On the packing number of antibalanced signed simple planar graphs of negative girth at least 5

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Abstract

The *packing number* of a signed graph (G, σ), denoted $\rho(G, \sigma)$, is the maximum number l of signatures $\sigma_1, \sigma_2, \ldots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets of negative edges $E^-_{\sigma_i}$ of (G, σ_i) are pairwise disjoint. A signed graph *packs* if its packing number is equal to its negative girth. A reformulation of some well-known conjecture in extension of the 4-color theorem is that every antibalanced signed planar graph and every signed bipartite planar graph packs. On this class of signed planar graph the case when negative girth is 3 is equivalent to the 4-color theorem. For negative girth 4 and 5, based on the dual language of packing T-joins, a proof is claimed by B. Guenin in 2002, but never published. Based on this unpublished work, and using the language of packing T-joins, proofs for girth 6, 7, and 8 are published. We have recently provided a direct proof for girth 4 and in this work extend the technique to prove the case of girth 5.
Keywords: signed graph, packing number, planar graphs.

1 Introduction

A graph G is called *planar* if it can be embedded in plane in such a way that its edges intersect only at their endpoints. A *signed graph* (G, σ) is a graph together with a signature σ which assigns a sign (i.e., $+$ or $-$) to each edge of G. We denote by E^-_σ the set of negative edges of a signed graph (G, σ). Given a graph G, the signed graph $(G, -)$ (respectively, $(G, +)$) is the signed graph where all edges are negative (positive). One of the key notions in the study of signed graphs is the concept of switching. A *switching* of a signed graph (G, σ) at a vertex v is the operation of multiplying the signs of all edges incident to v by a $-$. When e is a loop on a vertex v, then v will be viewed as the two ends of e, which means switching does not affect the sign of a loop. A *switching* of (G, σ) is a collection of switchings at each of the elements of a given set X of vertices. That is equivalent to switching the signs of all edges in the edge cut $(X, V \setminus X)$. Two signatures σ_1 and σ_2 on a graph G are said to be *equivalent* if one can be obtained from the other by a switching, in this case we say (G, σ_1) is *switching equivalent* to (G, σ_2).

Given a signed graph (G, σ) and a signature σ_i equivalent to σ, when there is no ambiguity, we may write E_i^- in place of $E_{\sigma_i}^-$. It is easily observed that (G, σ_1) and (G, σ_2) are switching equivalent if and only if the symmetric difference $E_1^- \triangle E_2^-$ is an edge cut of G.

The sign of a structure in a signed graph (G, σ) is the product of the signs of the edges in the given structure, counting multiplicity. Note that the signs of cycles are invariant under a switching operation and they determine some crucial properties of a signed graph. If every cycle in a signed graph (G, σ) is positive, then (G, σ) is said to be *balanced*. A signed graph (G, σ) is said to be *antibalanced* if $(G, -\sigma)$ is *balanced*.

Harary was first to show that a signed graph is balanced if it is equivalent to $(G, +)$. An extension of this is the following result of Zaslavsky which shows that the set of negative cycles of a signed graphs uniquely determines the equivalence class of signatures:

Theorem 1. *[10] Given two signatures σ_1 and σ_2 on a graph G, they are equivalent if and only if (G, σ_1) and (G, σ_2) have the same set of negative cycles.*

Given a signed graph (G, σ) and an element $ij \in \mathbb{Z}_2^2$, we define $g_{ij}(G, \sigma)$ to be the length of the shortest closed walk W whose number of negative edges modulo 2 is i and whose length modulo 2 is j. When there exists no such a closed walk, we define $g_{ij}(G, \sigma) = \infty$. Furthermore, the length of a shortest negative closed walk will be denoted by $g_-(G, \sigma)$ (i.e., $g_-(G, \sigma) = \min\{g_{10}(G, \sigma), g_{11}(G, \sigma)\}$). Given $ij \in \mathbb{Z}_2^2, ij \neq 00$, the class $\mathcal{G}_{ij}$ of signed graphs is defined as follows:

$$\mathcal{G}_{ij} = \{(G, \sigma) \mid g_{i'j'}(G, \sigma) = \infty \text{ for } i'j' \in \mathbb{Z}_2^2 - 00, i'j' \neq ij\}.$$

Hence, $\mathcal{G}_{01}$ is the class of signed graphs (G, σ) which can be switched to $(G, +)$ and $\mathcal{G}_{11}$ is the class of signed graphs (G, σ) which can be switched to $(G, -)$. The class $\mathcal{G}_{10}$ is the class of signed bipartite graphs.

Given signed graphs (G, σ) and (H, π), a *homomorphism* of (G, σ) to (H, π) is a mapping φ of the vertices and edges of G to the vertices and edges of H, respectively, such that adjacencies, incidences and signs of closed walks are preserved. When there exists such a homomorphism, we write $(G, \sigma) \to (H, \pi)$. The definition of homomorphisms of signed graphs implies the following no-homomorphism lemma:

Lemma 2. *If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for every $ij \in \mathbb{Z}_2^2$.*

The *packing number* of a signed graph (G, σ), denoted $\rho(G, \sigma)$, is the maximum number of signatures $\sigma_1, \sigma_2, \ldots, \sigma_l$ such that each σ_i is switching equivalent to σ and the sets E_i^- are pairwise disjoint. The *signed projective cube* of dimension d, denoted $\mathcal{SPC}_d$, is a signed graph with the vertex set $\mathbb{Z}_2^d$, two vertices of $\mathcal{SPC}_d$ being adjacent by a positive edge if they are at hamming distance 1 and by a negative edge if they are at hamming distance d. Let $\mathcal{SPC}_d^o$ be the signed graph obtained from $\mathcal{SPC}_d$ by adding a positive loop to each of its vertices. A relation between packing numbers and homomorphisms observed in [9] is as follows:

Theorem 3. *[9] Given a non-negative integer d, for a signed graph (G, σ), we have $\rho(G, \sigma) \geq d + 1$ if and only if $(G, \sigma) \to \mathcal{SPC}_d^o$.*

Combined with the main result of [1], this implies that given an integer $k \geq 2$, for the general class of signed graphs the problem of deciding if $\rho(G, \sigma) \geq k$ is an NP-complete problem. In contrast the following relaxation of the problem is easily tractable. Given a signed graph $(G, -)$ where all edges are negative and a set $\{\sigma_1, \ldots, \sigma_r\}$ of signatures each obtained by a switching of $(G, -)$, we can easily decide and find if a switching equivalent signed graph (G, σ') exists which has no common negative edge with any of σ_i's.

Theorem 4. *Given a signed graph $(G, -)$ and switching equivalent signed graphs $(G, \sigma_1), \ldots, (G, \sigma_r)$, there exists a switching equivalent signed graph (G, σ) which has no common negative edge with any of (G, σ_i)'s if and only if the set $\cup_{i=1}^{r} E_i^-$ induces a bipartite graph.*

Proof. For the only if part, observe that if (G, σ) is obtained from a switching of $(G, -)$, then the set of positive edges of (G, σ) is an edge cut of G. As all edges in $\cup_{i=1}^{r} E_i^-$ must be positive in (G, σ), the claim follows. For the if part, assume $\cup_{i=1}^{r} E_i^-$ is bipartite. A bipartition of the the subgraph induced by these edges can be viewed as an edge cut of G. A signature σ then is obtained from switching $(G, -)$ at this edge cut. □

Given a signed graph (G, σ), since every signature must assign at least one negative sign to the edges of a negative cycle, we know that: $\rho(G, \sigma) \leq g_-(G, \sigma)$. We say (G, σ) *packs* if $\rho(G, \sigma) = g_-(G, \sigma)$. It is straightforward to verify that $\rho(K_4, -) = g_-(K_4, -) = 3$, so $(K_4, -)$ packs. Note that $(K_5, -)$ is the smallest signed simple graph whose packing number is 1, in particular $(K_5, -)$ does not pack. Meanwhile, the smallest signed multigraph which does not pack has only three vertices, that is the signed graph K_3^2 as depicted in Figure 1.

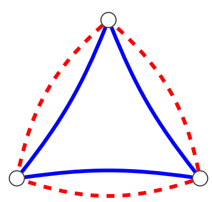

Figure 1: $\rho(K_3^2) = 1$, $g_-(K_3^2) = 2$

Restated in the language of packing signatures, a result of Gan and Johnson [2] in 1989 claimed that if a signed graph (G, σ) has no K_3^2-minor, then it packs. This result applies to signed graphs allowing multi-edges and loops. When restricted on signed simple graphs, the story becomes much more complicated, but it is still expected that in some sense $(K_5, -)$ is the minor-minimal element among certain classes of signed simple graphs that do not pack. More precisely, the following conjecture is strongly related to some of the central problems in graph theory and, in particular, to the four-color theorem and to several conjectures in extension of it.

Conjecture 5. Every signed simple planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ packs.

The restriction to the classes $\mathcal{G}_{11}$ and $\mathcal{G}_{10}$ of signed planar graphs is necessary. There exists a planar simple graph (thus negative girth 3) whose packing number is 1. This follows from a coloring result of [3], we refer to [9] for more details.

However, one may expect that, while remaining in the class $\mathcal{G}_{11} \cup \mathcal{G}_{10}$, the condition of planarity can be replaced with having no $(K_5, -)$-minor.

The restriction of the conjecture to the subclass with negative girth 3 is equivalent to the four-color theorem. The restriction to the class with negative girth 4 is known to imply the four-color theorem and is proved using it, historical background provided next.

In the subclass of planar graphs the conjecture can be restated using the dual notion of packing T-joins where T would be vertices of the dual that correspond to the negative faces of the planar embedding. The statement of the conjecture based on the notion packing T-join was first proposed by B. Guenin in early 2000's who then gave a proof of the next two cases. In our language that would be proving the conjecture for members of the class whose negative girth is 4 or 5. The T-join approach is extended in three follow up work which means that the conjecture is proved for the cases with negative girth at most 8. We note that proof for each case of girth condition relies on the proof for the earlier cases, thus dependent on the proof of the 4-color theorem. However, the work of Guenin remains unpublished and mostly not available.

