# Projective cubes: a coloring point of view

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# Motivation

The four-color theorem has played a central role in development of graph theory. It was originally stated as coloring (regions) of maps, but soon after was translated, through duality, to vertex-coloring of graphs which gave birth to notions such as the chromatic number and, in most general form, to homomorphisms of graphs. In its simplest form it states that every loop-free planar graph can be 4-colored. However, a large number of equivalent reformulations has been introduced since its introduction as a conjecture in 1852. One of its earliest reformulations is that of Tait's which says: every cubic birdgeless planar graph is 3-edge-colorable. This reformulation was then birth place of three of main topics in graph theory. The statement itself raised the study of edge-coloring and edge-chromatic number. False attempts in proving it introduced the notion of Hamiltonicity. And Tait's proof technique of the equivalence of this statement with the 4-color theorem was extended to the theory of nowhere-zero flows by Tutte.

Through its extension to the theory of homomorphisms of graphs, the notion of coloring can be viewed as algebraic concept of graphs, whereas the study of planarity through extensions such as minors and crossing number is viewed as the topological counterpart. Thus the 4-color theorem and its possible extensions lie at the heart of the study of correlation of algebraic and topological properties of discrete structures.

Slight strengthening of the theorem using the terminology of minor and homomorphisms claims that any graph with no  $K_5$ -minor admits a homomorphism to  $K_4$ . Hadwiger then proposed strengthening that any graph with no  $K_k$ -minor admits a homomorphism to  $K_{k-1}$ . The conjecture, known as "the Hadwiger conjecture", is widely believed to be the most difficult conjecture in graph theory. In many such attempts to better understand the four-color theorem, the 4-coloring is understood as a homomorphism to the complete graph on four vertices (each vertex being a distinct color).

In Tait's proof of the equivalence between the classic statement of the 4-color theorem and the edge-coloring formulation, he regarded the four colors as the four elements of the group  $\mathbb{Z}_2^4$  and then used the three non-zero elements (01, 10, 11) to color the edges. These three elements being the set of all non-zero elements of  $\mathbb{Z}_2^4$  lead Tutte to the study of nowhere zero flows on graphs.

The work here is based on the observation by the author that  $K_4$  in the statement of the 4-color theorem could be seen as the Cayley graph  $(\mathbb{Z}_2^4, \{e_1, e_2, J\})$  where  $e_i$ 's are the standard basis and J is the all-1 vector. Defining  $PC(k) = Cayley(\mathbb{Z}_2^k, \{e_1, e_2, \cdots, e_k, J\})$ as the projective cube of dimension k we have conjectured in [39] that:

**Conjecture 0.0.1.** Every planar graph whose shortest odd-cycle is of length at least 2g + 1 admits a homomorphism to PC(2g).

This view has proved to be powerful. Indeed the notion of homomorphism of signed graphs is introduced by B. Guenin [20] to capture the case of the homomorphism from planar graphs to projective cubes of odd-dimension which are bipartite graphs. Using the notion of signed graphs, where each edge is assigned one of the two signs  $\{+, -\}$ , the relation between minor and homomorphism is not only more intuitive, but also stronger. For example while the class of graphs with no  $K_3$ -minor are the class of all forest, the class of (signed) graphs with no negative signed  $K_3$ -minor are the class of all signed graphs

with underlying bipartite graphs. Thus while the chromatic number of both families of graphs is bounded by 2, the latter is a much larger class of graphs.

In this work then we study the projective cubes. They are known to be among must symmetric non-trivial graphs. They can be defined in various ways, with each definition leading to different properties of these graphs and providing new tools to work with them. They capture, as subgraphs, some of the most symmetric graphs such as the Petersen graph and the Coxeter graph. They also provide connection to other subject of mathematics such as algebraic surfaces which dates back to a time when there were no notion of graphs.

We have special interest in the question of homomorphism to the projective cubes. In general this question is related to a packing problem. A special case of this connection says: a graph G is 4-colorable if and only if its edge-set can be partitioned into 3 sets  $E_1$ ,  $E_2$  and  $E_3$  such that each odd-cycle intersects each  $E_i$  an odd number of times.

We show that questions on mapping planar graphs into projective cubes relates to the study of several other notions such as circular chromatic number, fractional chromatic and edge chromatic number of planar (multi)graphs.

In present format of this text, working with classic theories of graphs, the focus is on the projective cubes of even dimension which are 4-chromatic. This work will be completed later using the more general notion of signed graphs in which case the homomorphism problem to (signed) projective cubes of odd dimension is no longer matter of triviality.

Abstract. The projective cube of dimension k, denoted PC(k), is the graph obtained from the hypercube of dimension k + 1 by identifying antipodal pairs of vertices. It is equivalently the Cayley graph  $(\mathbb{Z}_2^k, \{e_1, e_2, \ldots, e_k, J\})$  where the  $e_i$ 's are the vectors of the standard basis and J is the all-1 vector. They can be defined in various ways and thus are known under the names such as folded cubes and augmented cubes. Not only these graphs themselves are among the most transitive graphs, but they also contain a number of highly symmetric graphs as (induced) subgraphs. PC(2) is the complete graph  $K_4$ , PC(3) is  $K_{4,4}$  and PC(4) is known as one of the most symmetric graphs. It has received different names for its appearance on different subjects: in most of the literature it is named after Clebsch, an algebraic geometrist from the 19th century, as it is the intersection graph of the 16 straight lines of a cubic surface obtained from blowing up of 5 points of the projective plane. It is also, independently, named after Greenwood and Gleason in some other literatures because of its appearance in Ramsey theory. In general, PC(2k + 1) is a bipartite graph while PC(2k) is a graph of odd-girth 2k + 1 and chromatic number 4.

The goal of this work is to study these graphs from a homomorphism point of view. To appreciate the full complexity of the study one must employ the notion of homomorphisms for signed graphs, however, due to limitation, current focus of this work will be PC(2k). The existence of a homomorphism from a graph G to PC(2k) is equivalent to partition of edges of G into 2k + 1 sets  $E_1, E_2, \ldots, E_{2k+1}$  such that each even-cycle (resp. odd-cycle) intersects each  $E_i$  an even (resp. odd) number of times. Thus the problem mapping graphs to the projective cubes is equivalent to a packing problem. Such mappings are conjectured to exist for all planar graphs, or more generally for all graphs without a  $K_5$ minor, provided that a simple (necessary) condition is satisfied:

**Conjecture.** Any  $K_5$ -minor free graph of odd-girth at least 2k + 1 admits a homomorphism to PC(2k).

With PC(2) being the graph  $K_4$ , this conjecture is a direct extension of the four colour theorem. More generally, assuming the conjecture is true, we ask what are subgraphs of PC(2k) to which planar graphs of odd-girth 2l + 1,  $l \ge k$  admit a homomorphism? We show that the question captures a wide range of studies, such as finding the best bounds for circular and fractional chromatic numbers of planar graphs of odd-girth 2k + 1.

We provide a necessary and sufficient conditions for a graph B of odd-girth 2k + 1 to bound the class of partial *t*-trees of odd-girth 2k + 1. The condition of theorem can be checked in polynomial time. Using this result we verify the conjecture for the class of partial 3-trees which is a large subclass of  $K_5$ -minor free graphs.

Independently we show that the surprising result of C. Payan, that there are no binary Cayley graph of chromatic number 3, is about properties of homomorphisms among projective cubes of even dimensions. We then study possible generalizations.

**Remark** Results from about 11 published papers of the author are mentioned, a number of them without a proof and some with sketches of proofs. Some of the presented results are yet to be written as a paper. In particular Theorem 4.3.1 on bounding partial *t*-trees and its application on bounding partial 3-trees are recent joint work with L. Beaudou, M. Chen and F. Foucaud which are yet to be written as paper. They are strengthening of results from a recently accepted paper [4].

# Chapter 1 Introduction

# 1.1 Graphs

A graph is a binary structure consisting of a (finite) set of vertices and collection of edges, where each edge connects two (not necessarily distinct) vertices. Vertices connected by an edge are adjacent and the edge connecting them is *incident* to each of them. An edge connecting a vertex to itself is a *loop*. Two edges connecting a same pair of vertices are referred to as *parallel* or *multi* edges. In this work we only consider graphs without loops. Multi-edges are of importance, but we will use the term multi-graph when we consider them. Thus a graph G consists of two sets: vertices (V) and edges (E) where the latter itself is a set of 2-subsets of V.

#### 1.1.1 Cayley graphs

A Cayley graph is a graph whose vertices are the elements of an (additive) group  $\Gamma$  where two such vertices are adjacent if their difference is in a given symmetric set S. Such a Cayley graph will be denoted by  $(\Gamma, S)$ . Considering the natural association of elements of S to the edge of  $(\Gamma, S)$  we have the following important observation on Cayley graphs.

**Observation 1.** The total sum of the values associated to the edges of any cycle or any closed walk of  $(\Gamma, S)$  is 0.

#### 1.1.2 Hypercubes

The hypercube of dimension n, denoted H(n), is the Cayley graph  $(\mathbb{Z}_2^n, \{e_1, e_2, \ldots, e_n\})$  where the  $e_i$ 's are the vectors of the standard basis. There are many equivalent ways of defining these graphs. We will consider some of such definitions in the next sections in order to have a newer definition of the projective cubes, the main objects of this work.

#### 1.1.3 Kneser graphs

Another class of graphs that is important for this work is the class of Kneser graphs. Given integers n and k,  $n \ge 2k$ , the Kneser graph K(n, k) is a graph whose vertices are the k-subsets of an n-set where two such subsets are adjacent if they do not intersect.

#### 1.1.4 *t*-trees

As a subclass of graphs, the class of t-trees is defined inductively as follows: A complete graph on t vertices is a t-tree. Any graph built from a (previously built) t-tree by adding a vertex which is adjacent to all vertices of a t-clique is also a t-tree. It is easy to observe that the class of 1-trees is the class of all trees, this is the justification for the name "t-trees".

A *partial t*-tree is a subgraph of any *t*-tree.

Readers familiar with the notion of *tree-width* can check that a graph is of tree-width at most t if and only if it is a partial t-tree (see [10]).

# 1.2 Clique sum

Given two graphs G and H on disjoint sets of vertices and each with a clique of order t, a t-sum of G and H is a graph obtained by identifying the vertice of a t-clique of H with the vertices of a t-clique of G.

The class of t-trees can be equivalently defined using the t-sum operation:

1. The complete graphs  $K_t$  and  $K_{t+1}$  are t-trees.

2. A *t*-sum of any two *t*-trees is also a *t*-tree.

### **1.3** Minor and planarity

A minor of a graph G is any graph H which is built from G by the use of the following operations (in any sequence): i. deleting edges or vertices ii. contracting edges, that is identifying two ends of an edge. If H cannot be obtained as a minor of G, then G is referred to as an H-minor free graph.

A class C of graphs is *minor-closed* if any minor of a graph in C is also in C. It is an easy exercise to show that the class of partial *t*-trees is minor-closed.

Given a finite set  $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$  of graphs, the class of graphs containing none of  $H_i$  as a minor is a minor-closed class of graphs, which is denoted by  $Forb_m(\mathcal{H})$ . The graph minor project of Robertson-Seymour showed that, except for the class of all graphs, every minor-closed family of graphs is  $Forb_m(\mathcal{H})$  for some finite set H of graphs. However for some minor-closed families of graphs, when defined by ulterior ways, it may not be an easy task to find such a (unique) set  $\mathcal{H}$ . Indeed the full class of forbidden minors for partial t-trees are only known for  $t \leq 3$ : The class of forests (i.e., partial 1-trees) is  $Forb_m(\{K_3\})$ . The class of partial 2-trees is the class  $Forb_m(\{K_4\})$ . Thus the class of edge-maximal graphs with no  $K_3$ -minor (respectively no  $K_4$ -minor) is the class of trees (respectively 2-trees). The four graphs defining the class of partial 3-trees using the notion of forbidden minor are given in Figure ?? (independent proofs are given in [3] and [54]).

A *planar* graph is a graph with an embedding on the plane where each vertex is presented by a point on the plane and each edge is a simple continuous curve which does not



Figure 1.1: Forbidden minors of partial 3-trees

intersect any other edge on an internal point. A planar graph together with a specific planar embedding is called a *plane* graph. Class of graphs embeddable on any other given surface is defined similarly. It is a well-known theorem of Wagner that the class of planar graphs is  $Forb_m\{K_5, K_{3,3}\}$ . A *planar triangulation* is a planar graph with maximum number of edges, thus each face being isomorphic to a  $K_3$ . Except for the plane and the projective plane, for no other surface the precise list of forbidden minors is known.

Recall that the class  $Forb_m(K_t)$  for t = 3, 4 is the class of partial t-trees. However a direct extension is far from being true. Indeed the class  $Forb_m(\{K_5\})$  contains the class of planar graphs which is not included in the class of partial t-tree for any choice of t. A classification of  $K_5$ -minor free graphs, based on planar graphs, is given by Wagner [61]: The Wagner graph, which is the second (from right) graph of Figure 1.1, is a graph with no  $K_5$ -minor which is also the Möbius ladder, and the circulant clique C(8,3). Wagner proved that an edge-maximal graph with no  $K_5$ -minor is built from planar triangulations and the Wagner graph by the use of operations 2-sum or 3-sum.

# 1.4 Colorings and Homomorphisms

A Homomorphism of a multi-graph G to a multi-graph H is a mapping of vertices and edges of G to (respectively) vertices and edges of H which preserves both adjacencies and incidences. When G and H have no multi-edge, the definition is simplified to: a mapping of vertices which preserves adjacencies. A (unique) mapping of edges is induced naturally in this case. When there exists at least one homomorphism from a multi-graph G to a multi-graph H we write  $G \to H$ .

Considering contraction as the main operation of producing a minor, homomorphism and minors can be regarded as dual concepts: To produce a minor we (repeatedly) identify adjacent pairs, to produce a homomorphic image we (repeatedly) identify non-adjacent pairs. However, note that in the formal definition of a homomorphism, the target is not the same as the image, there could be vertices or edges in the target that are not used by the given homomorphism.

#### 1.4.1 Core

Core of a graph G is a smallest subgraph of G to which there exists a homomorphism from G. It is easy to show that core of a graph is well defined, i.e., any two distinct such

subgraphs are isomorphic.

A *core* is a graph which is its own core.



Dual of map of France and its core

Figure 1.2: Example of a planar core

#### 1.4.2 Proper coloring

A proper coloring of a graph G is an assignment of colors to the vertices such that adjacent vertices receive distinct colors. From now on any coloring of vertices will be considered as a proper coloring unless we specifically say otherwise. When H has no loop, then, by viewing vertices of H as colors, any homomorphism of G to H is a coloring of G. The *chromatic number* of G, denoted  $\chi(G)$ , is the smallest number of colors needed to color vertices of G. It is thus the smallest number of vertices of a graph H to which G admits a homomorphism. It is easily observed that such an image H must be a complete graph.

#### 1.4.3 Interval coloring and circular chromatic number

Alternatively, the chromatic number of G is the length of a smallest interval on the real line from which unit (open) intervals can be associated to the vertices such that adjacent vertices receive nonintersecting intervals. Here the choice of the real line is not of importance, the interval can be on any not self-intersecting Jordan-curve on the plane. However, if we allow the two ends of the interval to be identified, thus using unit length intervals from a closed Jordan curve, then we arrive at the definition of the *circular chromatic* number of G, denoted  $\chi_c(G)$ .

It follows from this definition that circular chromatic number is a refinement of the chromatic number:

**Theorem 1.4.1.** For any graph G we have  $\chi(G) = \lceil \chi(G)_c \rceil$ .

Given integers p and q, where  $p \ge 2q$ , we define the *circulant graph* C(p,q) to be the graph on vertex set  $\{1, 2, \ldots, p\}$  where two vertices i and j are adjacent if  $q \le |i-j| \le p-q$ . One of the earliest results in the study of circular coloring is to prove that homomorphisms to circulant graphs determine the circular chromatic number. More precisely:

**Theorem 1.4.2.** [see for example [25]] The circular chromatic number of a graph G is the smallest value of  $\frac{p}{q}$  such that  $G \to C(p,q)$ .

The set of circulant graphs C(p,q),  $p \ge 2$ , induces a total order in the homomorphism order of graphs which can be taken as the embedding of rational numbers at least 2 in the homomorphism order. Then, while finding the chromatic number of a graph is to find its upper integer part, finding its circular chromatic number is to find its upper rational part.

#### 1.4.4 Integer programming and fractional chromatic number

Observing that in a coloring of a graph, each color induces an independent set of the graph, we may redefine the chromatic number of a graph as follow: an assignment  $\phi$  of  $\{0,1\}$  to independent sets of G is a coloring if for each vertex x the sum of values assigned to independent sets containing x is at least 1 (i.e., it is in at least one independent set which is assigned 1). Then  $\chi(G)$  is the smallest possible total sum of values assigned by a coloring  $\phi$ . If we relax the assignment of values from the discrete set  $\{0,1\}$  to the continuous interval of [0,1], the minimum sum over all feasible solutions is called the *fractional chromatic number* of G and is denoted by  $\chi_f(G)$ .

This definition of fractional chromatic number is an alteration of the original definition which was given, implicitly, in terms of homomorphisms to Kneser graphs:

**Theorem 1.4.3** (See [55]). Given a graph G we have  $\chi_f(G) = \inf\{\frac{p}{q} \mid G \to K(p,q)\}$ , furthermore the infimum is attained.

Note that a homomorphism to Kneser graph K(p,q) is to assign q colors to each vertex from a set of p colors such that adjacent vertices receive disjoint sets of colors. Denoting by  $\chi_q(G)$  the smallest p for which there exists such a coloring, we have  $\chi_f(G) = \inf\{\frac{\chi_q(G)}{q}\}$ . This is how the original definition of the fractional chromatic number (see [55]).

# 1.5 Edge-coloring

Given a multi-graph G, a matching is a subgraph in which every vertex is of degree 1. A proper edge-coloring of G is a partitioning of the edges of G such that each part, referred to as a color-class, induces a matching. A proper edge-coloring with k color-classes is a k-edge-coloring. Thus we may refer to edge-colorings that are not necessarily proper, but when we say "k-edge-coloring", then being proper is implicit. The smallest k for which G admits a k-edge-coloring is the edge-chromatic number of G, denoted  $\chi'(G)$ .

For a graph G of maximum degree  $\Delta(G)$ , at least  $\Delta(G)$  colors are needed for a possible proper edge-coloring, but such number of colors is not always sufficient. The well-known theorem of Vizing claims that

**Theorem 1.5.1.** [see [6]] For any multi-graph G we have  $\chi'(G) \leq \Delta(G) + \mu(G)$  where  $\mu(G)$  is the maximum multiplicity of an edge of G.

