Homomorphisms and Bounds

by

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Abstract

A class $\mathcal{C}$ of graphs is said to be *bounded* by a graph H if each graph in $\mathcal{C}$ admits a homomorphism to H. Given an integer m and a graph U we say a graph G is *m-locally U-colourable* if each induced subgraph of G with at most m vertices admits a homomorphism to U. We study the following general question: Does a given a class $\mathcal{C}$ of graphs admit a bound H which is m-locally U-colourable?

Let H be an m-locally U-colourable bound for a class $\mathcal{C}$ of graphs. Then every graph G in $\mathcal{C}$ admits an H-colouring which satisfies the following property:

($*$) Any subgraph of G induced by the union of a set of m colour classes is U-colourable.

This introduces a necessary condition for $\mathcal{C}$ to have an m-locally U-colourable bound. We prove that this necessary condition is also sufficient. Then we give several applications of this result.

Exploring the connection between homomorphisms and the four colour theorem we prove that the Hadwiger's conjecture is equivalent to the following conjecture, of J. Nešetřil and P. O. De Mendez: Any minor-closed class $\mathcal{C}$ of graphs admits a bound $H \in \mathcal{C}$.

We conjecture that the class of planar graphs of odd girth $2k+1$ is bounded by the Cayley graph $C(\mathbb{Z}_2^{2k+1}, S)$, where

$$S = \{(1,1,0,\cdots 0),(0,1,1,0,\cdots 0),\cdots (1,0,\cdots 0,1)\}.$$

We prove that this conjecture is equivalent to a certain case of a well known conjecture of P. D. Seymour. In support of our conjecture we prove that the class of planar graphs of odd girth $4k+1$ is bounded by a Cayley graph on $\mathbb{Z}_4$ (of odd girth $2g+1$).

Finally, we study the chromatic covering number of graphs, introduced by A. Amit, N. Linial and J. Matoušek. The chromatic covering number of a graph G, denoted by $F_\chi(G)$, is the smallest integer k for which there are k induced subgraphs $G_1, G_2, \cdots G_k$ of G, such that every vertex x of G satisfies the inequality

$$\sum_{x \in G_i} \frac{1}{\chi(G_i)} \geq 1.$$

We prove that

$$\chi(G) \leq \left\lfloor \left(\frac{F_\chi(G)+1}{2} \right)^2 \right\rfloor.$$

We also characterize the graphs for which equality holds, and show that in most cases the inequality is close to being tight.

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Contents

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Chapter 1

Background and Summary

1.1 Historical background and the four colour theorem

The Four Colour Theorem is one of the problems in mathematics which is most easily understood by non-mathematicians. It can even be explained to those who do not know how to count without using their fingers. However it is arguably one of the most difficult theorems in mathematics to prove. Heavy use of computers, together with a large number of cases, make both of the published proofs of this theorem almost unreadable.

The origin of the problem goes back to at least 1852. As quoted in [39], Kenneth O. May in his paper on the origin of the four colour theorem concludes that: "It was not the culmination of a series of individual efforts, but flashed across the mind of Francis Guthrie while colouring a map of England $\cdots$ his brother communicated the conjecture, but not the attempted proof to DeMorgan in October, 1852".

The four colour theorem in its original form simply states that:

The regions of every (simple) map can be *properly* coloured using at most four distinct colours.

It requires some work to put everything in exact mathematical form, for example one has to define boundaries using Jordan curves. Here we remark that a simple map is a map in which every country is in one contiguous piece. A proper colouring is what one should normally expect from colouring of a map, i.e., neighbouring countries must receive different colours (in order to be distinguishable). Based on the account in [39], the history of the four colour theorem continues as follows.

The first printed reference to the problem is due to Cayley, and was published in the Proceedings of London Mathematical Society in 1878, [11]. Since then the attempts to prove the problem have led to many beautiful theories. In 1879 Kempe published a paper, [41], in which he claimed to have a proof of the four colour problem. But 10 years later Heawood discovered a gap in Kempe's argument, [33]. He completed Kempe's idea, the Kempe chain method, to a proof of what is known as the Five Colour Theorem (every simple map can be coloured properly, using at most five distinct colours).

Tait was the first person to notice there was much beauty in the structure of the four colourings of planar graphs. Even at the time Kempe's argument was accepted Tait was trying different approaches to the four colour problem, [73]. His equivalent statement of the four colour theorem is the birth place of some important subjects in graph theory, such as the theory of edge colouring and the theory of flows in graphs. This approach, with some generalizations, will be studied in Chapter 3.

A direct approach to the four colour theorem led to the theory of graph colouring. This theory turned out to be a fruitful branch of mathematics with many applications in modern technology such as networking and communication. So it has attracted large number of researchers and mathematicians, who have developed numerous results and posed numerous open problems, such as Brook's theorem, the four colour theorem and Hadwiger's conjecture. Each of these examples will be described in details in the forthcoming chapter.

There also have been many different ways to generalize the theory; fractional colouring, list colouring, circular colouring, acyclic colouring and the theory of graph homomorphisms are examples of successful generalizations. Among the generalizations with an algebraic flavor, graph homomorphisms seems to be the most general approach.

The four colour theorem was finally proved by W. Haken and K. Appel, [4]. Roughly speaking, they introduced almost 1500 configurations, called "unavoidable configurations", and they showed that if a planar graph contains one of these unavoidable configurations, then any four-colouring of the subgraph obtained by removing the unavoidable configuration will be extendible to a four-colouring of the graph itself. They complete the proof by showing that any minimal counter example to the four-colour theorem must contain at least one of the unavoidable configurations. They use the computer to verify both statements. A similar proof with a smaller number of unavoidable configurations (exactly 633 of them) was found by N. Robertson, D. Sanders, P. Seymour and R. Thomas ([66]).

However the four colour theorem has not yet passed into history, and is still one of the main topics of research in graph theory. This is because, first of all, its proof is not satisfactory, secondly, the theories that are arising from this theorem are not fully developed, and also because these theories have been found to be very important for applications in modern technology.

1.2 About this thesis

Throughout this text a graph is always a simple finite graph (i.e., no loops or multiple edges). A finite graph with possible multiple edges but no loops will be called a multigraph. A set of size n will be called n-set. The set $\{1, 2, \cdots, k\}$ will be denoted by $[k]$.

For the classical and standard definitions and notation of graph theory we refer to [15] and [49]. The more advanced or the less standard ones will be given throughout the text. However since colouring and the chromatic number are the main subject of this text we should repeat their definition here.

Definition 1.1 Given a graph G, a proper k-colouring of the vertices of G or simply a *k-colouring* of G is an assignment of the colours 1, 2, ..., k to the vertices of G in such a way that any pair of adjacent vertices receive different colours. The minimum number k for which there exists a k-colouring of G is called the chromatic number of G and is denoted by $\chi(G)$. A graph G with chromatic number $\chi(G) = k$ will be called a k-chromatic graph. $\diamond$

We will study certain homomorphism problems, some closely related to the four colour theorem. In the next chapter, Chapter 2, we will consider problems of the following type:

Problem 1.2 *Given a class C of graphs, does every member of G admit a homomorphism to a graph F with certain properties?*

Brooks' theorem and the four colour theorem are examples of this type of the problems, (see Chapter 2 and Chapter 3 for more details). For some kind of properties, like the property of being K_k-free or the property of having high odd girth, we show that one can answer this question by answering a certain ordinary colouring problem. As an application we include a new proof of a theorem of R. Häggkvist and P. Hell. We also include a new proof (without using the four colour theorem) of the existence of a K_5-free graph F to which every planar graph admits a homomorphism. This can be viewed as an improvement of the five colour theorem and was first proved by J. Nešetřil and P. O. De Mendez in [60].