An independent proof for the case of girth 4 is recently given in [9]. This proof has extra advantage that works for any minor closed family that are 4-colorable. Thus, on the one hand it works for the larger family of K_5-minor free graphs, and, on the other hand, it provides a proof without the use of the 4-color theorem for subclasses such as graphs of treewidth at most 3. In this work we build up on our method from [9] to verify the case of girth 5 of this conjecture. The main idea of the proof is presented in [9]. We first provide a reformulation of the conjecture, then we do a double induction and use different statements for different directions of the induction. The restatement of the conjecture is based on the following definition.

Given a signed graph (G, σ) of negative girth k, a negative cycle C of it is said to be *super negative* with respect to σ if it has at most $k-2$ positive edges. The key property of a super negative cycle, relevant to this study, is in the following observation. Let σ' be a signature equivalent to σ but disjoint from it. One can easily find such a signature using Theorem 4. Let $G_{/\sigma}$ be the graph obtained from G by contracting the negative edges of σ and (by little abuse of notation) let σ' be the signature on $G_{/\sigma}$ where the negative edges of it are the images of the negative edges of (G, σ'). That $(G_{/\sigma}, \sigma')$ is well defined is because the two signatures do not share a negative edge. Now a negative cycle C in $(G_{/\sigma}, \sigma')$ is of length less than or equal to $k-2$ if and only if it is the image of a super negative cycle of (G, σ). In other words, if (G, σ) has no super negative cycle, then $(G_{/\sigma}, \sigma')$ has negative girth $k-1$ and thus one may apply induction on k. This is the key point in showing that the following is equivalent to Conjecture 5. We refer to [9] for more details. However, we note that since in this paper we will be working with signed graphs of the form $(G, -)$ of negative girth 5. In this case negative cycles are the same as odd cycles, a super negative cycle would be a cycle with either 0 or 2 positive edges.

Conjecture 6. Any signed planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ admits an equivalent signature σ' where (G, σ') has no super negative cycle.

We shall note that the property of having no super negative cycle is a homomorphism property in the following sense: Suppose (H, π) is a signed graph where every negative cycle has at least l positive edges. If a signed graph (G, σ) maps to (H, π), then there is a signature σ' equivalent to σ such that in (G, σ') each negative cycle has at least l positive edges. One such choice for σ' is by taking inverse image of π under the homomorphism of (G, σ) to (H, π).

This observation and Theorem 3 imply that given an integer k, a minimum counterexample (G, σ) of negative girth k to each of Conjecture 5 and Conjecture 6 must have no proper homomorphic image which satisfies all three conditions: It is of negative girth k, it is planar, and it is in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$. Then, combined with the folding lemma of [4] which applies to cases in $\mathcal{G}_{11}$ and the folding lemma of [7] that applies to cases in $\mathcal{G}_{10}$, we conclude that in every planar embedding of (G, σ) each face must be a negative k-cycle.

The rest of this paper is about proving the following theorem.

Theorem 7. *For any antibalanced signed simple planar graph (G, σ) of negative girth at least 5, we have $\rho(G, \sigma) \geq 5$.*

2 Proof of Theorem 7

Let us start with the full picture of the proof. We are assuming that each planar graph in $\mathcal{G}_{11} \cup \mathcal{G}_{10}$ with negative girth at most 4 packs. The case of negative girth 3 is equivalent to the 4-color theorem and the case of negative girth 4 is a stronger statement a proof of which can be found in [9].

Let us take a planar graph (G, σ) in $\mathcal{G}_{11}$ with negative girth at least 5. We want to prove that $\rho(G, \sigma) \geq 5$. Let us suppose we can find a switching equivalent signature σ' such that (G, σ') has no super negative cycle. As (G, σ) is in $\mathcal{G}_{11}$, a cycle is negative if and only if it is odd. If there is no super negative cycle, then each odd cycle of G has at least one positive edge in (G, σ). In other words E^-_σ induces a bipartite subgraph. Hence, applying Theorem 4, we can find a second equivalent signature σ'' such that (G, σ') and (G, σ'') have no negative edge in common. We then contract all the negative edges in (G, σ') and consider the negative edges of σ'' as a signature on this new graph. This would be a signed planar graph in $\mathcal{G}_{10}$ whose negative cycles are of length at least 4. Applying the case of negative girth 4, we have four disjoint signatures on the contracted graphs. Together with σ' we have a total of five signatures with no pair of them having a common negative edge.

So what remains to show is that (G, σ) admits an equivalent signature with no super negative cycle. At this point the second inductive step kicks in. We assume G is a smallest counterexample. That is to say: (G, σ) is a signed planar graph in $\mathcal{G}_{11}$ which has no loop and no triangle, it does not admit a packing of size five and among all such examples, it has (first) minimum number of vertices and (second) minimum number of edges. The order on the number of vertices together with the folding lemma implies that all faces are 5-cycles. The minimality of the number of edges means removing any edge e, the remaining signed graph must admit a 5-packing. Viewing each of these five signatures as a signature on G, equivalent to σ, we must have a super negative cycle with respect to each equivalent signature. However, each such cycle must contain e. This would be enough to stablish a rich enough structure around vertices of degree 2 and 3 to apply discharging technique and get a contradiction with

Euler's formula. Thus we split details of the proof to three parts: dealing with 2-vertices, 3-vertices and then discharging.

2.1 2-vertices

Let v be a vertex of degree 2 in G and let x and y be its two neighbours, furthermore, in the rest of this subsection e is the edge vx and e' is the edge vy.

Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be the five signatures equivalent to σ such that, when restricted on $G-e$, they have no common negative edge. Thus e is the only potentially common negative edge among some of these signatures. Each (G, σ_i) must contain a super negative cycle. If more than one, then we choose one and name it C_i. Moreover we denote by P_i the $x-y$ path in C_i that does not contain v. Furthermore, we assume σ_i's are minimal in the sense that there is no other signature on $G-e$ equivalent to σ such that all its negative edges are also negative in σ_i. Clearly replacing each signature with a minimal one does not affect the packing property. However, then we may have a set of edges each of which is positive in all five of $(G-e, \sigma_i)$. Let E_6 be such a set of edges of $G-e$. We proceed with a series of claims.

Claim 1. We have one of two:

- Either $\sigma_i(e) \neq \sigma_i(e')$ for each i, $i = 1, 2, \ldots, 5$, in which case all the positive edges of each C_i must be in E_6.
- Or for exactly one of the five signatures, say σ_5, we have $\sigma_i(e) = \sigma_i(e')$ in which case the positive edge of each P_i in (G, σ_i), $i = 1, 2, 3, 4$, is a negative edge in (G, σ_5).

Proof. First we show that we cannot have two such signatures satisfying $\sigma_i(e) = \sigma_i(e')$. Suppose to the contrary that two of them, say σ_1 and σ_2, assign the same sign to e and e'. By switching at v, if necessary, in each of (G, σ_1) and (G, σ_2) we may assume that $\sigma_1(e) = \sigma_1(e') = +$ and $\sigma_2(e) = \sigma_2(e') = +$. This implies that all the edges of P_1 are given a negative sign in (G, σ_1) and, similarly, all the edges of P_2 are given a negative sign in (G, σ_2), and thus a positive sign in (G, σ_1). Recall that, since each C_i is an odd cycle, each P_i is a path of odd length. Then the closed walk induced by $P_1 \cup P_2$, in (G, σ_1), and hence in (G, σ), is negative closed walk of even length. This contradicts the fact that $(G, \sigma) \in \mathcal{G}_{11}$.

Hence, and without loss of generality, we assume $\sigma_i(e) \neq \sigma_i(e')$ for $i = 1, 2, 3, 4$. Then for each i, $i = 1, 2, 3, 4$, the path P_i has a unique positive edge in (G, σ_i). Let us name this edge e_i. Then we first observe that e_i cannot be negative in any of (G, σ_j), $j = 1, 2, 3, 4$ as otherwise, C_i would be a positive cycle in (G, σ_j). If $\sigma_5(e) \neq \sigma_5(e')$, then for C_i, $i = 1, 2, 3, 4$, to be negative in (G, σ_5) we have $\sigma_5(e_i) = +$ which implies the first case of the claim. If $\sigma_5(e) = \sigma_5(e')$, then for C_i, $i = 1, 2, 3, 4$, to be negative in (G, σ_5) we must have $\sigma_5(e_i) = -$ in which case we have the second part of the claim. ◇

Suppose the first case of the claim happens. Then let σ'_5 be a signature whose negative edges in $G-e$ are those in $E_5^- \cup E_6$. We claim that (G, σ'_5) is also switching equivalent to $(G-e, -)$. That is because $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are equivalent disjoint signatures, thus every negative cycle, which is an odd cycle, has an even number of edges in $E_1^- \cup E_2^- \cup E_3^- \cup E_4^-$ and thus an odd number of edges in the remaining part which is $E_5^- \cup E_6$ and thus it is a negative cycle in (G, σ_5'). Similarly, every even cycle is positive in (G, σ_5'), proving that it is switching equivalent to $(G-e, -)$. Thus we may assume that the second item of the claim is always the case at the cost of allowing σ_5 not to be minimal. Under this assumption, we may also assume that $\sigma_5(e) = \sigma_5(e') = +$, as otherwise we may switch at v in (G, σ_5). This, in particular, means that for any super negative cycle C_5, e and e' are the only positive edges.

We should note that in choosing the super negative cycle C_i of (G, σ_i) one may have more than one choice. Next we aim at showing that among the possible choices, at least one should have a fair number of high degree vertices. Recall that in our case of negative girth 5 a super negative cycle has either 2 or 0 positive edges. Thus if a super negative cycle has at least one positive edges, then it has precisely two positive edges.

Claim 2. Assume σ' is a minimal signature equivalent to σ such that every super negative cycle of (G, σ') contains xvy with one positive edge and one negative edge, and that, moreover, the other positive edge is incident to either x or y. Then in one of the super negative cycles of (G, σ') every vertex which is not incident to a positive edge is of degree at least 4 in G.

Proof. That σ' is assumed to be a minimal signature implies, in particular, that no vertex is incident to only negative edges. Among all the signatures for which the conditions of Claim 2 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimized. To get a contradiction we need to show that this number must be 0.