Given an integer q, we define  $\chi'_q$  to be the smallest number of colors using which one can assign q colors to each edge such that adjacent edges receive disjoint sets of colors. The fractional edge-chromatic number of G, denoted  $\chi'_f(G)$ , is defined to be  $\inf\{\frac{\chi'_q(G)}{a}\}$ .

Let L(G) be the *line* graph of G, that is the graph whose vertices are the edges of G and whose edges are pairs of adjacent edges of G. It is then easily observed that  $\chi'(G) = \chi(L(G))$  and  $\chi'_f(G) = \chi_f(L(G))$ .

Two lower bounds for the fractional edge-chromatic number can be obtained easily. The first is  $\Delta(G)$ , as edges incident to a vertex of maximum degree need  $q\Delta(G)$  distinct colors altogether. The second lower bound is based on the following observation: Let X be a subset of vertices of odd order. Consider a proper edge-coloring  $\phi$  of G which assigns q colors to each edge. The restriction of  $\phi$  to G[X] is a proper edge-coloring which assigns q colors to each edge of this subgraph. Since X is a set of an odd order, each color class induces a matching of size at most  $\frac{|X|-1}{2}$ . Since each edge receives exactly q colors, we conclude that G[X] has at most  $\frac{\chi'_q(G)}{q} \times \frac{|X|-1}{2}$  edges. Hence,  $\chi'_f(G) \geq \min\{\frac{2|E(G[X])|}{|X|-1}\}$ . A result of P. Seymour [58] shows that these lower bounds combined determine the fractional edge-chromatic number of a multi-graph. We refer to [55] for a proof.

**Theorem 1.5.2.** For any multi-graph G we have  $\chi'_f(G) = \max\{\Delta(G), \frac{2E(G[X])}{|X|-1}\}$ , where the maximum is taken over all subsets of vertices of an odd order.

It's worth noting that, unlike the fractional chromatic number or the edge-chromatic number, the fractional edge-chromatic number can be computed in polynomial time (consequence of Theorem 1.5.2), see [55] for details.

The fractional edge-chromatic number provides a lower bound for the edge-chromatic number:

$$\chi'(G) \ge \lceil \chi'_f(G) \rceil. \tag{1.1}$$

Equality holds for all graphs of maximum degree at most 2. However, Petersen indeed built the well-known Petersen graph showing that this is not the case for cubic graphs (though such notions were not developed at his time). For a nice proof of the fact that the Petersen graph is not 3-edge-colorable we refer to [46].

Being the smallest example of a graph for which equality does not hold in (1.1), it is expected that any other such example contains the Petersen graph as a minor:

**Conjecture 1.5.3.** If G is a multi-graph with no Petersen minor, then  $\chi'(G) = \lceil \chi'_f(G) \rceil$ .

This is a deep conjecture which captures the four-color theorem as a very special case. It is originally formulated by P. Seymour for the subclass of planar graphs:

**Conjecture 1.5.4.** If G is a planar multi-graph, then  $\chi'(G) = \lceil \chi'_f(G) \rceil$ .

The restriction of this conjecture to cubic planar graphs is Tait's reformulation of the four-color theorem (see next section for details) and, therefore, was proved by Appel and Haken in 1977 [1]. There has been some progress in recent years on the subclasses of *i*-regular planar multi-graphs (see [8] for the latest result). It is proved for the subclass of  $K_4$ -minor free graphs by Seymour [59]. This has been extended to the subclass of graphs

with no  $K_{3,3}$  and  $K_5^-$  minor, where the latter is the complete graph on five vertices with one edge removed [35]. However, it remains largely open, even on the subclass of planar graphs of maximum degree 3.

# 1.6 Coloring and homomorphisms of planar graphs

Perhaps the most well-known theorem of graph theory is the four-color theorem. It simply claims that:

**Theorem 1.6.1.** For every planar graph G we have  $\chi(G) \leq 4$ .



Figure 1.3: Coloring map of France

It was originally asked by Francis Guthrie, who was a geographist. He asked if the regions of any simple map can be colored using at most four colors such that one can always distinguish neighbouring regions. Using the notion of duality, the question is translated to a vertex coloring problem of graphs.

Tait introduced an equivalent statement claiming:

**Theorem 1.6.2** (Tait's reformulation of the 4CT). Every bridgeless cubic planar graph can be properly 3-edge-colored.

The equivalence of this statement to the 4CT has given birth to several theories on graphs, such as the notion of edge-coloring and the theory of nowhere-zero flows. A proof of the equivalence between this statement and the four-color theorem will be given in Section 3.3 in a more general form.

In an attempt to prove the 4CT, normally the first step is to consider a minimum counterexample. Then the first observation is that every face is a triangle, as, otherwise, identifying a pair of non-adjacent vertices of a face would result in a smaller counterexample. To study homomorphism questions of similar nature for planar graphs of odd-girth higher than 3, Klostermyer and Zhang proved a similar tool which is known as the *folding lemma*: **Lemma 1.6.3** (Folding Lemma). [30] Let G be a 2-connected planar graph of odd-girth 2k + 1 and let F be a face whose boundary is not a (2k + 1)-cycle. Then there are three consecutive vertices x, y and z on F (in the order xyz) such that identifying x and z would not decrease the odd-girth of G.

That x and y are on a same face implies that the resulting graph is still planar. Thus after a repeated application, until there are no more such faces, we have:

**Lemma 1.6.4.** Given a planar graph G of odd-girth 2k + 1, there is a plane graph G' of odd-girth 2k + 1 all whose faces are (2k + 1)-cycles and  $G \rightarrow G'$ .

# 1.7 Walk power

Given a graph G and a positive integer k, the  $k^{th}$  walk power of G, denoted  $G^{(k)}$ , is a graph on vertex set V(G) where vertices u and v are adjacent if there is a walk of length (exactly) k connecting u and v. If G has at least one edge and k is an even number, then  $G^{(k)}$  has a loop. For odd values of k,  $G^{(k)}$  is loop free if and only if it has odd-girth larger than k.

The following fact, which is easy to observe, is what makes walk powers of special importance for this work:

**Lemma 1.7.1.** Given two graphs G and H, if  $G \to H$ , then  $G^{(k)} \to H^{(k)}$ .

# 1.8 Weighted graphs

A weighted graph is a pair  $(G, \psi)$  where  $\psi$  is an assignment of positive integers to the edges of G. A weighted subgraph of  $(G, \psi)$  is a subgraph H of G together with weights induced by  $\psi$ . Given a connected graph G, the complete G-weighted graph, denoted  $(K_{V(G)}, d_G)$ , is the complete graph on V(G) together with the weight assignment  $d_G$ , i.e., weight of a pair x and y of vertices is their distance in G. Then any weighted subgraph of  $(K_{V(G)}, d_G)$ is called a partially G-weighted graph.

Given an integer k, to each weighted graph  $(G, \psi)$  where  $1 \leq \psi(e) \leq k - 1$ , we associate a graph  $\overline{(G, \psi)}_k$  built from G as follows: first each edge e of G is replaced by two parallel edges e', e'', then e' is subdivided into a path of length  $\psi(e)$  and e'' is subdivided into a path of length  $k - \psi(e)$ . Then, for an odd-value of k, we say  $(G, \psi)$  is k-wide if  $\overline{(G, \psi)}_k$  is of odd-girth k. Thus, when we claim  $(G, \psi)$  is k-wide, we implicitly imply that  $1 \leq \psi(uv) \leq k - 1$  for any edge uv of G.

Observe that if a partially G-weighted graph  $(H, \psi)$  contains G as subgraph and is k-wide, then G is of odd-girth at least 2k + 1.

## 1.9 Homomorphism order

Since the composition of a homomorphism  $\psi: G \to H$  with a homomorphism  $\phi: H \to F$  is a homomorphism of G to F, the relation  $\to$  is a transitive one, and thus induces a

quasi-order on the class of all graphs. It becomes a partial order when restricted to the class of cores. See Figure 1.4 for a presentation of this order with a few well-known graphs.



Figure 1.4: A presentation of the homomorphism order

This order is a universal one in the following sense:

**Theorem 1.9.1.** [53] Given any countable partial order, one can find an isomorphic copy of it in the partial order of cores with respect to the homomorphism order.

Many notions in graph theory, especially the ones from the theory of coloring, follow the mainstream language of mathematics using this order. We present a few such examples here:

As a special case of Theorem 1.9.1, the (total) order of positive integers has a representation in the homomorphism order. The most natural one of such presentations is to consider the order induced by complete graphs, where  $K_i$  represents the integer *i*. Then, two of most studied parameters of graphs, the clique and the chromatic number, are to find the lower and upper integer parts of a graph. Thus, we may indeed write:  $\lfloor G \rfloor$  for the clique number of a graph and  $\lceil G \rceil$  for its chromatic number.

Similarly, by considering the circulant graphs C(p,q),  $p \ge 2q$ , as the natural representation of rational numbers  $\frac{p}{q}$ ,  $\frac{p}{q} \ge 2$ , we can define the circulant clique number and circular chromatic number of a graph as the lower and upper rational part of the graph.

Theorem 1.9.1 is based on incomparable pairs of graphs in the homomorphism order. To find a first such pair has already received quite a bit of attention. As  $K_1$  and  $K_2$  are comparable to any other graph,  $K_3$  is the first core to be considered as one of the elements of an incomparable pair. That leads to the study of the chromatic number of triangle-free graphs. The results of [9], which strengthen result of Harary [22], can then be translated as the pair {Grötzsch graph,  $K_3$ } being the first pair of incomparable graphs in the homomorphism order.

Two of the most famous coloring theorems on planar graphs, i.e., the four-color theorem and the Grözsch theorem, are presented in Figure 1.5. The former is to say that " $K_4$  is the maximum of the planar cores in the homomorphism order" and the latter is to say that " $K_3$  is a cut in the homomorphism order induced on planar graphs", i.e., that any planar graph either admits a homomorphism from  $K_3$  or that it admits a homomorphism to  $K_3$ .



Figure 1.5: Homomorphism order on planar cores

In [42] we have shown that  $K_3$  is also a cut in the class of  $K_5$ -minor free graphs. We end this section with following intriguing question:

**Problem 1.9.2.** Is it true that  $K_{n-2}$  is a cut in the homomorphism order induced on the class of  $K_n$ -free cores?

# 1.10 Bounds and maximum

Given a class C of graphs we say a graph B bounds C if every graph in C admits a homomorphism to B. Furthermore, if B is a core and itself also a member of C, then we say B is the maximum of C.

For example, using this terminology the 4-color theorem is to say that  $K_4$  is a bound for the class of planar graphs. As  $K_4$  itself is a planar graph, in this case we may even say  $K_4$  is the maximum of planar graphs.

In [41] we have shown that the Hadwiger conjecture can reformulated using this terminology as follows:

**Conjecture 1.10.1** (The Hadwiger conjecture reformulated). Every minor-closed family of graphs admits a maximum in the homomorphism order.

# 1.11 No-homomorphism lemmas and the odd-girth

Given a pair G and H of graphs, the question "does G map to H?" is, usually, easier to be handled when the answer would turn out to be YES. That is because, normally, a

YES answer comes with a mapping which is easily verified to be a mapping. In contrast, it is not so easy to verify a NO answer and a great deal of work is done to provide tools for verification of such a case. A key tool here is the transitivity of the homomorphism relation. That  $G \to H$  and  $H \to F$  implies  $G \to F$ . Using this main property of graph homomorphism, a number of so-called "no-homomorphism lemmas" are provided. The followings are the two main examples:

**Lemma 1.11.1.** If  $G \to H$ , then  $\chi(G) \leq \chi(H)$ .

**Lemma 1.11.2.** If  $G \to H$ , then  $\omega(G) \leq \omega(H)$ .

While both lemmas are quite important, the fact that  $\omega(G)$  and  $\chi(G)$  are themselves NP-hard to compute makes them not as practical as one would wish.

There are a few no-homomorphism lemmas based on parameter(s) of graph which are computable in polynomial time. The odd-girth of a graph is one such parameter and the corresponding no-homomorphism lemma would be of high importance in this text. It is based on the following observation:

**Lemma 1.11.3.** If  $C_{2k+1} \to C_{2r+1}$ , then  $k \ge r$ .

Since the image of any odd-cycle must contain an odd-cycle, we have:

**Lemma 1.11.4.** If  $G \to H$ , then odd-girth $(G) \ge odd$ -girth(H).

This lemma works similarly with the odd-girth of weighted graphs.

Another application of the fact that the image of any odd-cycle must contain an odd-cycle is that  $C_{2k+1}$  is a core and that any mapping of a (2k + 1)-cycle to a (2k + 1)-cycle is an isomorphism and, therefore, the distances between pairs of vertices are preserved. This observation is extended as follows:

**Theorem 1.11.5.** Let G and H be two graphs of odd-girth 2k + 1. Let  $(G', w_1)$  and  $(H', w_2)$  be, respectively, partially G-weighted and partially H-weighted graphs such that any edge of weight at least 2 of  $(G', w_1)$  (resp.  $(H', w_2)$ ) is in a (2k + 1)-cycle of G (resp. H). Then any homomorphism of G to H is also a homomorphism of  $(G', w_1)$  to  $(H', w_2)$ .

# Chapter 2

# **Projective cubes**

# 2.1 Definitions and basic properties

In the following, projective cubes are defined in several different ways. Each definition, given in a separate subsection, gives a new insight to these graphs.

#### 2.1.1 As projections of the hypercubes

The first definition is the one from which the name "projective cube" is derived.

**Definition 2.1.1.** The *projective cube* of dimension k is the graph obtained from the hypercube of dimension k + 1 by identifying antipodal pairs of vertices.

This projection is labeled as "folding" in earlier papers and thus the term "folded cube" is commonly used to refer to these graphs.

#### 2.1.2 As augmented cubes

An alternative definition, yet obtained from hypercubes, which has granted the less common name of "augmented cubes" is as follows.

**Definition 2.1.2.** The projective cube of dimension k is the graph built from the hypercube of dimension k by adding an edge between each pair of antipodal vertices.

To observe that the two definitions are equivalent, consider an inductive definition of H(k + 1). In such a definition, H(k + 1) is built from two disjoint copies  $H_1$  and  $H_2$  of H(k) by adding a matching between corresponding pairs of vertices from two disjoint copies. To find the antipodal of a vertex  $x_1$  in  $H_1$  part of H(k + 1), we first must find  $\overline{x}_1$ , the antipodal of  $x_1$  in  $H_1$ , then  $\overline{x}_2$ , the match of  $\overline{x}_1$  in  $H_2$ , is the antipodal of  $x_1$  in  $H_1$ , then  $\overline{x}_2$ , the match of  $\overline{x}_1$  in  $H_2$  is also an isomorphism of  $H_1$  to  $H_2$ . Thus, when the vertices of  $H_2$  are projected onto their antipodal vertices in  $H_1$ , the edges of  $H_2$  are mapped to the edges of  $H_1$ . Therefore, the resulting graph of this projection is the graph built on  $H_1$  where the only new edges are the images of the matching between  $H_1$  and  $H_2$ . Such images connect exactly the antipodal pairs of  $H_1$ .

### 2.1.3 As Cayley graphs

Definition 2.1.1 in turn invokes an algebraic definition of the projective cubes. Recall that given a group  $\Gamma$  and a subset S which is closed under taking inverses, the Cayley graph  $(\Gamma, S)$  is the graph on vertex set  $\Gamma$  where two vertices are adjacent if their difference is in S. Noting that the antipodal of a vertex x in the hypercube H(k) is the vertex x + J (where J is the all-1 vector) we have the following equivalent definition of the projective cubes.

**Definition 2.1.3.** The projective cube of dimension k is the Cayley graph  $(\mathbb{Z}_2^k, \{e_1, e_2, \ldots, e_k, J\})$  where the  $e_i$ 's are the standard basis and J is the all-1 vector.

This definition leads to a natural edge-coloring of PC(k), where each edge is colored with its corresponding element from  $\{e_1, e_2, \ldots, e_k, J\}$ . Observe that this is indeed a proper edge-coloring, i.e., edges incident to a same vertex receive distinct colors. It is important to note that in this definition, edges corresponding to J are not really different from those corresponding to  $e_i$ 's. Indeed, projective cubes are highly symmetric as we will see later. But to see this symmetry, in the following proposition, we give a Cayley graph presentation of PC(k) in a more general setting.

**Proposition 2.1.4.** Let S be a subset of size k + 1 of  $\mathbb{Z}_2^n$  such that the (binary) sum of all elements of S is 0, but such a sum over any proper subset is nonzero. Then the Cayely graph  $(\mathbb{Z}_2^n, S)$  consists of n - k + 2 disjoint copies of PC(k).

*Proof.* Observe that, since the only linear relation involves all elements of S, we have  $n \ge k-1$ , thus n-k+2 is a positive integer. When n = k-1, an isomorphism between  $(\mathbb{Z}_2^n, S)$  and  $(\mathbb{Z}_2^k, \{e_1, e_2, \ldots, e_k, J\})$  is obtained from any 1-1 correspondence between S and  $\{e_1, e_2, \ldots, e_k, J\}$ . If  $n \ge k$ , then  $(\mathbb{Z}_2^n, S)$  has n-k+2 disjoint copies each in a coset of the subgroup generated by S.

In particular, if we consider the set S of vectors in  $\mathbb{Z}_2^k$  with exactly two nonzero coordinates which are also (cyclically) consecutive, we get two disjoint copies of PC(k). In this view it is apparent that the graph is edge-transitive.

### 2.1.4 As power graphs of cycles

To give this definition we should first define the notion of power graph.

**Definition 2.1.5.** Given a graph H, the power graph of H, denoted Pow(H), is a graph whose vertices are the subsets of the vertices of H (including the empty set), where two vertices are adjacent if their symmetric difference is an edge of H.

Observe that for two vertices to be adjacent the symmetric difference is, in particular, of order 2. Therefore, the set of vertices of odd order and the set of vertices of even order induce two isomorphic graphs with no connection between their vertices. To obtain an isomorphism between the two subgraphs, consider a fixed set F of an odd order and take the symmetric difference with F. Each of these two parts is a connected component of the graph if and only if H itself is a connected graph. The first part of this claim, i.e., that if H is not connected, then Pow(H) has more than two components, is observed easily. For the inverse, we will prove a stronger claim based on the following proposition.

**Proposition 2.1.6.** Given a tree T of order n + 1, the power graph Pow(T) consists of two disjoint copies of H(n).

*Proof.* First we label the vertices of T by  $e_1, e_2, \ldots, e_{n+1}$  where the  $e_i$ 's are the vectors of the standard basis of  $\mathbb{Z}_2^{n+1}$ . Each subset of vertices then corresponds to a vector of  $\mathbb{Z}_2^{n+1}$  by adding its elements, furthermore this is a one-to-one correspondence. Taking the symmetric difference of two subsets then corresponds to the sum of their corresponding vectors in  $\mathbb{Z}_2^{n+1}$ .