At the end of Chapter 2 we will include a section on Hadwiger's conjecture. There we prove that Hadwiger's conjecture is equivalent to the fact that every minor-closed family $\mathcal{C}$ of graphs contains a graph H which admits a homomorphism from every member of $\mathcal{C}$.

Most of the new results of Chapter 2 can be found in [50]. The proof of the existence of a K_5-free bound for the class of planar graphs is from [51]. The results on the reformulation of Hadwiger's conjecture is from [53].

In Chapter 3 we will focus more on planar graphs and different approaches to the four colour theorem. We will introduce a generalization of the four colour theorem in the language of homomorphisms, and we will show equivalence between this generalization and some of the known generalizations. Using this equivalence a triangle-free bound on 16 vertices will be found for the class of triangle-free planar graphs. In support of our conjecture we will prove that class of planar graphs of odd girth at least $4g + 1$ is bounded by a graph of odd girth $2g + 1$. These results are from [52].

It will be shown that if a graph H of odd girth $2k + 1$ admits a homomorphism from every planar graph of odd girth $2k + 1$, then H can not be planar. This result is from [50]. We will also include a new proof of the fact that Petersen graph is not 3-edge colourable; this proof is from [54].

In the last chapter we will study the chromatic covering number of graphs, introduced in [3] . The chromatic covering number can be compared to the fractional chromatic number, but unlike the fractional chromatic number, the chromatic covering number is bounded by functions of the chromatic number from both sides. We will tighten the bounds given in [3] and we show that our bounds are the best possible. The results of this chapter are from [55].

Chapter 2

Homomorphisms and bounds

2.1 Introduction

Given two graphs G and H, a *homomorphism* of G to H is an edge preserving mapping $f : V(G) \to V(H)$, that is to say, for every edge xy of G, $f(x)f(y)$ is an edge of H. The existence of a homomorphism of G to H is denoted by $G \to H$. If $G \to H$, then we say G maps to H. If f is a homomorphism of G to H, then the *homomorphic image* of G in H is a subgraph H' of H such that for every vertex $x \in V(H')$ there is a vertex v of G with $f(v) = x$, and also for every edge $xy \in E(H')$ there is an edge uv with $f(u) = x$ and $f(v) = y$.

Sometimes a homomorphism of G to H is called an H-colouring; in this case vertices of H have been regarded as a set of colours and the homomorphism is an assignment of these colours to the vertices of G. This is a generalization of classical vertex colouring problem of graphs, as a graph G is k-colourable if and only if it admits a homomorphism to the complete graph K_k.

The notation $G \nrightarrow H$ will be used to represent the fact that there is no homomorphism of G to H. Normally it is more difficult to show the nonexistence of homomorphisms than the existence of a homomorphism. This is analogous to the problem of chromatic number, where the lower bounds are normally harder to prove than the upper bounds. The chromatic number, clique number and the odd girth are some parameters that can help us to prove the nonexistence of homomorphisms. More precisely we have the following two well known lemmas, each of which is sometimes called a "*no-homomorphism lemma*".

Lemma 2.1 [32] *Given two graphs G and H, if $\omega(G) > \omega(H)$ or if $\chi(G) > \chi(H)$ then $G \nrightarrow H$.*

Proof. To see this, first observe that if $k < l$ then there is no homomorphism of K_l to K_k. Now let $f : V(G) \to V(H)$ be a homomorphism. Then for a complete subgraph W in G, the homomorphic image of W, $f(W)$, is a complete graph in H which has the same size as W. This proves that if $\omega(G) > \omega(H)$ then $G \nrightarrow H$.

To see that $G \to H$ implies $\chi(G) > \chi(H)$, assume g is a homomorphism of G to H. If $\chi(H) = k$ then there is a homomorphism $f : H \to K_k$. Let g be a homomorphism of G to H. Now the composition $f \circ g$ is a homomorphism of G to K_k, i.e., G is k-colourable and $\chi(G) \leq k$. □

Another folklore no-homomorphism lemma is the following:

Lemma 2.2 *[35] Given two graphs G and H, if odd–girth$(G) <$ odd–girth(H) then $G \nrightarrow H$.*

Proof. First notice that any homomorphic image of an odd cycle, C_{2r+1} must contain an odd cycle of size smaller than or equal to $2r + 1$. Otherwise the image which has at most $2r + 1$ vertices would be bipartite which contradicts Lemma 2.1. Let *odd–girth*$(G) = 2g + 1$. If there is a homomorphism of G to H then the homomorphic image of the shortest odd cycle of G in H contains an odd cycle of size at most $2g+1$, i.e., *odd–girth*$(H) \leq 2g + 1$. This proves the lemma. □

A graph parameter for which a no-homomorphism lemma holds is called a *monotone* graph parameter. So another way of stating Lemma 2.1 and Lemma 2.2 is to say that each of the three graph parameters, clique number, chromatic number and odd girth, is a monotone graph parameter.

Graphs G and H are said to be *homomorphically equivalent* provided that each of them admits a homomorphism to the other one. If G and H are homomorphically equivalent then we write $G \sim H$. Any graph with the minimum number of vertices to which G is homomorphically equivalent is called a *core* of G. The following classical lemma proves that the core of a graph G is unique and that it is a subgraph of G.

Lemma 2.3 *[35] For any given graph G there is a unique core (up to isomorphisms). Moreover the core of G is isomorphic to a subgraph of G.*

Proof. Let H_1 and H_2 be two cores of G. Let f be a homomorphism of H_1 to G and g be a homomorphism of G to H_2. Then $g \circ f$ is a homomorphism of H_1 to H_2. Let H' be the homomorphic image of H_1 in H_2. H' maps to G because it is a subgraph of H_2, G maps to H' because it maps to H_1. So H' is also homomorphically equivalent to G, by the minimality of H_2, H' cannot be a proper subgraph of H_2, and by minimality of H_1, the mapping $g \circ f$ is one to one and therefore is an isomorphism of H_1 to $H' = H_2$.

To see that the core $c(G)$ of G is a subgraph of G, consider a mapping of $c(G)$ to G; in this mapping the homomorphic image of $c(G)$ is also homomorphically equivalent to G and by the minimality of $c(G)$, it must be isomorphic to $c(G)$. $\square$

In this proof it was crucial that the graphs are finite; the concept of core for infinite graphs have been studied by B. Bauslaugh in [5]. A *core* is a graph which is its own core. Equivalently, a core is graph which does not admit a homomorphism to any proper subgraph of itself. Complete graphs and odd cycles, or in general colour critical graphs, are examples of cores. By the definition a k-colour critical graph is a graph with chromatic number k such that every proper subgraph has chromatic number smaller than k. Now applying Lemma 2.1 it is easy to see that every colour critical graph is a core.

For an example of a core which is not a colour critical graph see the graph of Figure 2.1. To see that this graph is a core, note that every proper subgraph of it maps to

Figure 2.1: A core which is not colour critical

C_5 but the graph itself does not map to C_5. This is so because if this graph maps to C_5 then the subgraph induced by deleting the central vertex x, which is a 9-cycle, must map to C_5 surjectively. On the other hand, every vertex y in this 9-cycle is either adjacent to the central vertex x or there is a path of length three joining x and y. This means that identifying x with any of the vertices of the outer 9-cycle will produce a loop or a triangle, so there is no homomorphism of this graph to C_5.