Suppose not and let $C_1, C_2, \ldots C_r$ be the set of super negative cycles of (G, σ') and assume that C_1 is a shortest one among these cycles. Since the conclusion does not hold, C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges (with respect to the signature σ') and $d_G(z) \leq 3$. Since not all edges incident to a vertex are negative, we must have $d_G(z) = 3$ and that the third neighbour of z, say z', is adjacent to it with a positive edge. We first claim that $z' \notin \{x, y\}$. Let P_1' be the $x - z$ path in C_1 which does not contain v and P_1'' be the $z - y$ path which does not contain v. Observe that only one of P_1' and P_1'' have a positive edge. We continue the proof assuming that P_1'' has a positive edge, which then must be incident to y. The other case would be symmetric. If $z' = x$, then P_1' together with xz induces a cycle with exactly one positive edge, depending on the parity of the length, that would either be a negative even cycle or a positive odd cycle both of which are forbidden in a member of $\mathcal{G}_{11}$. If $z' = y$, then the cycle C_1' obtained from C_1 by replacing P_1'' with the zy is also a super negative cycle of (G, σ_1) whose length is less than C_1, contradicting the choice of C_1.

Since $z' \notin \{x, y\}$, and by our assumption that in every super negative cycle of (G, σ') each positive edge is either incident to x or to y, we conclude that the edge zz' does not belong to any super negative cycle of (G, σ'). We now consider the signature σ'' obtained from (G, σ') by a switching at z. Then each super negative cycle of (G, σ'') is also a super negative cycle of (G, σ') with the same signature. Thus (G, σ'') also satisfies the conditions of the claim, but it has less super negative cycles than (G, σ'), contradicting the choice of σ'. $\diamond$

To take a better advantage σ_5, we consider a signature σ_5' where the negative edges are those of σ_5 and the edges in E_6. It is already mentioned that σ_5' is an equivalent signature. We have following claim on (G, σ_5').

Claim 3. In (G, σ'_5) there exists a super negative cycle C in which all vertices, but possibly x, v and y, have degree at least 4 in G.

Proof. Observe that in (G, σ'_5) the edges xv and vy are of the same sign. Thus if needed, by a switching at v we may assume they are both positive. This implies that in every super negative cycle of (G, σ'_5) all edges not incident to v are negative. Let $C_1, C_2, \ldots, C_r$ be the set of super negative cycles of (G, σ'_5). If each of them has a vertex of degree 2 or 3 except v, by switching at all those vertices we will get a signature with no super negative cycle, contradicting the minimality of the counterexample. The details that such switching does not create new super negative cycles and that each switching kills of the corresponding super negative cycle is similar to the previous claim. ◇

Claim 4. In each of (G, σ_i), $i = 1, 2, 3, 4$, one of the following holds:

- Either x or y has a negative neighbour whose degree in G is at least 4.
- Each of x and y has a negative neighbour of degree 3.

Proof. Suppose to the contrary that one of them, say (G, σ_1) does not satisfy the claim. That means, for one of x and y, say y, all negative neighbours (possibly none) are of degree 2. Let (G, σ'_1) be obtained from (G, σ_1) by switching at all negative neighbours of y. Since each of these vertices are of degree 2 and each is incident to at least one negative edge, the switching does not create a new super negative cycle. As y has no negative neighbour in (G, σ'_1), the condition of Claim 2 holds for (G, σ'_1). Thus (G, σ'_1) has a super negative cycle C where each vertex not incident to a positive edge is of degree at least 4. Let x' be the neighbour of x in C, $x \neq v$. Since C must be of length at least 5 and both positive edges are incident to y, both edges of C incident with x' are negative and thus x' has degree at least 4. Moreover, as x is not adjacent to y, and switchings were done only at neighbours of y, the sign of the edge xx' is negative in (G, σ_1) as well. This means x' is a negative neighbour of x whose degree is at least 4, thus the first case of the claim holds. ◇

Claim 5. Suppose that u and v are two adjacent 2-vertices with u' and v' being the other neighbour, respectively. Then both u' and v' have degree at least 6 and have at least 5 4^+-neighbours.

Proof. We give the proof for u' and the proof for v' is analogous. By minimality of the counterexample we have a signature packing $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ of $(G - \{u, v\}, \sigma)$. Each of these signatures can be extended to G such that first of all (G, σ_i) is equivalent to (G, σ), secondly, in each of them, both uv and vv' are positive. The latter, can be achieved, if not already the case, by switching at v, u or both.

Since each (G, σ_i) has to have a super negative cycle, then uu' must be a negative edge in all of them and this would be the only common negative edge between any pair of them. Each of these five signatures, however, satisfies the conditions of Claim 2, thus there is a super negative cycle C_i in (G, σ_i) where vertices not incident to positive edges are 4^+-vertices. In C_i the neighbour u_i of u', $u_i \neq u$, is not incident to a positive edge. Since $u'u_i$ is negative only in (G, σ_i), the vertices u_i are 5 distinct 4^+-neighbours of u'. As u is also a distinct neighbour of u', u' has a total of at least six neighbours. ◇

Claim 6. Suppose that u is a 2-vertex with u' and v as neighbours and that v is 3-vertex with its two other neighbours being v_1 and v_2. Then, first of all, u' has at least four 4^+-neighbours. Secondly, among v_1 and v_2 either one has at least four 4^+-neighbours or together they have at least five 4^+-neighbours.

Proof. We consider induced signed subgraph by deleting the edge uu' and as before define σ_1, σ_2, σ_3, σ_4 to be four minimal signatures with no common negative edge and let σ'_5 be the signature which assigns negative to the edges that are not negative in any of $(G - uu', \sigma_i)$, $i \leq 4$. As before, we consider σ_i and σ'_5 as signatures on G rather than $G - uu'$, thus some of them have uu' as (the only) common negative edge.

By our choice of σ'_5 only the second case of Claim 1 can happen. Then if necessary, in (G, σ'_5) we switch at u to get a (G, σ''_5) where uu' and uv are both positive, noting that each super negative cycle of (G, σ''_5) is also a super negative cycle of (G, σ'_5). As there must be at least one such cycle, and as there are already two positive edges, all other edges must be negative. That implies that, in particular, at least one of the two edges vv_1 and vv_2 is negative in (G, σ'_5). We consider two cases depending on if only one is negative or both.

First assume the case that $\sigma'_5(vv_1) = -$ and $\sigma'_5(vv_2) = +$. Since each edge beside uu' is negative in only one of the signatures, we may assume $\sigma'_i(vv_1) = -$. Then for each $j \neq i$, in (G, σ_j) all the positive edges of each of the super negative cycle are incident to v, and, thus, by Claim 2 for $j \leq 4$ and by Claim 3 in the case of $j = 5$ we have a super negative cycle in (G, σ_j) in which the neighbour of u' distinct from u is of degree at least 4. We note moreover that the positive edges of any super negative cycle in (G, σ_j) are negative in σ'_5. This implies that vv_2 cannot be a positive edge in these cycles. Thus the second positive edge of any super negative cycle in (G, σ_j), $j \leq 4$, $j \neq i$ is vv_1. Again using Claim 2 the neighbour of v_1 in each of these cycles must be at least of degree 4. Since that is the case for the super negative cycle of (G, σ'_5) as well, v_1 must have at least four such neighbours.

Now we consider the case that $\sigma'_5(vv_1) = \sigma'_5(vv_2) = -$. In this case then for all j's, $j = 1, 2, \ldots, 5$ every super negative has two positive edges incident with v. Thus, first of all u' will have at least five 4^+-neighbours, secondly, each of the signatures will imply a 4^+-neighbour for either v_1 or for v_2, giving a total of at least five such neighbours for the two of them. ◇

2.2 3-vertices

Similar to the last subsection, let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma'_5$ be the five signatures equivalent to σ such that, when restricted on $G - e$, for a fixed edge e, each edge in $G - e$ is negative in exactly one of these five signatures, and σ_i's are minimal for $i = 1, 2, 3, 4$. Thus e is the only potentially common negative edge among some of these signatures. As (G, σ_i) is a counterexample to Theorem 7, and by the equivalence to Conjecture 6, each (G, σ_i) contains at least one super negative cycle, one of which is named C_i.

Claim 7. Every 4-cycle of G contains a vertex of degree at least 4.

Proof. Suppose not, let $C = v_1v_2v_3v_4v_1$ be a 4-cycle that all its vertices have degree at most 3. By the folding lemma, every face of G is of length 5. Thus C is not a facial cycle, hence it is a separating cycle. We note that G is a 2-connected graph. This can be observed either by using Theorem 3 and the fact that $\mathcal{SPC}_d^o$ is vertex transitive. Or by considering a vertex cut x, and applying induction on two subgraphs $G - G_1$ and $G - G_2$ where G_1 and G_2 each has all vertices of a connected component of $G - x$.

Therefore, at least two of v_1, v_2, v_3, v_4 have neighbours inside of C, and similarly at least two of them have neighbours outside. But since each v_i is a 3^--vertex, it follows that they are all 3-vertices and that precisely two of them have neighbours inside and two of them have neighbours outside. By symmetry, we consider two case: (1) v_1, v_2 have neighbours inside C, (2) v_1, v_3 have neighbours inside C. In case (1), the path $v_1v_4v_3v_2$ is part of a facial cycle inside C. As every facial cycle is a 5-cycle, there is a common neighbour of v_1 and v_2. But that would make triangle with v_1v_2. In case (2), considering the faces inside C formed by $v_1v_2v_3$ and $v_1v_4v_3$, we conclude that the neighbours x, y of v_1 and v_3 inside C are themself adjacent and that the edge xy is part of both mentioned faces. That implies that x and y are adjacent 2-vertices. But we have already seen that for adjacent 2-vertices x, y their other neighbours must be of degree at least 6. ◇

Claim 8. If C is a shortest super negative cycle, then C contains no chord.

Proof. Observe that a chord on a negative cycle creates one positive cycle and one negative cycle. Let C be a shortest super negative cycle with a chord e. Let C' be the negative cycle created by C and e. We claim that C' is a shorter super negative cycle, contradicting the choice of C. That C' is negative is by our choice. That it is shorter is by the fact that there are no parallel edges and e is a chord of C. It remains to show that C' is super negative, i.e. it has at most two positive edges. Since C has at most two positive edges, in $C \cup \{e\}$ there are at most three positive edges. But as (G, σ) is switching equivalent to $(G, -)$, every negative cycle (which is an odd cycle of G) has an even number of positive edges, thus C' has at most two positive edges. ◇

Claim 9. Let v be a vertex of degree 3 in G and $N(v) = \{v_1, v_2, v_3\}$, such that both v_2 and v_3 have degree 3. Let σ' be a signature equivalent to σ in which every super negative cycle contains vv_1, noting that such a signature exists by the minimality of (G, σ). If (G, σ') has the extra property that every super negative cycle has two positive edges each of which is incident to at least one of v, v_2 or v_3, then there exists a super negative cycle $C_{\sigma'}$ such that every vertex not incident to a positive edge is of degree at least 4 in G.