Next we label edge of T: An edge  $e_i e_j$  of T is labeled by  $e_i + e_j$  (addition is done in  $\mathbb{Z}_2^{n+1}$ ). Based on the fact that T has no cycle, it is easily observed that the set of labels of edges is linearly independent in  $\mathbb{Z}_2^{n+1}$ . Since there are n edges, they form a basis for a subspace of dimension n. Since every vector generated by edges must have an even number of 1's, this subspace is contained in the subspace of  $\mathbb{Z}_2^{n+1}$  formed by vectors with an even number of 1's. But considering the dimension, it follows that these two subspaces are exactly the same.

Then, to build the power graph of T on subsets of even order, is to take edges as a basis, and build the Hamming graph based on this basis. The result then is then hypercube of dimension n.

**Corollary 2.1.7.** If H is a connected graph, then Pow(H) consists of two connected components. Furthermore, each components contains the hypercube of dimension |V(H)| - 1 as a spanning subgraph.

*Proof.* Immediately from the definition it follows that if  $H \subset H'$ , then  $Pow(H) \subset Pow(H')$ . To complete the proof it is enough to consider a spanning tree of H.  $\Box$ 

We now give another definition of the projective cube of dimension n, we leave it as an exercise to show that this definition is equivalent to anyone of the other definitions.

**Definition 2.1.8.** Given a cycle  $C_n$ , a connected component of the power graph  $Pow(C_n)$  is the projective cube of dimension n.

#### 2.1.5 Constructed from posets

When hypercubes are viewed as poset graphs, the projective cubes, built from the projection of hypercubes, find a new definition which we will frequently use. Recall that in the definition by projection, antipodal vertices of the hypercube H(k + 1) are identified in order to form PC(k). In the classic definition of the hypercube H(n) each vertex vis a vector in  $\mathbb{Z}_2^n$ . By associating the set of coordinates at which v is 1, we obtain a presentation of H(n) where vertices are all subsets of an n-set where vertices A and B are adjacent if  $A \subset B$  and |A| = |B| - 1. We refer to this as the poset representation of H(n). In such a representation, the antipodal of a vertex (of H(k+1)) is its complement. Thus, when PC(k) is formed from H(k+1) by identifying antipodal pairs of vertices, each new vertex receives a label of two complementary subsets of the reference (k + 1)-set (the set which is used to view H(k+1) as a poset). The transition to the projective cubes is as follows: **Definition 2.1.9.** Let S be a set of size k + 1. Then PC(k) is a graph whose vertices are pairs of complementary subsets  $\{A, \overline{A}\}$  of S where  $\{A, \overline{A}\}$  and  $\{B, \overline{B}\}$  are adjacent if  $A \subset B$  and |B| = |A| + 1.

We normally will take the set  $S_k = \{e_1, e_2, \dots, e_k, J\}$  to be the reference (k+1)-set.

For odd values of k, i.e., k = 2i + 1, the poset  $2^{S_k}$  has a middle layer consisting of (i + 1)subsets. As the antipodal of a vertex in this layer resides in the same layer, identification of antipodals preserves the natural bipartition of the graph (every second layer is colored black). In contrast, for even values of k, k = 2i, the two middle layers will be identified. Since the two middle layers induce an (i + 1)-regular bipartite graph, this will create a new layer which is indeed (i + 1)-regular, thus making the resulting graph nonbipartite. The subgraph induced on this layer is introduced in the next proposition, whose proof we leave as an exercise.

**Proposition 2.1.10.** In the poset presentation of PC(2i), the subgraph induced by vertices  $\{A, \overline{A}\}, |A| = i$ , is isomorphic to the Kneser graph K(2i + 1, i).

In this presentation of PC(k), a vertex  $\{A, \overline{A}\}$  may be simply represented by the smaller of the two sets A and  $\overline{A}$ . With such a labeling, the vertices of PC(2i) are subsets of order at most i of  $S_{2i} = \{e_1, e_2, \ldots, e_{2i}, J\}$ . However, to present vertices of PC(2i+1), when  $|A| = |\overline{A}| = i + 1$  one must make a choice between A and  $\overline{A}$ , but indeed any choice would be ok.

With such a labeling of vertices, the distance between two vertices can be computed by the following formula.

**Proposition 2.1.11.** The distance between vertices A and B of the projective cube PC(k) is min{ $|A \oplus B|, k + 1 - |A \oplus B|$ }.

Moreover, the set of vertices at distance *i* from *A* and *j* from *B*, d(A, B) = i+j determines  $A \oplus B$ . To simplify, we assume  $A = \emptyset$ . For the general case it would suffice to take the symmetric difference with *A* (see Section 2.5 on the automorphisms of PC(k)).

**Proposition 2.1.12.** Given a vertex  $\{B, \bar{B}\}$   $(|B| < |\bar{B}|)$  and positive integers *i* and *j* such that i + j = |B|, the union of vertices at distance *i* from  $\emptyset$  and *j* from *B* is the set *B*. Furthermore, if  $|B| = |\bar{B}|$ , then the internal vertices of all  $\frac{k+1}{2}$ -paths connecting  $\emptyset$  and *B* induce two connected components. The union of vertices at distance *i*,  $(i \le \frac{k+1}{2})$ , from  $\emptyset$  in one component is *B* and in the other is  $\bar{B}$ .

#### 2.1.6 Inductive definition

Given PC(k) together with edges corresponding to J, the graph PC(k+1) is defined as follows: For each vertex u of PC(k) there are two vertices  $u_1$  and  $u_2$  in PC(k+1). If uvis an edge of PC(k) not corresponding to J, then  $u_1v_1$  and  $u_2v_2$  are edges of PC(k+1). If uv is an edge corresponding to J, then  $u_1v_2$  and  $u_2v_1$  are edges of PC(k+1). Finally for each vertex u of PC(k) we add an edge  $u_1u_2$ , these edges being the edges corresponding to J in PC(k+1). Thus each cycle with an odd number of edges corresponding to Jis replaced by a Möbius ladder. This definition is of higher interest when the notion of signed graph is employed (to write more in Part II).

# 2.2 Examples and properties

For k = 1, following the Cayley graph definition of the projective cube and noting that in  $\mathbb{Z}_2^1$  the vector J is the same as  $e_1$ , we get  $K_2$  as the projective cube. However, in this case, we allow multiple edges and let PC(1) be a complete (multi)graph on two vertices with an edge of multiplicity two (one corresponding to  $e_1$ , another corresponding to J). This will be of importance when we consider signed projective cubes. The projective cubes of dimension 2 and 3 are, respectively  $K_4$  and  $K_{4,4}$ . The next case, i.e., PC(4) is a well studied graph known as the Clebsch graph, also as GreenWood-Gleason graph and PC(5) is known as the Kummer graph.

Considering the Cayley graph definition of the projective cubes, and noting that the sum of colors of edges on each cycle is 0 (Observation 1), we have the following property which separates the cycles of G into two essentially different types:

**Proposition 2.2.1.** Given a cycle or a closed walk C of PC(k) either all elements of  $\{e_1, e_2, \ldots, e_k, J\}$  appear on edges of C an odd number of times, or they all appear an even number of times.

*Proof.* Assume that an element x of  $S_k = \{e_1, e_2, \ldots, e_k, J\}$  appears an odd number of times. Then in the sum of the values associated to edges of C (which equals to 0), x appears with a coefficient of 1. As the only linear relation among elements of  $S_k$  is that the total sum is 0, to null x each other element of  $S_k$  must also appear with a coefficient 1, that is to say it appears an odd number of times on C.

**Corollary 2.2.2.** Given a cycle C of PC(2i), a color appears an odd number of times if and only if the cycle C is of an odd length. Furthermore, the length of a shortest odd-cycle of PC(2i) is 2i + 1.

On the other, hand if  $\{e_1, e_2, \ldots, e_k, J\}$  is of an even order, i.e., if k is odd, the graph PC(k) has no odd-cycle, in other words PC(2i + 1) is bipartite for any values of i. However, we keep in mind that PC(2i) has two essentially different types of cycles. The notation of signed graph allows us to take advantage of this difference and develop an analogue theory for PC(2k + 1).

Considering the Cayley graph definition of the projective cube and the corresponding edge-coloring we have the following property.

**Proposition 2.2.3.** Given two vertices u and v of PC(k) and any shortest path P connecting them, all edges of P are colored distinctly.

*Proof.* Let  $c_{j_1}, c_{j_2}, \ldots, c_{j_l}$  be the set of colors appearing on P an odd number of times. Thus,  $v = u + \sum_{x=1}^{l} c_{j_x}$ . Then, the set  $\{u, u + c_{j_1}, u + c_{j_1} + c_{j_2}, \ldots + u + \sum_{x=1}^{l} c_{j_x}\}$ , induces a path of length l connecting u and v. As P is a shortest path connecting these u and v, P contains exactly one edge of color  $c_{j_x}, x = 1, 2, \ldots, l$  and it contains no other edge.  $\Box$ 

# **2.3** Subgraphs of PC(2i)

#### 2.3.1 Cycles

By Corollary 2.2.2, the shortest odd-cycle which is a subgraph of PC(2i) is  $C_{2i+1}$ . Therefore any such subgraph is induced. Furthermore we have:

**Theorem 2.3.1.** Given a pair u, v of vertices of PC(2i), there is a 2(i + 1)-cycle of PC(2i) containing both u and v.

*Proof.* Suppose u and v are at distance d, and let P be a shortest path connecting u and v. By Proposition 2.2.3, the edges of P are colored distinctly. Let  $c'_{j_1}, c'_{j_2}, \ldots, c'_{j_l}$  be the set of colors that do not appear on P. The set  $\{u, u + c'_{j_1}, u + c'_{j_1} + c'_{j_2}, \ldots, u + \sum_{x=1}^{l} c'_{j_x}\}$  induces a path P' of length 2i + 1 - d whose edges are colored all distinctly and none of which is used by edges of P. This path together with P induces a walk of length 2i + 1 whose edges have received all possible colors, exactly one of each. As a walk of odd-length must contain a cycle of odd-length, and as PC(2i) is of odd-girth 2i + 1, this walk is indeed a cycle of length 2i + 1.

**Corollary 2.3.2.** For any positive integer *i* the graph PC(2i) is a core.

*Proof.* If a homomorphism  $\rho$  of PC(2i) to itself (or to any other graph) identifies two vertices u and v, then the image of any (2i + 1)-cycle passing through both u and v will contain an odd-cycle of length strictly smaller than (2i + 1). As PC(2i) has odd-girth 2i + 1, this implies that any mapping of PC(2i) to itself is one-to-one.

#### 2.3.2 Möbius ladders and circulant graphs

Using the Cayley presentation of PC(2i), let  $v_j = \sum_{l=1}^{j} e_l$  (thus  $v_{2i} = J$ ).  $\{v_j\}_{j=1}^{2i}$  induces a path of length 2*i*. These sets of vertices together with their antipodals (that is  $v_j + J$ ) induces a 4*i*-cycle together with a perfect matching that connects antipodal pairs. This is the graph known as the Möbius ladder  $M_{4k}$ . It is also isomorphic to the circulant graph C(4i, 2i - 1). Another way of observing Möbius ladders in PC(k) is to consider the inductive definition of PC(k). By further analysis we observe that PC(4) can be (vertex)-decomposed into two induced isomorphic copies of  $M_8$ .

#### 2.3.3 Augmented toroidal grids

Given positive integers a and b, the toroidal (a, b)-grid, denoted T(a, b), is the cartesian product  $C_a \Box C_b$ . In a toroidal (2a, 2b)-grid, each vertex u has unique vertex at (maximum) distance a+b, which therefore is referred to as the antipodal of u. The augmented toroidal (2a, 2b)-grid, denoted AT(2a, 2b), is the graph obtained from T(2a, 2b) by adding an edge between each pair of antipodal vertices. Figure 2.1 depicts a representation of T(24, 24)and Figure 2.2 presents AT(6, 6).

We note that AT(4, 4) is isomorphic to the projective cube PC(4), but in general AT(2a, 2b) is a proper subgraph of a corresponding projective cube as proved in the following theorem.



Figure 2.1: A representation of the  $24 \times 24$  toroidal grid.



Figure 2.2: The augmented toroidal grid AT(6, 6).

**Theorem 2.3.3.** Given positive integers a and b such that a + b = k, AT(2a, 2b) is a subgraph of PC(k).

Proof. Considering the definition of PC(k) as the augmented hypercube, it would be enough to show that the hypercube of dimension k contains a copy of  $C_{2a} \Box C_{2b}$  in such a way that the antipodal of each vertex in the subgraph is the same as its antipodal in the hypercube. To this end in fact we present an isometric embedding of  $C_{2a} \Box C_{2b}$  in H(k). Considering H(k) as the Cartesian product of k  $K_2$ 's we observe that  $H(k) = H(a) \Box H(b)$ . Now consider the Cayley graph definition of H(k) and assume  $e_1, e_2, \ldots, e_a$  are the basis for H(a) and  $e_{a+1}, e_{a+2}, \ldots, e_k$  are the bases for H(b). Let  $v_0$  be the zero vertex of H(a), and define  $v_1 = v_0 + e_1, v_2 = v_1 + e_2, \ldots, v_a = v_{a-1} + e_a = J$ , similarly using reverse order on  $e_i$ , we define  $v'_1 = v_0 + e_a, v'_2 = v'_1 + e_{a-1}, \ldots, v'_a = v'_{a-1} + e_1 = J$ . Thus  $v_a = v'_a$  but all other vertices are distinct. Consider a similar 2b-cycle of H(b). Then in the subgraph induced by the Cartesian product of these two cycles, the distances are the same as the hamming distances, hence this is an isometric subgraph of the hypercube H(k).  $\Box$ 

#### 2.3.4 Generalized Mycielski constructions

Consider PC(2i) as a power graph of  $C_{2i+1}$  whose vertices are labeled  $v_1, v_2, \ldots, v_{2i+1}$  in the cyclic order. We consider the component induced by subsets of odd order. For given j and  $r, 1 \leq j \leq 2i+1, 0 \leq r \leq k$  let  $v_j^r$  be the set of vertices at distance at most r from  $v_j$ . Thus in particular,  $v_j^0 = \{v_j\}, v_j^k = \{v_1, v_2, \ldots, v_{2i+1}\}$ . Note that  $v_j^r$  is of (odd) order 2r + 1. It can then be checked:

**Theorem 2.3.4.** The subgraph induced by the set  $\{v_j^r | 1 \le j \le 2i+1, 0 \le r \le k\}$  is the generalized Mycielski graph  $M_i(C_{2i+1})$ .

**Corollary 2.3.5.** For every value of *i* we have  $\chi(PC(2i)) = 4$ .

Proof. Payan ?? showed that generalized Mycielski  $M_i(C_{2i+1})$  has chromatic number 4. Since PC(2i) contains the  $M_i(C_{2i+1})$  as subgraph, it has chromatic number at least 4. In Section 3.1 we will see that  $PC(2i) \rightarrow PC(2k)$  for any *i* satisfying  $i \ge k$ . As PC(2) is isomorphic to the complete graph  $K_4$ , mapping PC(2i) to PC(2) is a 4-coloring of PC(2i).

#### 2.3.5 Kneser graphs

Given positive integers n and k,  $n \ge k$ , the Kneser graph K(n,k) is a graph whose vertices are all k-subsets of an n set where vertices A and B are adjacent if  $A \cap B = \emptyset$ . By labeling vertices of PC(2i) as in Definition 2.1.9, it is easy to show that the set of vertices  $\{\{A, \overline{A}\} | |A| = i\}$  induces a subgraph isomorphic to K(2i + 1, i). In particular, K(5, 2), which is the well-known Petersen graph, is a subgraph of PC(4), the Clebsch graph.

#### 2.3.6 Coxeter graphs

The Kneser graphs K(2i + 1, i) in particular contains a number of highly symmetric subgraphs which in turn are also subgraphs of PC(2i). One way of producing such subgraphs is by considering vertices forming a design. Indeed, if we remove the seven triples of a Fano plane from thirty five vertices of K(7,3), the resulting graph is the well-known Coxeter graph, which is a cubic graph of girth 7 on 28 vertices and, thus, an induced subgraph of PC(6).

# 2.4 Binary Cayley graphs

A binary Cayley graph is any Cayley graph on a binary group (that is any subgroup of  $\mathbb{Z}_2^k$ ). In 1990, C. Payan [52] presented a somewhat surprising result that any nonbipartite binary Cayley graph has chromatic number at least 4. Here, we show that this result is strongly related to the projective cubes of even dimension.

**Theorem 2.4.1.** Given a nonbipartite binary Cayley graph  $G = (\Gamma, S)$ , if 2i + 1 is the length of a shortest odd-cycle of G, then PC(2i) is a subgraph of G.

Proof. Let C be a cycle of length 2i+1 of G. Then the edges of C are labeled  $\gamma_1, \gamma_2, \ldots, \gamma_{2i+1}, \gamma_j \in S$ . Since C is a cycle we have  $\gamma_1 + \gamma_2 + \ldots + \gamma_{2i+1} = 0$ . That C is a shortest odd-cycle of G implies that the only linear relation among the  $\gamma_j$ 's is the one mentioned above. In particular, this means that all  $\gamma_j$ 's are distinct. Furthermore, the assignment  $\rho(X) = \sum_{\gamma_j \in X} \gamma_j$  for subsets X of  $\{\gamma_1, \gamma_2, \ldots, \gamma_{2i+1}\}$  is a two-to-one assignment, where X and its complement are assigned the same value (due to the fact that  $\gamma_1 + \gamma_2 + \ldots + \gamma_{2i+1} = 0$ ). It can then be readily checked that  $Pow(C_{2i+1})$ , after an identification of two components, is a subgraph of G.

**Corollary 2.4.2.** If G is a nonbipartite Cayley graph, then it has chromatic number at least 4.

# 2.5 Automorphisms of the projective cubes

It is easier to find a full set of automorphisms of a given projective cube by considering a labeling of vertices by pairs  $\{A, \bar{A}\}, A \subset S_k$  as in Definition 2.1.9. Viewed such, it is easy to verify that any nontrivial permutation of  $S_k$  is a nontrivial automorphism of PC(k). Moreover, for a fixed nontrivial subset X of  $S_k$ , taking the symmetric difference with X is also a nontrivial automorphism of PC(k). However, as taking the symmetric difference with X and then with  $\bar{X}$  changes each set A to its complement  $\bar{A}$ , the automorphism induced by taking the symmetric difference with X is the same as the one obtained by taking the symmetric difference with  $\bar{X}$ . For clarification, we then only consider the symmetric difference with X where  $J \notin X$ . Finally, the composition of any two such automorphisms is also an automorphism. In the next theorem it is shown that this is the full set of automorphisms of PC(k).