Homomorphisms allow us to treat many colouring problems in a more general setting. One way of doing this is the following definition of *homomorphism order* ($\preceq$) on the class of graphs:

$$G \preceq H \quad \text{if and only if} \quad G \to H.$$

Following this notation we say G is *smaller* than H if $G \preceq H$.

The homomorphism order, which sometimes it is also called *colouring order* (for example see [57]), is a quasi order. Below we give a proof of this folklore lemma.

Lemma 2.4 *The homomorphism order ($\preceq$) on the class of graphs is a reflexive and transitive binary relation.*

Proof. This order is reflexive because the identity is a homomorphism of G to G. To see that homomorphism order is transitive let f be a homomorphism of G to H and g be a homomorphism of H to K, then $g \circ f$ is a homomorphism of G to K. $\square$

The homomorphism order does not have the antisymmetric property and therefore is not a partial order on the class of all graphs. As an example consider two bipartite graphs G and H, then the inequalities $G \preceq H$ and $H \preceq G$ both hold but G and H are not necessarily isomorphic. However if we identify all homomorphically equivalent graphs, the induced order on the new class will be a partial order.

Lemma 2.5 *The homomorphism order induced on the class of cores is a partial order.*

Proof. Obviously this order inherits the properties of being reflexive and transitive. To prove that it also antisymmetric let G and H be two cores for which $G \preceq H$ and $H \preceq G$. This means G and H are homomorphically equivalent and therefore by Lemma 2.3 they have isomorphic cores, but each of G and H are isomorphic to their own cores, because they are core themselves, so they must be isomorphic. $\square$

The concept of a bound for a family of graphs is a natural consequence of having the homomorphism order. Given a class $\mathcal{C}$ of graphs, we say $\mathcal{C}$ is *bound*ed by a graph H if and only if every member of $\mathcal{C}$ admits a homomorphism to H. In other words H is said to be a bound for $\mathcal{C}$ if for every $G \in \mathcal{C}$ we have $G \preceq H$. If the bound H is also in $\mathcal{C}$ then we say H is a *maximum* of $\mathcal{C}$. Moreover if H is a bound for the class $\mathcal{C}$ of graph such that for any other bound $\mathcal{F}$ for C we have $H \preceq F$ then we say H is a supremum for the class $\mathcal{C}$.

We would like to remark that the terminology of bound, maximum and supremum and as well as the project of studying graphs in this order-theoretic approach has been initiated by J. Nešetřil, see [50, 57, 60, 61]. The concept of bound has also been introduced in [30] under the name of *universal graph.*

This notation allows us to put many important colouring and homomorphism results in a form understandable to a broader group of mathematicians. For example if we define C_Δ to be the class of graphs with maximum degree Δ, then Brooks' theorem can be stated as follows:

Brooks' theorem [10]

1) The class $C_2 \backslash \{\text{odd cycles}\}$ is bounded by K_2, and

2) The class $C_\Delta \backslash \{K_{\Delta+1}\}$ is bounded by K_Δ, for $\Delta \geq 3$.

There are different ways of generalizing Brooks' theorem in the literature, one simple generalization is done by introducing the concept of k-degenerate graph. A graph G is said to be *k-degenerate* if every subgraph of G contains a vertex of degree at most k. The smallest integer k for which G is k-degenerate will be called the *degeneracy number* of G and will be denoted by $D(G)$. This concept has been introduced by various authors in different equivalent forms, the oldest reference we found is a paper of V. G. Vizing, [80]. Degeneracy number sometimes is called Szekeres-Wilf number because of the following theorem of G. Szekeres and H. S. Wilf:

Theorem 2.6 *[72] Let $\mathcal{C}_d$ be the class of all d-degenerate graphs. Then $\mathcal{C}_d$ is bounded by K_{d+1}.*

A nicer example of a reformulation, using this new terminology, is the reformulation of Hadwiger's conjecture which was introduced in [60]. To introduce Hadvigwer's conjecture we first define the concept of a minor.

Given two graphs G and H we say H is a *minor* of G if H can be obtained from G by a series of operations: contracting edges, deleting vertices and deleting edges. Contracting an edge xy means removing the edge xy and then identifying x and y. A class $\mathcal{C}$ of graphs is said to be *minor-closed* if for every graph G in $\mathcal{C}$ and every minor H of G, H is also in C. Moreover we say $\mathcal{C}$ is a *proper minor-closed* family of graphs if it is not the class of all graphs. Note that a minor-closed family of graphs is proper if and only if it does not contain all complete graphs K_n.

H. Hadwiger in [29] conjectured that every k-chromatic graph contains the complete graph K_k as a minor. This is an almost trivial statement for $k = 1, 2, 3$. For $k = 4$ it was proved by G. A. Dirac in [16]. For $k = 5$ it implies the four colour theorem, and was in fact shown by K. Wagner to be equivalent to the four colour theorem in this case, [81]. For $k = 6$ it also has been proved by N. Robertson, P. D. Seymour and R. Thomas to be equivalent to the four colour theorem. It remains open for $k \geq 7$ and is one of the most attractive conjectures in graph theory. One of the remarkable results toward Hadwiger's conjecture is due to W. Mader who proved the following fact.

Lemma 2.7 [47] *For every positive integer k there exists an integer $h(k)$ such that if the minimum degree of a graph G is at least $h(k)$, then G contains K_k as a minor.*

An equivalent way of stating this lemma is to say that for every proper minor-closed family $\mathcal{C}$ of graphs there is an integer k such that every graph in $\mathcal{C}$ is k-degenerate. In view of Theorem 2.6 the following is an immediate corollary of Lemma 2.7. This result was originally proved by Wagner in [81].

Lemma 2.8 [81] *For every proper minor-closed family $\mathcal{C}$ of graphs, there is an integer k, such that each graph in $\mathcal{C}$ is k-colourable.*

Using Lemma 2.8 it is now easy to see that the following conjecture introduced in [60] is a reformulation of Hadwiger's conjecture.

Conjecture 2.9 [60] *Every proper minor-closed family $\mathcal{C}$ of graphs contains a complete graph as a maximum with respect to colouring order.*

This formulation of the conjecture splits the problem into two different problems, each of which has its own interest.

Conjecture 2.10 [60] *Any bounded minor-closed family of graphs has a maximum.*

Conjecture 2.11 *The core of the maximum of a minor-closed family of graphs is a complete graph.*

Conjecture 2.10 has been introduced in [60] as a weaker form of Hadwiger's conjecture. However at the last section of this chapter we will prove that Conjecture 2.11 is implied by Conjecture 2.10, therefore proving that Hadwiger's conjecture is equivalent to Conjecture 2.10. We believe Conjecture 2.11 should not be very difficult to prove; an evidence for this belief is a theorem of P. Hell which will be introduced in the last section.

In the rest of this chapter and also in the next chapter, we will frequently refer to the class of planar graphs with odd girth at least $2g+1$. To simplify our notation we will denote this class by $\mathcal{P}_{2g+1}$. Therefore $\mathcal{P}_3$ is the class of all planar graphs which we will be simply denoted by $\mathcal{P}$. The class $\mathcal{P}_5$ is the class of triangle-free planar graphs, and so on.

Aside from the four colour theorem, one of the important results in the theory of colouring of planar graphs is Grötzsch's theorem which states that every triangle-free planar graph is 3-colourable, [25]. With our terminology this theorem can be restated as follows:

Theorem 2.12 *The class $\mathcal{P}_5$ is bounded by K_3.*

This statement of Grötzsch theorem has been depicted in Figure 2.2, where it is contrasted to the four colour theorem. The four colour theorem states that the class of planar graphs has a maximum, namely K_4. But Grötzsch theorem only provides a bound, K_3, for the class of triangle-free planar graphs, $\mathcal{P}_5$.