Proof. Among all the signatures for which the conditions of Claim 9 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimum. To get a contradiction we would like to show that this number must be 0. Let $N(v_i) = \{v, x_i, y_i\}$ for $i = 2, 3$.

Let $C_1, C_2, \ldots, C_r$ be the set of super negative cycles of (G, σ') and assume that C_1 is a shortest one among these cycles. Since the conclusion of the claim on (G, σ') does not hold, C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges (with respect to the signature σ') and $d_G(z) \leq 3$. If all edges incident to z are negative, then we consider (G, σ'') obtained from (G, σ) by switching at z. We observe that super negative cycles of (G, σ'') are exactly those super negative cycles of (G, σ') which do not contain z.

Thus (G, σ'') also satisfies the conditions of the claim, but it has less super negative cycles than (G, σ'), contradicting the choice of σ'.

Since both edges of C_1 incident to z are negative we must have $d_G(z) = 3$ and that the third neighbour of z, say z', is adjacent to it with a positive edge. We claim zz' belongs to some super negative cycle of (G, σ'). Suppose not. Let π be the signature obtained from (G, σ') by switching at z. Then, first of all, there is still no super negative cycle in (G, π) containing zz', because for cycles containing this edge the number of positive edges is the same in (G, σ') and (G, π). Secondly, any super negative cycle of (G, σ') containing z has two more positive edges in (G, π). Since we assume every super negative cycle of (G, σ') has two positive edges, those containing z, in particular C_1, are not super negative in (G, π). This contradicts with the number of super negative cycles of (G, σ') being minimum. Thus zz' is in a super negative cycle, say C_i, $2 \leq i \leq r$.

Next we claim that $z \notin \{v, v_1\}$. We assume to contrary and first consider the case that $z = v$. Recall that vv_1 is an edge of C_1. Between v_2 and v_3, by symmetry, assume $vv_3 \in C_1$. As edges of C_1 incident to z are negative we have $\sigma'(vv_1) = \sigma'(vv_3) = -$ and since not all edges incident to z are negative we have $\sigma'(vv_2) = +$. Since C_1 must have two positive edges, and they must be incident to v or v_2 or v_3, the vertex v_2 should be on C_1 and, moreover, should be incident to a positive edge of C_1. Noting that vv_2 is not an edge of C_1, $x_2v_2y_2$ should be a part of C_1. This implies that vv_2 is a chord of C_1, contradicting Claim 8. Next we consider the case that $z = v_1$. In this case, since both edges of C_1 incident to z are negative, and since vv_1 is an edge of every super negative cycle, vv_1 is a negative edge. Thus, noting that zz' is a positive edge, $z' \neq v_1$. However, we have already noted that v_1z' must be in super negative cycles, say C'. But then by the assumption on the positive edge of super negative cycles, $z' \in \{v_2, v_3\}$ in either case then G must have a triangle.

Since zz' must be a positive edge of the super negative cycle C_i and since all such edges are incident to one of v, v_2, v_3 we must have $z \in \{v_2, v_3, x_2, y_2, x_3, y_3\}$. By symmetries we consider only two possibilities of $z = v_2$ or $z = x_2$. First let $z = v_2$. If $z' = v$, then C_1 contains the edge zz' as a chord and we have contradiction with Claim 8. If $z' \in \{x_2, y_2\}$, say $z' = y_2$, then $\sigma'(vv_2) = \sigma'(v_2x_2) = -$, since vv_1 is also an edge of C_1, vv_3 is not. As all positive edges are incident to v, v_2 or v_3 and since there are two such edges in C_1, v_3 is a vertex of C_1, but then again vv_3 is a chord contradicting Claim 8.

Finally assume $z = x_2$ and let $N(x_2) = \{v_2, x_2', x_2''\}$. If $z' = x_2'$ or $z' = x_2''$, since zz' belongs to a super negative cycle, positive edges of super negative cycles are incident to v, v_2 or v_3 and as G contains no triangle, $z' = v_3$, in which case $vv_2x_2v_3v$ is a 4-cycle which contains four 3-vertices, contradicting Claim 7. Therefore, we must have $z' = v_2$, then $\sigma'(x_2x_2') = \sigma'(x_2x_2'') = -$ and $x_2'x_2x_2''$ is a part of C_1. If $vv_2 \in C_1$, then zz' is again a chord of C_1, which contradicts Claim 8. So $vv_2 \notin C_1$, by symmetry we may write C_1 as $v_1vv_3x_3P_1x_2'x_2x_2''P_2v_1$. If $\sigma'(vv_2) = -$, then again vv_2x_2 creates two cycles from C_1 one of which is negative. And this negative cycle has at most two positive edges, therefore is a super negative cycle and thus contains the edge vv_1. By Claim 7, path $vv_3x_3P_1x_2'x_2$ must have length at least 3, as otherwise together with v_2 we will have a 4-cycle all whose vertices are of degree 3. Replacing this path with vv_2x_2 we find a shorter super negative cycle, contradicting the minimality of C_1. Next let $\sigma'(vv_2) = +$. By the assumption on C_1, the fact that every cycle has even number of positive edges, and the fact that the edges of the path

$x_2x_2''P_2v_1$ are all negative, we must have $\sigma'(vv_1) = -$, as otherwise the cycle $v_1vv_2x_2x_2''P_2v_1$ has three positive edges. Since positive edges of C_1 must be incident to v, v_2 or v_3, we have $\sigma'(vv_3) = \sigma'(v_3x_3) = +$. Furthermore $x_2''P_2v_1$ has an even number of edges since otherwise $v_1vv_2x_2x_2''P_2v_1$ is a shorter super negative cycle. Recall that zz' is also in the super negative cycle C_i. We consider the following two cases.

1. If $vv_2 \in C_i$, then by our assumption $vv_3 \notin C_i$, thus we may write C_i as $v_1vv_2x_2P_3v_1$, but then the cycle obtained from two paths from x_2 to v_1 of C_1 and C_i forms a super negative cycle which does not contain vv_1 and contains no positive edge, a contradiction.

2. Otherwise $vv_2 \notin C_i$, let $C_i = v_1vv_3y_3P_3x_2'x_2v_2y_2P_4v_1$. But then the cycle $vv_3y_3P_3x_2'x_2v_2v$ contains exactly three positive edges, which never happens in (G, σ'). $\diamond$

Claim 10. Let v be a vertex of degree 3 in G and $N(v) = \{v_1, v_2, v_3\}$, such that both v_2 and v_3 have degree 3. Then v_1 has at least two neighbours of degree at least 4.

Proof. Let $G' = G - vv_1$ and $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5'$ be the five signatures equivalent to σ such that, when restricted on G', they have no common negative edge. Thus vv_1 is the only potentially common negative edge among some of these signatures. Each (G, σ_i) must contain a super negative cycle using the edge vv_1. If more than one, then we choose one and name it C_i. Let $N(v_i) = \{v, x_i, y_i\}$ for $i = 2, 3$.

We first claim that among all these five signatures, there are at least two, in which, after switching at v, v_2 and v_3 (if necessary), we have the following: first of all in each of the four paths $v_1vv_ix_i$ and $v_1vv_iy_i$, $i = 2, 3$, there are at least two positive edges, secondly, we do not create any new super negative cycle by the said switching. To see this we consider two cases.

Case 1. Assume vv_2 and vv_3 belong to the same signature, say $E^-_{\sigma_5'}$. Then in each of the other signatures, namely $\sigma_1, \sigma_2, \sigma_3$ and σ_4, vv_2 and vv_3 are both positive. If vv_1 is also positive in at least two of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, say σ_1 and σ_2, then these two signatures are the desired ones, without a need for switching.

So we may assume that vv_1 is negative in σ_1, σ_2 and σ_3. We discuss how to do the switching on (G, σ_1) in order to build (G, σ_1') have the required property. The same approach would work to build (G, σ_2') (and (G, σ_3') though not needed). To this end, in (G, σ_1), we switch at v to have a signature σ_1''. As $\sigma_1(vv_1) = -$ and $\sigma_1(vv_2) = \sigma_1(vv_3) = +$, the number of positive edges of a cycle containing vv_1 does not change. A negative cycle containing both vv_2 and vv_3 is a negative cycle of $G - vv_1$, and thus must have at least one negative edge in each of $\sigma_2, \sigma_3, \sigma_4$ noting that none of these edges can be in $\{vv_2, vv_3\}$. Thus not such a negative cycle is super negative in (G, σ_1''). Now for $i = 2, 3$, if one of v_ix_i or v_iy_i is negative in σ_1'', then we switch at v_i. Let σ_1' be the resulting signature. Since v_i, which is of degree three, is adjacent to at least two negative edges in (G, σ_1''), there is no new super negative cycle in (G, σ_1'). Moreover, in the final signature, (G, σ_1'), each of $v_1vv_ix_i$ and $v_1vv_iy_i$ has at least two positive edges as claimed.

Case 2. Assume vv_2 and vv_3 do not belong to the same E^-_i, by symmetries, say $vv_2 \in E^-_1$ and $vv_3 \in E^-_2$. Then in (G, σ_i), $i = 1, 2$, we could first switch at v (if necessary) to make vv_1 positive, since exactly one of vv_2 and vv_3 is positive, this switching will not create new super negative cycle. One vv_1 is positive, we could switch at either, both or none of v_2 or v_3 to obtain the required conditions.

