Bounding signed series-parallel graphs and cores of signed K_4-subdivisions

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Abstract

We study homomorphism properties of signed K_4-minor-free graphs. On the one hand we give a necessary and sufficient condition for a signed graph B to admit a homomorphism from any signed K_4-minor-free graph and we determine the smallest of all such signed graphs. On the other hand, we characterize the minimal cores that do not belong to the class of signed K_4-minor-free graphs. This, in particular, gives a classification of odd-K_4's that are cores. Furthermore, we show some applications of our work.

Keywords: graph homomorphisms, signed graphs, minor of graphs, series-parallel graphs

1 Introduction

The class of K_4-minor-free graphs, also known as series-parallel graphs, is of special importance in the study of the homomorphism properties of graphs. On the one hand, restriction to closely related subclasses such as outerplanar graphs reduces the homomorphism relation to the study of finding odd-girth of graphs; indeed Gerards proved in [7] that the core of any outerplanar graph is its shortest odd cycle. On the other hand, Hubička and Nešetřil proved, in [8], that the homomorphism order on the class of K_4-minor-free graphs is a universal countable order, i.e., it contains an isomorphic copy of any countable order.

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The study of homomorphism properties of K_4-minor-free graphs becomes even more appealing by the observation that this is precisely the class of graphs of tree-width at most 2.

Recall that the property of no K_4-minor for a graph is equivalent to the property of no K_4-subdivision [5]. This leads to a study of the homomorphism properties of minimal graphs that do not belong to the class of graphs of tree-width at most 2.

A development of homomorphisms of signed graphs has begun recently in [11] (see also [12]). This is of high interest because it provides a framework for stronger relation between minor theory and homomorphisms of graphs. Thus in this paper we are interested in the study of homomorphism properties for K_4-minor-free signed graphs or the minimal graphs not belonging to this family.

On the one hand, we provide a necessary and sufficient condition for a signed (simple) graph to admit a homomorphism from any K_4-minor-free signed graph and we determine the smallest of such graphs. On the other hand, we determine which signed subdivisions of K_4 are cores. The latter could be used as a tool in the study of some conjectures such as Jaeger-Zhang [14] conjecture or its bipartite analogue [11] (see in Section 5). Characterizing subdivisions of K_4 which are cores as graphs (without a signature) is a special case of our result. Because of an independent interest of this case, it is considered in Section 4.2.1 without using the notion of sign or signature.

The paper is organized as follows. In the next section, we settle our choice of terminology and mention basic tools in the study of homomorphisms of signed graphs. We note that there are competing terminologies as the theory of signed graphs is being developed from different areas. In Section 3, we prove a necessary and sufficient condition for a signed graph to bound the class of signed K_4-minor-free graphs and we find the smallest of such bounds. This is a notion of chromatic number of signed graphs, noting that there are other interesting notions of chromatic number of signed graphs. In Section 4, we classify all the signed subdivisions of K_4 that are cores. In Section 5, we explain how this could be used as a tool in other homomorphism problems. Finally in Section 6, we address the future direction of study and mention a few open problems.

2 Preliminary

We consider simple graphs. Given a graph G, a *walk* in G is an alternating sequence of vertices and edges which starts and ends at vertices, where an edge e of the sequence lies in between its two endpoints. A *closed walk* is a walk where the start and end vertices are the same. Closed walks are rather viewed in a cyclic order, thus losing the importance of the start point. A *path* in G is a walk of G where all vertices are distinct (thus all edges are distinct as well). A *thread* in G is a path where all the vertices are of degree 2 except possibly the starting and ending points of the path. For a graph G on at least 3 vertices, a *cycle* in G is a closed walk with distinct edges and vertices in the sequence except the staring and ending points.

Given an edge $uv \in E(G)$, to *subdivide an edge uv of a graph G* is to replace it with a u-v thread P. The length of the subdivision of an edge uv is the length of P. A *subdivision of a graph* G is a graph G' which is obtained from G by subdividing some or all edges of G. A subdivision of G sometimes is referred to as a *G-subdivision*.

Given an edge $uv \in E(G)$, to *contract an edge uv of a graph G* is to remove it and merge its two endpoints. A *minor* of G is a graph H obtained from G by a sequence of operations: deleting vertices, deleting edges and contracting edges in any order. We say that G has an *H-minor* if it admits H as its minor, otherwise G is said to be *H-minor free*.

It is easily observed that if G is a subdivision of H, then H is a minor of G.

The converse is not always true, but in the following special case it is.

Lemma 2.1. [5] *Let H be a graph with $\Delta(H) \leq 3$. A graph G has an H-minor if and only if G has an H-subdivision as a subgraph.*

2.1 signed graph

A *signed graph* (G, σ) is a pair consisting of a graph G and an assignment $\sigma : E(G) \to \{+, -\}$. In our figures, we use a dashed line to present a negative edge and a solid line to present a positive edge. A signed graph (H, π) is a *subgraph* of signed graph (G, σ) if H is a subgraph of G and $\pi = \sigma|_{E(H)}$. A signed graph where all edges are negative is denoted by $(G, -)$ and a signed graph where all edges are positive is denoted by $(G, +)$. The *sign of a closed walk* is the product of signs of its edges (allowing repetition).

Given a signed graph (G, σ) and a vertex v, a *switch* at vertex v is to switch the signs of edges which are incident to v. A signed graph (G, σ) is a *switch* of (G, σ') if it is obtained from (G, σ') by a sequence of switchs at vertices, in which case we may also say σ' is a switch of σ and (G, σ) is *switching equivalent* to (G, σ').

Observe that the sign of a closed walk, and in particular the sign of a cycle is invariant under switch. Zaslavski proved that the inverse is also true in the following sense:

Proposition 2.2. [13] *Two signed graphs (G, σ_1) and (G, σ_2) are switching equivalent if and only if they have the same set of negative cycles.*

Closed walks, as the key structures of a signed graph, are of four possible types based on their sign and the parity of their length. As sign itself is the parity of the number of negative edges, one may use elements of $\mathbb{Z}_2^2$ to denote these four types. More precisely: a *positive odd closed walk* in (G, σ) is of type 01, a *negative odd closed walk* in (G, σ) is of type 11, a *positive even closed walk* in

(G,σ) is of type 00 and a *negative even closed walk* in (G,σ) is of type 10. Given a signed graph (G,σ), $g_{ij}(G,\sigma)$ denotes the length of the smallest closed walks of type ij in (G,σ). When there is no closed walk of type ij we set $g_{ij}=\infty$.

Observe that $g_{00}(G,\sigma)=2$ as long as G has at least one edge. It is shown in [12] that of the other three values, $g_{01}(G,\sigma)$, $g_{10}(G,\sigma)$ and $g_{11}(G,\sigma)$ if one is ∞, then at least two of them are ∞. This leads to three special subclasses of signed graphs: the $\mathcal{G}_{01}$ consisting of signed graphs (G,σ) which can be switched to $(G,+)$ (thus $g_{10}=g_{11}=\infty$ for every member of this family), the $\mathcal{G}_{11}$ consisting of signed graphs (G,σ) which can be switched to $(G,-)$ (then $g_{10}=g_{01}=\infty$) and the $\mathcal{G}_{10}$ consisting of signed bipartite graphs (where $g_{01}=g_{11}=\infty$). These subclasses are of special importance in the study of homomorphisms of graphs and signed graphs, we refer to [12] for more details.