It is an interesting question to ask whether the class $\mathcal{P}_5$ has a maximum or a supremum. In the next chapter we will show that $\mathcal{P}_5$ does not have a maximum, but it does not seem to be an easy task to decide whether $\mathcal{P}_5$ admits a supremum or not. Problems of similar type has been studied in [62].

The following problem is the first step in the direction of deciding whether $\mathcal{P}_5$ admits a supremum.

Problem 2.13 *Does $\mathcal{P}_5$ admit a bound smaller than K_3?*

This question (in an equivalent form), was asked by J. Nešetřil in [57]. He then raised two similar questions, in [17]. To state the Problem 2.13 in the equivalent form we should first give a definition of the categorical product of two graphs. Categorical products of graphs are discussed, e.g., in [18].

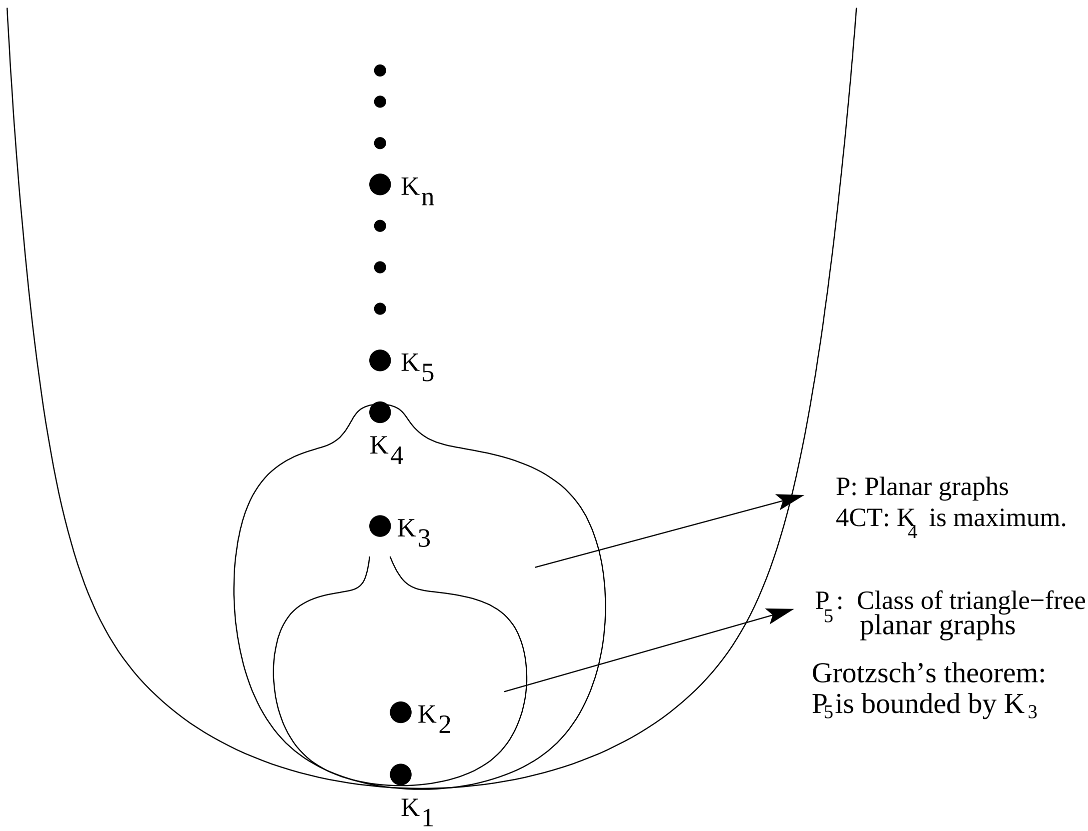

Figure 2.2: Order of graphs

Definition 2.14 Given two graphs G and H, the *categorical* product of G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ where two vertices (x, u) and (y, v) are adjacent if and only if x is adjacent to y in G and u is adjacent to v in H. $\diamond$

The following two well known propositions are among the first important properties of the categorical product of graphs. For more on the lattice theoretic aspects of the categorical product we refer to [19] and [57]

Proposition 2.15 *The categorical product of graphs, $G \times H$, admits homomorphisms to both G and H.*

Proof. It is easy to check that the projection f_1 defined by $f_1(x, u) = x$ is a homomorphism of $G \times H$ to G. Similarly the projection f_2 defined by $f_2(x, u) = u$ is a homomorphism of $G \times H$ to H. $\square$

Proposition 2.16 *Let F be a graph which admits homomorphisms to both G and H then F admits a homomorphism to the categorical product $G \times H$.*

Proof. To see this let f_g and f_h be homomorphisms of F to G and H respectively, then $f : V(F) \to V(G \times H)$ defined by $f(x) = (f_g(x), f_h(x))$ is easily seen to be a homomorphism of F to $G \times H$. $\square$

These two propositions together show that $G \times H$ has the following two important properties: First of all $G \times H$ is smaller than G and H both. Secondly if a graph F is smaller than G and H both, then F is also smaller than $G \times H$. Therefore in the lattice obtained from homomorphism order on the class of cores, $c(G \times H)$ is the meet of G and H in the homomorphism order.

The following proposition, which is a special case of a general phenomenon to be detailed below, will help us to reformulate Problem 2.13.

Proposition 2.17 [59] *If $\mathcal{P}_5$ admits a triangle-free bound then it admits a bound smaller than K_3.*

Proof. Let H be a triangle-free bound for $\mathcal{P}_5$, we will show that $H \times K_3$ is also a bound for $\mathcal{P}_5$ which is smaller than K_3. To see this let G be triangle-free planar graph, then $G \to H$ because $\mathcal{P}_5$ is bounded by H. On the other hand because of the Grötzsch theorem $G \to K_3$. Now by Proposition 2.16 $G \to H \times K_3$. This shows that $H \times K_3$ is also a bound for $\mathcal{P}_5$.

To see that $H \times K_3$ is smaller than K_3, note that by Proposition 2.15 $H \times K_3 \to K_3$, but K_3 does not admit a homomorphism to $H \times K_3$, as if it did then by Proposition 2.16 we would have $K_3 \to H$, which contradicts the fact that H is a triangle-free graph. $\square$

In general, if $\mathcal{C}$ is a class of graphs bounded by B then to find a bound for $\mathcal{C}$ smaller than B, it will be enough to find a bound B' such that $B \not\to B'$. If we find such a bound, then $B \times B'$ will be a bound for $\mathcal{C}$, which is guaranteed to be smaller than B.

On the other hand, if there is a bound B for $\mathcal{P}_5$ with $B \prec K_3$ then B must be a triangle-free graph. This observation, together with Proposition 2.17, shows that the following problem of [57] is a new formulation of Problem 2.13.

Problem 2.18 [57] *Does there exist a triangle-free bound for the class of all triangle-free planar graphs?*

This problem was studied in [17], where the following two similar problems were also posed:

Problem 2.19 [17] *Does there exist a K_4-free bound for the class of all K_4-free planar graphs?*

Problem 2.20 [17] *Does there exist a K_5-free bound for the class of all (K_5-free) planar graphs?*

The answer to the Problem 2.20 is positive by virtue of the four colour theorem, but in view of the difficulty of the four colour theorem it would be interesting to find an independent proof. Each of these three problems have been answered positively by P. O. De Mendez, and J. Nešetřil, in [60] and [61].