In conclusion, we have switching equivalent signatures σ'_1 and σ'_2 obtained from σ_1 and σ_2 by potentially switching at v, v_2 and v_3, such that all the super negative cycles in (G, σ'_1) and in (G, σ'_2) use the edge vv_1 and their two positive edges are incident to v, v_2 or v_3. Thus by Claim 9, there exists a super negative cycle C'_1 of (G, σ'_1) (similarly C'_2 in (G, σ'_2)) such that every vertex not incident to a positive edge is of degree at least 4 in G. Let v'_1 be the neighbour of v_1 in C'_1 which is distinct from v. Since v_1 is not adjacent to v_2 or v_3, $v_1v'_1 \in E_1^-$. Then we claim that $d(v'_1) \geq 4$. If not, v'_1 is incident to a positive edge of C'_1, which means $v'_1 \in \{x_2, y_2, x_3, y_3\}$. By symmetries, we assume $v'_1 = x_2$ which means x_2v_2 is a positive edge of C'_1. Since each super negative cycle has length at least 5 and C'_1 uses the edge vv_1, we have $vv_2 \notin C'_1$ and $x_2v_3 \notin C'_1$, for the latter if it exists. Therefore both v_2y_2 and vv_3 are edges of C'_1. Thus one of v_3x_3 or v_3y_3, say v_3x_3, is also an edge of C'_1. By what we proved above, there are at least two positive edges in $x_3v_3vv_1$ part of C'_1. But x_2v_2 is also a positive edge of C'_1,having a total of at least three positive edges, contradicting the fact that C'_1 is a super negative cycle. Therefore $d(v'_1) \geq 4$. The second neighbour v'_2 of v having $d(v'_2) \geq 4$ can be found by similar argument using (G, σ'_2). ◇

Claim 11. Let u and v be two adjacent vertices of degree 3 in G. Assume σ' is a signature equivalent to σ, such that every super negative cycle of (G, σ') contains uv and contains two positive edges which are incident to either u or v. Then there exists a super negative cycle in which every vertex not incident to a positive edge is of degree at least 4 in G.

Proof. As in the proof of Claim 9 among all the signatures for which the conditions of Claim 11 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimum, and, moreover, we take σ' to be a minimal signature. Let $N(u) = \{u_1, u_2\}$ and $N(v) = \{v_1, v_2\}$.

Let $C_1, C_2, \ldots, C_r$ be the super negative cycles of (G, σ'), and assume w.l.o.g. that $|C_1| \leq |C_j|$, $2 \leq j \leq r$. If the conclusion does not hold, then C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges and $d_G(z) \leq 3$. Minimality of σ' implies that $d_G(z) = 3$ and that the third neighbour of z, say z', is adjacent to it with a positive edge. Furthermore, zz' is in a super negative cycle, say C_i, $2 \leq i \leq r$, as otherwise by switching at z we have less super negative cycles.

As each of u and v is incident to a positive edge of C_1, $z \notin \{u, v\}$. Considering the super negative cycle C_i, zz' is a positive edge, thus by our assumption one of the end point is u or v. As $z \notin \{u, v\}$, we have $z' \in \{u, v\}$ and hence $z \in \{u_1, u_2, v_1, v_2\}$. W.l.o.g. let $z = u_1$ and $z' = u$. But then uu_1 is a chord of C_1 and we have a contradiction with Claim 8. ◇

Claim 12. Let v_1, v_2, v_3 and v_4 be four vertices of degree 3, and σ' be a signature equivalent to σ such that the following holds.

1. v_i is adjacent to v_{i+1}, $i = 1, 2, 3$.
2. $\sigma'(v_1v_2) = -$, $\sigma'(v_2v_3) = \sigma'(v_3v_4) = +$.
3. Each of v_2 and v_3 is incident to exactly two positive edges.
4. Either v_1 is incident to two positive edges or v_4 is incident to three positive edges.
5. Every super negative cycle of (G, σ') contains the positive edge v_2v_3.
6. The other positive edge of any other super negative cycle must be incident to one of the v_i ($i \in \{1, 2, 3, 4\}$).

Then there exists a super negative cycle of (G, σ') in which every vertex not incident to a positive edge is of degree at least 4 in G.

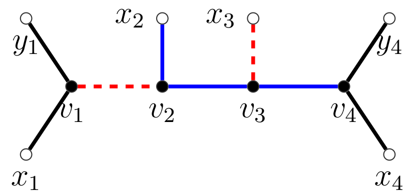

Figure 2: Four vertices of degree 3 in Claim 12

Proof. Again among all the signatures for which the conditions of Claim 12 hold but the conclusion does not, we take σ' to be one where the number of super negative cycles of (G, σ') is minimum. Let the other two neighbours of v_i be x_i, y_i for $i = 1, 4$, and the third neighbour of v_j be x_j for $j = 2, 3$, as shown in Figure 2. Suppose to the contrary that $C_1, C_2, \ldots, C_r$ are super negative cycles of (G, σ') and assume that C_1 is a shortest one among these cycles. Thus C_1 has a vertex z whose two neighbours on C_1 are connected to it by negative edges and $d_G(z) \leq 3$. It follows that $d_G(z) = 3$ and that the third neighbour of z, say z', is a positive neighbour, moreover, zz' is in a super negative cycle, say C_l, $2 \leq l \leq r$.

Since in (G, σ') every super negative cycle contains v_2v_3 as a positive edge, $z \notin \{v_2, v_3\}$. And since each positive edge of any super negative cycle is incident to some v_i, we have $z \in \{v_1, v_4, x_1, x_2, x_3, x_4, y_1, y_4\}$. By symmetries we consider following possibilities.

1. $z = v_1$. Since $\sigma'(v_1v_2) = -$, $v_1v_2 \in C_1$ and at least one of v_1x_1, v_1y_1 is negative. By assumption v_4 is incident to three positive edges, and thus $v_4 \notin C_1$. This contradicts with the fact that positive edges of every super negative are incident to v_i.

2. $z = v_4$. Since $\sigma'(v_3v_4) = +$, and edges in C_1 incident to z are both negative, we have that $v_4x_4, v_4y_4 \in C_1$ and $\sigma'(v_4x_4) = \sigma'(v_4y_4) = -$, and hence $z' = v_3$. By the original assumption of the claim, $\sigma'(v_1x_1) = \sigma'(v_1y_1) = +$. Recall that every super negative cycle of (G, σ') must contain v_2v_3. But any cycle that contains both v_2v_3 and v_3v_4 must contain at least one more positive edge. This is a contradiction with the fact that zz' is in a super negative cycle.

3. $z = x_2$. Since $\sigma'(v_2x_2) = +$ and $v_2x_2 \notin C_1$ we must have $v_1v_2 \in C_1$. But then v_2x_2 is a chord of C_1 which contradicts the Claim 8.

4. $z = x_3$. Since the super negative cycle C_l contains the positive edges zz' and v_2v_3, it contains no other positive edges, in particular $v_2x_2 \notin C_l$. Thus $v_1 \in C_l$. So each of v_1 and x_3 is incident with at least two negative edges, and they are not connected by a negative edge, since otherwise $v_1v_2v_3x_3$ induces a negative 4-cycle. Recall that if a vertex of degree 3 is incident with at least two negative edges, then a switching at it may eliminate some super negative cycle, but will never create a new one. Thus if we switch at both v_1 and x_3, then the remaining set of super negative cycles all must still contain the edge v_2v_3. But then in the new signature all edges incident to v_2 and v_3

are positive, which implies that every cycle containing v_2v_3 has at least three positive edges and there can be no super negative cycle.

5. $z = x_4$. First suppose $v_4x_4 \in C_1$, then $\sigma'(v_4x_4) = -$ and by the assumption $\sigma'(v_1x_1) = \sigma'(v_1y_1) = +$. Therefore $v_3v_4 \notin C_1$, since otherwise there will be three positive edges in C_1. However in this case v_3v_4 is a chord of C_1, this contradicts the Claim 8. Now suppose $v_4x_4 \notin C_1$, then $z' = v_4$ and $\sigma'(v_4x_4) = +$. We now consider the super negative cycle C_l containing v_4x_4 $(= zz')$. As it must contain the positive edge v_2v_3 as well, it can have no other positive edge. In particular, v_3v_4 is not in C_l. This implies that first of all $v_4y_4 \in C_l$ and, secondly, that v_4y_4 is a negative edge in (G, σ'). But then, by the assumption of $\sigma'(v_1x_1) = \sigma'(v_1y_1) = +$, the cycle C_l contains three positive edges, contradiction with C_l being a super negative cycle because it must contain at least one of v_1x_1, v_1y_1 and v_2x_2, all of whom are positive.

6. $z = x_1$. If $v_1x_1 = v_1z$ is in C_1, then it must be a negative edge of C_1. Thus v_1 is incident to at most one positive edge. The assumption of the claim implies that all edges incident to v_4 are positive. That implies $v_4 \notin C_1$ as otherwise C_1 will have at least three positive edges. As each positive edge of C_1 should be incident to one of v_1, v_2, v_3, the second positive edge of C_1 can only be either v_1y_1 or v_2x_2, in either case it follows that v_1v_2 is not an edge of C_1, and hence it is a chord of C_1 which contradicts Claim 8.

   So we may assume $v_1x_1 \notin C_1$. This implies that $z' = v_1$ and that $\sigma'(v_1x_1) = +$. As C_1 has no chord, $v_1 \notin C_1$. This implies that $v_2x_2 \in C_1$ and since $\sigma'(v_2x_2) = +$, it is the only other positive edge of C_1. Thus $v_3v_4 \notin C_1$ and hence, $v_3x_3 \in C_1$. Furthermore, $x_1 \neq x_3$, because otherwise $v_1v_2v_3x_3v_1$ is a 4-cycle where all vertices are of degree 3, contradicting Claim 7. We now claim that $x_1x_2 \in C_1$. If not, then the part of C_1 which connects x_1v_2 and is of even length is of length at least 4, but then in the union of C_1 and the path $x_1v_1v_2$ we will find a shorter super negative cycle. Moreover, we observe that $\sigma'(x_1x_2) = -$. We now consider the cycle C_1' obtained from C_1 by replacing $x_1x_2v_2$ with $x_1v_1v_2$. This cycle is also a super negative cycle of (G, σ') and is of the same length as C_1, i.e. it is one of the shortest super negative cycles of (G, σ'). Hence we could restart the analysis with C_1', based on which we conclude that there must be a vertex z_1 of C_1' which is of degree three in G, and both edges of C_1' incident to z_1 are negative. Then we conclude that $z_1 \in \{x_1, y_1\}$. The case $z_1 = y_1$ is not possible as otherwise C_1' contains a chord, and the case $z_1 = x_1$ is not possible because $\sigma'(v_1x_1) = +$. $\diamond$

Claim 13. Let v_1, v_2, v_3, v_4 be vertices of degree 3 in G such that v_i is adjacent to v_{i+1}, $i = 1, 2, 3$, where other neighbours of v_i's are labelled as in Figure 2. Then either each of x_2 and x_3 has at least three neighbours of degree at least 4, or one of x_2 and x_3 has at least four neighbours of degree at least 4.

