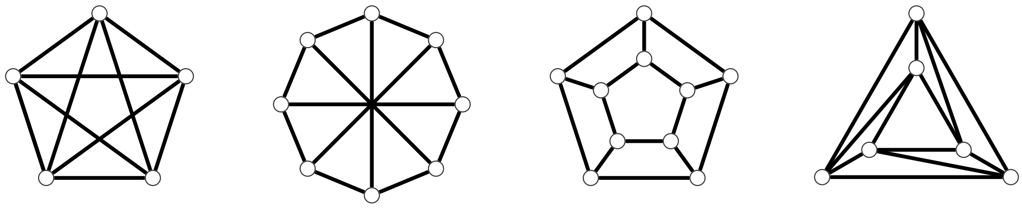

Figure 6: Forbidden minors for graphs of treewidth at most 3

On the other hand, it would be expected that a stronger version of Theorem 31 would hold. Such a strengthening would be based on the notion of minor of signed graphs rather than minor of graphs. More precisely the following conjecture is stronger than Conjecture 22.

Conjecture 36. *Given a signed graph $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, if (G, σ) has no $(K_5, -)$-minor, then it packs.*

The idea of induction on the negative girth would work here as well. That is because if σ_1 and σ_2 are two disjoint signatures each equivalent to (G, σ), then $(G/E_1, E_2)$ is a minor of (G, σ), and if $(G, \sigma) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$, then $(G/E_1, E_2) \in \mathcal{G}_{11} \cup \mathcal{G}_{10}$.

However, the class of signed graphs with no $(K_5, -)$-minor is not a sparse family and contains signed graphs with $O(n^2)$ number of edges. Thus one cannot expect the discharging technique we used here to work directly. However, one may look for decomposition results where the planar case studied here would work as a base class.

Finally we note a generalization on a different direction. While in this work we asked for finding maximum number of disjoint signatures all equivalent to a given signature, in a more general setting one might be given a set of signatures, say $\{\sigma_1, \sigma_2, \ldots, \sigma_l\}$ on a graph G, and the task then would be to find signatures $\sigma'_1, \sigma'_2, \ldots, \sigma'_l$ where σ'_i is equivalent to σ_i and the sets of negative edges $E'_1, E'_2, \ldots, E'_l$ are pairwise disjoint. When $\sigma_1, \sigma_2, \ldots, \sigma_l$ are all equivalent to a signature σ, then the maximum l is the packing number we studied here. This general version is studied in [14].

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