Theorem 2.21 [60] *The class of all triangle-free planar graphs is bounded by a triangle free graph.*

Theorem 2.22 [61] *The class of all K_4-free planar graphs is bounded by a K_4-free graph.*

Theorem 2.23 [61] *The class of all (K_5-free) planar graphs is bounded by a K_5-free graph.*

The proof of each of these theorems uses the constructive method of [50]. These constructions are the main subjects of this chapter. So our goal in this chapter is to show how one can answer any of the questions mentioned above by answering certain other problem on ordinary colouring. This will be a very useful tool as it is normally easier to deal with ordinary colouring problems than with general homomorphism problems. But before going any further we need to introduce some notation. For further details on homomorphism and related topics we refer to [31].

We will talk about graph properties. The type of property we will be interested in is the property of U-colourability, which we will denote it by $P(U)$. In other words, G has the property $P(U)$ if it admits a homomorphism to U. As has been mentioned earlier, when $U \cong K_k$ the property $P(U)$ is the property of being k-colourable. For simplicity this property will be denoted by $P(k)$. The following definition will help us to recognize certain kinds of colourings:

Definition 2.24 Given a graph property P, we say that a proper colouring of a graph G is an (m, P)-colouring, if the subgraph of G induced by the union of any m colour classes has property P. $\diamond$

For some properties P, the concept of (m, P)-colouring has been studied in the literature. For example, if P is the property of being independent, i.e., the property $P(1)$, then a $(1, P)$-colouring is just a proper colouring of graphs. If P is the property of being acyclic then a $(2, P)$-colouring of a graph is an acyclic colouring. (An acyclic colouring of a graph G, introduced by B. Grünbaum in [25], is a proper colouring of the vertices of G, in which every pair of colours induces an acyclic graph, i.e., a forest.)

For certain types of (m, P)-colouring the existence of (m, P)-colouring for G will depend on the nonexistence of certain structures in G. The following two propositions are examples of this phenomenon:

Proposition 2.25 *A graph G admits a $(k, P(k-1))$-colouring if and only if it is K_k-free.*

Proof. If G admits a $(k, P(k-1))$-colouring then any set of k colour classes induces a $(k-1)$-colourable graph, in particular G does not contain a K_k as a subgraph. Conversely if G does not contain a K_k then one can colour (for example) all the vertices with different colours to obtain a $(k, P(k-1))$-colouring. $\square$

Proposition 2.26 *Given a graph G, it admits a $(m, P(2))$-colouring if and only if odd–girth$(G) \geq m+1$.*

Proof. If G admits an $(m, P(2))$-colouring, then the union of any m colour classes (therefore any set of m vertices) induces a bipartite graph, hence *odd–girth*$(G) \geq m+1$. Conversely if *odd–girth*$(G) \geq m+1$ then it is enough to colour (for example) all the vertices with different colours to obtain an $(m, P(2))$-colouring of G. $\square$

We will also use the following notation:

For a finite set $\mathcal{F}$ of graphs we denote by $Forb_h(\mathcal{F})$ the class of all graphs G satisfying $F \nrightarrow G$ for all $F \in \mathcal{F}$. For example, if $\mathcal{F}$ consist of only K_3, then $Forb_h(\mathcal{F})$ is the class of all triangle-free graphs. Or if $\mathcal{F} = \{C_5\}$ then $Forb_{\mathcal{F}}$ is set all graphs with odd girth at least 7. More generally if $\mathcal{F}$ consists of all odd cycles, then $Forb_h(\mathcal{F})$ is the set of bipartite graphs.

Now the following theorem is a reformulation, in our terminology, of a theorem of R. Häggkvist and P. Hell from [30].

Theorem 2.27 *For a given positive integer d and a finite set of connected graphs $\mathcal{F}$, let $\mathcal{C}_d$ be the class of all graphs in $Forb_h(\mathcal{F})$ with maximum degree at most d. Then $\mathcal{C}_d$ is bounded by a graph H in $Forb_h(\mathcal{F})$.*

A new proof of this theorem will be given in Section 2.3. The next section is devoted to constructing a family of graphs with certain properties. We will show that for a given class C of graphs these constructions can be used to form a bound with certain properties, provided that each graph in $\mathcal{C}$ admits a certain type of colouring. In section 2.3, in addition to a new proof of Theorem 2.27, we also solve Problem 2.20 with a method different from the one in [61]. Section 2.3 is devoted to providing a better understanding of our constructions. In the last section we will prove that Hadwiger's conjecture is equivalent to Conjecture 2.10.

2.2 Construction of bounds

For a graph property P, we have introduced the notion of an (m, P)-colouring of a graph G (namely, it is a colouring where the subgraph induced by the union of any m colour classes has the property P). We now define the (m, P)-chromatic number of a graph G, denoted by $\chi_{m,P}(G)$, to be the minimum number of colours in an (m, P)-colouring of G (provided that one exists).

Given a set A and an integer m, the notation of $\binom{A}{m}$ is used to denote the set all subsets of A with m elements. Moreover $\binom{A}{m}_i$ will denote the set of all subsets of A with m elements containing the particular element i. Recall that the set of integers from 1 to n, i.e., $\{1, 2, \dots n\}$, will be denoted by $[n]$ and that an m-set is a set of m elements. Now the following definition introduces a key construction. It can be compared to [2, 63, 65] for constructions of similar flavor:

Definition 2.28 Let m and n be two positive integers and let U be a graph. Then the graph $\Pi = \Pi(n, m, U)$ is defined as below:

The vertex set $V(\Pi)$ of Π is the set of all ordered pairs (i, ϕ), where $i \in [n]$ and ϕ is a function from $\binom{[n]}{m}_i$ to $V(U)$.

The edge set $E(\Pi)$ of Π consists of all the unordered pairs $\{(i, \phi), (j, \psi)\}$ for which the following two conditions hold:

1. $i \neq j$;
2. $\phi(S)$ is adjacent to $\psi(S)$ in U whenever $\phi(S)$ and $\psi(S)$ both are well defined, that is for all the m-subsets S of $[n]$ which contain both i and j.

$\diamond$

To have a better understanding of this construction notice that if we colour each vertex (i, ϕ) by the first coordinate, i, we obtain an n colouring of $\Pi(n, m, U)$. The graphs obtained from this construction, which we will call Π-graphs, are normally very large. In general $\Pi(n, m, U)$ has $n \times |V(U)|^{\binom{n-1}{m-1}}$ vertices.

The smallest non-trivial Π-graph is the graph $\Pi(3, 2, K_2)$, which has 12 vertices and is depicted in Figure 2.3. In this figure the label ixy represents the vertex (i, ϕ) with $\phi(\{i, j_1\}) = x$ and $\phi(\{i, j_2\}) = y$ where $j_1 < j_2$. For example 3ba means the vertex $(3, \phi)$ where $\phi(\{13\}) = b$ and $\phi(\{23\}) = a$.

Our interest in these graphs is due to their homomorphism properties. But we may restrict a discussion of graphs in the context of homomorphism to graphs which are cores, since the core of a graph inherits all the homomorphism properties of the graph. In the case of Π-graphs this may become very handy as the core of a Π-graph can be much smaller than the Π-graph itself. As an example, the core of $\Pi(3, 2, K_2)$ is just a K_3, because it is 3-colourable and it contains K_3 as a subgraph. However we do not have an easy method to find the core of a Π-graphs in general. We will say more about the chromatic number and the fractional chromatic number of $\Pi(n, m, U)$ later on in this chapter.