Proof. Let $G' = G - v_2v_3$ and let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be the five signatures equivalent to σ such that, when restricted on G', they have no common negative edge. So v_2v_3 can be the only common negative edge among some of these signatures. Each (G, σ_i) must contain a super negative cycle using the edge v_2v_3, one of which is named C_i.

In each signature σ_i, $i = 1, \ldots, 5$, if v_2x_2 and v_1v_2 have the same sign, then by switching at v_2 and v_3 (if necessary), we can either be sure that all the super negative cycles have exactly two positive edges and that each of them is incident to either v_2 or v_3. Then by Claim 11, there exists a super negative cycle C_1 such that every vertex not incident to a positive edge is of degree at least 4 in G. It is easy to observe that in at least three of the signatures $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, say σ_1, σ_2, and σ_3, the edges v_2x_2 and v_1v_2 have the same sign. Since we only switch at v_2 and v_3, in each of the signatures σ_1, σ_2, and σ_3, either v_1 or x_2 has a negative neighbour of degree at least 4 in each of (G, σ_1), (G, σ_2), and (G, σ_3). If in each of (G, σ_4) and (G, σ_5) either the pair v_1v_2 and v_2x_2 have the same sign or the pair v_3v_4 and v_3x_3 have the same sign, then in total v_1 and x_2, as well as v_4 and x_3 have five neighbours that are of degree at least 4 in G. As either v_1 or v_4 can have at most two such neighbours, each of x_2 and x_3 must have at least 3 of them. We note that the conclusion holds.

Hence we suppose $\sigma_4(v_1v_2) = -\sigma_4(v_2x_2)$ and $\sigma_4(v_3v_4) = -\sigma_4(v_3x_3)$. Since we can switch at either v_2 or v_3 (if necessary), we assume v_2v_3 is positive. If $\sigma_4(v_2x_2) = \sigma_4(v_3x_3)$, then we first make v_1v_2 and v_3v_4 to be negative by switching at v_2 and v_3 (if necessary). If at least one of v_1x_1 and v_1y_1 (resp. v_4x_4 and v_4y_4) is negative, then after switching at v_1 (resp. v_4), we will not create any new super negative cycle. Otherwise, both v_1x_1 and v_1y_1 (resp. v_4x_4 and v_4y_4) are positive. In either case, each cycle containing v_2v_3 has at least three positive edges, which is a contradiction. Therefore, we may suppose $\sigma_4(v_2x_2) = -\sigma_4(v_3x_3)$, and w.l.o.g. assume $\sigma_4(v_2x_2) = +$. By switching at v_1, if necessary, we can make sure that v_1 is incident to at least two positive edges, let the obtained signature be σ'_4. Then the positive edges of each super negative cycle in (G, σ'_4) must be incident to either v_1 or v_2. Since $\sigma'_4(v_3v_4) = +$, every super negative cycle of (G, σ'_4) contains the edge v_3x_3. By Claim 12, there exists a super negative cycle such that every vertex not incident to a positive edge is of degree at least 4. Therefore, either x_3 has a negative neighbour (in (G, σ_4)) of degree at least 4, or x_3 is adjacent to one of x_1 and y_1. If we switch at v_2 and v_3, by symmetry, we have that either x_2 has a negative neighbour (in (G, σ_4)) of degree at least 4, or x_2 is adjacent to one of x_4 and y_4. Now it suffices to consider two cases based on σ_5.

Case 1: Either $\sigma_5(v_1v_2) = \sigma_5(v_2x_2)$ or $\sigma_5(v_3v_4) = \sigma_5(v_3x_3)$. Applying the same argument as before that for each (G, σ_i), $i = 1, 2, 3$, either v_1 or x_2 has a negative neighbour of degree at least 4. Similarly either v_4 or x_3 have a negative neighbour of degree at least 4. Since $d(x_1) = d(x_4) = 3$, both x_2 and x_3 have at least two neighbours of degree at least 4. Suppose the conclusion of the claim does not hold, assume x_2 has at most two neighbours of degree at least 4, w.l.o.g. assume x_2 is adjacent to x_4. Since x_3 can have at most three neighbours of degree at least 4, x_3 is adjacent to either x_1 or y_1, which implies x_2 has at least three neighbours of degree at least 4, a contradiction.

Case 2: $\sigma_5(v_1v_2) = -\sigma_5(v_2x_2)$ and $\sigma_5(v_3v_4) = -\sigma_5(v_3x_3)$. Applying the same argument as for σ_4, either x_2 has a negative neighbour (in (G, σ_5)) of degree at least 4, or x_2 is adjacent to one of x_4 and y_4. Similarly either x_3 has a negative neighbour (in (G, σ_5)) of degree at least 4, or x_3 is adjacent to one of x_1 and y_1. Again we suppose the conclusion does not hold, and assume x_2 has at most two neighbours of degree at least 4. W.l.o.g. let x_2 be adjacent to x_4 and $\sigma_4(x_2x_4) = -$. Since x_3 can have at most three neighbours of degree at least 4, w.l.o.g. we assume x_3 is adjacent to x_1 and $\sigma_4(x_1x_3) = -$. Therefore, both x_2 and x_3 have at least two neighbours of degree at least 4. Hence, it must be the case that x_2 is adjacent to y_4 and $\sigma_5(x_2y_4) = -$, which implies that x_3 has three neighbours of degree at least 4. But then x_3 must be adjacent to y_1 and $\sigma_5(x_3y_1) = -$, which implies x_2 has three neighbours of degree at least 4, a contradiction. $\diamond$

2.3 Discharging

In the following, we will use the discharging technique to get a contradiction. The *initial charge* ω on $V(G) \cup F(G)$ is defined as follows: $\omega(x) = d(x) - 4$ for every $x \in V(G) \cup F(G)$. By the relation $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$ and Euler's formula, the initial total charge of the vertices and faces satisfies the following:

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} (d(x) - 4) = -4|V(G)| + 4|E(G)| - 4|F(G)| = -8.$$

Since any discharging procedure preserves the total charge of G, after applying appropriate discharging rules to change the initial charge ω to the final charge ω^* such that $\omega^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$, we can have the contradiction below:

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -8,$$

and thus completes the proof.

For brevity, we call a 4^+-vertex *big*, and a 3^--vertex *small*. For a vertex v, by $n_k(v)$ we denote the number of k-neighbours of v and by $n_b(v)$ the number of big neighbours of v. Given a face f, $n_k(f)$ is the number of k-vertices incident to f. For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the charge transferring from x to y. A pair of f and v_2 where f is a 5-face $f = [v_1 v_2 \cdots v_5 v_1]$ is said to be *special* if following conditions hold: *i.* $n_b(v_2) = 3$, *ii.* v_2 is not adjacent to any 3-vertex, *iii.* $d(v_1) = d(v_3) = 2$, finally *iv.* v_4 ad v_5 are big vertices. We will do discharging in three stages. Below are our needed discharging rules for first stage:

(R1) Let $d(v) \geq 5$. If $n_b(v) \geq 4$, then v sends 1 to each adjacent small vertex. Otherwise if $n_3(v) + n_b(v) \geq 4$, then v sends 1 to each 2-neighbour, and $\frac{d(v)-4-n_2(v)}{n_3(v)}$ to each 3-neighbour.
(R2) Let $d(f) = 5$. If $n_{3^-}(f) = 1$, then f sends 1 to the incident small vertex.

After the first round of discharging, each 3-vertex which is adjacent to a 5^+-vertex v with $n_b(v) \geq 4$ or incident to a face f with $n_{3^-}(f) = 1$, has a non-negative charge. If a 2-vertex is incident or adjacent to at least two of the following, then it would end up with a non-negative charge: face with only one small vertex or 5^+-vertex with four 3^+-neighbours. We call these small vertices *rich*. In the following rules, if not specified, the small vertices that we consider are those who remain negative, and refer to them as *poor* vertices. We use 5^i-face to denote 5-face incident to i poor vertices. A 3-vertex is called $3_{k,l}$-vertex, if it is adjacent to k vertices each of which has at least three big neighbours, at least two 3-neighbours, and is incident to l 5^2-faces.

(R3) For the 5^+-vertex v such that $n_b(v) \leq 3$ and $n_3(v) + n_b(v) \leq 3$, v sends $\frac{d(v)-4}{n_2(v)}$ to each 2-neighbour.

(R4) Suppose f is a non-special 5-face. Then

(R4.1) If f is a 5^1-face, then f sends 1 to incident small vertex;

(R4.2) If f is a 5^2-face, then f sends $\frac{1}{2}$ to each small vertex incident to f.

(R4.3) If f is a 5^3-face then

(R4.3.1) If $n_2(f) = 2$ and $n_3(f) = 1$, then f sends $\frac{1}{2}$ to each incident 2-vertex.

(R4.3.2) If $n_2(f) = 1$ and $n_3(f) = 2$, then f sends $\frac{1}{2}$ to the incident 2-vertex. First suppose f is incident to a $3_{k,l}$-vertex. If $k + l \geq 2$, then f sends $\frac{1}{2}$ to the other 3-vertex; If $k = 1$ and $l = 0$, then f sends $\frac{1}{6}$ to $3_{1,0}$-vertex and $\frac{1}{3}$ to the other 3-vertex; If $k = 0$ and $l = 1$, and moreover it is incident to a 5^3-face which contains no 2-vertex, then f sends $\frac{1}{6}$ to this $3_{0,1}$-vertex, and $\frac{1}{3}$ to the other 3-vertex. Otherwise, f sends $\frac{1}{4}$ to each incident 3-vertex.

(R4.4) If f is a 5^3-face such that $n_3(f) = 3$ or a 5^{4^+}-face, then f does not give charge to any incident $3_{k,l}$-vertex such that $k + l \geq 2$, but sends $\frac{1}{6}$ to each incident $3_{1,0}$-vertex, then distribute its remaining charge equally among the other incident 3-vertices.

A charge pot at (F, v) is a set of consecutively adjacent special faces whose special vertex is v, as shown in Figure 3, noting that the number of face is unspecified. After carrying out (R1)-(R4), we apply (R5) as follows.