Observe that for any signed graph in $\mathcal{G}_{ij}$, every cycle of it is either of type 00 or of type ij. A positive (negative) cycle of length l will be denoted by C_l^+ (C_l^-).

2.2 Homomorphisms and bounds

A *homomorphism* of a signed graph (G,σ) to another signed graph (H,π) is a mapping φ from $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that the adjacencies, the incidences and the signs of closed walks are preserved. When there exists a homomorphism of (G,σ) to (H,π), we write $(G,\sigma)\to(H,\pi)$. A homomorphism of (G,σ) to (H,π) is said to be an *edge-sign-preserving homomorphism* if it also preserves the signs of the edges. For more on homomorphisms of signed graphs we refer to [11] and [12].

The *core* of a signed graph (G,σ) is the smallest subgraph (H,π) of (G,σ) such that $(G,\sigma)\to(H,\pi)$. A signed graph (G,σ) is a *core* if its core is itself. Cycles which are not of type 00 are examples of signed graphs that are cores.

The following no-homomorphism lemma is an immediate consequence of the definition:

Lemma 2.3. *If $(G, \sigma) \to (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$.*

For most other applications of homomorphism of signed graphs, we use the condition from the following theorem.

Theorem 2.4. [12] *A signed graph (G, σ) admits a homomorphism to a signed graph (H, π) if and only if there exists a switch σ' of σ such that (G, σ') admits an edge-sign-preserving homomorphism to (H, π).*

Using this reformulation, one observes that restriction of homomorphisms of signed graphs to the class $\mathcal{G}_{01}$ (or $\mathcal{G}_{11}$) is the classic notion of homomorphisms of graphs. Furthermore, it is shown, in [11], that restriction on the class $\mathcal{G}_{10}$ already captures the notion of graph homomorphism via a simple graph operation.

Given a signed graph B and a class $\mathcal{C}$ of signed graphs, we say B *bounds* the class $\mathcal{C}$ if every member of $\mathcal{C}$ admits a homomorphism to B. The most well-known example of a bound is a reformulation of the four color theorem which says $(K_4, +)$ (resp. $(K_4, -)$) bounds the class of planar graphs in $\mathcal{G}_{01}$ (resp. $\mathcal{G}_{11}$).

A main question of importance then is the following. Given a class $\mathcal{C}$ of signed graphs and a signed graph B, are the necessary conditions of Lemma 2.3 also sufficient? In other words, does every signed graph $(G, \sigma) \in \mathcal{C}$ satisfying $g_{ij}(G, \sigma) \geq g_{ij}(B)$ admits a homomorphism to B? If so, then we say B is $\mathcal{C}$-easy. It is not hard to check that the four color theorem is equivalent to claiming that:

Theorem 2.5 (4CT, restated)**.** *The signed graph $(K_4, +)$ is planar-easy, and so is $(K_4, -)$.*

Some other conjectures and results can also be nicely presented in this language.

For SSPG being the class of signed series-parallel graphs and for a signed graph B which is in one of the three special classes $\mathcal{G}_{ij}$, $ij \in \{01, 10, 11\}$, a necessary and sufficient condition for B to be SSPG-easy is given in [2] and [3].

Our work in the next section is a first step towards a generalization of this work where B can be chosen among all signed graphs. More precisely, we give a necessary and sufficient condition for a simple signed graph B to bound the class of signed K_4-minor-free graphs. On the other hand it is shown in [6] that if B is in one of $\mathcal{G}_{ij}$ and B is SSPG-easy, then so is the extended double cover $\mathrm{EDC}(B)$ (see Section 6 or [12] for a definition of $\mathrm{EDC}(B)$). We will show that this is not true in general.

In Section 4, we characterize the minimal signed graphs that are not in the class of signed K_4-minor-free graphs. Those are the signed subdivisions of K_4 which are cores. In Section 5 then we mention possible applications of this characterization.

3 Coloring of signed K_4-minor-free graphs

In this section, we study the homomorphism of signed K_4-minor-free graphs. We recall a characterization of edge-maximal K_4-minor-free graphs:

Proposition 3.1. [5] *A graph with at least three vertices is edge-maximal K_4-minor-free if and only if it can be constructed recursively from triangles by pasting along K_2s.*

Here, similar to the results of [2] and [3], we give a characterization of simple signed graphs to which every signed K_4-minor-free graph admits a homomorphism.

Theorem 3.2. *A signed graph (H, π) bounds the class of signed K_4-minor-free graphs if and only if there exists a nontrivial subgraph $(H', \pi') \subset (H, \pi)$ such that each edge of (H', π') belongs to at least one positive triangle and at least one negative triangle in (H', π').*

Proof. Suppose that (H, π) contains a subgraph (H', π') such that every edge of (H', π') belongs to at least one positive triangle and at least one negative triangle.

Given a signed K_4-minor-free graph (G, σ), let (G^*, σ^*) be an edge-maximal signed K_4-minor-free graph containing (G, σ) where $\sigma^*(e)$ is chosen arbitrarily for each edge e not in (G, σ). We will show that (G^*, σ^*) maps to (H', π').

Let $v_1, v_2, \ldots, v_n$ be an ordering of vertices of G^* obtained by following Proposition 3.1. Thus v_1, v_2, v_3 induces a triangle in G^*. As H' is nontrivial, it has at least one edge and thus by the assumption on the edges it has at least one positive triangle on this edge and one negative triangle. Taking a triangle T in (H', π'), of the same sign of the triangle $v_1v_2v_3$ in (G^*, σ^*), and after a suitable switch if needed we may map v_1, v_2, v_3 to the vertices of T. To complete this mapping to a mapping of (G^*, σ^*) to (H', π') we use induction: having ϕ which maps vertices $v_1, v_2, \ldots, v_i$ to (H', π'), from the structure of G^* we know v_{i+1} forms a triangle with two vertices, say v_{j_1}, v_{j_2}, in $v_1, v_2, \ldots, v_i$ and is connected to no other. Consider that the sign of the triangle $v_{i+1}v_{j_1}v_{j_2}$. As $\phi(v_{j_1})\phi(v_{j_2})$ is an edge of (H', π'), and according to the main property of (H', π'), there exists a triangle $\phi(v_{j_1})\phi(v_{j_2})u$ which is of the same sign as the sign of the triangle $v_{i+1}v_{j_1}v_{j_2}$. Extend ϕ by mapping v_{i+1} to u (after a switch at v_{i+1} if necessary). This proves one direction of the theorem.

It remains to show that if (H, π) bounds the class of signed K_4-minor-free graphs, then there exists such a subgraph satisfying the property. Suppose that (H, π) bounds the class of signed K_4-minor-free graphs. Let (H^*, π^*) be a minimal subgraph of (H, π) which bounds the class of signed K_4-minor-free graphs. Here minimality is with respect to taking subgraphs. Thus, in particular, for each edge $e_i \in E(H^*)$, $i \in \{1, \ldots, |E(H^*)|\}$, there exists one K_4-minor-free graph G^i such that $(G^i, \sigma^i) \not\to (H^* - e, \pi^*|_{E(H^*-e)})$.