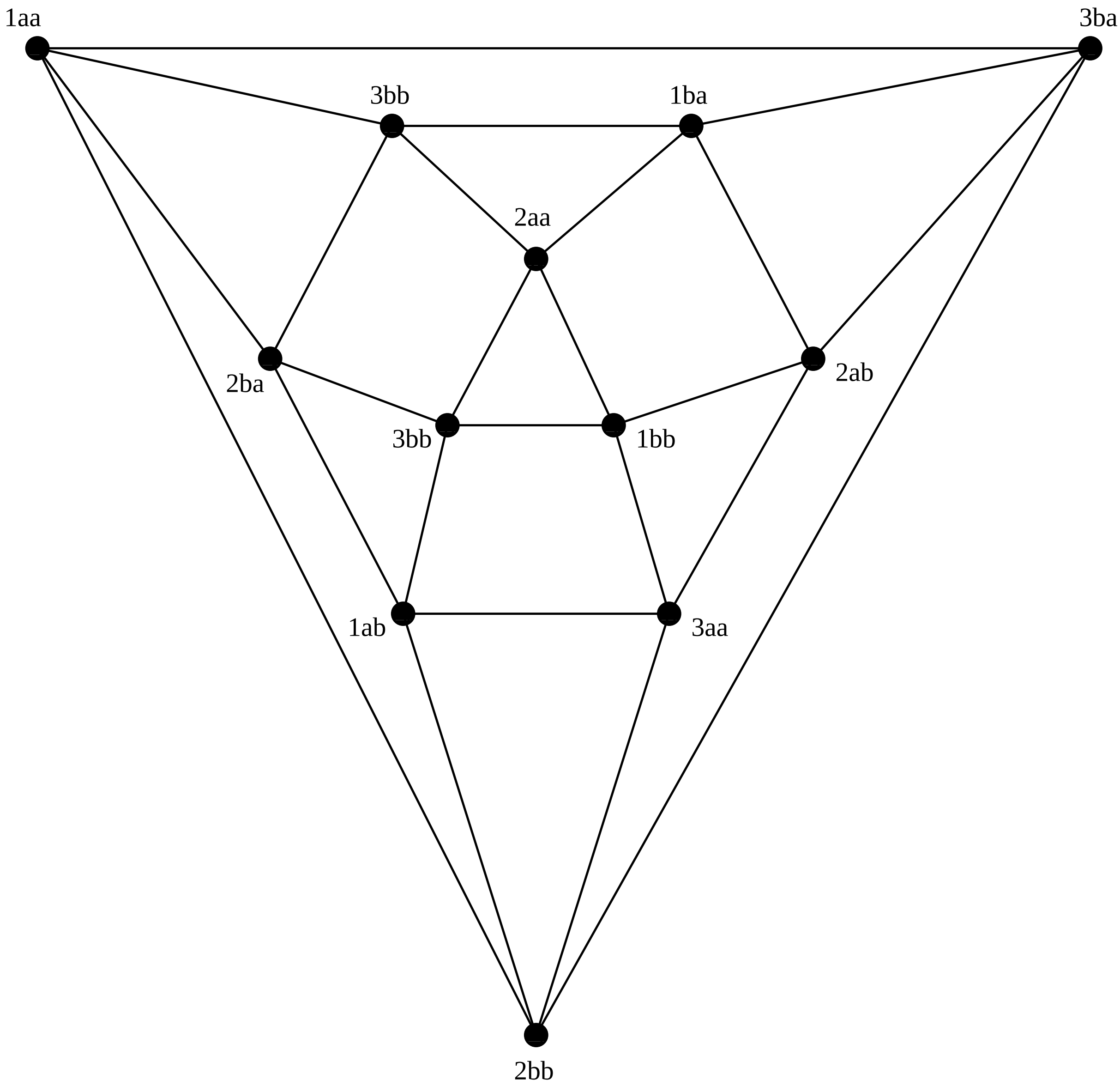

Figure 2.3: $\Pi(3, 2, K_2)$

To give a taste of the importance of Π-graphs, consider the typical problem of finding a K_k-free bound for some given family $\mathcal{C}$ of graphs. The point is that if there is such a K_k-free bound then $\Pi(n, k, K_{k-1})$ is also such a bound, for n large enough. In fact it will be sufficient to choose n large enough such that every graph G in $\mathcal{C}$ admits a $(k, P(k-1))$-colouring using at most n colours. Each of these statements will be proven in detail later on, but here we should remark that what makes it to work nicely, is replacing the property of being K_k-free with the equivalent property of $(k, P(k-1))$-colourability (see Proposition 2.25). The following then is a generalization of this concept:

Definition 2.29 Let G and U be graphs and m a positive integer. The graph G is said to be *m-locally U-colourable* if every induced subgraph of G with at most m vertices admits a homomorphism to U. $\diamond$

The following lemma is natural consequence of the definition.

Lemma 2.30 *Let U be a graph and let $\mathcal{F}$ be the set of all cores on at most m vertices which do not admit a homomorphism to U. Then a graph G is m-locally U-colourable if and only if it belongs to $Forb_h(\mathcal{F})$.*

Proof. Let G be a graph in $Forb_h(\mathcal{F})$, and let G' be any subgraph of G on at most m vertices. We show that G' maps to U. By contradiction, suppose G' does not admit a homomorphism to U. Then the core $c(G')$ of G' does not admit a homomorphism to U either, so it must be in $\mathcal{F}$. But $c(G') \to G'$ and therefore $c(G') \to G$ which contradicts the choice of G.

For the converse, assume G is m-locally U-colourable, and suppose for some $F \in \mathcal{F}$ there exists a homomorphism f of F to G . By the definition of $\mathcal{F}$, F and therefore the homomorphic image $f(F)$ of F in G, has at most m vertices. Since G is m-locally U-colourable, $f(F)$ maps to U by a homomorphism g. But then $g \circ f$ maps F to U which contradicts the choice of F. $\square$

Corollary 2.31 *Let U be a graph and let $\mathcal{F}$ be the set of all cores on at most m vertices which do not admit a homomorphism to U. Then a graph G is m-locally U-colourable if and only if it does not contain any member of $\mathcal{F}$ as subgraph.*

Proof. Let G be a graph which is not m-locally U-colourable. Then by Lemma 2.30 there is an $F \in \mathcal{F}$ which admits a homomorphism f to G. But then the core $c(f(F))$ of the image of F in G, is also a member of $\mathcal{F}$ and a subgraph of G. □

Obviously in applying this corollary we only need to consider the minimal elements of $\mathcal{F}$ with respect to taking subgraphs, in other words we only consider the elements of $\mathcal{F}$ which do not contain any other element as a subgraph.

Example 2.32 :
(a) m-locally K_2-colourable graphs are precisely the graphs not containing an odd cycle of length $\leq m$.
(b) k-locally K_{k-1}-colourable graphs (equivalently $(k+1)$-locally K_{k-1}-colourable graphs) are precisely the graphs not containing K_k.
(c) 9-locally C_5-colourable graphs are precisely graphs which do not contain any of the three graphs shown in the Figure 2.4 as a subgraph. ◇

Parts (a) and (b) of the example can be seen easily by Corollary 2.31. For (a), $\mathcal{F}$ is set to be the set of odd cycles of size at most m, and for (b) $\mathcal{F}$ is set to be $\{K_k\}$. For part (c) note that K_3 does not admit a homomorphism to C_5 and that any 4-chromatic graph on at most 9 vertices must contain K_3 as a subgraph. So applying Corollary 2.31 it will be enough to find the set of 3-colourable cores on at most 9 vertices which do not admit homomorphism to C_5. By an exhaustive search, aside from K_3 we found four such cores, which have been depicted in Figure 2.4 and Figure 2.5. The two graphs of Figure 2.5 contain the graph on the right side of the Figure 2.4. So in order to see if a graph G is 9-locally C_5-colourable we only need to check if it does not contain any of the three graphs in Figure 2.4.