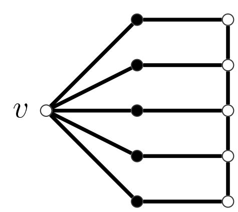

Figure 3: Charge pot

(R5) Each special face contributes 1 to its charge pot. These charges then will be redistributed to 2-vertices of the special pot as follows. If a 2-vertex is in a unique charge pot, then it will take as much charge as needed until its final charge is non negative. Assume a 2-vertex v is in two different charge pots, say (F_1, v_1) and (F_2, v_2) (where v_1 and v_2 are the two neighbours of v). Suppose v_i has given a charge of c_i, $c_i < 1$, to v. Then (F_i, v_i) gives a charge of $1 - c_i$ to v.

First, we observe that the following facts are true.

Fact 1. Let $d(v) = 2$ and $N(v) = \{v_1, v_2\}$, then $n_b(v_1) + n_b(v_2) + \frac{n_3(v_1)+n_3(v_2)}{2} \geq 6$.

Proof. By Claim 3, each of v_1 and v_2 has a big neighbour where the connecting edge is in $E^-_{\sigma'_5}$. By Claim 4, in each of (G, σ_i), $i = 1, 2, 3, 4$, either v_1 or v_2 has a big neighbour connected to it by an edge in $E^-_{\sigma_i}$, or each of v_1 and v_2 has a 3-neighbour connected to them by an edge in $E^-_{\sigma_i}$. Therefore, $n_b(v_1) + n_b(v_2) + \frac{n_3(v_1)+n_3(v_2)}{2} \geq 6$. ◇

Fact 2. A non-special 5-face sends charge at least $\frac{1}{2}$ to its incident 2-vertex.

In what follows, we are going to show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$ and the charge pot is also non-negative. Let $v \in V(G)$.

Case 1. $d(v) \geq 5$. If $n_b(v) \geq 4$, then $\omega^*(v) \geq d(v) - 4 - (d(v) - 4) = 0$ by (R1). Or if $n_3(v)+n_b(v) \geq 4$, then $\omega^*(v) \geq d(v)-4-n_2(v)-n_3(v) \times \frac{d(v)-4-n_2(v)}{n_3(v)} = 0$ by (R1). Otherwise, by (R3), $\omega^*(v) \geq d(v) - 4 - n_2(v) \times \frac{d(v)-4}{n_2(v)} = 0$.

Case 2. $d(v) = 4$. Since 4-vertex v does not participate in the discharging procedure, $\omega^*(v) = \omega(v) = d(v) - 4 = 0$.

Case 3. $d(v) = 3$. If v is rich, then it has non-negative charge. Suppose v is not rich. If v is incident to at least one 5^1-face, then $\omega^*(v) \geq 3 - 4 + 1 = 0$ by (R4.1). Otherwise let $N(v) = \{v_1, v_2, v_3\}$, denote by f_i the face that is incident to v such that vv_i and vv_{i+1} are its two boundary edges (indices modulo 3).

If v is adjacent to one 2-vertex, say $d(v_1) = 2$, then by Claim 6, v_2 or v_3 has at least 4 big neighbours or in total they have at least 5 big neighbours. Since v is poor, w.l.o.g., assume $n_b(v_2) = 3$ and $n_b(v_3) = 2$. And the other neighbour of v_1, say v_1' has at least 4 big neighbours. Since $n_3(v_2) + n_b(v_2) \geq 4$, f_1 is incident to at most two poor vertices possibly v and v_1, thus $\tau(f_1 \to v) \geq \frac{1}{2}$ by (R4.2). If $d(v_3) = 3$, then f_3 is a 5^2-face, and $\tau(f_3 \to v) \geq \frac{1}{2}$ by (R4.2). Thus $\omega^*(v) \geq -1 + 2 \times \frac{1}{2} = 0$. Suppose $d(v_3) \geq 4$. Let v_2' and v_3' be the other two vertices of f_2. By Claim 5 and Claim 6, either both of them have degree at least 3 or one of them has degree 2 and the other has degree at least 4. Therefore, either $\tau(v_2 \to v) = \frac{1}{2}$ by (R1) or f_2 is a 5^2-face and $\tau(f_2 \to v) = \frac{1}{2}$, both imply that $\omega^*(v) \geq -1 + 2 \times \frac{1}{2} = 0$.

Suppose now v is not adjacent to any 2-vertices.

Case 3.1. v is also not adjacent to any 3-vertex. Then by the fact that v is poor, Claim 5 and Claim 6, for $i = 1, 2, 3$, f_i is either incident to three 3-vertices or it is a 5^2-face, therefore $\tau(f_i \to v) \geq \frac{1}{3}$ by (R4.2) and (R4.4), $\omega^*(v) \geq -1 + 3 \times 1/3 = 0$.

Case 3.2. v is adjacent to exactly one 3-vertex, say v_1. Then again f_2 is either incident to three 3-vertices or f_2 is a 5^2-face, therefore $\tau(f_2 \to v) \geq \frac{1}{3}$ by (R4.2) and (R4.4). Let $f_1 = vv_1x_1y_1v_2$ and $f_3 = vv_1x_2y_2v_3$. Since f_i, $i = 1, 2, 3$, cannot contain two 2-vertices, each of them sends at least $\frac{1}{6}$ to v by (R4.3.2) and (R4.4). First if v is a $3_{k,l}$-vertex, such that $k+l \geq 2$, then $\omega^*(v) \geq -1+2\times\frac{1}{2} = 0$ by (R1) and (R4.2). For cases that $k+l \leq 1$, if $k = 1$, then we have $\omega^*(v) \geq -1+\frac{1}{2}+\frac{1}{3}+2\times\frac{1}{6} = \frac{1}{6}$ by (R1). If $l = 1$, we consider following cases. If v is incident to a 5^3-face which contains no 2-vertex, then $\omega^*(v) \geq -1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6} = 0$ by (R4.2) and (R4.3.2). Therefore we could always assume that $\tau(f_1 \to v) \geq \frac{1}{4}$ and $\tau(f_3 \to v) \geq \frac{1}{4}$ by (R4), which implies that $\omega^*(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R4.2). Thus $k = l = 0$, we suppose f_1, f_3 are 5^{3^+}-faces and f_2 is a 5^3-face that contains no 2-vertex. By (R4), we still have that $\tau(f_1 \to v) \geq \frac{1}{4}$ and $\tau(f_3 \to v) \geq \frac{1}{4}$.

1. First suppose both f_1 and f_3 do not contain any 2-vertex, if they are both 5^3-faces, then both of them send a charge of $\frac{1}{3}$ to v by (R4.4) and thus $\omega^*(v) \geq -1+3\times\frac{1}{3} = 0$. So let f_1 be a 5^4-face, then by Claim 13, $n_b(x_2) = 3$. Then when $d(y_2) = 3$, $\tau(f_3 \to v) \geq \frac{2}{3}$ by (R4.4), and when $d(y_2) \geq 4$, $\tau(f_3 \to v) = \frac{1}{2}$ by (R4.2). Thus $\omega^*(v) \geq -1+\frac{1}{3}+\frac{1}{2}+\frac{1}{4} = \frac{1}{12}$.

2. Either $d(x_1) = 2$ or $d(x_2) = 2$, by symmetry, we assume $d(x_1) = 2$. Then by Claim 6, x_2 has at least three big neighbours and y_1 has at least four big neighbours. If $d(y_2) = 3$,

then by (R4.4), $\tau(f_3 \to v) \geq \frac{2}{3}$, and thus $\omega^*(v) \geq -1 + \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{1}{3}$ by (R4.3.2). Assume now that $d(y_2) = 2$. If $n_3(x_2) \geq 2$ or $n_b(x_2) \geq 4$, then by (R4.3.2) and (R4.2), each of f_1 and f_3 will send v at least $\frac{1}{3}$, thus $\omega^*(v) \geq -1 + 3 \times \frac{1}{3} = 0$. So suppose $n_3(x_2) = 1$ and $n_b(x_2) = 3$. Then by Claim 4, $n_b(v_3) \geq 2$ and f_3 is a 5^2-face, which contradicts with our assumption that f_3 is a 5^{3^+}-face.

3. Either $d(y_1) = 2$ or $d(y_2) = 2$, by symmetry, we assume $d(y_1) = 2$. Then $d(x_1) \geq 4$. And we know that $d(x_2) \geq 3$. By Claim 4 and the fact that f_1 is a 5^{3^+}-face, $n_b(x_1) = 3$ or $n_b(v_2) = 3$. First suppose $n_b(v_2) = 3$, then by (R1), $\tau(v_2 \to v) \geq \frac{1}{2}$ and thus $\omega^*(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} + \frac{1}{3} = \frac{1}{3}$. Suppose now that $n_b(x_1) = 3$, then $n_b(v_2) = 1$ since otherwise y_1 is rich. By Claim 4, $n_3(x_1) \geq 2$. By (R4.3.2), $\tau(f_1 \to v) \geq \frac{1}{3}$. If f_3 is also a 5^3-face, then we have $\tau(f_3 \to v) \geq \frac{1}{3}$. Otherwise f_3 is a 5^4-face such that all the small vertices have degree 3, then by Claim 13, the third neighbour of x_2 has at least three big neighbours. The third face of v_1 is either a 5^2-face or a 5^3-face with no 2-vertex, therefore, either v_1 is a $3_{1,1}$-vertex or both v_1 and x_2 are $3_{1,0}$-vertices, by (R4.4), we always have $\tau(f_3 \to v) \geq \frac{1}{3}$. And $\omega^*(v) \geq -1 + 3 \times \frac{1}{3} = 0$.

Case 3.3. v is adjacent to two 3-vertices, say v_2 and v_3. Let the other two vertices of f_i be x_i and y_i in the clockwise order, $i = 1, 2, 3$. By Claim 10, $n_b(v_1) \geq 2$. Again, since f_i, $i = 1, 2, 3$, cannot contain two 2-vertices, each of them sends at least $\frac{1}{6}$ to v by (R4.3.2) and (R4.4). Similarly if v is a $3_{k,l}$-vertex, such that $k + l \geq 2$ or $k = 1$, then $\omega^*(v) \geq 0$. Suppose $k = 0$ and $l = 1$, if v is incident to a 5^3-face which contains no 2-vertex, then $\omega^*(v) \geq -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$ by (R4.2) and (R4.3.2). By (R4.3.2), f_i sends charge at least $\frac{1}{4}$ to v, except f_2 is a 5^5-face.