**Theorem 2.5.1.** Each automorphism of PC(k) is a composition of a permutation of  $S_k$ and a symmetric difference with a set  $X \subset S_k$ ,  $J \notin X$ .

*Proof.* We use the labeling of vertices by pairs  $\{A, A\}$  and, furthermore, we use the smaller of A and  $\overline{A}$  to denote the vertex  $\{A, \overline{A}\}$  (when  $|A| = |\overline{A}|$ , we may choose A or  $\overline{A}$  arbitrarily).

Let  $\psi$  be an automorphism of PC(k). First, we consider the case where  $\emptyset$  is a fixed point of  $\psi$  (i.e.  $\psi(\emptyset) = \emptyset$ ). Then, the neighbours of  $\emptyset$ , i.e., the singletons, are mapped to singletons in a one-to-one manner. This induces a permutation  $\pi_{\psi}$  on  $S_k = \{e_1, e_2, \ldots, e_k, J\}$ . We claim that  $\psi$  is the automorphism induced by this permutation. Consider a vertex A and let  $A' = \psi(A)$ . As the distance from  $\emptyset$  is solely determined by the order of A, A and A' are of the same order. Furthermore, by Proposition 2.1.12, A is the union of singletons on shortest paths connecting it to  $\emptyset$ . The automorphism  $\psi$  maps these shortest paths to shortest paths from A' to  $\emptyset$ . Thus, the singletons of shortest paths from A' to  $\emptyset$  are determined by the permutation  $\pi_{\psi}$ , but by the same proposition, A' is the union of these singletons.

Next, we consider a general automorphism  $\psi$ . Assume  $\psi(\emptyset) = B$  and let  $\sigma_B$  be an automorphism obtained by taking symmetric difference with B. Then,  $\psi \circ \sigma_B$  is an automorphism which stabilizes  $\emptyset$ . Thus,  $\psi \circ \sigma_B = \pi$  where  $\pi$  is an automorphism induced by a permutation of  $S_k$ . Then, noting that  $\sigma_B \circ \sigma_B$  is the identity, we have:  $\psi = \pi \circ \sigma_B$ .  $\Box$ 

# 2.6 PC(k)-weighted complete graphs

By the definition of a G-weighted complete graph, the PC(k)-weighted complete graph is a weighted complete graph on  $2^k$  vertices labeled by the elements of  $\mathbb{Z}_2^k$  where the weight of an edge xy is the distance of x and y in PC(k). We will denote this weighted graph by  $K_{PC(k)}$ . We will use a finer notation to work with  $K_{PC(k)}$ , to present this notation we need a strengthening of Proposition 2.1.11. To this end, in the rest of this section, we consider the poset presentation of PC(k), thus vertices are labeled by pairs of complementary subsets of the set  $S_k = \{e_1, e_2, \ldots, e_k, J\}$ .

**Proposition 2.6.1.** Given vertices  $u = \{A, \bar{A}\}$  and  $v = \{B, \bar{B}\}$ , let  $X_{uv}$  be the smaller of the two subsets  $A \oplus B$ ,  $A \oplus \bar{B}$ . Then edges of any shortest path connecting u and v are colored from the set  $X_{uv}$  in a one-to-one manner, unless  $|A \oplus B| = k + 1 - |A \oplus B|$  in which case the color set  $\bar{X}_{uv}$  could be employed in place of  $X_{uv}$ .

It should be noted that while  $X_{uv}$  corresponds to a shortest path  $P_{X_{uv}}$  connecting u and v, the complement of  $X_{uv}$  corresponds to a path  $P_{\bar{X}_{uv}}$  of length  $k+1-|X_{uv}|$  whose edges are colored, in a one-to-one manner, from  $\bar{X}_{uv}$ . The union of  $P_{X_{uv}}$  and  $P_{\bar{X}_{uv}}$  is a (k+1)-cycle which uses each color (that is elements of  $\{e_1, e_2, \ldots, e_k, J\}$ ) exactly once. In particular, in PC(2k) this is a (2k+1)-cycle containing both u and v.

Given the PC(k)-weighted complete graph, we label an edge uv by the pair set  $\{X_{uv}, X_{uv}\}$ where  $X_{uv}$  is defined as above. We may then choose (freely) one of  $X_{uv}$  or  $\bar{X}_{uv}$  to denote the label of this edge. The weight of an edge labeled  $\{X_{uv}, \bar{X}_{uv}\}$  is  $\min\{|X_{uv}|, |\bar{X}_{uv}|\}$ which is the same as  $\min\{|X_{uv}|, k+1-|X_{uv}|\}$ .

We may now give a classification of triples of the PC(2k)-weighted complete graph.

**Theorem 2.6.2.** Given a triple A, B and C of subsets of  $S_k$ , there is a triangle in the PC(2k)-weighted complete graph whose edges are labeled A, B and C if and only if one of the following happens:

- Every element of  $\{e_1, e_2, \ldots, e_{2k}, J\}$  appears in exactly two of A, B and C.
- Every element of  $\{e_1, e_2, \ldots, e_{2k}, J\}$  appears either in one or all three of A, B and C,

*Proof.* We observe that the two conditions are actually the same. A triple A, B and C satisfies the first condition if and only if the triple A, B and  $\overline{C}$  satisfies the second condition.

Consider a triangle of the PC(2k)-weighted complete graph whose edges are labeled A, Band C. By replacing C with  $\overline{C}$ , if necessary, we may assume that |A| + |B| + |C| is odd. That A, B and C correspond to edges of a cycle implies that the sum of the elements of these three sets is 0 (in  $\mathbb{Z}_2^{2k}$ ). As there are an odd number of elements, and similar to the proof of Proposition 2.2.1, all elements must appear an odd number of times in the closed walk corresponding to A, B and C passing through vertices of the triangle. Thus, each element either appears only in one of the three sets, or in all three of them.

Conversely, assume A, B and C are so that each element of  $\{e_1, e_2, \ldots, e_{2k}, J\}$  appears either in exactly one of the three sets or in all three of them. Then the sum (in  $\mathbb{Z}_2^{2k}$ ) of the elements is 0. Thus, for a given vertex x of the PC(2k)-weighted complete graph, if y and z are chosen so that xy and xz labeled A and B, respectively, then zy is labeled  $A \cup B$  which is either C or  $\overline{C}$ .

# **2.7** Transitivity of PC(k)

As with any Cayley graph on a commutative group, PC(k) is a vertex-transitive graph, that is to say that for any pair u and v of vertices, there is an automorphism of PC(k) which maps u to v. Indeed, assuming that uv is labeled X in the PC(k)-weighted complete graph as in previous subsection, and considering the poset representation of PC(k), the automorphism obtained by taking the symmetric difference with X exchanges u and v.

This implies higher levels of symmetries on PC(k). As discussed in the previous section, the distance of a pair u, v of vertices is a function of the order of the set which the edge uv is labeled by in the PC(k)-weighted complete graph. Thus, given a pair uv and xy of edges of a same weight in the PC(k)-weighted complete graph, an automorphism that maps x to u composed with a suitable permutation of  $\{e_1, e_2, \ldots, e_k, J\}$  would map vertices u and v to vertices x and y respectively. This means that the PC(k)-weighted graph is edge-transitive, or that, equivalently, PC(k) is distance-transitive. Yet, this is not full strength of the symmetries of PC(k). It is known that PC(k) is triple-transitive, see [37]. We give a proof of this using the terminology we have developed.

**Theorem 2.7.1.** Let  $\{u, v, w\}$  and  $\{x, y, z\}$  be two triples of vertices of PC(k) such that  $d_{PC(k)}(u, v) = d_{PC(k)}(x, y)$ ,  $d_{PC(k)}(u, w) = d_{PC(k)}(x, z)$ ,  $d_{PC(k)}(v, w) = d_{PC(k)}(y, z)$ . Then, there exists an automorphism  $\psi$  of PC(k) such that  $\psi(u) = \psi(x)$ ,  $\psi(v) = \psi(y)$ ,  $\psi(w) = \psi(z)$ .

Proof. Consider the labeling  $X_{uv}, X_{uw}$  and  $X_{vw}$  of the edges of a triangle induced by u, v and w in the PC(k)-weighted complete graph and consider the corresponding paths  $P_{X_{uv}}, P_{X_{uw}}$  and  $P_{X_{vw}}$ . Since they form a closed walk, the sum (in  $\mathbb{Z}_2^k$ ) of elements from  $X_{uv}, X_{uw}$  and  $X_{vw}$  must be 0. Thus, by Proposition 2.2.1, either all elements of  $\{e_1, e_2, \ldots, e_k, J\}$  appear an even number of times in this sum, or they all appear an odd number of times. If it is the former case, we replace  $X_{uv}$  by its complement, thus we assume each element appears an odd number of times. That there is no repeated color on each path means each element of  $S_k$  appears either in exactly one of  $X_{uv}, X_{uw}$  and  $X_{vw}$  or in all three of them. More precisely, exactly  $t = \frac{|X_{uv}| + |X_{uw}| + |X_{vw}| - 2k + 1}{2}$  of the elements appear in all three of them, the rest appearing in exactly one of the three sets.

Thus, the positioning of a triple is only a function of their mutual distances. To find an automorphism mapping vertices of a first triple to the second, we first choose a permutation of  $S_k$  which maps the *t* common elements of the first triple to the second and maps remaining elements of each path in the first triangle to its corresponding path in the second triangle. To complete the automorphism, all we need is to take a symmetric difference which maps one of the three vertices of the first triple to its corresponding vertex in the second set of triples.  $\Box$ 

# Chapter 3

# Homomorphisms to projective cubes

The existence of a homomorphism from a graph G to a projective cube is of special interest (compared to other targets) as it captures a packing problem. This property is better expressed using the language of signed graphs. Here we express the graph version for projective cubes of even dimension, for general version and for a proof we refer to [?].

**Theorem 3.0.1.** A graph G admits a homomorphism to PC(2i) if and only if E(G) can be partitioned into 2i + 1 sets  $E_1, E_2, \ldots, E_{2i+1}$  such that each intersection of an  $E_j$  and a cycle of G has an order of the same parity as the order of the cycle.

This theorem together with the fact that  $PC(2) \cong K_4$  distinguishes 4-coloring from all other k-colorings: To (properly) 4-color vertices of a graph is equivalent to 3-color its edges such that each even cycle uses any of the three colors an even number of times and each odd cycle uses each color an odd number of times. This is the hidden fact behind Tait's reformulation of the four-color theorem (see Section 1.6).

We study homomorphisms to projective cubes from two points of view; first we consider homomorphisms from binary Cayley graphs to projective cubes. Then we consider homomorphisms from planar graphs.

# 3.1 Homomorphisms of binary Cayley graphs to projective cubes

We consider the Cayley presentation of PC(2i), thus each vertex is also a vector in  $\mathbb{Z}_2^{2i}$ . A natural mapping of PC(2i+2) into PC(2i) is as follows: for a vector v of PC(2i+2) let  $v_{12}$  be the vector of length 2 (in  $\mathbb{Z}_2^2$ ) whose coordinates are (respectively) the first and second coordinate of v, then let v' be a vector of length 2i (in  $\mathbb{Z}_2^{2i}$ ) whose coordinates are the remaining coordinates of v in the same order. The mapping  $\rho$  is defined as follows:

$$\rho(x) = \begin{cases}
\rho(v) = v' & \text{if } v_{12} \in \{(00), (11)\} \\
\rho(v) = v' + J & \text{if } v_{12} \in \{(01), (10)\}
\end{cases}$$

where J is the all-1 vector of the same length as v'.

Suppose that u and v are two adjacent vertices of PC(2i). Then u + v = x where  $x \in S_{2i}$ . If  $x \in \{e_3, e_4, \ldots, e_{2i}\}$ , then  $\rho(u)$  and  $\rho(v)$  differ only at the coordinate corresponding to x. If  $x \in \{e_1, e_2\}$  then by symmetry of u and v we have  $u_{12} \in \{(00), (11)\}$  and  $v_{12} \in \{(01), (10)\}$  thus  $\rho(u) + \rho(v) = J$  with J being of length 2i - 2. If u + v = J (of length 2i), then  $\rho(u) + \rho(v) = J$  (of length 2i - 2). Thus, in all cases,  $\rho(u)$  is adjacent to  $\rho(v)$  in PC(2i - 2). Hence, considering the transitivity of homomorphism and odd-girth of PC(2i), we have proved:

**Theorem 3.1.1.** There is a homomorphism of PC(2i) to PC(2j) if and only if  $i \ge j$ .

In particular, for any value of i, PC(2i) maps to PC(2) which is isomorphic to  $K_4$ . Hence every PC(2i) is 4-colorable, and thus 4-chromatic (this completes the proof of Corollary 2.3.5).

The result of C. Payan [52], proving that there are no binary Cayley graph of chromatic number 3, can be restated as: if a non-bipartite Cayley graph maps to PC(2), then any such mapping must be surjective. This then suggests the following generalization by replacing PC(2i) with PC(2) which we proposed in [5].

**Conjecture 3.1.2.** If G is binary Cayley graph mapping to PC(2i), then any such mapping is surjective.

This in particular claims that any mapping of PC(2j) to PC(2i),  $j \leq i$ , is surjective. However, by Theorem 2.4.1, if a binary Cayley graph G is non-bipartite, then it contains PC(2j) as a subgraph, where 2j + 1 is the odd-girth of G. Therefore, Conjecture 3.1.2 is equivalent to the following restricted version:

**Conjecture 3.1.3.** Given  $j \leq i$ , any mapping of PC(2j) to PC(2i) is surjective.

The case i = j of the conjecture is to say that PC(2i) is a core, which is proved in Proposition 2.3.2. For i = 1, the claim is to say that for any *i* the projective cube PC(2i)is 4-chromatic, which is the case because it contains  $M_i(C_{2i+1})$  which are shown to be 4-chromatic by Payan [52]. The case j = 3 and i = 2 is proved in [5].

**Theorem 3.1.4.** Any homomorphism of PC(6) to PC(4) is surjective.

In [5] we proposed a stronger conjecture as a possible method of proving Conjecture 3.1.3. That was to claim that  $PC(2j)^{2i-1}$ ,  $(j \ge i)$ , is of chromatic number  $2^{2i}$ . If it was true, it would have implied, in particular, that any homomorphism of PC(2j) to PC(2i) is onto, as  $PC(2i)^{2i-1}$  is the complete graph on  $2^{2i}$  vertices. However, computational work of Gordon Royle showed that this stronger conjecture fails even for j = 4 and i = 2, indeed a 12-coloring of  $PC(6)^3$  is given by G. Royle.

# 3.2 Homomorphisms of planar graphs into projective cubes

Extending the four-color theorem, the following conjecture plays a central role in the development of the theory of homomorphisms to projective cubes.

**Conjecture 3.2.1.** Any planar graph of odd-girth 2g + 1 maps to PC(2g).

If we denote the class of planar graphs of odd-girth 2g + 1 by  $\mathcal{P}_{2g+1}$ , then the conjecture claims that:  $\mathcal{P}_{2g+1}$  is bounded by PC(2g) in the homomorphism order of graphs. Since  $PC(2) \cong K_4$ , the case g = 1 is the 4-color theorem. The cases g = 2, 3 are verified (using the 4-color theorem) as we will explain in Section 3.3. It remains open for larger

values of g.

A bipartite analogue of the conjecture, using the notion of signed graphs, is introduced By B. Guenin, we refer to [?]. This general view using notion of signed graphs is of high importance because, among other reasons, it would provide room for a finer inductive approach: stepping from PC(k) into PC(k+1) rather than PC(k+2).

In the following subsection, we will see the construction of a planar graph of odd-girth 2k + 1 for which any homomorphic image of odd-girth 2k + 1 would have at least  $2^{2k}$  vertices. We will also see that if B is a minimal graph of odd-girth 2k + 1 which admits a homomorphism from any planar graph of odd-girth at least 2k + 1, then B has minimum degree no less than 2k + 1. Thus Conjecture 3.2.1, if true, provides an optimal solution; PC(2k) would be the smallest graph (in terms of both number of vertices and number of edges) of odd-girth 2k + 1 to which every planar graph of odd-girth 2k + 1 admits a homomorphism.

This, in particular, implies that no subgraph of PC(2k) bounds the class  $\mathcal{P}_{2k+1}$ . However, if we consider mapping planar graphs of odd-girth at least 2r + 1,  $r \geqq k$ , into PC(2k), the situation changes and we may not need the whole PC(2k). We will show that finding minimal such subgraphs captures some classic theories and gives birth to new theories. More formally we ask:

**Problem 3.2.2.** Given  $r \ge k$ , what are the minimal subgraphs of PC(2k) to which every planar graph of odd-girth 2r + 1 admits a homomorphism?

For r = k, or for values of r closer to k, the question would only make sense if Conjecture 3.2.1 holds. We give suggestions of such possible solutions in special cases.

#### 3.2.1 The case r = k

In this case there would be an answer only if Conjecture 3.2.1 holds. However, if so, then the whole graph PC(2k) would be needed, in fact we will show a stronger result, proving that if the conjecture holds, then PC(2k) is a smallest graph (in terms of both number of edges and number of vertices) of odd-girth 2k + 1 to which any planar graph of odd-girth 2k + 1 admits a homomorphism. This claim in turn is a corollary of a yet stronger claim:

**Theorem 3.2.3.** There exists a planar graph G of odd-girth 2k + 1 whose  $(2k - 1)^{th}$  walk power contains a  $2^{2k}$ -clique. Furthermore, such a graph G can be chosen to be a partial 3-tree as well.

We explicitly build such a graph, here we give the construction and refer to [45] for details. Let  $G_0$  be a subdivision of  $K_4$  where two parallel edges are subdivided each 2k - 2 times. Thus, we have a planar graph with four faces, each of which is of length 2k + 1 and moreover the odd-girth of the graph is also 2k + 1, i.e., there is no odd-cycle of length less than 2k + 1. Let  $S_0$  be the set of two threads of length 2k - 1 of  $G_0$ . In what follows, we build graph  $G_i$  from the graph  $G_{i-1}$  using the threads in  $S_{i-1}$  and we update our list of threads to have a new list  $S_i$  of (shorter) threads. When the length of threads in  $S_i$  is 1, that is when we have built  $G_k$ , we have our construction.

To build  $G_i$  from  $G_{i-1}$  and  $S_{i-1}$ . we introduce two operations:

**Copy thread:** For each thread in  $S_{i-1}$ , add a new thread of same length which connects its two ends but all internal vertices are new and distinct.

**Shorten thread:** For each thread  $T = v_0 v_1 \dots v_i$ ,  $T \in S_{i-1}$ , and its copy  $T = v_0 v'_1 \dots v'_{i-1} v_i$ add a new thread of length 2k + 1 - i connecting  $v_1$  and  $v'_{i-1}$ .