Given an edge $e_i = xy$, to prove the property for this edge, we denote all the homomorphisms of (G^i, σ^i) to (H^*, π^*) by $\varphi_1, \varphi_2, \ldots, \varphi_r$. For each φ_s, $s \in \{1, \ldots, r\}$, there exists an edge $x_sy_s \in E(G^i)$ such that $\varphi_s(x_s)\varphi_s(y_s) = xy$. We now construct a signed K_4-minor-free graph (G^i_*, σ^i_*) based on (G^i, σ^i) in order to show the edge property of (H^*, π^*) for the edge e_i. For any edge $x_s y_s$ with $s \in \{1, \ldots, r\}$, we add vertices w_s and z_s to (G^i, σ^i), and then add a negative edge $w_s y_s$ and three positive edges $w_s x_s, z_s x_s, z_s y_s$. By Proposition 3.1, we know that G^i_* is still a K_4-minor-free graph, so there exists $\Phi : (G^i_*, \sigma^i_*) \to (H^*, \pi^*)$. Furthermore, we know that $\Phi|_{(G^*,\sigma^*)} = \phi_{s_0}$ for an $s_0 \in \{1, \ldots, r\}$. Thus $\Phi(x_s)\Phi(y_s) = xy$, and $\Phi(w_s), \Phi(z_s)$ are vertices of (H^*, π^*). By the definition of homomorphism, we conclude that one of $\Phi(w_s)xy$ and $\Phi(z_s)xy$ is a positive triangle while the other is a negative triangle. This prove the property for the edge e_i. □

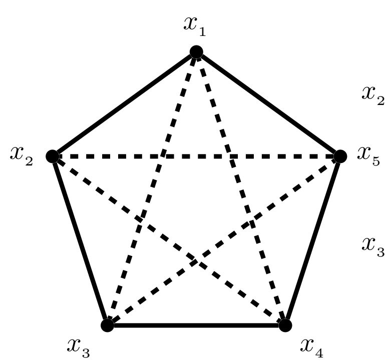

Figure 1: $SPal_5$

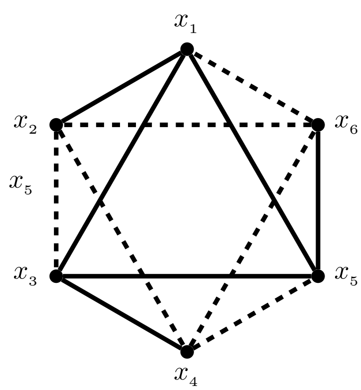

Figure 2: A 6-vertices bound for the class SSPG

An example of a signed graph on 5 vertices satisfying Theorem 3.2 is a graph of Figure 1 known as *Signed Paley graph* on 5 vertices, denoted by $SPal_5$. This was used to prove the chromatic number of signed K_4-minor-free graph (in the sense of [11]) is at most 5. We show that 5 is the best and moreover $SPal_5$ is the only bound on 5 vertices. This graph has 10 edges which is also the minimum number of edges among signed graphs bounding signed K_4-minor-free graphs. However, in terms of the number of triangles this graph is not the optimal one as the graph of Figure 2 is also a bound having only eight triangles. Furthermore, note that the bound of Figure 2 is a planar one.

Theorem 3.3. *Up to a switch, the signed graph $SPal_5$ is the smallest signed graph (in terms of both number of vertices and number of edges) which bounds the class of signed K_4-minor-free graphs.*

Proof. First we show that no signed graph on 4 vertices satisfies the property in Theorem 3.2. Toward a contradiction, assume a signed graph on four vertices x_1, x_2, x_3 and x_4 satisfies the condition of the theorem. Assume x_1x_2 is a positive edge, then for x_1x_2 to satisfy the condition, and by the symmetry of the labels, $x_1x_2x_3$ is a negative triangle and $x_1x_2x_4$ is a positive triangle. By switching, if needed, we may then assume x_1x_3 is a negative edge and x_1x_2, x_1x_4, x_2x_4 and x_2x_3 are positive edges. At this point only the edge x_1x_2 satisfies the property. We are allowed to add one more edge (x_3x_4) with a sign of our choice. But with each of the two choices for the sign of x_3x_4, either the two disjoint edges x_1x_3, x_2x_4 or the two disjoint edges x_1x_4, x_2x_3 will not have the property. This proves that we need at least five vertices.

Next we want to show that $SPal_5$ is the only signed graph on five vertices (up to a switching) that satisfies the property. Start from a signed graph on $\{x_1, \dots, x_4\}$ with the triangle $x_1x_2x_3$ being positive and the triangle $x_1x_2x_4$ being negative, so that the edge x_1x_2 satisfies the property. Following the previous part of the proof, and without loss of generality, we assume that x_2x_3 and x_1x_4 does not satisfy the property before adding a fifth vertex. By a switch at some of vertices x_1, x_2, x_3, if needed, we may assume that x_1x_2, x_1x_3, x_2x_3, x_2x_4 are positive edges and that x_1x_4 is a negative edge. This means after the above mentioned switch the pair x_3x_4 is either not an edges or a positive edge. Currently x_2x_3 is in a positive triangle and x_1x_4 is in a negative triangle. Thus after adding the vertex x_5 we must form a negative triangle $x_2x_3x_5$ and a positive one triangle $x_1x_4x_5$. Using a switch at x_5, if needed, we may assume that x_4x_5 is a positive edge and that x_1x_5 is a negative edges. Then we have two possibilities: (i) x_3x_5 is a positive edge and x_2x_5 is a negative edge, or (ii) x_2x_5 is a positive edge and x_3x_5 is a negative edge. In the case (i) the edge x_2x_4 has yet to satisfy the condition, as it is already in two negative triangles, the pair x_3x_4 must form a positive edge. In the case (ii) the edge x_1x_3 has yet to satisfy the condition, as it is already in two positive triangles, the pair x_3x_4 must form a positive edge. In the case (i) after switches at x_2 and x_4 and in the case (ii) after switches at x_3 and x_4 we get an isomorphic copy of $SPal_5$.

Thus, so far, we have shown that any bound for the class of signed K_4-minor-free graphs is of order at least 5 and that up to a switch, $SPal_5$ is the only such a bound on five vertices. It remains to show that any such bound has also at least 10 edges. To this end we observe that in a minimal bound, as each edge is in at least 2 triangles, the degree of each vertex should be at least 3 but furthermore if all vertices are of degree 3, the graph must be K_4 which we have already seen not to be possible. As the only bound on five vertices has indeed 10 edges, we may consider a (minimal) bound on at least six vertices. Such a graph where all vertices are of degree 3 or higher and at least one vertex is of degree 4 or higher must then have at least 10 edges. □

4 Signed K_4-subdivisions

As mentioned before, each cycle of a signed graph, based on its parity and its sign, is of one of the four possible types: 00, 01, 10 and 11. Considering a planar embedding of a signed K_4-subdivision based on the types of the faces we have total of 11 possibilities. To see that is the full list of possibilities, we first observe that for each signed subdivision (G, σ) of K_4 if we take σ' to be a signature which is a switch of σ and has the least possible number of negative edges, then σ' will have at most two negative edges, and that if there are two negative edges, then they must be on the subdivisions of disjoint edges of K_4. We may then consider possible parities and signs of the facial cycles to get the list of 11 possibilities. These possibilities are presented in Figure 3 where they are classified into three groups, each group is presented in a row. In this figure, for each subdivision, the type of each bounded face is indicated inside the face, then the type of the outer face is determined as the binary sum of the types of the three bounded faces. In the following figures, a zigzag line represents a subdivision of an edge inside which each edge is of the positive sign and a dashed line represents a single edge of the negative sign.