If one of x_2 and y_2 has degree 2, then by Claim 6, f_2 is a 5^2-face. Then $\omega^*(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R4.2). If one of x_2 and y_2 has degree 3, say $d(x_2) = 3$, then by Claim 13, either one of v_1 or y_1 has at least four big neighbours, or both of them have at least three big neighbours. Therefore either f_1 is a 5^2-face, or $n_b(v_1) \geq 3$ and $n_3(v_1) \geq 2$, both imply that $\omega^*(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$. Therefore, we may assume both x_2 and y_2 have degree at least four, and $\tau(f_2 \to v) \geq \frac{1}{3}$ by (R4.4).

If either y_1 or x_3 has degree at most 3, by symmetry say $d(y_1) \leq 3$, then by Claim 6 and Claim 13, either f_2 is a 5^2-face which implies $\omega^*(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R4.2), or both v_1 and x_2 have at least three big neighbours. If $n_3(v_1) \geq 2$, then $\tau(v_1 \to v) \geq \frac{1}{2}$ by (R1) and thus $\omega^*(v) > 0$. So we suppose $n_3(v_1) = 1$ and $n_b(v_1) = 3$. Then $d(x_1) \geq 4$ and $\tau(f_1 \to v) \geq \frac{1}{3}$. If $d(x_3) \leq 3$, then similarly either f_2 is 5^2-face or $\tau(f_3 \to v) \geq \frac{1}{3}$, we have $\omega^*(v) \geq 0$. Therefore, we assume $d(x_3) \geq 4$. If $d(y_3) \geq 4$, then f_3 is a 5^2-face which gives v enough charge. Otherwise $d(y_3) = 2$ since $n_3(v_1) = 1$, then by Claim 4, either f_3 is again a 5^2-face or f_3 is a 5^3-face and v_3 is incident to a 5^2-face and a 5^3-face which contains no 2-vertex, in both cases $\tau(f_3 \to v) \geq \frac{1}{3}$ by (R4.2) and (R4.3.2). So we always have $\omega^*(v) \geq -1 + 3 \times \frac{1}{3} = 0$.

In the following we may assume both y_1 and x_3 have degree at least 4. If both x_1 and y_3 have degree at least 3, then each f_i sends at least $\frac{1}{3}$ to v by (R4.2) or (R4.4), and $\omega^*(v) \geq -1 + 3 \times \frac{1}{3} = 0$. Assume $d(x_1) = 2$, if f_1 is a 5^2-face, then $\tau(f_1 \to v) \geq \frac{1}{2}$ and $\omega^*(v) \geq -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{12}$. Suppose f_1 is a 5^3-face. If v_2 is incident to a 5^2-face, since

v_2 is incident to another 5^3-face which contains no 2-vertex, by (R4.3.2), $\tau(f_1 \to v) \geq \frac{1}{3}$. Otherwise we may assume the face $f' = v_2x_2x'_2y'_1y_1$ that v_2 incident is a 5^3-face, both x'_2 and y'_1 must have degree at most 3. We first derive that $d(y'_1) = 3$, since otherwise by Claim 5 and Claim 6, y_1 has at least four big neighbours. By Claim 4 and the fact that both x_1 and v are poor, $n_b(y_1) = 3$ and thus $\tau(f_1 \to v) \geq \frac{1}{3}$ by (R4.3.2). By symmetry, either $d(y_3) = 2$ or $d(y_3) \geq 3$, $\tau(f_3 \to v) \geq \frac{1}{3}$. Thus we have $\omega^*(v) \geq -1 + 3 \times \frac{1}{3} = 0$.

Case 3.4. Finally suppose v is adjacent to three 3-vertices. Then by Claim 10, for $i = 1, 2, 3$, each f_i is a 5^3-face that contains no 2-vertex, thus sends at least $\frac{1}{3}$ to v by (R4.4), and $\omega^*(v) \geq -1 + 3 \times \frac{1}{3}$.

Case 4. Assume now $d(v) = 2$ and let $N(v) = \{v_1, v_2\}$. If v is rich, or it is incident or adjacent to at least two of the following, then it would end up with a non-negative charge by (R1) and (R4.1): 5^+-vertex which has at least four 3^+-neighbours, and face with only one 3^--vertex, and 5^1-face.

Otherwise first suppose that v is adjacent to a 2-vertex v_1. Then by Claim 5, v_2 has at least five big neighbours and both incident faces are 5^2-faces and not special. So we have $\omega^*(v) \geq 2 - 4 + 1 + 2 \times \frac{1}{2} = 0$ by (R1) and (R4.2).

Suppose now v is adjacent to a 3-vertex v_1. By Claim 6, v_2 has at least four big neighbours and both incident faces are 5^{3^-}-faces and not special, thus by (R1), (R4.2) and (R4.3), $\omega^*(v) \geq 2 - 4 + 1 + 2 \times \frac{1}{2} = 0$.

Case 4.1. Suppose $d(v_1) = d(v_2) = 4$. Then by the Fact 1, all the neighbours of v_1 and v_2 except v are big vertices. Thus the incident faces of v only contains one small vertex and v is rich.

Case 4.2. Suppose $d(v_1) \geq 5$ and $d(v_2) = 4$. If v_1 has at least four big neighbours, then by definition the incident faces of v are not special since otherwise v_2 is a special vertex, thus $\omega^*(v) \geq 2 - 4 + 1 + 2 \times \frac{1}{2} = 0$ by (R1) and Fact 2. Otherwise since $d(v_2) = 4$, by Fact 1, $n_b(v_1) = 3$. If $n_b(v_2) \leq 2$ or $n_3(v_1) \geq 1$, then $n_b(v_1) + n_3(v_1) \geq 4$, which implies that $\tau(v_1 \to v) = 1$ by (R1) and the incident faces of v are not special, so $\omega^*(v) \geq 2 - 4 + 1 + 2 \times \frac{1}{2} = 0$. Suppose now $n_b(v_2) = 3$ and $n_3(v_1) = 0$. If the incident faces of v are not special, then both of them contain only one small vertex which implies that v is rich. Otherwise by (R5) v would get enough charge such that $\omega^*(v) \geq 0$.

Case 4.3. Suppose both v_1 and v_2 have degree at least 5. If one of the incident faces of v is special, then, by (R5), $\omega^*(v) \geq 0$. Otherwise, if at least one of v_1 and v_2 has at least four 3^+-neighbours, then $\omega^*(v) \geq 2 - 4 + 1 + 2 \times \frac{1}{2} = 0$ by (R1) and Fact 2. Suppose now both v_1 and v_2 have at most three 3^+-neighbours. Then by Fact 1, $n_b(v_1) = n_b(v_2) = 3$, thus for $i = 1, 2$ $\tau(v_i \to v) \geq \frac{1}{2}$ by (R3). Since the incident faces are not special, each of them sends a charge of at least $\frac{1}{2}$ to v. Therefore $\omega^*(v) \geq 2 - 4 + 4 \times \frac{1}{2} = 0$.

Let $f \in F(G)$ and $d(f) = 5$. If f is special, then by (R5) $\omega^*(f) \geq 5 - 4 - 1 = 0$. Thus we may assume f is not special. If f is incident to at most one small vertex, then by (R2) $\omega^*(f) \geq 5 - 4 - 1 = 0$. If f is a 5^1-face, then $\omega^*(f) \geq 5 - 4 - 1 = 0$ by (R3). If f is a 5^2-face, then $\omega^*(f) \geq 5 - 4 - 2 \times \frac{1}{2} = 0$ by (R4.2). Suppose f is a 5^3-face. If $n_2(f) = 2$ and $n_3(f) = 1$, then by (R4.3.1) $\omega^*(f) \geq 5 - 4 - 2 \times \frac{1}{2} = 0$. If $n_2(f) = 1$ and $n_3(f) = 2$, then by

(R4.3.2), either $\omega^*(f) \geq 5 - 4 - \frac{1}{2} + 2 \times \frac{1}{4} = 0$ or $\omega^*(f) \geq 5 - 4 - \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$. Finally suppose f is a 5^{4^+}-face, it has non-negative charge by (R4.4).

It remains to show that every charge pot has non-negative charge. Observe that in a special face, every 4^+-vertex except the special vertex has at least 3 big neighbours by Fact 1. Let P be a charge pot with special vertex v which is obtained by k consecutive special faces $f_1, f_2, \cdots, f_k$. Let $v_1, v_2, \ldots, v_{k+1}$ be the consecutive 2-vertices on the special faces. Then by (R5) $\omega(P) = k$ and there are $k+1$ 2-vertices which will take charge from P. By (R3), v in total sends charge at least $(k+1) \times \frac{k+1+3-4}{k+1} = k$ to these 2-vertices. Let $N(v_1) = \{v, v_1'\}$, f_0 and f_1 be the incident faces of v_1. If $d(v_1') = 4$, then f_0 contains only one small vertex and thus $\tau(f_0 \to v) = 1$ by (R2). Suppose $v_1' \geq 5$, then $\tau(v_1' \to v_1) \geq \frac{1}{2}$ by (R3). If f_0 is not special with respect to v_1', then $\tau(f_0 \to v) \geq \frac{1}{2}$ by (R4.2). Otherwise by (R5) v_1 gets charge 1 from v_1' and the charge pot with respect to v_1'. By symmetry v_{k+1} gets charge at least 1 which is not from v or the charge pot with respect to v. Thus $\omega^*(P) \geq k - (2(k+1) - k - 2) = 0$. This completes our proof.

3 Concluding remarks

In this work, using the result of [9] which itself is based on the 4-color theorem, we showed for every triangle free planar simple graph G, the signed graph (G, σ) has a packing number at least 5. Unlike the result of [9], the discharging technique used here is based on a planar embedding of G and thus cannot be applied to the class of K_5-minor-free graphs directly. However, an extension from planar graphs to K_5-minor-free graphs is already shown in [5].

It is unclear if this result can be proved independent of 4-color theorem. It is also not clear how important is the choice of the all negative signature. More precisely we would like to ask:

Question 8. What is the best possible lower bound on the packing number of planar signed graph of girth at least g?

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4 Compliance with Ethical Standards

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