Note that, in operation Shorten thread, each original thread  $v_0v_1 \ldots v_{i-1}v_i$  and its copy,  $v_0v'_1 \ldots v'_{i-1}v_i$ , is shortened by two vertices. Add these two shortened threads  $(v_1 \ldots v_{i-1})$  and  $v'_1 \ldots v'_{i-1}$  to  $S_i$  (but not the thread which connects  $v_1$  and  $v'_{i-1}$ ).

The following three claims whose proofs we do not include imply the claim of the theorem:

1. The graph  $G_k$  is of odd-girth 2k + 1.

2. There is an odd-walk of length at most 2k - 1 between any pair u and v where u and v are either vertices of the original  $K_4$ , or a vertex of a selected thread at some step. 3. There are a total of  $2^{2k}$  such vertices.

Remark. Observe that the construction given here is also a partial 3-tree.

#### **3.2.2** The case r = k + 1

We propose the following subgraph of PC(2k) as one of possible answers for this case:

**Conjecture 3.2.4.** Every planar graph of odd-girth at least 2k + 3 admits a homomorphism to K(2k + 1, k).

For k = 1, since K(3, 1) is the complete graph on three vertices, the conjecture is equivalent to the Grötzsch theorem. For k = 2 the best known result is that of [12] where it is proved that every planar graph of odd-girth at least 9 admits a homomorphism to the Petersen graph (our conjecture claims that odd-girth at least 7 suffices). For larger values of k, the question is yet to study and for most cases the best known results are implied from stronger claims of mapping into  $C_{2k+1}$  (to be discussed in the next sections).

This conjecture would imply, in particular, the slightly weaker conjecture that:

**Conjecture 3.2.5.** If G is a planar graph of odd-girth at least 2k+3, then  $\chi_f(G) \leq 2+\frac{1}{k}$ .

The bound of  $2 + \frac{1}{k}$  is the best possible if possible at all. Indeed, extending a result of K. Jones [29], we have built planar graphs of odd-girth 2k + 1 whose independence ratio (i.e.,  $\frac{|V(G)|}{\alpha(G)}$ ) is arbitrarily close to  $2 + \frac{1}{k}$ . As this ratio is a lower bound for the fractional chromatic number, Conjecture 3.2.5, if true, would prove the best possible bound.

# **3.2.3** The case (r, k) = (5, 3) and the Coxeter graph

For the special case of (r, k) = (5, 3) we introduce the Coxeter graph as a possible answer. That is, a highly symmetric cubic graph of girth 7, first built by Coxeter. We follow a definition given in [19].

Recall that K(7,3) is a subgraph of PC(6), and if Conjecture 3.2.4 is true, then K(7,3) would, in particular, be a bound for  $\mathcal{P}_{11}(\subseteq \mathcal{P}_9)$ . The Coxeter graph, a subgraph of K(7,3) defined below, is a candidate for being a smaller bound in this case.

**Definition 3.2.6.** A *Fano plane* is a collection of (seven) triples, called lines, from a set of seven points where each pair of points appears exactly on one line. Given a Fano plane, the Coxeter graph is a graph built from K(7,3) by removing seven triples of the Fano plane.

**Conjecture 3.2.7.** Any planar graph of odd-girth at least 11 admits a homomorphism to the Coxeter graph.

Intuitively, Conjecture 3.2.4 suggests that each planar graph of odd-girth at least 9 admits a (7, 3)-coloring, that is to assign a set of three colors to each vertex from a set of seven available colors, such that each pair of adjacent vertices receive disjoint sets of colors. For the subclass of planar graphs of odd-girth at least 11, Conjecture 3.2.7 strengthens this by suggesting that: if the set of seven available colors are the vertices of a Fano plane, then such a (7, 3)-coloring can be given so that the triple of colors at each vertex are not co-linear.

So far, the best result toward Conjecture 3.2.7 is our result in [23] proving that  $\mathcal{P}_{17}$  is bounded by the Coxeter graph.

Since this conjecture depends on the Fano plane, we do not have a generalization for each value of k. However, as pointed out by P. Cameron, we may propose another conjecture based on the existence of an (11, 5, 2) design. That is a collection of 5-tuples from a set of 11 points where each pair appears exactly once. Removing such collection of 5-tuples from vertices of K(11, 5) we may expect that the remaining induced subgraph bounds the class  $\mathcal{P}_{15}$ .

These examples show how the study of Problem 3.2.2 may lead to interesting new theories on coloring graphs.

# **3.2.4** The case $r \ge 2k - 1$ and the circular chromatic number

Recall that PC(2k) contains, as a subgraph, the circulant cliques C(4k, 2k - 1) and C(2k+1, k). The former is the Möbius ladder of length 2k and the latter is the odd-cycle of length 2k + 1.

We propose them as answers of Problem 3.2.2 for r = 2k - 1 and r = 2k, noting that the latter is a well-known conjecture of Jaeger-Zhang.

**Conjecture 3.2.8.** Every planar graph of odd-girth at least 4k - 1 admits a homomorphism to C(4k, 2k - 1).

**Conjecture 3.2.9.** [30] Every planar graph of odd-girth at least 4k + 1 admits a homomorphism to C(2k + 1, k).

Conjecture 3.2.9 is well-known as the Jaeger-Zhang conjecture. Jaeger conjectured that every 4p connected graph admits an orientation where the difference between the number of incoming edges and the number of outgoing edges mod 2p + 1 is either +1 or -1. The restriction of this conjecture to planar graphs would be that every planar graph of girth at least 4p admits a homomorphism to  $C_{2p+1}$ . Zhang then proposed that odd-girth suffices for this conjecture. Several projects in support of this conjecture are completed. The best result so far is that of [32]; introducing the notion of odd-connectivity, the authors strengthen the original conjecture of Jaeger and then prove it for the smaller class of 6p-oddly-connected graphs. The restriction to planar graphs then implies:

**Theorem 3.2.10.** Every planar graph of odd-girth at least 6k+1 admits a homomorphism to C(2k+1,k).

The two conjectures introduced here can be combined to formulate a single conjecture in bounding the circular chromatic number of planar graphs of odd-girth 2k + 1.

**Conjecture 3.2.11.** Given a planar graph G of odd-girth 2k+1 we have  $\chi_c(G) \leq 2 + \frac{2}{k}$ .

If true, this bound would be the best possible as (recently) shown by H. Qi and X. Zhu.

# 3.3 Relation to edge-coloring

In this section we show that Conjecture 3.2.1 is equivalent to a restricted version of Conjecture 1.5.4.

Given a *d*-regular multi-graph G, d is an obvious lower bound for the edge-chromatic number of G. What we are interested in, is conditions which would imply that G has edge-chromatic number exactly d. This case is of special interest as it would imply a partition of the edges of G into perfect matchings. Consider a subset X of vertices which is of an odd order. Then, as discussed in Section 1.5, the value  $\frac{2|E(X)|}{|X|-1|}$  is a lower bound for the edge-chromatic number. Thus, one necessary condition would be  $\frac{2|E(X)|}{|X|-1|} \leq d$  for every X where X is subset of V(G) of an odd order. Denoting by  $\delta(X)$  the set of edges having exactly one end in X, and considering that G is d-regular, this is equivalent to  $|\delta(X)| \geq d$ . Thus a restricted version of Conjecture 1.5.4 is to say:

**Conjecture 3.3.1.** If G is a planar d-regular multi-graph such that any subset X of an odd number of vertices satisfies  $\delta(X) \ge d$ , then  $\chi'(G) = d$ .

The case d = 3 of this conjecture is Tait's reformulation of the four-color theorem (Theorem 1.6.2). Being a special case of Seymour's conjecture (Conjecture 1.5.4) it is widely known as Seymour's edge-coloring conjecture. Applying induction on d, thus using the four-color theorem, the cases  $d \leq 8$  are settled in [21, 11, 7, 8].

Here we show the following equivalence. A similar equivalence, using the notion of homomorphism of signed graphs, is given for even values of d in [?].

**Theorem 3.3.2.** The case d = 2k + 1 of Conjecture 3.3.1 is equivalent to the case 2k + 1 of Conjecture 3.2.1.

Proof. Let d = 2k + 1 and consider a *d*-regular planar multi-graph *G*. Then, since the number of odd-degree vertices is always even, |X| and  $|\delta(X)|$  are of same parity. Since  $\delta(X)$  corresponds to a cycle of  $G^D$ , the dual of *G*, the condition of Conjecture 3.3.1 are satisfied if and only if  $G^D$  is of odd-girth 2k + 1. Then, assuming Conjecture 3.2.1 holds for this value of *k*, there exists a mapping  $\phi$  of  $G^D$  to PC(2k). The mapping  $\phi$  induces an edge-coloring on  $E(G^D)$  and, therefore, by correspondence, also on E(G). Since edges incident to a same vertex correspond to a (2k + 1)-cycle in  $G^D$ , and by Corollary 2.2.2, the *d*-edge-coloring induced by  $\phi$  on *G* is a proper one.

For the converse, suppose G is a planar graph of odd-girth 2k + 1. Assuming Conjecture 3.3.1 holds for d = 2k + 1, we want to show that G admits a homomorphism to PC(2k). Applying the Folding lemma (Lemma 1.6.4), if necessary, we may assume that each face of G is a (2k+1)-cycle. Thus  $G^D$  is a (2k+1)-regular planar multi-graph which satisfies the conditions of Conjecture 3.3.1 and, therefore, admits a (2k+1)-edge-coloring. In such an edge-coloring, given an odd set X of vertices, each color class must intersect  $\delta(X)$  an odd number of times. Thus, associating the corresponding colors to the edges of G, we have a partition of the edges of G which satisfies the conditions of Theorem 3.0.1 (for PC(2k)). Hence, G admits a homomorphism to PC(2k).

# Chapter 4

# Bounding minor-closed families

### 4.1 Intersection of $Forb_m$ and $Forb_h$

In this chapter we consider a general question on homomorphism of minor-closed families of graphs. Our goal is to develop tools to answer the question in the case the minor-closed family is the class of partial *t*-trees.

Consider a finite set  $H = \{H_1, H_2, \ldots, H_i\}$  of connected graphs. Let  $Forb_h(H)$  be the class of graphs G such that no  $H_i$  admits a homomorphism to G. Observe that for  $H = \{K_n\}$  the class  $Forb_h(H)$  is the class of all  $K_n$ -free graphs and for  $H = \{C_{2k-1}\}$  it is the class of graphs of odd-girth at least 2k + 1.

Let  $\mathcal{C}$  be a minor-closed family of graphs. Note that by the graph minor theorem of Robertson and Seymour,  $\mathcal{C}$  can be presented in the form of  $Forb_m(M)$  for some finite set  $M = \{M_1, M_2, \ldots, M_j\}$  of graphs. A general question is to find a smallest graph in  $Forb_h(H)$  to which every graph in  $Forb_h(H) \cap Forb_m(M)$  admits a homomorphism.

For  $H = M = \{K_n\}$ ,  $Forb_h(H) \cap Forb_m(M)$  is the class  $Forb_m(M)$  and the question is to find a  $K_n$ -free graph to which any graph from  $Forb_m(M)$  admits a homomorphism to. The smallest of such graphs would be  $K_{n-1}$  if the Hadwiger conjecture is true.

When  $H = \{C_{2k-1}\}$  and  $M = \{K_{3,3}, K_5\}$ , the intersection  $Forb_h(H) \cap Forb_m(M)$  is the class of planar graphs of odd-girth at least 2k + 1 for which the best bound of odd-girth 2k + 1 is proposed to be the projective cube PC(2k) (see Theorem 3.2.3).

The fact that such a bound always exists follows from a general result of J. Nešetřil and P. Ossona De Mendez:

**Theorem 4.1.1.** [49] Given a finite set  $H = \{H_1, H_2, \ldots, H_i\}$  of connected graphs, and any finite set  $M = \{M_1, M_2, \ldots, M_j\}$  of graphs, there exists a graph in  $Forb_h(H)$  to which any graph in  $Forb_h(H) \cap Forb_m(M)$  admits a homomorphism.

For an illustration of Theorem 4.1.1 see Figure 4.1.

A natural question is then the following:

**Problem 4.1.2.** What is the smallest order of a bound in Theorem 4.1.1?

As mentioned above, the question captures some of the most difficult theorems and conjectures in graph theory. Our goal in this chapter is to give an algorithm which, given



Figure 4.1: Illustration of Theorem 4.1.1

a graph B of odd-girth 2k + 1 (i.e.,  $B \in Forb_h(C_{2k-1})$ ) and for  $\mathcal{PT}_t$  being the class of partial *t*-trees, decides, in polynomial time (with respect to the order of B), if B bounds  $Forb_h(C_{2k-1}) \cap \mathcal{PT}_t$ . The algorithm is an application of an "if and only if" theorem which can also be checked theoretically.

### 4.2 A subdivision of $2K_t$

By the definition, complete graphs of order t are the building blocks of t-trees. They will then play an important role in our development of the theory of homomorphism bounds for these classes in the format of weighted blocks.

A  $2K_t$  is a multi-graph where there are exactly two pairs of (multi) edges between each pair of vertices. Given a weight function w on the edges of  $K_t$  satisfying  $1 \le w(e) \le 2k$ for each edge e, recall that  $\overline{(K_t, w)}_{2k+1}$  is the graph built from  $2K_t$  by subdividing edges where the two edges corresponding to e are subdivided into threads of lengths w(e) and 2k + 1 - w(e) respectively. Furthermore, by claiming that  $(K_t, w)$  is (2k + 1)-wide first of all we, implicitly, imply that  $1 \le w(e) \le 2k$  for all edges, secondly that  $\overline{(K_t, w)}_{2k+1}$  is of odd-girth 2k + 1.

Observe that replacing w(uv) by 2k + 1 - w(uv) does not change the graph  $\overline{(K_t, w)}_{2k+1}$ 

and, therefore, for  $(K_t, w)$  to be (2k + 1)-wide is invariant under such a change. Given a cycle C of  $K_t$ , each edge uv could be replaced by either a path of length w(uv) or a path of length 2k + 1 - w(uv) in order to find a cycle corresponding to C in  $\overline{(K_t, w)}_{2k+1}$ . Thus, there are  $2^{|V(C)|}$  cycles in  $\overline{(K_t, w)}_{2k+1}$  corresponding to C, exactly half of which are odd-cycles. However, the length of a shortest odd-cycle of  $\overline{(K_t, w)}_{2k+1}$  can be determined much simpler than calculating the length of all such odd-cycles. We first establish couple of checks for a complete weighted triangle to be (2k + 1)-wide, then we show that the shortest odd-cycle of a complete weighted graph can be determined by considering its triangles only.

**Proposition 4.2.1.** Let  $(K_3, w)$  be a weighted triangle with weights  $1 \le a \le b \le c \le k$ . Then it is (2k + 1)-wide if and only if one of the following holds:

 $i \ a+b+c \ is \ odd \ and \ a+b+c \ge 2k+1, \ or$ 

ii a + b + c is even and  $a + b \ge c$ .

Proof. Let x, y and z be three vertices of  $K_3$ . Consider a cycle of  $\overline{(K_3, w)}_{2k+1}$ . If it uses only two of x, y, z, then it is of length exactly 2k + 1 by construction. Otherwise, of the eight cycles using all three of x, y, z the shortest cycle is of length a + b + c and exactly four are odd-cycles. If a + b + c is an odd number, then [i] applies. If a + b + c is an even number, then, since we have assumed  $a \le b \le c$ , the shortest odd-cycle of  $\overline{(K_3, w)}_{2k+1}$ containing all three vertices x, y, z is of length a + b + (2k + 1 - c) which is of length at least 2k + 1 as [ii] applies.  $\Box$ 

To apply Proposition 4.2.1 the weight of each edge must be at most k (rather than at most 2k). If a weighted triangle is given such that  $w(e) \leq 2k$  for all edges but for some edges we have  $w(e) \geq k + 1$ , then to apply the proposition we may simply replace the weight of each such edge e by 2k + 1 - w(e).

**Proposition 4.2.2.** Let  $(K_3, w)$  be a weighted triangle with weights  $1 \le a, b, c \le 2k$  such that a+b+c is odd. Let  $f_{2k+1}(a, b, c) = \frac{1}{2}(a+b+c-(2k+1))$ . Then  $(K_3, w)$  is (2k+1)-wide if and only if  $0 \le f_{2k+1}(a, b, c) \le \min\{a, b, c\}$ .

*Proof.* As before let x, y and z be three vertices of  $K_3$ . Assume  $\overline{(K_3, w)}_{2k+1}$  is of odd-girth 2k + 1, then, since a + b + c corresponds to length of an odd-cycle containing all three of x, y and z, we have  $f_{2k+1}(a, b, c) \ge 0$ . The three other odd-cycles of G containing all three of x, y and z have length (2k + 1) - a + (2k + 1) - b + c, a + (2k + 1) - b + (2k + 1) - c, (2k + 1) - a + b + (2k + 1) - c. Each of these cycles must be of length at least 2k + 1. Considering the first one, we have  $(2k + 1) - a + (2k + 1) - b + c \ge 2k + 1$  which is to say  $2c \ge a + b + c - (2k + 1)$  thus  $c \ge f_{2k+1}(a, b, c)$ . By considering the other two cycles, we get the other two required inequalities.

For the inverse, assume that  $f_{2k+1}(a, b, c)$  is within the given interval. If a + b + c is the shortest odd-cycle, then  $f_{2k+1}(a, b, c) \ge 0$  implies that the odd-girth of  $\overline{(\overline{K_3}, w)}_{2k+1}$  is 2k + 1. Otherwise, and without loss of generality, we may assume  $1 \le a \le b \le c$ . Then the length of a shortest odd-cycle of  $\overline{(\overline{K_3}, w)}_{2k+1}$  containing all three vertices x, y and z is  $a + (2k+1) - b + (2k+1) - c = 2a + (2k+1) - 2f_{2k+1}(a, b, c)$  which is at least 2k + 1since  $a \ge f_{2k+1}(a, b, c)$ .

To apply this Proposition 4.2.2, if the sum of the three weights of edges of a triangle uvw is not an odd number, then it would suffice to replace one of the weights with the complementary value of 2k + 1 - w(uv).

Next, we want to check if  $(K_t, w)$  is (2k+1)-wide. In the next theorem, we show that it is enough to apply either one of the previous propositions for all  $K_3$ -subgraphs of  $K_t$  together with the induced weights. Note that, when applying Proposition 4.2.2 on triangles of a  $(K_t, w)$ , given an edge e of weight a, one may consider the weight a in some triangles containing e and the weight 2k + 1 - a in some other triangles containing e in order to have an odd sum of weights in each of the triangles containing e.

