K_5-free Bound for the Class of Planar Graphs

Reza Naserasr
Department of mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6, Canada

Abstract

We define k-diverse colouring of a graph to be a proper vertex colouring in which every vertex x, sees $min\{k, d(x)\}$ different colours in its neighbors. We show that for given k there is an $f(k)$ for which every planar graph admits a k-diverse colouring using at most $f(k)$ colours. Then using this colouring we obtain a K_5-free graph H for which every planar graph admits a homomorphism to it, thus another proof for the result of J. Nešetřil, P. Ossona de Mendez.

1 Introduction

Let G and H be graphs. A homomorphism f of G to H is an edge preserving mapping of $V(G)$ to $V(H)$. Homomorphism defines a quasiorder (a reflexive and transitive binary relation) on the class of graphs, by $G \preccurlyeq H$ if and only if there is a homomorphism from G to H. Given a class $\mathcal{C}$ of graphs we say that $\mathcal{C}$ is bounded by H if for every $G \in \mathcal{C}$ we have $G \preccurlyeq H$. In this setting the four colour theorem simply states that the class P of planar graphs is bounded by K_4. Similarly Grötsch theorem (every triangle free planar graph is 3 colourable) states that the class of triangle free planar graphs is bounded by K_3.

The problem of the existence of a bound with some special properties, for a given class of graphs has been a subject of study in graph homomorphism. Some particular cases of this problem has been studied in [4, 5, 6, 7]. In [7] authors have proved the existence of K_k-free bound for any minor closed class of K_k-free graphs. This indeed provides a K_5-free graph H which bounds the class of planar graphs, therefore answering a question of authors posed in [1]. Note that if we take the categorical product of H with K_5 we will find a bound for the set of planar graphs which is smaller than K_5 in the sense of homomorphism, this is a stronger result than the five colour theorem whose proof does not use the four colour theorem.

In this note we define a new type of vertex colouring which we call it diverse colouring. We give an algorithm to find a k-diverse colouring for any planar graph using a fixed number of colours. Using the colouring obtained from this algorithm we construct a K_5-free bound for the class of planar graphs. This will be a completely different method for proving the existence of a K_5-free bound for the class of planar graphs.

The following theorem of Kotzig [3] will play an important role in our algorithm.

Theorem 1 *(c. [3]) For every planar graph G with minimum degree at least 3 there is an edge $e = uv$ with $d(u) + d(v) \leq 13$.*

We say that a graph G is k-critical if it is k-chromatic but every proper subgraph of G is $k - 1$ colourable. Obviously every k-chromatic graph contains a k-critical subgraph. An alternative form of the four colour theorem is to state that "there is no 5-critical planar graph". In the absence of the four colour theorem the following lemma will help us to achieve our goal of constructing a K_5-free bound for the class of planar graphs. The proof of this lemma is inherited from the proof of the 5-colour theorem, the one which uses the Kempe's chain method.

Lemma 2 *If G is a 5-critical planar graph then $\delta(G) \geq 5$.*

2 Diverse colouring

Definition For the given integers k and l we say that an l-colouring c, of a given graph G is a *k-diverse colouring* if at least $min\{d(x), k\}$ different colours appear on the neighbors of x. A k-diverse colouring which uses at most l colours will be denoted by (k, l)-colouring. $\diamond$

Theorem 3 *Given an integer $k \geq 11$, every planar graph admits a $(k, 5k + 8)$-colouring.*

Proof. We will prove this by induction on the number of vertices of G. For graphs on at most $5k + 8$ vertices we can colour all the vertices with different colours. Suppose we have found a $(k, 5k + 8)$-colouring for every planar graph on at most n vertices and let G be a planar graph on $n + 1$ vertices. We may assume G is connected, because otherwise $(k, 5k+8)$-colourings of the components of G all together will give a $(k, 5k + 8)$-colouring of G.

If G has a vertex x with $d(x) = 1$ then any $(k, 5k + 8)$-colouring of $G\backslash x$ can be extended to a $(k, 5k + 8)$-colouring of G. To see this let y be the only neighbour of x. If there are at least k-different colours on the neighbours of y, then any colour different from the colour of y will work. Otherwise $d(y) \leq k - 1$ and we choose a colour which has not appeared on y or any of its neighbours, this is indeed possible because there are more than k colours available.

If G does not have a vertex of degree 1 but it has a vertex x of degree 2 then we identify x with one of its neighbours, remove the loop and the possible multiple edge. We call the new graph G_x. By induction G_x admits a $(k, 5k + 8)$-colouring c_x. Colour all the vertices of G except x with the same colour as in the colouring c_x. Notice that neighbours of x have taken two different colours, and in order to extend c_x to a $(k, 5k + 8)$-colouring of G all we need to do is to choose a colour for x different from colours of its neighbours in such a way that the requirement of diversity for the neighbours of x still holds.

For each neighbour y of x, either y already has k different colours on its neighbours or $d(y) \leq k-1$. In the first case the only restriction for the colour of x, coming from y, is to have a colour different from the colour of y, (in order to have a proper colouring). In the second case, i.e., if $d(y) \leq k-1$, the vertex x must take a colour different from the colours of y and all of its neighbours. In either of the cases, each neighbour of x introduces at most k-colours not admissible for x. Since x has two neighbours there are maximum of $2k$ colours not admissible for x, so c_x can be extended to a $(k, 5k+8)$-colouring of G.