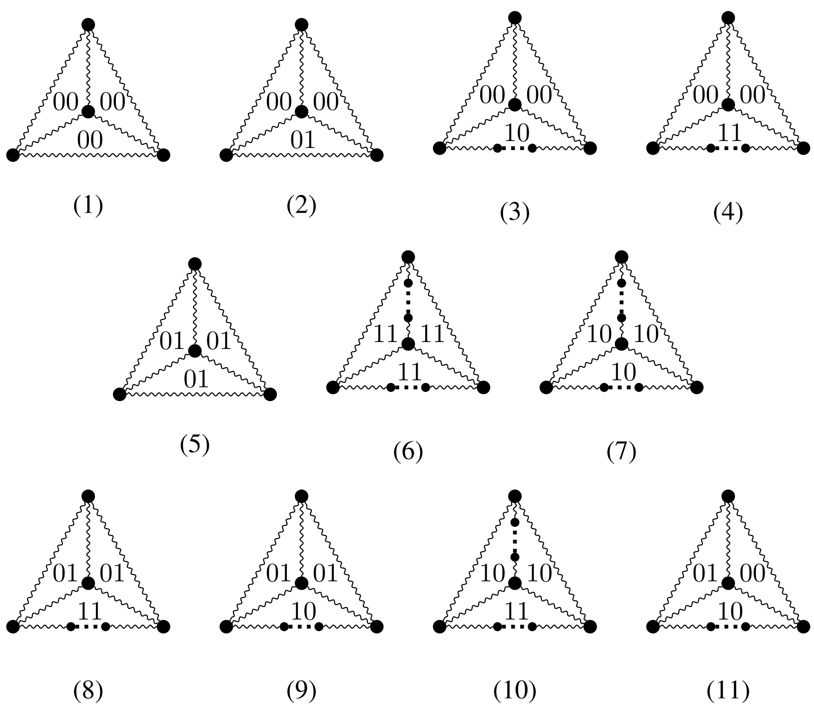

Figure 3: Classification of signed K_4-subdivisions

In the first row of Figure 3 we have subdivisions with (at least) two facial cycles of type 00, i.e., two positive even facial cycles. We will see that in these cases the signed subdivision can never be a core. The three kinds of subdivisions presented in the second row are the signed subdivisions (G, σ) when $(G, \sigma) \in \mathcal{G}_{ij}$, $ij \in \{01, 11, 10\}$. As the first (from left) can be switched to all positive and the second can be switched to all negative, these two cases are about graphs (signature being of no importance), thus to deal with these two cases we will consider subdivisions of K_4 which are cores as a graph. The third case of this row represents subdivisions that are signed bipartite graphs. For this family of signed subdivisions of K_4, while the core of the underlying graph is K_2, as a signed graph the core of the subdivision is either the graph itself, or the core of it is the shortest negative cycle of it. We will then classify, based on the lengths of subdivisions, when the subdivision is a core. The three cases of the second row are of special importance in the study of a number of important conjectures.

The remaining cases are when all four types of closed walks exist in (G, σ). We will show for a member of this class to be a core, the triangular inequality must hold on the unique cycle of type 00 of (G, σ).

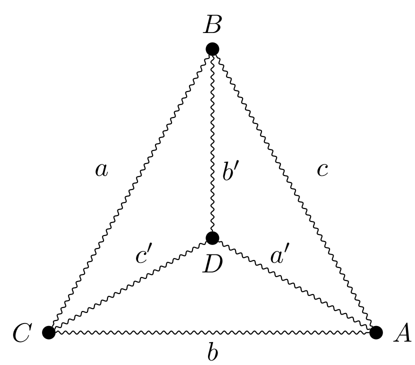

Figure 4: Labeling a K_4-subdivision

To continue we adapt the labeling of Figure 4 for all cases. Thus the four main vertices of a K_4-subdivision are labeled A, B, C and D where a, b, c, a', b', c' denotes lengths of the subdivisions of the corresponding edges. We may call such a subdivision an (a, b, c, a', b', c')-K_4.

4.1 Signed (a, b, c, a', b', c')-K_4 with at least two positive even cycles

It is an analogue of the handshake lemma that: in every planar graph the number of facial cycles of odd length is even. A similar argument for signed planar graphs implies that: in every planar signed graph the number of negative facial cycles is even. The combination of these two facts implies that if two of the faces of a signed K_4-subdivision are positive and even (i.e., of type 00), then the other two are of a same type. As there are four possible types of cycles, we have the four possibilities of the first row of Figure 3. In the first of these four all faces are even and positive thus all cycles (including non-facial cycles) are of type 00. Thus after a switch, if needed, all edges are positive, and we have a bipartite graph whose core is K_2, or $(K_2, +)$ as a signed graph. In each of the other three cases of the first row we have a closed walk (or more precisely a cycle) of type ij, $ij \in \{01, 11, 10\}$ and do not have any closed walk of the other two types. We show that, in each case, a shortest closed walk of type ij, $ij \neq 00$, which is necessarily a cycle of the graph, is the core of it. This in fact is obtained by repeating applications of the folding Lemma of [9] for the cases of $ij = 01$ and $ij = 11$ and of the folding lemma of [11] for the case of $ij = 10$. We state a combined version of these two lemmas in the following and we leave the remaining details to the reader.

Lemma 4.1 ([9], [11]). *Let (G, σ) be a plane signed graph in $\mathcal{G}_{ij}$ with $g_{ij}(G, \sigma) = g$, for $ij \in \{01, 10, 11\}$. If $C = v_0 \cdots v_{r-1} v_0$ is a facial cycle of (G, σ) which is either of type ij with $r > g$ or a cycle which is not of type ij (thus of the type 00), then there is an integer $i \in \{0, \dots, r-1\}$ such that the signed graph (G', σ') obtained from (G, σ) by identifying v_{i-1} and v_{i+1} (subscripts are taken modulo r) after a possible switch is a homomorphic image of (G, σ) satisfying*

$g_{ij}(G,\sigma) = g$.

We have thus proved the following theorem.

Theorem 4.2. *Suppose that (G,σ) is a signed (a,b,c,a',b',c')-K_4 with at least two facial cycles of type 00, then either all the four facial cycles are of type 00 in which case the core of (G,σ) is $(K_2,+)$ or (G,σ) has a cycle which is not of type 00 and the shortest of all such cycles is the core of (G,σ).*

4.2 Signed (a,b,c,a',b',c')-K_4 in $\mathcal{G}_{ij}$

Here we consider the three cases of the second row of Figure 3. In this row, case (5) is an element of $\mathcal{G}_{01}$ where after a switch, if needed, all edges are positive. The case (6) is an element of $\mathcal{G}_{11}$ where after a switch, if needed, all edges are negative. Thus in these two cases, (i.e., (5) and (6)) the problem of the subdivision being a core is reduced to whether the underlying graph is a core. Therefore we will address this case without using signature so that a reader interested only in the case of a graph can read it independently. Then we will consider case (7) which is about signed bipartite graphs. It is shown in [11] that restriction of homomorphism to this class is already richer than homomorphisms of graphs.