**Theorem 4.2.3.** A complete weighted graph  $(K_t, w)$  is (2k + 1)-wide if and only if each of its induced triangles is (2k + 1)-wide.

*Proof.* If  $(K_t, w)$  is (2k + 1)-wide, then every induced triangle is also (2k + 1)-wide, so we only need to prove the converse.

By contradiction, assume every induced triangle of  $(K_t, w)$  is (2k + 1)-wide but itself is not. Then  $\overline{(K_t, w)}_{2k+1}$  has an odd-cycle C of length smaller than 2k + 1. Let C' be the cycle of  $K_t$  corresponding to C and let  $x_1, x_2, \ldots, x_r$  be the vertices of C' in cyclic order. Among all odd-cycles in  $\overline{(K_t, w)}_{2k+1}$  of length less than 2k + 1, let C be one for which ris minimized. By the assumption on the triangles of  $(K_t, w)$ , we have  $r \ge 4$ . Thus, C'has a proper chord (that is, a pair  $x_i x_j$  which is not an edge of C'). Let  $P_1$  and  $P_2$  be the two paths connecting  $x_i$  and  $x_j$  in C'. Let  $Q_1$  and  $Q_2$  be the two threads of  $\overline{(K_t, w)}_{2k+1}$ connecting  $x_i$  and  $x_j$ , noting that the total length of  $Q_1$  and  $Q_2$  is 2k + 1. Since the sum of lengths of  $P_1$  and  $P_2$  is odd (that is the length of C), we may assume (by relabeling indices, if needed) that the length of  $P_i$  together with the length of  $Q_i$  is an odd number for i = 1, 2. Thus, we obtain two odd-cycles in  $\overline{(K_t, w)}_{2k+1}$ , corresponding to  $P_i \cup Q_i$ , i = 1, 2. The total length of these cycles is 2k + 1 + |V(C)|. Since C is of length smaller than 2k + 1, at least one of these two cycles is of length smaller than 2k + 1 but it uses less vertices of  $K_t$ , contradicting the minimality of r.

Let (B, w) be a weighted graph and let  $\mathcal{W}$  be a set of weighted (t + 1)-cliques of (B, w). Let  $\mathcal{W}'$  be the collection of all weighted t-cliques each a subclique of a weighted (t + 1)clique of  $\mathcal{W}$ . Observe that a member of  $\mathcal{W}'$  could be derived as a subclique of more than one clique in  $\mathcal{W}$ .

Given a t-clique X of  $\mathcal{W}'$ , we say X is fully (t, k)-observed in  $\mathcal{W}$  if for any extension of X into a weighted (t + 1)-clique which is (2k + 1)-wide, there is a member of  $\mathcal{W}$  which realizes this extension. If all members of  $\mathcal{W}'$  are fully (t, k)-observed, then  $\mathcal{W}$  is said to be (t, k)-closed.

**Theorem 4.2.4.** Let (B', w) be a partially B-weighted graph which is (2k + 1)-wide. If a nonempty collection  $\mathcal{W}$  of (t + 1)-cliques of B' is (t, k)-closed, then for any (2k + 1)-wide complete graph  $(K_t, w)$  there is an isomorphic copy of  $(K_t, w)$  in  $\mathcal{W}'$ . Proof. Let  $X = (K_t, w)$  be an element of  $\mathcal{W}'$ . We construct  $X^+$  from X by adding a vertex u to X which is joined to each vertex of X by an edge of weight k. We claims that  $X^+$  is (2k + 1)-wide. That is because any cycle using vertex u (in the corresponding graph) is of length at least 2k + 1 and every other cycle is also a cycle of X, thus by the assumption on B, is of length at least 2k + 1. Thus  $X^+$  is also a member of  $\mathcal{W}$ . Let X' be a weighted complete graph on t vertices obtained from  $X^+$  by removing a vertex other than u.

Let  $K_{t+1}^k$  be weighted complete graph on t+1 vertices where all edge-weights equal to k. By repeated application of building  $X^+$  and X' from X we conclude that: First of all the weighted  $K_{t+1}^k$  is a member of  $\mathcal{W}$ . Secondly that from any choice of X, as long as X is (2k+1)-wide, by repeated application of the operation described above, one can get to  $K_{t+1}^k$ , and, furthermore, that this is a reversible operation.

Therefore, as  $\mathcal{W}'$  is not empty, by starting from an arbitrary element X we can get to each  $(K_t, w)$  which is (2k + 1)-wide.

# 4.3 Bounding partial *t*-trees

Let  $\mathcal{PT}_t$  be the class of partial *t*-trees, and  $\mathcal{PT}_{t,2k+1}$  be the subclass of partial *t*-trees of odd-girth at least 2k + 1. As a special case of Problem 4.1.2 we would like to search for a smallest graph B of odd-girth 2k + 1 to which there exists a homomorphism from every member of  $\mathcal{PT}_{t,2k+1}$ . To this end we provide, in this section, a necessary and sufficient condition for a graph B of odd-girth 2k + 1 to bound  $\mathcal{PT}_{t,2k+1}$ .

**Theorem 4.3.1.** A graph B of odd-girth 2k + 1 bounds the class of partial t-trees of odd-girth at least 2k + 1 if and only if there exists a (2k + 1)-wide partially B-weighted graph  $\hat{B}$  with a nonempty set  $\mathcal{W}$  of (t + 1)-cliques of  $\hat{B}$  that is (t, k)-closed.

Proof. First assume that  $\hat{B}$  and a (t, k)-closed collection  $\mathcal{W}$  of its (t+1)-cliques exist. Let G be a partial t-tree of odd-girth 2k+1. We want to prove that G admits a homomorphism to B. We prove a stronger statement as follows. Let  $\tilde{G}$  be a t-tree which contains G as a spanning subgraph. Consider the weighted graph  $(\tilde{G}, \varphi)$  where  $\varphi(xy) = \min\{d_G(x, y), k\}$ . We claim that  $(\tilde{G}, \varphi)$  admits a (weight preserving) homomorphism to  $\hat{B}$ .

To prove this, we first show that  $(\tilde{G}, \varphi)$  is (2k + 1)-wide. Consider  $\overline{(\tilde{G}, \varphi)}_{2k+1}$  and let  $\tilde{C}$  be an odd-cycle of this graph. We need to show that  $\tilde{G}$  is of length at least 2k + 1. Let C be a cycle of G which corresponds to  $\tilde{C}$ . Recall that each edge of C corresponds to two threads in  $\overline{(\tilde{G}, \varphi)}_{2k+1}$ , one of length  $\varphi(e)$ , another of length  $2k + 1 - \varphi(e)$ . Thus there are  $2^{|C|}$  cycles in  $\overline{(\tilde{G}, \varphi)}_{2k+1}$  corresponding to C. If  $\tilde{C}$  is a cycle where each edge of C is replaced by a thread of length strictly less than k, then the length of each such thread is the distance (in G) of its end points. In this case length of  $\tilde{C}$  is equal to the length of C, which implies that C is an odd-cycle of G. Since G has no odd-cycle of length less than 2k + 1,  $\tilde{C}$  is also of length at least 2k + 1. If two edges of C are replaced by threads of length at least k, then the length of  $\tilde{C}$  is at least 2k + 1. Finally suppose exactly one thread, connecting x and y, corresponds to a path of length at least k. If this thread is of length  $2k + 1 - \varphi(xy)$ , then the sum of lengths of other threads forming  $\tilde{C}$  is, by the

triangular inequality, at least  $\varphi(xy)$  and thus C is of length at least 2k + 1. If the x - y thread is of length exactly k or k + 1, then  $\varphi(xy) = k$  which means the distance of x and y is at least k, and the same argument as before applies.

Now, to prove our claim, consider a sequence of (t+1)-cliques which are used to build  $\hat{G}$ . Let  $X_1$  be the first of these cliques. As  $X_1$  together with the edge-weights induced by  $\varphi$  is (2k+1)-wide, and by Theorem 4.2.4, there is an isomorphic copy of  $(X_1, \varphi)$  in  $\mathcal{W}'$ . Let  $\rho$  be this isomorphism. Then  $\rho$  can easily be extended to a homomorphism of  $(\tilde{G}, \varphi)$  to  $\hat{B}$  using the sequence of (t+1)-cliques and the fact that  $\mathcal{W}$  is (t, k)-closed. This proves that the conditions of the theorem are sufficient for B to be a bound.

Next, we shall prove that the conditions are also necessary. Consider a graph B of odd-girth 2k + 1 which admits a homomorphism from any partial t-tree of odd-girth at least 2k + 1. Our aim is to prove the existence of a nonempty set  $\mathcal{W}$  of (t + 1)-cliques of the complete B-weighted graph on V(B) which is a (t, k)-closed set. The partially B-weighted graph  $\hat{B}$  is then obtained from the edges in the cliques of  $\mathcal{W}$ .

Let  $\mathcal{PT}_{t,2k+1}$  be the collection of all partial *t*-trees of odd-girth at least 2k + 1. Define  $\mathcal{WPT}_{t,2k+1}$  to be the class of weighted partial *t*-trees (G, w) satisfying the following conditions:

- The subgraph  $G_1$  of edges of weight 1 is a connected spanning subgraph of G which is of odd-girth at least 2k + 1.
- (G, w) is partially  $G_1$ -weighted graph, i.e., for each edge xy of G, w(xy) is the distance in  $G_1$  of x and y.
- For each edge  $xy \in E(G)$  of weight at least 2, there exists a (2k + 1)-cycle of  $G_1$  which contains both x and y.

Observe that each (2k+1)-wide weighted complete graph  $(K_t, w)$  is a member of  $WPT_{t,2k+1}$ , thus this is a nonempty set of graphs.

Let  $B^*$  be the complete *B*-weighted graph on V(B). Given a member (G, w) of  $\mathcal{WPT}_{t,2k+1}$ , since  $G_1$  is of odd-girth at least 2k+1 and by our assumption, it admits a homomorphism to *B*. By Theorem 1.11.5 any such homomorphism is also a homomorphism of (G, w) to  $B^*$ .

Observe that most members of  $WPT_{t,2k+1}$  contain many (t+1)-cliques and that in the mapping of an element (G, w) of  $B^*$ , the image of any such clique must be a (t+1)-clique of  $B^*$ . Let W be a minimal set of (t+1)-cliques of  $B^*$  satisfying the following:

Each element (G, w) of  $\mathcal{WPT}_{t,2k+1}$  admits a homomorphism  $\varphi$  to  $B^*$  where the images of all (t+1)-cliques of G belong to the set  $\mathcal{W}$  of (t+1)-cliques of  $B^*$ .

Let us say a mapping of (G, w) is a mapping to  $(B^*, W)$  if the images of all (t+1)-cliques of G belong to W. Minimality of W implies that for any (t+1)-clique W of W, there exists a weighted graph  $(G_W, w_W)$  of  $W \mathcal{PT}_{t,2k+1}$  for which any mapping to  $(B^*, W)$  must map at least one (t+1)-clique of  $G_W$  onto W.

We claim that  $\mathcal{W}$  is (t, k)-closed. Let W be a (t+1)-clique of G. For a vertex v of W, let W' be an extension of W - v into a (2k + 1)-wide (t + 1)-clique. Consider the weighted graph  $(G_W, w_W)$  as defined above. For any t-clique of this graph isomorphic to W - v,

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# (B is not a bound)

and for any such an isomorphism, add a vertex v and using it, build an isomorphic copy of W'. Furthermore, for each newly added edge e = xy of weight p, add two x - y paths, one of weight p and another of weight 2k + 1 - p.

Let  $(\hat{G}, \hat{w})$  be the resulting weighted graph. It can be checked that  $(\hat{G}, \hat{w}) \in \mathcal{WPT}_{t,2k+1}$ . Thus, by our choice of  $\mathcal{W}$ , there exists a homomorphism  $\varphi$  of  $(\hat{G}, w)$  to  $(B^*, \mathcal{W})$ . Recall that, this is a homomorphism of  $(\hat{G}, w)$  to  $B^*$  where all (t+1)-cliques of  $(\hat{G}, w)$  are mapped to elements of  $\mathcal{W}$ . By the choice of  $(G_W, w_W)$ , at least one such (t+1)-clique is mapped to W. Considering all extensions W' over this copy, the image of at least one extension W' provides the extension we are looking for.

#### 4.3.1 Algorithm

Theorem 4.3.1 implies an algorithm which, for a fixed pair t and k of positive integers, decides, in polynomial-time, if a given graph of odd-girth 2k+1 bounds the class  $\mathcal{PT}_{t,2k+1}$  of partial t-trees of odd-girth 2k + 1. The algorithm is given in the following whose verification of validity is left to the reader.

```
Deciding whether a graph B of odd-girth 2k + 1 bounds \mathcal{PT}_{t,2k+1}.
```

**Input:** A graph B of odd-girth 2k + 1.

- 1: Compute the distance function  $d_B$  of B.
- 2: Let  $(B, d_B)$  be the k-partial distance graph of B obtained from the complete distance graph of B by removing all edges of weight more than k.
- 3: Compute the set  $\mathcal{W}$  of all weighted (t+1)-cliques.
- 4: if  $\mathcal{W} = \emptyset$  then
- 5: return NO
- 6: end if
- 7: for  $W \in \mathcal{W}$  and any vertex v of W do
- 8: Check if every extension of W v to a (t + 1)-clique of odd-girth 2k + 1 has a realization in W.
- 9: if not then
- 10: Delete W from the list of cliques, reset W and go back to Step 4.
- 11: **end if**
- 12: **end for**
- 13: if  $\mathcal{W}$  is nonempty then
- 14: return YES # (B is a bound, and in the weighted graph induced by all edges in cliques of W, the edges of weight 1 induce a subgraph of B which is also a bound.)
  15: end if

# 4.4 Bounding partial 3-trees

As an application of Theorem 4.3.1, we show, in this section, that PC(2k), the projective cube of dimension 2k, bounds the class  $\mathcal{PT}_{3,2k+1}$ , that is the class of partial 3-trees of odd-girth at least 2k + 1. Since the graphs built in the proof of Theorem 3.2.3 are also

partial 3-trees, we conclude that projective cubes are indeed the optimal homomorphism bound of odd-girth 2k + 1 for  $\mathcal{PT}_{3,2k+1}$ .

Let  $PC(2k)^*$  be the complete PC(2k)-weighted graph.

**Theorem 4.4.1.** The set  $\mathcal{W}$  of all weighted 4-cliques of  $PC(2k)^*$  is (3,k)-closed.

*Proof.* Considering Theorem 2.7.1 it is enough to show that  $PC(2k)^*$  contains an isomorphic copy of each (2k + 1)-wide  $(K_4, w)$ .

Recall that each edge xy of  $PC(2k)^*$  is associated with a pair of complementary subsets of  $\{e_1, e_2, \ldots, e_{2k}, J\}$ . If, in the poset labeling of PC(2k), vertices x and y are labeled  $\{A, \overline{A}\}$  and  $\{B, \overline{B}\}$ , then the edge xy is labeled  $\{A \oplus B, A \oplus \overline{B}\}$ . Furthermore, note that the orders of  $A \oplus B$  and its complement correspond to  $d_{PC(2k)}(x, y)$  and  $2k + 1 - d_{PC(2k)}(x, y)$ . Consider a (2k + 1)-wide  $(K_4, w)$  on vertices x, y, z, u and assume w(xy) = a', w(xz) = b', w(xu) = c', w(zu) = a, w(yu) = b, w(yz) = c. By changing the weight of an edge with the complementary weight of 2k + 1 - w, if necessary, we assume that: (i) The elements of each pair  $\{a, a'\}, \{b, b'\}$   $\{c, c'\}$  are of the same parity, (ii) the sum a + b + c is odd. Thus, the four triangles of  $(K_4, w)$  correspond to a + b + c, a + b' + c', a' + b + c' and a' + b' + c, each being an odd number.

Let  $f_x = \frac{1}{2}(a+b+c-2k-1)$ ,  $f_y = \frac{1}{2}(a+b'+c'-2k-1)$ ,  $f_z = \frac{1}{2}(a'+b+c'-2k-1)$  and  $f_u = \frac{1}{2}(a'+b'+c-2k-1)$ . Note that since each of the four triangles are of odd-girth 2k+1 and by Proposition 4.2.2, each of the four values  $f_v, v \in \{x, y, z, u\}$  is nonnegative and is smaller than the minimum of the three elements defining it.

Our goal is to associate a subset  $A_{vv'}$  of  $\{e_1, e_2, \ldots, e_{2k}, J\}$  to each edge vv' of the  $K_4$  such that: 1.  $|A_{vv'}|$  is either w(vv') or 2k + 1 - w(vv') and 2. for each triangle of uvw of the  $K_4$ , each element of  $\{e_1, e_2, \ldots, e_{2k}, J\}$  appears either in exactly one of  $A_{uv}$ ,  $A_{uw}$  and  $A_{vw}$  or in all three of them (in order to satisfy Theorem 2.6.2).

Assume without loss of generality that  $f_x = \min\{f_x, f_y, f_z, f_u\}$ . Let  $S_x$  be a subset of order  $f_x$  of  $S = \{e_1, e_2, \ldots, e_{2k}, J\}$ . In what follows we will add elements of  $S_x$  to all six sets. We first define  $A_{yz}$ ,  $A_{yu}$  and  $A_{zu}$ . To do this we partition the set  $S \setminus S_x$  into three sets  $S_{yz}$ ,  $S_{yu}$ ,  $S_{zu}$  of order  $|S_1| = a - f_x$ ,  $|S_2| = b - f_x$  and  $|S_3| = c - f_x$  respectively. Then we define  $A_{yz} = S_{yz} \cup S_x$ ,  $A_{yu} = S_{yu} \cup S_x$  and  $A_{zu} = S_{zu} \cup S_x$ . Next we consider three subsets  $S'_y$ ,  $S'_z$  and  $S'_u$  such that  $S'_y \subset A_{zu}$  and  $|S'_y| = f_y - f_x$ ,  $S'_z \subset A_{yu}$  and  $|S'_z| = f_z - f_x$ ,  $S'_u \subset A_{zu}$  and  $|S'_u| = f_u - f_x$ . The choice of these three sets implies, in particular, that these three sets are disjoint. Then, the set  $S_y = S'_y \cup S_x$  is of order  $f_y$ ,  $S_z = S'_z \cup S_x$  is of order  $f_z$  and  $S_u = S'_u \cup S_x$  is of order  $f_u$ .

We now complete our assignment of labels to the edges as follows:  $A_{ux} = (A_{yz} - S'_u) \cup S_y \cup S_z$ . Since  $c - c' = f_y + f_z - (f_x + f_u)$ , and since  $A_{yz}$  is of order c,  $A_{ux}$  is of order c'. Similarly we define  $A_{zx} = (A_{yu} - S'_z) \cup S_y \cup S_u$  implying  $|A_{zx}| = b'$  and  $A_{yx} = (A_{zu} - S'_y) \cup S_z \cup S_u$  with  $|A_{zx}| = a'$ .