We have the following important property of Π-graphs:

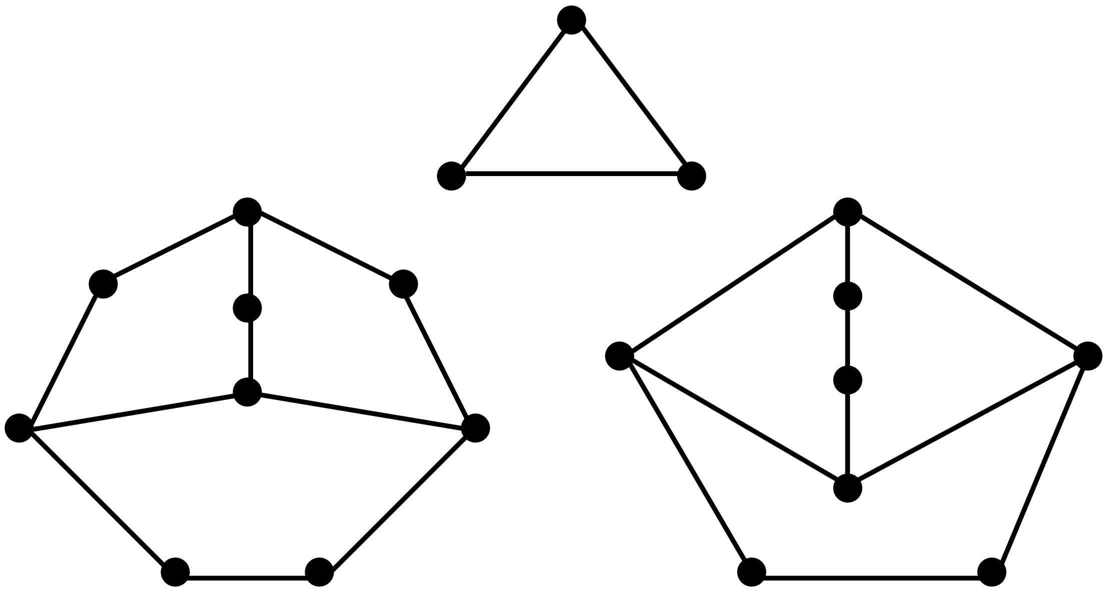

Figure 2.4: Forbidden subgraphs of 9-locally C_5-colourable graphs

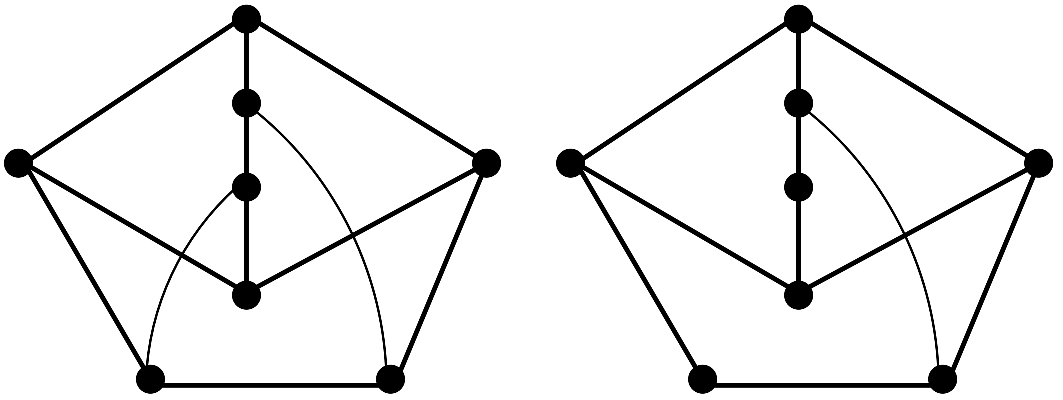

Figure 2.5: More 3-colourable cores

Proposition 2.33 *The graph $\Pi(n,m,U)$ is m-locally U-colourable.*

Proof. Let $\{(i_k,\phi_k)\}_{k=1}^m$ be any set of m vertices and let Π_1 be the subgraph of $\Pi(n,m,U)$ induced by these vertices. If S is any m-set containing $\{i_1,i_2,\cdots i_m\}$ (note that i_j's are not necessary distinct and $\{i_1,i_2,\cdots i_m\}$ may have fewer than m elements), then the mapping $(i_k,\phi_k)\to\phi_k(S)$ is a homomorphism of Π_1 to U. To see this let (i_k,ϕ_k) be a vertex adjacent to (i_l,ϕ_l), then S contains both i_k and i_l, and by definition $\phi_k(S)$ must be adjacent to $\phi_l(S)$. □

We are now ready to introduce the conditions under which one can construct an m-locally U-colourable bound. The first step is the following proposition.

Proposition 2.34 *Let $P(U)$ be the property of U-colourability. If $\chi_{m,P(U)}(G)\leq n$ then there is a homomorphism of G to $\Pi(n,m,U)$.*

In other words if G admits an n-colouring in which the union of every m-colour classes induces a U-colourable graph, then G admits a homomorphism to $\Pi(n,m,U)$.

Proof. Let $c:V(G)\to[n]$ be an $(m,P(U))$-colouring of G and let S be any subset of $[n]$ of cardinality m. The vertices coloured by colours from S induce a subgraph of G which we denote by G_S. By the definition of $(m,P(U))$-colouring, the subgraph G_S must be U-colourable. Let ρ_S be the homomorphism of G_S to U. We now define:

$$f:V(G)\to V(\Pi(n,m,U));\quad f(v)=(c(v),\phi_v),$$

where $\phi_v:\binom{[n]}{m}_{c(v)}\to V(U)$ is defined by $\phi_v(S)=\rho_S(v)$. The mapping $\phi(S)$ is well defined because it is only defined for subsets of size m which contain $c(v)$.

We now show that f is a homomorphism. Let u and v be two adjacent vertices in G and set $f(u)=(c(u),\phi_u)$ and $f(v)=(c(v),\phi_v)$. To see that $f(u)$ is adjacent to $f(v)$ in $\Pi(n,m,U)$ first of all note that $c(u)\neq c(v)$, because c is a proper colouring. Secondly, if S is any set containing both $c(u)$ and $c(v)$ then $\phi_u(S)=\rho_S(u)$ is adjacent to $\phi_v(S)=\rho_S(v)$ in U, because ρ_S preserve the adjacency. Therefore $f(u)$ and $f(v)$ are adjacent in $\Pi(n,m,U)$ and f is a homomorphism of G to $\Pi(n,m,U)$. □

If we are searching for an m-locally U-colourable bound for a given family of graphs, $\mathcal{C}$, then using Proposition 2.33 and Proposition 2.34 it will be enough to bound the $(m, P(U))$-chromatic number of graphs in $\mathcal{C}$. The following theorem shows that this sufficient condition is necessary too.