4.2.1 odd-K_4

Let G be a subdivision of K_4. As mentioned in the previous section, if G has an even face, then it has at least two even faces and then core of G is either a K_2 or its shortest odd cycle. Thus in the rest of this subsection we assume G is a subdivision of K_4 together with a plane embedding where all faces are odd cycles. Such a subdivision of K_4 is normally referred to as an *odd*-K_4. Furthermore, and similar to the general case, an odd-(a,b,c,a',b',c')-K_4 refers to an odd-K_4 where lengths of the paths resulted from subdivided edges are a,b,c,a',b',c' where the pair i, i' refers to subdivisions of disjoint edges. This implies that $a = a' \pmod 2$, $b = b' \pmod 2$, $c = c' \pmod 2$ and $a+b+c = 1 \pmod 2$.

It is easily observed that after removing an edge of G, the resulting subgraph maps to its shortest odd cycle. Thus we have the following claim.

Proposition 4.3. *The core of an odd-K_4 is either itself or its shortest odd cycle.*

Towards a characterization of odd-K_4's that are cores, we need the following lemma which is based on a subdivision of the graph presented in Figure 5. In this figure the planar embedding is of importance and, with respect to this embedding, the facial cycle bounded by the two TZ threads is the only even one. We may allow r' or l' to be 0 in which case vertex T is identical to Y or to Z. When for example $l' = 0$, then AT path of length l is a cycle.

Lemma 4.4. *Given a graph G of Figure 5 and assuming that the outer face is a shortest odd cycle of length $2k+1$, we have $G \to C_{2k+1}$ if and only if the following three conditions hold:*

- $l \geq l'$,
- $r \geq s_1 + s_3 + l'$,
- $s_2 \geq s_1 + r'$

Proof. First observe that any pair u and v of vertices of an odd cycle C_{2k+1} partitions it into two paths, one of even length and another of odd length. If G is a graph obtained from C_{2k+1} by adding a u-v path P where all internal vertices of P are distinct, then G maps to C_{2k+1} if and only if the length of P is at least as the length of the u-v part of C_{2k+1} which is of the same parity as length of P. The three conditions of lemma then are applications of this with the outer face being C_{2k+1} for three different choices of P. □

Theorem 4.5. *Let G be an odd-(a, b, c, a', b', c')-K_4 whose shortest odd cycle is of length $a + b + c$. Then G is a core if and only if the following three conditions are satisfied:*

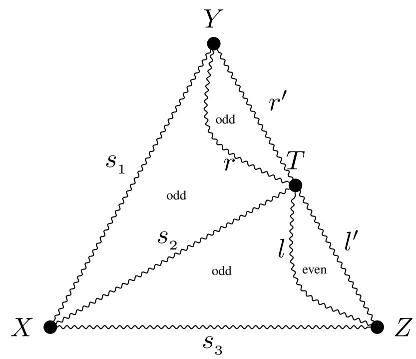

Figure 5: Subdivision of a C_{2k+1}-cycle with 3 chords

(i) $b' + c' - a < a + b + c$;

(ii) $a' + b' - c < a + b + c$;

(iii) $a' + c' - b < a + b + c$.

Proof. We will use the notation of Figure 4 and, furthermore, we assume, without loss of generality, that the outer face is a shortest odd cycle of G and that it is of length $2k+1$. We will rather work with counter opposite statement: G is not a core if and only if (at least) one of the three conditions are not satisfied.

By Proposition 4.3, if G is not a core, then G must map to the outer cycle which is a $(2k+1)$-cycle. We use C_{ABC} to denote this cycle.

We first prove the "if" part of the statement. Assume G is not a core, then $G \to C_{ABC}$ and let f be one such mapping. Depending on which of the three parts of C_{ABC} the vertex D is mapped to, we will have a contradiction with one of the conditions (i), (ii) or (iii). By the symmetry, we may assume that $f(D)$ is on AB part of C_{ABC}, furthermore, we may assume that $f(D)$ is at distance c_1 from A on the AB path and distance $c_2 = c - c_1$ from B on the AB path, allowing c_1 or c_2 to be 0. We then choose the unique vertex D' on the AB path of G which is at distance c_1 from A and distance c_2 from B on the path AB. Let G' be the graph obtained by identifying vertices D and D'. Observe that G' is a graph of Figure 5. As the mapping f also identifies D and D', the graph G', using remaining identification of f, maps to its outer cycle, thus the three conditions of Lemma 4.4 must be satisfied. By adding the second and third inequality of Lemma 4.4 we will have $b' + c' \geq 2a + b + c_1 + c_2$ which contradicts condition (i) of the theorem.

For the "only if" part, we assume that condition (i) is not satisfied, and the other two cases are analogous. Thus we are assuming that $b' + c' - a \geq a + b + c$ and we need to show that G is not a core. We first observe that if one of a', b' or c' is at least as $a + b + c = 2k + 1$, then G is not a core. That is because, assuming $a' \geq a + b + c$, after deleting the AD-path the remaining graph maps to its shortest odd cycle which is C_{2k+1}, and then no matter where A and D are mapped to, we can extend the mapping. Thus in the rest of the proof we may assume $a', b', c' < a + b + c$.

As we have assumed $b' + c' - a \geq a + b + c$ and by the fact that b and c are positive integers, we have $b' + c' > 2a$ which implies either $b' > a$ or $c' > a$. By the symmetry we assume that $b' > a$. In what follows, characterizing three possibilities based on how large b' is, in each case we will find a vertex of the outer cycle to which we may map the vertex D, after which, applying the Lemma 4.4 to the image, we will conclude that G is not a core. The three cases we consider are as follows: (1) $a < b' \leq a + b$, (2) $a + b < b' \leq a + b + a'$, (3) $a + b + a' < b'$.

Case (1) In this case we choose a vertex u on the path AC such that u is at distance $a + b - b'$ from A and at distance $b' - a$ from C where both distances are considered on the path AC. As we are in the case (1), both numbers $a + b - b'$ and $b' - a$ are nonnegative integers, thus such a choice of u is possible. We now consider a homomorphic image G' of G obtained from identifying D with u. Observe that a' and $a + b - b'$ are of a same parity because $a + b + a' + b'$ is the length of an even cycle $BCAD$ in G. We may then observe that G' is a graph of Figure 5 where A represents vertex Z, B represents X, C represents Y and D represents T and $s_1 = a$, $s_2 = b'$, $s_3 = c$, $l = a'$, $l' = a + b - b'$, $r = c'$ and $r' = b' - a$. To complete the proof we will show that G' satisfies the three conditions of Lemma 4.4.

First claim is $l \geq l'$: that is to say $a' \geq a+b-b'$ or equivalently $a'+b' \geq a+b$. This is the result of assumption that $a + b + c$ is the length of the shortest odd cycle of G, in this graph ABD is an odd cycle of length $a' + b' + c$, thus proving that $a' + b' \geq a + b$.