Our choice of labels for the edges implies that the sum of the elements of the labels of the edges of a cycle in  $K_4$  is 0 (in  $\mathbb{Z}_2^{2k}$ ). Thus, to embed the given  $(K_4, w)$  in the PC(2k)-weighted complete graph, we let x be any vertex. To define y, z and u, respectively, we add to x all elements of  $A_{xy}$ ,  $A_{xz}$  and  $A_{xu}$ . Then y, z and u are well-defined and their distance from each other or from x is determined by the size of the corresponding set, completing the proof.

The first claim of the following theorem is then an immediate corollary of Theorem 4.3.1 and the previous theorem. The second part follows from further analysis of the constructions given in the proof of Theorem 3.2.3.

**Theorem 4.4.2.** The projective cube PC(2k) bounds the class of partial 3-trees of oddgirth at least 2k + 1. Furthermore, among such bounds of odd-girth 2k + 1, the projective cube has smallest number of vertices and edges.

Remark. In the proof of Theorem 4.4.1, we gave a solution where the labels of all edges contain all elements of  $S_x$ . This is not a necessity and in general other kinds of labeling can be provided. This implies that being triple-transitive is the limit of transitivity of PC(2k), meaning that one can find isomorphic 4-cliques of the PC(2k)-weighted complete graph which cannot map to each other by an automorphism of PC(2k).

# 4.5 Bounding $K_4$ -minor-free graphs

Recall that the class of  $K_4$ -minor graphs is also known under two other notations: 1. It is the class of series parallel graphs, hence we may use SPG to denote the whole class and  $SPG_{2k+1}$  to denote the class of  $K_4$ -minor-free graphs of odd-girth at least 2k + 1. 2. It is the class of partial 2-trees. Thus applying Theorem 4.3.1 for t = 2 we have the following special case:

**Theorem 4.5.1.** A graph B of odd-girth 2k + 1 bounds the class  $SPG_{2k+1}$  if and only if there exists a partially B-weighted graph (B', w) such that each edge e = xy of weight w(e) = p of B' satisfies the following condition:

If the triangle  $(K_3, w)$  of edge weights  $\{p, q, r\}$ ,  $p, q, r \leq k$ , is of (2k + 1)-wide, then B' contains vertices z and w adjacent to both x and y such that w(zx) = q, w(zy) = r, w(wx) = r and w(wy) = q.

*Proof.* It is enough to note that the set of triangles of B' used to fulfill the condition of the theorem forms a set of 3-cliques (of (B', w)) which is (2, k)-closed. Thus Theorem 4.3.1 can be applied.

As an application we give two classes of bounds for  $K_4$ -minor-free graphs of odd-girth 2k+1. The first, being a Kneser graph, helps determining the exact value of the fractional chromatic number of a  $K_4$ -minor-free graph of given odd-girth. The second, being nearly an optimal bound, gives an application to edge-coloring of  $K_4$ -minor-free graphs. We note since  $K_4$ -minor-free graphs are in particular of partial 3-trees, and by Theorem 4.4.2, PC(2k) bounds the class of  $K_4$ -minor-free graphs of odd-girth 2k+1. The bounds we give next are both subgraphs of PC(2k). In each case we prove that the set of all triangles of the corresponding complete weighted graph is (2, 2k + 1)-closed. In an optimal bound of odd-girth 2k + 1 perhaps only a proper subclass of triangles satisfy this property, indeed we will see that this is the case for  $SPG_7$ .

# **4.5.1** Bounding by K(2k + 1, k)

**Theorem 4.5.2.** The Kneser graph K(2k+1,k) bounds  $SP_{2k+1}$ .

*Proof.* Consider the K(2k+1, k)-weighted complete graph and let  $\mathcal{W}$  be the set of triangles of this graph. It would be enough to show that  $\mathcal{W}$  is (2, k)-closed. Given vertices A and B of K(2k+1, k) their distance is only a function of the size of  $A \cap B$  and, therefore, K(2k+1, k) is distance-transitive. Hence, it is enough to show that for each (2k+1)-wide  $(K_3, w)$  there is an isomorphic copy of it in the K(2k+1, k)-weighted complete graph.

Given any such triangle  $(K_3, w)$  consider an extension by adding a vertex u at distance k from all three vertices. Observe that this is a weighted complete graph on four vertices and its corresponding graph  $(K_4, w)$  is also (2k+1)-wide. Therefore, by Theorem 4.4.1, it is a subgraph of the PC(2k)-weighted complete graph. Since all three vertices of  $(K_3, w)$  are at distance k from u, and since the set of vertices at distance k from a fixed vertex of PC(2k) induces a copy of K(2k+1, k), there is an isomorphic copy of any (2k+1)-wide  $(K_3, w)$  in K(2k+1, k).

By the definition of fractional chromatic number, the following is an immediate corollary of Theorem 4.5.2 and the fact that  $\chi_f(C_{2k+1}) = 2 + \frac{1}{k}$ .

**Corollary 4.5.3.** If G is a K<sub>4</sub>-minor-free graph of odd-girth 2k+1, then  $\chi_f(G) = 2 + \frac{1}{k}$ .

#### 4.5.2 Bounding by augmented toroidal grids

Recall that the augmented toroidal gird, AT(2k, 2k) is built from the Cartesian product  $C_{2k} \Box C_{2k}$  by adding an edge between each pair of antipodal vertices (vertices at distance 2k). We saw in Theorem 2.3.3 that this is a subgraph of PC(2k). We have proved in [4] that the set of all weighted triangles of the AT(2k, 2k)-weighted complete graph is (2, k)closed. Thus implying that AT(2k, 2k) is a bound of odd-girth 2k + 1 for  $SPG_{2k+1}$ . The key properties of AT(2k, 2k) needed for such a proof are stated below.

Recall that the toroidal grid T(2a, 2b) is of diameter a+b and, furthermore, given a vertex v, there is a unique vertex at distance a+b of v which is, therefore, called the *antipodal* of v; we denote it by  $\overline{v}$ .

**Lemma 4.5.4.** Given a positive integer k, the augmented toroidal grid AT(2k, 2k) has the following properties:

- (i) It is vertex-transitive.
- (ii) Any two vertices of it belong to a common (2k + 1)-cycle, it is of odd-girth 2k + 1and hence has diameter exactly k.
- (iii) For any pair u and v of vertices:

 $d_{AT(2k,2k)}(u,v) = \min\{d_{T(2k,2k)}(u,v), 2k+1 - d_{T(2k,2k)}(u,v)\}.$ 

(iv) For any vertex u and any integer d,  $1 \le d \le k$ , the neighborhood of u at distance d in AT(2k, 2k) is the set

$$N^{d}_{AT(2k,2k)}(u) = N^{d}_{T(2k,2k)}(u) \cup N^{d-1}_{T(2k,2k)}(\overline{u}).$$

**Theorem 4.5.5.** Given a positive integer k, the set of all weighted triangles in the AT(2k, 2k)-weighted complete graph is (2, k)-closed.

Therefore we have:

**Theorem 4.5.6.** For any positive integer k, the augmented toroidal grid AT(2k, 2k) bounds SP2k + 1.

In a joint project with W. He and Q. Sun, we have built  $K_4$ -minor-free graphs of odd-girth 2k + 1 whose  $(2k - 1)^{th}$ -walk power has a clique of order  $\binom{k+1}{2}$ . Thus, a smallest graph of odd-girth 2k + 1 bounding  $SPG_{2k+1}$  is of order at least  $\binom{k+1}{2}$ . As AT(2k, 2k) has  $4k^2$  vertices, it is nearly an optimal bound.

# 4.5.3 Optimal bounds for SPG<sub>5</sub> and SPG<sub>7</sub>

The graph  $C_8^{++}$  obtained from  $C_8$  by adding two main diagonals as in the Figure 4.3 is a triangle-free bound for  $SPG_5$ . Furthermore, this is the optimal triangle-free bound for this class (see [4]) in terms of number of vertices and edges. Being a subgraph of C(8,3)this claim is a strengthening of the fact that  $\frac{8}{3}$  is the best upper bound for the circular chromatic number of graphs in  $SPG_5$  (see [26, 50, 51]). On the other hand, this graph is also obtained from the Petersen graph (K(5,2)) by deleting two adjacent vertices. Thus, this claim is also a strengthening of Theorem 4.5.2 for the specific case of k = 2.

An optimal bound of odd-girth 7 for the class  $SPG_7$  is the graph  $X_{15}$  on 15 vertices presented in two different ways in Figure 4.2 (see [4] for a proof). As  $\chi_c(X_{15}) = \frac{5}{2}$  and  $\chi_f(X_{15}) = \frac{7}{3}$ , this result is a simultaneous improvement of the best upper bounds for circular and fractional chromatic numbers of series-parallel graphs of odd-girth 7.



Figure 4.2: Two drawings of the graph  $X_{15}$ , an optimal 15-vertex bound for SP7.

For larger values of odd-girth we do not, yet, know the optimal bound of odd-girth 2k + 1 for  $SPG_{2k+1}$ . But we expect that: (i.) It would be of order  $(k + 1)^2 - 1$ , (ii.) That it would provide a simultaneous improvement on the best known bounds for the fractional and circular chromatic numbers of series-parallel graphs of odd-girth 2k+1. These bounds also provide improvements on edge-coloring results as we will see in the next section.

### 4.6 Applications to edge-colourings

Using the proof technique of Theorem 3.3.2 and since we have verified Conjecture 3.2.1 for partial 3-trees, we have the following result.

**Theorem 4.6.1.** Let G be a planar (2d + 1)-regular multi-graph such that any subset X of an odd number of vertices satisfies  $\delta(X) \ge d$  and that its dual is a partial 3-tree, then  $\chi'(G) = 2d + 1$ .

As  $K_4$  is self-dual, dual of a (plane)  $K_4$ -minor-free graph is also  $K_4$ -minor-free. Thus, Theorem 4.6.2 in particular implies:

**Theorem 4.6.2.** If G is a (2d+1)-regular plane multi-graph with no  $K_4$ -minor such that any subset X of an odd number of vertices satisfies  $\delta(X) \ge d$ , then  $\chi'(G) = 2d + 1$ .

This theorem is only an application of the fact that  $SPG_{2k+1}$  is bounded by PC(2k). However, as pointed out in the previous subsection, a number of subgraphs of PC(2k) bounds  $SPG_{2k+1}$ . One may employ these bounds to obtain improved results on edgecoloring of  $K_4$ -minor-free regular multi-graphs. In the following, we give examples of such improvements.

Consider an edge-colouring of  $C_8^{++}$  given in Figure 4.3. Note that this is a coloring induced by PC(4) viewed as the Cayley graph  $(\mathbb{Z}_2^4, S_4 = \{e_1, e_2, e_3, e_4, J\}).$ 



Figure 4.3: A 5-edge-coloured  $C_8^{++}$ .

Note that there are only four 5-cycles in  $C_8^{++}$ , and there are only two different cyclic orders of edge-colours induced by these four 5-cycles. Then, using the technique of the proof of Theorem 3.3.2, we can prove the following.

**Theorem 4.6.3.** Let G be a 5-regular plane and  $K_4$ -minor-free multi-graph where for each set X of odd number of vertices  $\delta(X) \ge 5$ . Let  $\{c_1, c_2, c_3, c_4, c_5\}$  be a set of five colours. Then, one can colour the edges of G such that at each vertex, the cyclic order of colours is either  $c_1, c_2, c_3, c_4, c_5$  or  $c_1, c_4, c_5, c_2, c_3$ .

*Proof.* As in the proof of Theorem 3.0.1 it is enough to consider a mapping of the dual of G (which is of odd-girth 5) to  $C_8^{++}$ .

A similar result can be stated for 7-edge-coloring of 7-regular  $K_4$ -minor-free multi-graphs using the mapping to  $X_{15}$ . For the general case of (2k + 1)-regular multi-graphs with no  $K_4$ -minor, we consider the mapping to augmented toroidal grids. We observe that any (2k + 1)-cycle of AT(2k, 2k) uses exactly k horizontal edges, k vertical edges and an edge connecting antipodal pairs of T(2k, 2k). Furthermore, the set of horizontal edges, in their order of appearance on the cycle, induces a cyclic order of  $e_1, e_2, \ldots, e_k$  and similarly the set of vertical edges induces a cyclic order of  $e_{k+1}, e_{k+2}, \ldots, e_{2k}$ . Thus, we can derive the following definition of special (2k + 1)-edge-colourings.

Given k, let  $B = b_1, b_2, \ldots, b_k$  be a sequence of k distinct colours in the family of blue colours and let  $R = r_1, r_2, \ldots, r_k$  be a sequence of k distinct colours in the family of red colours. Given a (2k + 1)-regular plane multi-graph G, we say that G is (B, R)-edgecolourable if it can be properly edge-coloured using colours from B, R and a unique green colour such that at each vertex v, the cyclic ordering of the blue colours (respectively red) around v always induces the same cyclic order as in B (in R, respectively).

**Theorem 4.6.4.** Let B and R be two sequences of blue and red colours such that |B| = |R| = k. Then, every plane  $K_4$ -minor-free (2k + 1)-regular multi-graph which is (2k + 1)-edge-colorable is also (B, R)-edge-colourable.

# Chapter 5

# Experience, possible projects and future work

In terms of my experience of working with graduate students I would like to point out that I have collaborated with five ph. D. students during my post doctoral positions (J. Macdonald, B. Seamone, F. Foucaud, P. Valicov and A. Parreau). Since I have began my position at CNRS, I have co-supervised two Ph.D. students: Q. Sun and M. Abi Aad. One of my students (Q. Sun) wrote part of his thesis on the subject of this work. Furthermore, S. Sen who did a joint master with me and E. Sopena, and continued Ph.D. with Eric, did also work with me during his Ph.D. on the subject of this work and our results formed part of his thesis.

In my own view, the beauty of this work is that in one or two simple questions it captures a number of classic theories and it relates to several other theories. Furthermore, by introducing the analogue question for the case of signed bipartite graphs (to briefly mention in the next section) we reach to an ocean of untouched, but highly motivational, problems each of which can be subject of a Ph.D. thesis.

In the following I present only a selection of projects that I already have ideas on how to lead them to success. Of course the development of new ideas along the work would be the highest point of achievement.

# 5.1 Project on graphs

One of the motivating conjectures of this project is the following:

**Conjecture.** Any  $K_5$ -minor free graph of odd-girth at least 2k + 1 admits a homomorphism to PC(2k).

Being the four-color theorem for k = 1, the difficulty level of the conjecture itself is at least as high as the 4CT. But below are some outline of possible projects motivated by such an ambitious conjecture.

1. To prove a support for the conjecture by relaxing the girth condition. From the result of [32] we now know that planar graphs of odd-girth 6k + 1 map to  $C_{2k+1}$  which is a subgraph of PC(2k). Thus we would look for a value of odd-girth, noticeably smaller

than 6k, which would work on the place of 2k + 1.

2. In this work we have shown that the conjecture is true if the condition of no  $K_5$ -minor is relaxed by considering partial 3-trees, thus in particular, the conjecture is verified for the class of  $K_4$ -minor free graphs. On the other hand it is believed that condition of no  $K_5$ -minor can be replaced with a weaker condition of no odd- $K_5$ -minor, thus claiming same result for a larger class of graphs. A suitable project in this regard is to prove the conjecture for the class of graphs with no odd- $K_4$ . The definition of odd- $K_n$ -free is based on the notion of signed graphs which we only mention briefly in the next section. But we can define an odd- $K_4$  as any subdivision of  $K_4$  where each of the four triangles have become an odd-cycle. Then a graph is odd- $K_4$ -free if it has no odd- $K_4$  as a subgraph. A main idea here would be to use decomposition theorem of B. Gerard for the class of graphs with no odd- $K_4$ . This decomposition is based on notion of signed graphs.

**3**. We have shown in this work that PC(2k) is the smallest graph of odd-girth 2k+1 which bounds the class of partial 3-trees of odd-girth 2k + 1. The similar question of finding smallest graph of odd-girth 2k + 1 which bounds the class of partial 2-trees of odd-girth 2k + 1 is still unsolved. We know the order of such an optimal bound is quadratic in k, and we guess that the exact values is  $(k + 1)^2$  and that the optimal bound would be a subgraph of PC(2k). Finding optimal bounds for small values of k can lead to a pattern of construction for the general case. Once a correct pattern is found, a proof could be possible using Theorem 4.3.1. To find solutions for such small values, one may search for small subgraphs of PC(2k) which satisfies the condition of Theorem 4.3.1. This would require some theoretical work to minimize the number of subgraphs to look at, then a programming to look at all remaining subgraphs systematically.

4. Furthermore, the similar question for t-trees,  $t \ge 4$ , is yet to be considered, and it is a promising subject of research.

A more general question we considered in this work which also captures the above mentioned conjecture is the following.

**Problem.** Given  $r \ge k$ , what are the minimal subgraphs of PC(2k) to which every planar graph of odd-girth 2r + 1 admits a homomorphism?

Beside capturing the previous conjecture (for the case of r = k) this leads to following conjectures:

**Conjecture** If G is a planar graph of odd-girth at least 2k + 3, then  $\chi_f(G) \leq 2 + \frac{1}{k}$ .

**Conjecture** Any planar graph of odd-girth at least 11 admits a homomorphism to the Coxeter graph.

**Conjecture** Given a planar graph G of odd-girth 2k + 1 we have  $\chi_c(G) \leq 2 + \frac{2}{k}$ .

A prefect answer to each of the questions seems to be very challenging. In particular last one (on circular chromatic number) is related to Jaeger conjecture on flows which is tried by a number of well-known mathematicians. Here we give a few promising ideas on how to proceed.

5. As in idea number 1 one possible approach is approximation results in support of

these conjectures. The most classic idea for such work is to use discharging techniques. However, by considering extended notion of list-coloring one may also hope to use recent technique of C. Thomassen in proving such results. The notion of list-coloring is generated in trying to extend a given partial coloring to the coloring of the whole graph. This then naturally provides a better room for induction. However normally for such a claim to be carried on by induction, it must be carefully formulated. C. Thomassen has given a few such proofs on coloring planar graphs. The result of [32] seems to be of similar nature but on the theory of flows. An application of their result is that planar graphs of odd-girth at least 6k + 1 map to  $C_{2k+1}$  (i.e.  $\chi_c \leq 2 + \frac{1}{k}$ ). I believe a translation of this proof to coloring formulation would give ideas on improved approximation for the other questions, in particular for the question of mapping planar graphs to the the circulant graph  $C(4k, \{2k-1, 2k, 2k+1\})$ . Furthermore, as this graph is richer than  $C_{2k+1}$ , I expect a better approximation in this case and thus an improved bound on the circular chromatic number.