If neither of the previous two cases happens, then $\delta(G) \geq 3$ and by Theorem 1 there is an edge $e = uv$ with $d(u) + d(v) \leq 13$. Without loss of generality assume $d(u) \leq d(v)$. Therefore $d(u) \leq 6$. Identify u and v, remove loops and possible multiple edges and call the new graph G_e. Let v' to be the new vertex in G_e (obtained from identifying u and v), then $d(v') \leq 11$. By induction G_e admits a $(k, 5k+8)$-colouring, we denote this colouring by c_e. Note that all the neighbours of v' have taken different colours (this is because $k \geq 11$).

To find a $(k, 5k+8)$-colouring of G, colour every vertex $x \notin \{u, v\}$ with $c_e(x)$ and colour v with $c_e(v')$. To complete this colouring all we need is to find an admissible colour for u. Notice that all the neighbours of u have already received different colours. Let $t \neq v$ be a neighbour of u, if $d(t) > k$ then t already has k neighbours with k distinct colours and the only restriction coming from t is that $c(t)$ be different from the colour which we choose for u.

If $d(t) \leq k$ then the colour we would like to choose for u has to be different from colours of t and all of its neighbours. This will remove at most k colours from the list of available colours for u. Similarly there will be also at most $d(v)$ forbidden colours because of the diversity condition for v. In total there will be at most $k(d(u)-1) + d(v) = (k-1)d(u) - k + d(u) + d(v) \leq 5k+7$ forbidden colours for u. Since there are $5k+8$ possible colours, we can find an admissible colour for u. □

Notice that in the proof of the last theorem we introduced an inductive algorithm to find a $(k, 5k+8)$-colouring of any planar graph. We will call this algorithm *k-diverse colouring algorithm*. In the next section we will show that the colouring obtained from 11-diverse colouring algorithm satisfies the condition of the Theorem 7 of [4] and hence provides us with a K_5-free bound for the class of planar graphs.

3 K_5-free bound

Theorem 4 *Let G be a planar graph and c an $(11, 63)$-colouring of G obtained from 11-diverse colouring algorithm. Then c has the property that every 5-chromatic subgraph of G takes at least 6 different colours.*

Proof. It will be enough to show that every 5-critical subgraph has taken 6 different colours. We prove this by contradiction. Suppose this is not true and algorithm fails at some point. Let G be the smallest graph for which the 11-diverse colouring algorithm fails, i.e., for every graph on at most $|V(G)|-1$ vertices the $(11,63)$-colouring obtained from 11-diverse colouring algorithm has the required property but the colouring obtained for G by this algorithm uses only five colours on some 5-critical subgraph H of G.

It is easy to see that G does not contain vertices of degree 1 or 2. In fact if $\delta(G)=1$ or 2 then $(11,63)$-colouring of G has been obtained from $(11,63)$-colouring of some G_x where x is a vertex of degree 1 or 2. But then every 5-critical subgraph of G is also a subgraph of G_x and therefore takes at least 6 different colours.

So we may assume $\delta(G)\geq 3$. Let u and v be the vertices of G as in the algorithm. Recall that to obtain the colouring c we basically used an $(11,63)$-diverse colouring of G_e and we found an admissible colour for u. By the minimality of G the 11-diverse colouring of G_e has used at least 6 different colours on any 5-critical subgraph of G_e. So H could not be a subgraph of G_e, therefore it must contain both u and v. By lemma 2 degree of u in H must be at least 5. But all the neighbours of u have been given different colours. By adding the colour of u itself to this collection we will find at least 6 different colours on the vertices of H which is a contradiction.

□

Using this theorem and the results of [4] one can construct a K_5-free bound for the class of planar graphs. For the sake of completeness we will repeat the construction here.

Constructing $H_{(11,63)}$: The vertices of this graph are the pairs (i,φ) where $i\in[63]$ and φ is any function from $\{S|S\subset[63], i\in S \textit{ and } \ |S|=5\}$ to $\{1,2,3,4\}$. For the edge set; (i,φ) is adjacent to (j,ψ) if $i\neq j$ and $\varphi(S)\neq\psi(S)$ whenever both $\varphi(S)$ and $\varphi(S)$ are well defined, i.e. $\{i,j\}\subset S$.

Lemma 5 *(c. [4]) $H_{(11,63)}$ is K_5-free.*

Theorem 6 *$H_{(11,63)}$ bounds class of planar graphs.*

Proof. Let G be a planar graph and c a $(11,63)$-colouring of G obtained from the algorithm. By Theorem 4, for any set S of 5 colours, the subgraph G_S induced on these colours is 4-colourable. Let ρ_S to be a 4-colouring of G_S. The homomorphism of G to $H_{(11,63)}$ is defined by:

$$\text{for } x\in V(G) \text{ let } f(x)=(c(x),\varphi_x) \text{ where } \varphi_x(S)=\rho_S(x).$$

Notice that for every 5-subset S containing $c(x)$, φ_x is well defined. To see that f is a homomorphism let x and y to be two adjacent vertices of G then $c(x)\neq c(y)$ and if $\{c(x),c(y)\}\subset S$ then $\varphi_x(S)=\rho_S(x)\neq\rho_S(y)=\varphi_y(S)$. Therefore by the definition of $H_{(11,63)}$, $f(x)$ is adjacent to $f(y)$. □

4 Remarks

The K_5-free bound for the class of planar graphs we provided here, has a large number of vertices. The following problem, if it is answered without using the four colour theorem, will provide us with a better K_5-free bound which has fractional chromatic number $\frac{24}{25}$. Notice that the best known bound for the fractional chromatic number of planar graphs without using the four colour theorem is that it is strictly smaller than 5.

Problem 7 *Show that every planar graph can be coloured using 6 colours in such a way that every 5-chromatic subgraph of it receives all 6 different colours.*

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