Second claims is $r \geq s_1 + s_3 + l'$: that is to say $c' \geq a + c + (a + b - b')$ or equivalently $c' + b' - a \geq a + b + c$ and that is the case because we assume that condition (i) of the theorem does not hold.

The third claim is $s_2 \geq s_1 + r'$: that is to say $b' \geq a + (b' - a)$ which is trivially true.

Thus, by Lemma 4.4, G' maps to its outer face, which is the $(2k + 1)$-cycle ABC, thus G also maps to its shortest odd cycle.

Case (2) In this case we choose a vertex u on the path AB such that u is at distance $b'-(a+b)$ from A and at distance $a+b+c-b'$ from B with both distances being on the AB path. Since we are in case (2) we have $a + b < b'$ implying that $b' - (a + b)$ is nonnegative, and $a + b + c - b'$ is positive as we saw at the start of only if part. Thus we indeed have a choice of such a vertex u. As in the previous case we consider the graph G' obtained from G by mapping D to u. Once again we will show that G' satisfies the conditions of Lemma 4.4 which will complete the proof of this case as before. To apply this lemma, vertices A, B, C, D will be Z, Y, X and T respectively, and lengths are given by $s_1 = a$, $s_2 = c'$, $s_3 = b$, $l = a'$, $l' = b' - (a + b)$, $r = b'$ and $r' = a + b + c - b'$. It remains to show that the three conditions of Lemma 4.4 are satisfied. The condition $l \geq l'$ follows from assumption of the case (2). The second condition (that $r \geq s_1 + s_3 + l'$) holds as an identity. The third condition ($s_2 \geq s_1 + r'$) is the consequence of our assumption that condition (i) of the theorem does not hold.

Case (3) In this case first we delete internal vertices of the path BD. The result must map to its shortest odd cycle which is C_{ABC}. Then observing that b' is of a same parity as $a+b+a'$ (because the cycle corresponding to a, b, a', b' is an even cycle), we may map the path BD to path $BCAD$. $\square$

4.2.2 Signed bipartite subdivisions

In this subsection we consider the case (7) of Figure 3. Thus in this subsection (G, σ) is a signed bipartite subdivision of K_4 together with a planar embedding where all facial cycles are negative. Observe that in this case an equivalent signature with minimum number of negative edges has two negative edges, one on each of two paths corresponding to two disjoint edges of K_4. As before we will use the labeling of Figure 4 and we may refer to (G, σ) as the (a, b, c, a', b', c')-K_4. Furthermore, without loss of generality, we will always assume that the outer face is a shortest negative cycle of (G, σ), in other word $g_{10}(G, \sigma) = a + b + c$.

It is easy to observe that if we delete an edge of (G, σ), then the resulting signed graph maps to its shortest negative cycle. This implies that:

Proposition 4.6. *The core of a signed bipartite subdivision of K_4 whose faces are all negative cycles, is either itself or its shortest negative cycle.*

In this case we have the following analogue of Lemma 4.4.

Lemma 4.7. *Given a signed graph (G, σ) of Figure 6 if the outer face is a shortest negative cycle and is of length $2k$, then $(G, \sigma) \to C_{2k}^-$ if and only if the following three conditions hold:*

- $l \geq l'$,
- $r \geq s_1 + s_3 + l'$,
- $s_2 \geq s_1 + r'$

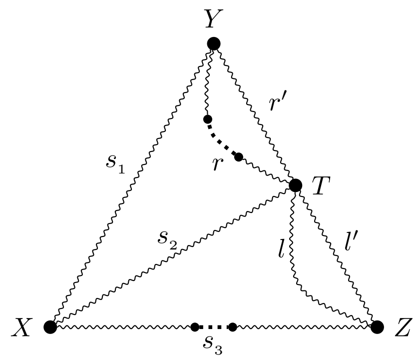

Figure 6: Bipartite subdivision of a negative even cycle with 3 chords

Proof. Given a negative even cycle C^-_{2k}, any pair u and v of vertices of the cycle partitions it into two paths P_1 and P_2 one of which has an even number of negative edges, and the other has an odd number of negative edges. Thus if we add a new signed $u - v$ path P', then the parity of the number of negative edges of P' is the same as either P_1 or P_2, say P_1 without loss of generality. If, furthermore, P' is such that after adding this signed path the underlying graph remains bipartite and if in the resulting signed graph there is no negative cycle shorter than $2k$, then P' must be of length at least as P_1 and it should have a same parity as P_1. Then after a suitable switch, if needed, one may map P' onto P_1. To prove the lemma, considering the outer cycle of (G, σ) and applying this general observation to the paths ZT, YT and XT, each would correspond to one of the three conditions of the Lemma. □

We now proceed with a proof similar to that of Theorem 3.2 to give an analogous necessary and sufficient condition for a signed bipartite subdivision of K_4 to be a core (see Theorem 4.9). However, we rather partially simplify proof by connecting this case to the case of odd-K_4 which we have considered in the previous subsection. To this end, we first have the following definition.

Let (G, σ) be an (a, b, c, a', b', c')-K_4 subdivision of type (7). Moreover, assume that σ has the smallest number of negative edges. Thus there are two negative edges which are on subdivisions of two disjoint edges of K_4. Assuming that the negative edges are on BC and AD, the graph obtained from G by subdividing these two negative edges each once is defined to be G_a. Observing that the signed graph with two negative edges on AC and BD is switch equivalent to (G, σ) we may define G_b and G_c similarly. We observe that each of G_a, G_b and G_c is an odd-K_4 where G_a is an $(a+1, b, c, a'+1, b', c')$-$K_4$, G_b is an $(a, b+1, c, a', b'+1, c')$-$K_4$ and G_c is an $(a, b, c+1, a', b', c'+1)$-$K_4$.

The following lemma provides connections between (G, σ) and G_a, G_b, G_c.

Lemma 4.8. *Let (G, σ) be a signed (a, b, c, a', b', c')-K_4 of type (7) of Figure 3 and assume $g_{10}(G, \sigma) = a + b + c = 2k$. Then $(G, \sigma) \to C_{2k}^-$ if and only if one of G_a or G_b or G_c maps to C_{2k+1}.*

Proof. We first consider the "if" part and by symmetry of a, b, c we assume $G_a \to C_{2k+1}$. We need to show that $(G, \sigma) \to C_{2k}^-$.

As the outer cycle of G_a is a $(2k+1)$-cycle, the mapping of G_a to C_{2k+1} identifies D with a vertex u of the outer cycle. We consider G'_a as the image of G_a obtained by identifying (only) D with u. Observing that u being on AB is symmetric with u being on AC, depending on the position of u and the parity of the new faces that are created by this identification, we have four possible cases to consider.

Case 1. u is on AB and the length of uA is of the same parity as $a'+1$.

Case 2. u is on AB and the length of uB is of the same parity as b'.

Case 3. u is on BC and the length of uB is of the same parity as b'.

Case 4. u is on BC and the length of uC is of the same parity as c'.

In each case we would like to give a homomorphic image of (G, σ) by identifying D with a vertex v on AB or AC in the Case 1, Case 2 and on BC in the Case 3, Case 4 for which we may apply Lemma 4.7.