6. To consider the general case of the question one must study structures in the projective cubes. Since the Coxeter graph is redefined as the complement of the blocks of the Fano plane in K(7,3), one may consider similar structures using known block designs. Indeed P. Cameron has proposed one such design for the case of K(11,5), but it has yet to be considered. Further such study, not only may answer some cases of our question, it may also lead to discovery of highly symmetric graphs.

7. From an algorithmic point of view many questions can be considered. A most natural one is: for which graphs H the H-coloring problem of planar graphs is polynomial time? Indeed our main conjecture, if true, would imply that this is the case for the projective cubes. But the question becomes NP-hard on subgraphs such as the Petersen graph. To prove NP-completeness results we may either build cross-over gadgets to capture the H-coloring problem on the class of all graphs or we may try the take advantage of recently developed techniques from universal algebra.

# 5.2 Employing the notion of signed graphs

A signed graph is a graph equipped with an assignment  $\sigma$  which assigns to each edge a sign (positive or negative). The set of negative edges is called the *signature* and is usually denoted by  $\Sigma$ , we use  $(G, \Sigma)$  to denote this signed graph.

A key notion in the study of signed graphs, which separates it from 2-edge-colored graphs, is the notion of *resigning*, that is to change the signs of all edges incident to a given vertex. One may resign at more than one vertex, that would be equivalent to change the signs of all the edges of an edge-cut of G. Two signatures  $\Sigma_1$  and  $\Sigma_2$  on a same graph G are said to be *equivalent* if one can be obtained from the other by a resigning. Given a connected graph G it is easily observed that there are  $2^{|E(G)|}$  signatures on G and  $2^{|E(G)|-n+1}$  nonequivalent signatures.

A closed walk of G is called *unbalanced* if it contains an odd number of negative edges, i.e., if the product of the signs of its edges is negative. Observe that resigning does not change the balance of a cycle or a closed walk. This also applies to loops which are closed walks of length 1. A resigning at a vertex incident to a loop would change the sign of

the corresponding loop twice, thus leaving it unchanged. Hence, if  $\Sigma_1$  is equivalent to  $\Sigma_2$ , then the set of balanced (resp. unbalanced) cycles of  $(G, \Sigma_1)$  and  $(G, \Sigma_2)$  are the same. Zaslavsky proved that the converse is true as well:

**Theorem 5.2.1.** Two signed graphs  $(G, \Sigma_1)$  and  $(G, \Sigma_2)$  have a same set of balanced cycles if and only if  $\Sigma_1$  and  $\Sigma_2$  are equivalent.

*Consistent signed graphs* are signed graphs in which all balanced cycles are of even length and all unbalanced cycles are of a same parity. Thus they consist of two main subclasses:

I. The class of *antibalanced* signed graphs, these are signed graphs  $(G, \Sigma)$  where  $\Sigma \equiv E(G)$ . Thus a cycle of G is unbalanced in  $(G, \Sigma)$  if and only if its length is odd.

II. The class of signed bipartite graphs. Thus all cycles, including the unbalanced ones, are of even length.

Signed projective cube of dimension k, denoted SPC(k) is obtained from PC(k), the projective cube of dimension k, by assigning – to the edges corresponding to J and + to all other edges. It can be easily checked that SPC(k) is a consistent signed graph. For even values of k, SPC(k) can be resigned to be an antibalanced signed graph and for odd values of k we have already seen that PC(k) is bipartite.

A minor of  $(G, \Sigma)$  is any signed graph obtained by operations of: deleting vertices and/or edges, contracting positive edges, and resigning.

Observe that, while the parity of the length of a cycle changes after contraction of a single edge, the balance remains stable. This is a key property that creates a stronger relation between minor and homomorphism of signed graphs (compared to graphs). For example, we have:

**Proposition 5.2.2.** A consistent signed graph (G, E(G)) is  $(K_3, E(K_3))$ -minor-free if and only if G is bipartite.

As mentioned above, balance of closed walks are considered among basic structures of the signed graphs. Thus it is only fitting to define homomorphisms of signed graphs to be mappings which not only preserves adjacency, but also preserves balance of closed walks. More formally:

A homomorphism of a signed graph  $(G, \Sigma)$  to a signed graph  $(H, \Pi)$  is a mapping of the vertices of G to the vertices of H and the edges of G to the edges of H which preserves adjacencies, incidences and balance of each closed walk. This general definition works for signed graphs where loops and digons are allowed. When multi-edges are forbidden, a homomorphism maybe given by mapping of vertices only, as such a mapping, when it is a homomorphism, induces a unique mapping of edges. When there exists a homomorphism of  $(G, \Sigma)$  to  $(H, \Pi)$  we write  $(G, \Sigma) \to (H, \Pi)$ .

In practice we use a simpler definition given in the following theorem whose proof is based on Theorem 5.2.1.

**Theorem 5.2.3.** There exists a homomorphism of a signed graphs  $(G, \Sigma)$  to  $(H, \Pi)$  if and only if there is a signature  $\Sigma'$  equivalent to  $\Sigma$  and a homomorphism of G to H which preserves the sign of edges of G with respect to  $\Sigma'$  (and  $\Pi$ ). The definition of homomorphism of signed graphs is due to B. Guenin who defined it as given in the theorem in order to introduce the following strengthening of Conjecture 3.2.1 (see [43] for more details).

**Conjecture 5.2.4.** The class of consistent signed planar graphs of unbalanced-girth k+1 is bounded by SPC(k).

The development of the theory of homomorphisms of signed graphs has then began as part of my post doc project with E. Sopena. In our first paper on the subject, among other results, we have shown that the notion of homomorphisms and coloring of graphs are captured by the notion of homomorphisms of signed bipartite graphs. Thus we have have a theory in our hand to develop which is barely touched, and hence a large number of projects. In particular each of the ideas 1-7 may also apply to bipartite cases where basically not much is known. But taking advatage of this theory, I would propose further ideas next.

#### 5.2.1 Further ideas for progress on signed graphs

7. Conjecture 5.2.4 for a fixed k is shown to be equivalent to an edge-coloring conjecture of Seymour. Through recent verifications of the edge-coloring version, this conjecture is verified for  $k \leq 7$ . An accessible but challenging project is to give a direct proof of these cases. The advantage here is that one can apply a direct induction on k, a technique which was not possible for the edge-coloring conjecture. Developments here would may then lead to either proof of further cases, or to support of the conjecture as proposed in idea **1**,

8. The analogue question of "finding minimal subgraphs of SPC(2k-1) to which every signed bipartite graph of unbalanced-girth at least  $2r, r \ge k$ ," not only leads to all analogue studies, but it also propose the study of highly symmetric signed bipartite graphs. Such subgraphs of SPC(2k-1) could form nomination of optimal bounds. An example of highly symmetric signed subgraph is the unbalanced cycle of length 2k. We conjecture that every signed bipartite graph of unbalanced-girth 4k - 2 admits a homomorphism to unbalanced cycle of length 2k. We hope to find an adaptation of the result of [32] for this conjecture.

**9**. We may now formally define the notion of  $\text{odd}-K_n$ . Given a graph G, it is said to be  $\text{odd}-K_n$  free if (G, E(G)) has no  $(K_n, E(K_n))$ -minor. The claim of Conjecture 5.2.4 is believed to be true even for the larger class of consistent signed graphs with no  $\text{odd}-K_n$ . Furthermore, many of the questions studied here are to be considered using a finer notion of minor of signed graphs. A particular one to consider is Theorem 4.1.1. We would like to know if one can extend this result to the class of fully negative signed graphs. While the homomorphism implication of such an extension would be the same, the class to which it may apply would normally be much larger. Indeed this theorem, as it is, can only be applied to sparse families of graphs, while such an extension would allow application to dense classes of graphs. We have some decomposition theorem for such notions of minor form Kawarabayashi which may help achieving such goal.

10. Finally, a couple of ambitious projects for myself to consider: i. The first is to take advantage of a natural graph operation on signed graphs. The notion of signed projective

cube allows a natural definition of projective cube to build SPC(k+1) from SPC(k). The operation is homomorphism preserving. Thus, assuming that SPC(2) bounds the class of fully negative planar signed graphs (the four color theorem), the operation builds a family of consistent signed graphs which are bounded by SPC(k) and are of average degree at least k. Perhaps one possible approach to Conjecture 5.2.4 is to say that each consistent signed planar graph is a subgraph of one member of this family. ii. The second is a surprise connection with Algebraic geometry. That PC(4) is the intersection graph of the straight lines of a cubic surface, suggests that in the case k = 4 of the conjecture, i.e., mapping triangle-free planar graphs to the Clebsch graph, perhaps ideas from Algebaric geometry can be employed. On one hand one may look for a homeomorphism of the plane to the cubic surface so that vertices of a given triangle free plane graph map to points on straight line such that adjacent ones map to intersecting line. On the other hand, considering the construction of the cubic surface by notion of "blowing up" one may reduce the homomorphism problem to an embedding problem. More important subject of investigation on this subject: is this relation unique for k = 4 or are there ways to extend this relation? Perhaps one may have to surfaces of higher degree.

# Bibliography

- K. Appel, and W. Haken. Every Planar Map is Four Colorable. I. Discharging. *Illinois Journal of Mathematics* 21(3):429–490, 1977.
- [2] K. Appel, W. Haken and J. Koch. Every Planar Map is Four Colorable. II. Reducibility, *Illinois Journal of Mathematics*, 21(3):491–567, 1977.
- [3] S. Arnborg, A. Proskurowski, D.G. Corneil. Forbidden minors characterization of partial 3-trees. *Discrete Mathematics* 80(1):1–19, 1990.
- [4] L. Beaudou, F. Foucaud and R. Naserasr. Homomorphism bounds and edgecolourings of  $K_4$ -minor-free graphs, *Journal of Combinatorial Theory, Series B* accepted.
- [5] L. Beaudou, R. Naserasr and C. Tardif. Homomorphisms of projective cubes, *Discrete Mathematics*, 338(12): 2539–2544, 2015.
- [6] J.A. Bondy, U.S.R. Murty, Graph theory with applications, American Elsevier Publishing Co., New York, (1976).
- [7] M. Chudnovsky, K. Edwards, K. Kawarabayashi and P. D. Seymour. Edge-coloring seven-regular planar graphs. *Journal of Combinatorial Theory, Series B* 115:276–302, 2015.
- [8] M. Chudnovsky, K. Edwards and P. D. Seymour. Edge-colouring eight-regular planar graphs. Journal of Combinatorial Theory, Series B 115:303–338, 2015.
- [9] V. Chvátal. The minimality of the Mycielski graph. Graphs and combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973), Lecture Notes in Mathematics, Vol. 406, Springer-Verlag, pp. 243–246, 1974.
- [10] R. Diestel. *Graph Theory*, 4th edition, Springer, 2010.
- [11] Z. Dvořák, K. Kawarabayashi and D. Král'. Packing six T-joins in plane graphs. Journal of Combinatorial Theory, Series B 116 :287–305, 2016.
- [12] Z. Dvořák, R. Skrekovski, and T. Valla. Planar graphs of odd-girth at least 9 are homomorphic to the Petersen graph. SIAM J. Discrete Math. 22(2):568–591 2008.
- [13] J. Edmonds. Maximum matching and a polyhedron with (0,1)-vertices. Journal of Research of the National Bureau of Standards, Section B 69:125–30, 1965.

- [14] K. Edwards. Optimization and packings of T-joins and T-cuts. M.Sc. Thesis, McGill University, Canada, 2011.
- [15] T. Feder and C. Subi. Bipartite subgraphs of large odd girth series-parallel graphs. Manuscript, 2015. http://theory.stanford.edu/~tomas/girth.ps
- [16] C. G. Fernandes and R. Thomas. Edge-coloring series-parallel multigraphs. Manuscript, 2011. http://arxiv.org/abs/1107.5370
- [17] A. M. H. Gerards. Homomorphisms of graphs into odd cycles. Journal of Graph Theory 12(1):73–83, 1988.
- [18] W. Goddard and H. Xu. Fractional, circular, and defective coloring of series-parallel graphs. Journal of Graph Theory 81(2):146–153, 2016.
- [19] C. Godsil and G. Royle. Algebraic graph theory, Springer-Verlag, New York, Graduate Texts in Mathematics 207, 2001.
- [20] B. Guenin. Packing odd circuit covers: a conjecture. Manuscript, 2005.
- [21] B. Guenin. Packing T-joins and edge-colouring in planar graphs. *Mathematics of Operations Research*, to appear.
- [22] F. Harary. Graph Theory. Addison-Wesley, Reading, Mass., 1969, p. 149, Exercise 12.19.
- [23] A. Harutyunyan, R. Naserasr, M. Petrusevski, R. Skrekovski and Q. Sun. Mapping planar graphs to coxeter graph, *Discrete Mathematics* 339(2): 839–849, 2016.
- [24] W. He, R. Naserasr and Q. Sun. Walk-powers and homomorphism bound of  $K_4$ -minor-free graphs. Manuscript, 2015.
- [25] P. Hell and J. Nešetřil. Graphs and Homomorphisms. Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, 2004.
- [26] P. Hell and X. Zhu. The circular chromatic number of series-parallel graphs. Journal of Graph Theory 33(1):14–24, 2000.
- [27] I. Holyer. The NP-completeness of edge-coloring. SIAM Journal on Computing 10(4):718–720, 1981.
- [28] T. R. Jensen and B. Toft. Graph Coloring Problems. Series in Discrete Mathematics and Optimization, 1995.
- [29] K.F. Jones. Independence in graphs with maximum degree four. Journal of Combinatorial Theory, Series B 37:254–269, 1984.
- [30] W. Klostermeyer and C. Q. Zhang. 2 + ε-colouring of planar graphs with large oddgirth. Journal of Graph Theory 33(2):109–119, 2000.
- [31] A. N. Letchford and N. A. Pearson. A fast algorithm for minimum weight odd circuits and cuts in planar graphs. *Operations Research Letters* 33(6):625–628, 2005.

- [32] L.M. Lovász, C. Thomassen and Y. Wu, C.Q. Zhang. Nowhere-zero 3-flows and modulo k-orientations. Journal of Combinatorial Theory, Series B 103(5):587–598, 2013.
- [33] C. Lund and M. Yannakakis. On the hardness of approximating minimization problems. Journal of the ACM 41(5):960–981, 1994.
- [34] G. MacGillivray and M. Siggers. On the complexity of H-colouring planar graphs. Discrete Mathematics 309(18):5729–5738, 2009.
- [35] O. Marcotte. Optimal edge-colourings for a class of planar multigraphs. Combinatorica 21(3):361–394, 2001.
- [36] T. Marshall, R. Naserasr and J. Nešetřil. Homomorphism bounded classes of graphs. European Journal of Combinatorics 27(4):592–600, 2006.
- [37] G.H.J Meredith. Triple transitive graphs. J. London Math. Soc. 13(2):249–257, 1976.
- [38] R. Naserasr. *Homomorphisms and bounds*. Ph.D. thesis, Simon Fraser University, Canada, 2003.
- [39] R. Naserasr. Homomorphisms and edge-coloring of planar graphs. Journal of Combinatorial Theory, Series B 97(3):394–400, 2007.
- [40] R. Naserasr. Mapping planar graphs into projective cubes. Journal of Graph Theory 74(3):249–259, 2013.
- [41] R. Naserasr and Y. Nigussie. On a new reformulation of Hadwiger's conjecture. Discrete Mathematics 306(23):3136–3139, 2006.
- [42] R. Naserasr, Y. Nigussie and R. Škrekovski. Homomorphisms of triangle-free graphs without a K<sub>5</sub>-minor. Discrete Mathematics 309(18):5789–5798, 2009.
- [43] R. Naserasr, E. Rollová, É. Sopena. Homomorphisms of planar signed graphs to signed projective cubes. Discrete Mathematics & Theoretical Computer Science 15(3): 1-12, 2013.
- [44] R. Naserasr, E. Rollová and E. Sopena. Homomorphisms of signed graphs. J. Graph Theory 79(3):178-212, 2015.
- [45] R. Naserasr, S. Sen and Q. Sun. Walk-powers and homomorphism bounds of planar signed graphs. *Graphs and Combinatorics* 32(4):1505–1519, 2016.
- [46] R. Naserasr and R. Skrekovski. The Petersen graph is not 3-edge-colorable a new proof. Discrete Mathematics, 268(1-3):325–326, 2003.
- [47] J. Nešetřil and Y. Nigussie. Density of universal classes of series-parallel graphs. Journal of Graph Theory 54(1):13–24, 2007.
- [48] J. Nešetřil and Y. Nigussie. Finite dualities and map-critical graphs on a fixed surface. Journal of Combinatorial Theory, Series B 102(1):131–152, 2012.

- [49] J. Nešetřil and P. Ossona de Mendez. Sparsity. Graphs, structures, and algorithms. Algorithms and Combinatorics, 28. Springer, Heidelberg, 2012.
- [50] Z. Pan and X. Zhu. The circular chromatic number of series-parallel graphs of large odd girth. *Discrete Mathematics* 245(1-3):235–246, 2002.
- [51] Z. Pan and X. Zhu. Tight relation between the circular chromatic number and the girth of series-parallel graphs. *Discrete Mathematics* 254(1-3):393–404, 2002.
- [52] C. Payan. On the chromatic number of cube-like graphs. *Discrete Mathematics* 103(3)271–277, 1992.
- [53] A. Pultr and Trnková. Algebraic and Topological Representations of Groups, Semigroups and Categories. North-Holland, Amsterdam, 1980.
- [54] A. Satyanarayana, L. Tung. A characterization of partial 3-trees. Networks 20(3):299– 322, 1990.
- [55] E. R. Scheinermann and D. H. Ullman. Fractional graph theory: a rational approach to the theory of graphs. Wiley, 1997.
- [56] P. D. Seymour. Matroids, Hypergraphs and the Max.-Flow Min.-Cut Theorem. Ph.D. Thesis, Oxford University, United Kingdom, 1975.
- [57] P. D. Seymour. Unsolved problem in "Graph Theory and Related Topics" (J. A. Bondy and U. S. R. Murty, Eds.), pp. 367–368, Academic Press, New York, 1979.
- [58] P. D. Seymour. On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. Proceedings of the London Mathematical Society 38(3):423–460, 1979.
- [59] P. D. Seymour. Colouring series-parallel graphs. *Combinatorica* 10(4):379–392, 1990.
- [60] P. G. Tait. Remarks on the colouring of maps. Proceedings of the Royal Society of Edinburgh 10:501–503, 1880.
- [61] K. Wagner. Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 144, 570–590, 1937.
- [62] A. L. Wells. Regular generalized switching classes and related topics. Ph.D. thesis, University of Oxford, United Kingdom, 1983.