In Case 1, if u is a vertex of G_a which is distinct from A, then we choose v to be a vertex on path AB in (G, σ) whose distance to A is one less than the distance of u to A in G_a. If $u = A$, then we choose v to be the neighbour of A on the AC path of (G, σ). Assuming (G, σ) has two negative edges which are on AC and BD, let (G', σ') be the signed graph obtained by identifying D with v. As we have assumed that G'_a maps to C_{2k+1}, the conditions of Lemma 4.4 must apply on G'_a. It can then readily be verified that conditions of Lemma 4.7 apply on (G', σ') which means (G', σ') and, therefore, (G, σ) maps to C^-_{2k}.

For the other Case 2 and Case 3, we choose v such that its distance from B on AB and BC correspondingly is the same as the distance of u from B. For Case 4, we choose v such that its distance from C on BC is the same as the distance of u from C. In these cases, we leave the verification of the details to the reader. This completes the proof of "if" direction of the theorem.

For the "only if" part of the theorem we assume $(G, \sigma) \to C_{ABC}$ and let ϕ be such a mapping. We need to show that one of G_a or G_b or G_c maps to C_{2k+1}. Let us assume, without loss of generality, that $\phi(D)$ is on the AB part of C_{ABC}. Then either the path AD has the same parity of number of negative edges with the path $A\phi(D)$ or BD has the same parity of number of negative edges with the path $B\phi(D)$, without loss of generality, we assume the former. Let (G', σ') be a signed graph obtained from (G, σ) by identifying (only) D with $\phi(D)$. It should be reminded that, since the final image, C^-_{2k}, has a bipartite underlying graph, G' is also a bipartite graph. We may now applying Lemma 4.7 to (G', σ'). Since it maps to its shortest negative cycle, the three inequalities of this lemma must be satisfied. Now in the graph G_b if we identify D with a vertex of AB whose distance on AB from A is the same as that of $\phi(D)$ from A (in (G, σ)), the resulting graph G'_b satisfies conditions of Lemma 4.4, implying that $G_b \to C_{2k+1}$. Similar conclusion can be made for G_c as well. This completes the only if part of the proof. □

We are now ready to state and prove the conditions for a signed K_4-subdivision of type (7) of Figure 3 to be a core.

Theorem 4.9. *Let (G,σ) be a signed bipartite (a,b,c,a',b',c')-K_4 where all facial cycles are negative. Furthermore, assume $g_{10}(G,\sigma) = a+b+c$. Then (G,σ) is a core if and only if the following conditions are satisfied.*

- $b' + c' - a < a + b + c$;
- $a' + b' - c < a + b + c$;
- $a' + c' - b < a + b + c$.

Proof. Suppose that (G,σ) is not a core and the shortest negative even cycle is ABC. By Lemma 4.8, one of G_a, G_b or G_c is not a core. Without loss of generality, we assume that G_a which is an odd $(a+1,b,c,a'+1,b',c')$-K_4 is not a core. By Theorem 4.5, one of the following three inequality does not hold: (1) $b'+c'-(a+1) < (a+1)+b+c$ or (2) $(a'+1)+b'-c < (a+1)+b+c$ or (3) $(a'+1)+c'-b < (a+1)+b+c$.

If (1) does not hold, i.e., $b'+c'-(a+1) \geq (a+1)+b+c$, then we have $b'+c'-a \geq a+b+c$, which means the first inequality of the theorem does not hold. Similarly, the case (2) implies that the second inequality of the theorem does not hold. And the case (3) implies the third one is not satisfied.

Conversely, suppose that one of the three conditions, say the first, is not satisfied. Thus, $b'+c'-a \geq a+b+c$. We consider G_b which is an odd-$(a,b+1,c,a',b'+1,c') - K_4$. As $(b'+1)+c'-a \geq a+(b+1)+c$ and by Theorem 4.5, G_b is not a core. By Lemma 4.8 then (G,σ) is not a core either. This completes the proof. □

4.3 Signed K_4-subdivisions of the third group

These four cases have the common property that each of the signed K_4-subdivisions of type (8), (9), (10) and (11) has a bounded $g_{ij}(G,\sigma)$ for all values of $ij \in \mathbb{Z}_2^2$. As a K_4 has four 3-cycles and three 4-cycles, a common property of these four types of subdivisions is that: in each case there is a unique cycle of type 00. Given (G, σ) of one of these types, we denote the unique positive even cycle of (G, σ) by $C_{00}(G, \sigma)$.

The core of the signed graph in each of these cases must also have the same value of g_{ij} for any given $ij \in \mathbb{Z}_2^2$. We will use this property and the cycle $C_{00}(G, \sigma)$ to characterize the cases that are cores. We will give a proof in the case that (G, σ) is of type (8), and due to the similarities of the proofs, we will leave the verification of the other three cases to the readers.

Theorem 4.10. *Let (G, σ) be a signed (a, b, c, a', b', c')-K_4 of type either (8) or (9) or (10). Then (G, σ) is a core if and only if the lengths of subdivisions corresponding to the edges of $C_{00}(G, \sigma)$ satisfy triangular inequality with respect to this 4-cycle. That is to say, if lengths of subdivided paths of $C_{00}(G, \sigma)$ correspond to a, c, a', c', then we must have*

- $c < c' + a + a'$;
- $c' < c + a' + a$;
- $a < a' + c + c'$;
- $a' < a + c + c'$.

Proof. We consider the case when (G, σ) is of type (8) with the other two cases being similar. Furthermore, we label vertices as in Figure 4. Observe that deleting a vertex on BD or AC will eliminate two types of closed walks, thus all these vertices must be present in the core. If a vertex of another path, say AB is not in the core, then no vertex of AB is in the core. But then all other vertices must be present. For this to happen the path AB must have mapped to a path of same parity with a same parity of the number of negative edges, meaning it forms a positive even cycle with its image. As there is a unique cycle of this type (that is C_{ABCD}), the path AB must map onto the path P_{ADCB}, thus we must have $c \geq c' + a + a'$. In general, mapping of a part P of the cycle C_{ABCD} to the $C_{ABCD} - P$ corresponds to one of the inequalities of the theorem. □

The classification of cores for subdivisions corresponding to the case (11) of Figure 3, given below, is similar to that of the previous theorem, with a sole difference being that $C_{00}(G, \sigma)$ in this case corresponds to a 3-cycle rather than a 4-cycle.

Theorem 4.11. *Let (G, σ) be a signed (a, b, c, a', b', c')-K_4 of type (11). Then (G, σ) is a core if and only if the lengths of subdivisions corresponding to the edges of $C_{00}(G, \sigma)$ satisfy triangular inequality with respect to this 3-cycle. That is to say, if lengths of subdivided paths of $C_{00}(G, \sigma)$ correspond to a, b, c, then we must have*

- $a < b + c$;
- $b < a + c$;
- $c < b + a$.

5 Possible applications

Theorem 4.5 and Theorem 4.9 are of special interest in the study of questions that are about mapping a class of graphs to an odd cycle and mapping a class of signed bipartite graphs to a negative even cycle. Two prime examples of such problems are the Jaeger-Zhang conjecture [14] and a bipartite analogue of it introduced in [11] and studied in [4]. They claimed the following.

Conjecture 5.1. [14] *Every planar graph whose shortest odd cycle is of length at least $4k + 1$ admits a homomorphism to C_{2k+1}.*

Conjecture 5.2. [11, 4] *Every planar signed bipartite graph* (G, σ) *satisfying* $g_{10}(G, \sigma) \geq 4k-2$ *admits a homomorphism to* C_{2k}^{-}.

A classical approach to such conjectures is the discharging technique that is based on the following corollaries of Theorems 4.5 and 4.9.

Corollary 5.3. *Given an odd-*(a, b, c, a', b', c')*-*K_4*,* G*, if* $a+b+c = 2k+1$ *is the length of its shortest odd cycle and* $a'+b'+c' > 4k$*, then* $G \to C_{2k+1}$.

Proof. Let G be such a subdivision. If G is a core, then by Theorem 4.5 we must have, $b'+c'-a \leq a+b+c-1$, $a'+b'-c \leq a+b+c-1$ and $a'+c'-b \leq a+b+c-1$. Adding up the three inequality we have $2(a'+b'+c') \leq 4(a+b+c)-3$ but then since all values are integers and considering parity we have $2(a'+b'+c') \leq 4(a+b+c)-4$, and finally using the assumption that $a+b+c = 2k+1$ we have $a'+b'+c' \leq 4k$. □

Corollary 5.4. *Given a signed* (a, b, c, a', b', c')*-*K_4*,* (G, σ)*, if all faces are of type* 10*,* $a+b+c = 2k$ *is the length of its shortest negative even cycle and* $a'+b'+c' \geq 4k-1$*, then* $(G, \sigma) \to C_{2k}^{-}$.

The proof is similar to the previous one and we leave the details to the reader.

Corollary 5.3 is a tool that is used in earliest approaches to Conjecture 5.1, see for example [15]. It should be noted that the best current result on this conjecture is that of [10] which is based on the theory of module k orientations. However, Conjecture 5.2 is rather new and the best support so far is a result of [4] that is indeed based on our Corollary 5.4 and is referred to this work.

Other connections to our work can be found in the study of smallest C_{2k+1}-critical graphs, that is a graph that does not map to C_{2k+1}, but every proper subgraph maps. We refer to [1] for more on this subject.

6 Further discussion

Combining results from [2] and [3], given a signed graph in one of the three classes $\mathcal{G}_{ij}$, $ij \in \{01, 10, 11\}$, a necessary and sufficient condition is given to decide whether B has the property that it admits a homomorphism from any signed K_4-minor-free graph (G, σ) as long as we have $g_{ij}(G, \sigma) \geq g_{ij}(B)$. In this work we have had a first look at the general case of the problem where B is not necessarily a member of $\mathcal{G}_{ij}$. We gave a necessary and sufficient condition for the case when B is a simple signed graph, that to say $g_{10}(B) = 4$, $g_{01}(B) = g_{11}(B) = 3$. We therefore ask for a possible generalization to all choices of B.

The notion of *Extended Double Cover* of a signed graph (G, σ) is presented in [12]. Given a signed graph (G, σ), $\mathrm{EDC}(G, \sigma)$ is a signed graph on vertex set $V^+ \cup V^-$, where $V^+ := \{v^+ : v \in V(G)\}$ and $V^- := \{v^- : v \in V(G)\}$. Vertices x^+ and x^- are connected by a negative edge; all other edges, that are described next, are positive. If vertices u and v are adjacent in (G, σ) by a positive edge, then v^+u^+ and v^-u^- are two positive edges of $\mathrm{EDC}(G, \sigma)$, if vertices u and v are adjacent in (G, σ) by a negative edge, then v^+u^- and v^-u^+ are two positive edges of $\mathrm{EDC}(G, \sigma)$.

It can be checked readily that if $B \in \mathcal{G}_{10}$ then EDC(B)$\in \mathcal{G}_{11}$ and vice versa. Then in [6] it is shown that if $B \in \mathcal{G}_{10}$ or $B \in \mathcal{G}_{11}$ has the property that it admits a homomorphism from any signed K_4-minor-free graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq g_{ij}(B)$ for all $ij \in \mathbb{Z}_2^2$, then $\mathrm{EDC}(B)$ has the same property.

However, this is not true for a general signed graph B, an example for which is $SPal_5$. As this is a simple graph we can easily check that $g_{10}(SPal_5) = 4$ and $g_{01}(SPal_5) = g_{11}(SPal_5) = 3$, and we saw in Theorem 3.3 that every signed K_4-minor graph admits a homomorphism to $SPal_5$. However the following is an example of a signed K_4-minor-free graph which while satisfies $g_{ij}(G, \sigma) \geq g_{ij}(EDC(SPal_5))$ for all $ij \in \mathbb{Z}_2^2$ does not admit a homomorphism to $\mathrm{EDC}(SPal_5)$.

Proposition 6.1. *There exists a signed K_4-minor-free graph (G,σ) satisfying $g_{11}(G,\sigma) \geq 5, g_{10}(G,\sigma) \geq 4$ and $g_{01}(G,\sigma) \geq 3$ such that $(G,\sigma) \not\to \mathrm{EDC}(\mathrm{SPal}_5)$.*

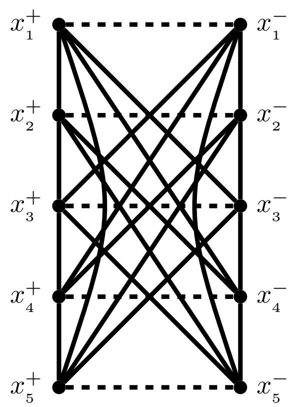

Figure 7: $\mathrm{EDC}(SPal_5)$

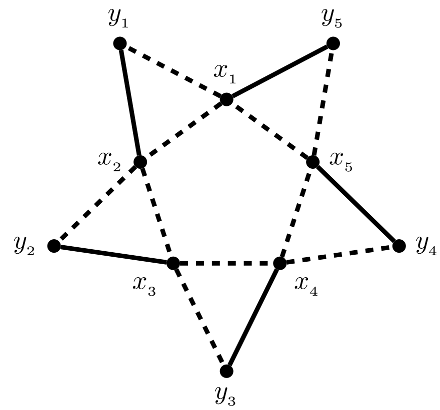

Figure 8: A signed K_4-minor-free graph which does not map to $\mathrm{EDC}(SPal_5)$

Proof. Signed graph (G,σ) of Figure 8 is such an example. It is easily observed that $g_{10}(G,\sigma) \geq 4$, $g_{01}(G,\sigma) \geq 3$ and $g_{11}(G,\sigma) \geq 5$. Assume to the contrary that $(G,\sigma) \to \mathrm{EDC}(SPal_5)$. By Theorem 2.4, there exists a switch σ' of σ and an edge-sign-preserving homomorphism of (G,σ') to $\mathrm{EDC}(SPal_5)$. As $x_1x_2x_3x_4x_5$ is a negative cycle of (G,σ), at least one of its edges is negative in (G,σ'), by symmetry, assume that $\sigma'(x_1x_2) = -$. Then the triangle $x_1x_2y_1$, which is a positive triangle, must have two negative edges. However, there exists no such a triangle in $\mathrm{EDC}(SPal_5)$ with the given signature. $\square$

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