Theorem 2.35 *A class $\mathcal{C}$ of graphs is bounded by an m-locally U-colourable graph if and only if $\{\chi_{m,P(U)}(G) |\ G \in \mathcal{C}\}$ is bounded above by an integer.*

Proof. Let $\{\chi_{m,P(U)}(G) |\ G \in \mathcal{C}\}$ be bounded by an integer n. Then for every G in $\mathcal{C}$, $\chi_{m,P(U)}(G) \leq n$ and therefore by Proposition 2.34 G admits a homomorphism to $\Pi(n, m, U)$. In other words $\Pi(n, m, U)$ is a bound for $\mathcal{C}$, but by Proposition 2.33 $\Pi(n, m, U)$ is an m-locally U-colourable graph. Therefore $\Pi(n, m, U)$ is an m-locally U-colourable bound for the class $\mathcal{C}$.

For the converse, suppose that $\mathcal{C}$ is bounded by an m-locally U-colourable graph H. We claim that $|V(H)|$ is an upper bound for $\{\chi_{m,P(U)}(G) |\ G \in \mathcal{C}\}$. To prove our claim, take a graph G in $\mathcal{C}$, and consider a homomorphism $\phi : G \to H$. We show that ϕ is an $(m, P(U))$-colouring, this will complete the proof. Let G' be a subgraph of G which takes at most m colours. Then the image $\phi(G')$ of G' in H has at most m vertices. Since H is an m-locally U-colourable graph, $\phi(G')$ maps to U. So ϕ is an $(m, P(U))$-colouring and we are done. $\square$

We will be mainly concerned with applying following two versions of this theorem (Proposition 2.36 and Proposition 2.37).

Proposition 2.36 *Let n be a positive integer and $\mathcal{C}$ be a class of graphs. Then the following statements are equivalent.*

(a) $\mathcal{C}$ is bounded by a K_n-free graph.

(b) $\{\chi_{n,P(n-1)}(G) |\ G \in \mathcal{C}\}$ is bounded above by an integer.

(c) $\{\chi_{n+1,P(n-1)}(G) |\ G \in \mathcal{C}\}$ is bounded above by an integer.

Proof. By Theorem 2.35 $\{\chi_{n,P(n-1)}(G)|\ G \in \mathcal{C}\}$ is bounded above by an integer if and only if $\mathcal{C}$ is bound by an n-locally K_{n-1}-colourable graph. Similarly the set $\{\chi_{n+1,P(n-1)}(G)|\ G \in \mathcal{C}\}$ is bounded by an integer if and only if $\mathcal{C}$ is bounded by an $(n+1)$-locally K_{n-1}-colourable graph. But both the properties of being n-locally K_{n-1}-colourable and being $(n+1)$-locally K_{n-1}-colourable are equivalent to the property of being K_n-free (see part (b) of Example 2.32). $\square$

The second application of Theorem 2.35 is to the property of being bipartite.

Proposition 2.37 *Let n be a positive odd integer and $\mathcal{C}$ be a class of graphs. Let B be the property of being bipartite. Then the following statements are equivalent.*

(a) $\mathcal{C}$ is bounded by a graph of odd girth n.

(b) $\{\chi_{n-2,B}(G)|\ G \in \mathcal{C}\}$ is bounded above by an integer.

(c) $\{\chi_{n-1,B}(G)|\ G \in \mathcal{C}\}$ is bounded above by an integer.

Proof. The property of having odd girth at least n is equivalent $(n-1)$-local K_2-colourability, and also to $(n-2)$-local K_2-colourability. Now apply Theorem 2.35. $\square$

2.3 Applications of Theorem 2.35

In this section, in two typical examples, we will show how one can apply Theorem 2.35, or Proposition 2.36 in particular, to construct bounds with certain properties. The first application is a new proof of Theorem 2.27 given below:

Theorem [30] For a given positive integer d and a finite set of connected graphs $\mathcal{F}$, let $\mathcal{C}_d$ be the class of all graphs in $Forb_h(\mathcal{F})$ with maximum degree at most d. Then $\mathcal{C}_d$ is bounded by a graph H in $Forb_h(\mathcal{F})$.

Proof. By replacing every graph in $\mathcal{F}$ with its core, we may assume $\mathcal{F}$ is a finite set of connected cores. Put $m = \max\{|V(F)|\ ; F \in \mathcal{F}\}$ and let U be the disjoint union of all (non-isomorphic) graphs in $Forb_h(\mathcal{F})$ which have at most m vertices.

We claim that $Forb_h(\mathcal{F})$ is precisely the class of all m-locally U-colourable graphs. To see this first consider a graph G which is in $Forb_h(\mathcal{F})$. Then any subgraph G' of G on at most m vertices is also in $Forb_h(\mathcal{F})$, and therefore, by the definition of U, G' is a component of U, hence $G' \to U$. For the other direction, let G be an m-locally U-colourable graph. We shall prove that G is in $Forb_h(\mathcal{F})$. By contradiction, suppose there is an $F \in Forb_h(\mathcal{F})$ which maps to G. Since F is connected, the image of F in G is a connected subgraph of G on at most m-vertices. This subgraph, and therefore F, must map to U. So there is mapping of F to one component of U, but this contradicts the definition of U.

We shall prove that for any $G \in \mathcal{C}_d$ we have $\chi_{m,P(U)}(G) \leq d^{2m+1}$. This will complete the proof as, by Theorem 2.35, the class $\mathcal{C}_d$ is then bounded by an m-locally U-colourable graph H, but as we have shown above any m-locally U-colourable graph is in $Forb_h(\mathcal{F})$.

Given $G \in \mathcal{C}_d$ we define a new graph $G^{(m)}$ as follows: $G^{(m)}$ has the same set of vertices as G, and two distinct vertices are joined by an edge if and only if they are joined in G by a path of length at most m. Note that $\Delta(G^{(m)}) < d^{m+1}$ and therefore by Brooks' theorem, $G^{(m)}$ admits a d^{m+1}-colouring c. The colouring c is also a vertex colouring of G with the property that any two distinct vertices of G in distance at most m are coloured differently.

We now prove that c is an $(m, P(U))$-colouring of G. Let G' be a subgraph of G induced by any m colour classes and let G'' be one of its components. Every pair of vertices in G'' is joined by a path in G''. If any of these paths is of length at least m then its vertices take at least $m+1$ colours (on the first $m+1$ vertices of the path). By the choice of G' this is impossible, so every two distinct vertices in G'' are joined by a path of length at most $m-1$, and so take distinct colours. Therefore G'' has at most m vertices. But G'' is in $Forb_h(\mathcal{F})$ because it is a subgraph of G and $G \in Forb_h(\mathcal{F})$.

Hence by the definition of U, G'' is U-colourable, and so is G'. □

Given positive integers d and k let $\mathcal{C}_d^k$ be the class of all k-colourable graphs with maximum degree at most d. The next theorem is a generalization of Theorem 2.27 which was first proved in [17].

Theorem 2.38 [17] *Let d and k be positive integers and let $\mathcal{F}$ be a family of connected graphs. The class $\mathcal{C} = \mathcal{C}_d^k \cap Forb_h(\mathcal{F})$ is bounded by a k-colourable graph in $Forb_h(\mathcal{F})$.*

Proof. Let H be the bound obtained from Theorem 2.27, then $H \in Forb_h(\mathcal{F})$. By definition $\mathcal{C}$ is also bounded by K_k. Now by Proposition 2.15 and Proposition 2.16 $H \times K_k$ is a k-colourable bound for $\mathcal{C}$ from $Forb_h(\mathcal{F})$. □

Remark As shown in [30], the condition of connectedness for the forbidden graphs in Theorem 2.27 and Theorem 2.38 is necessary. For example, let F be a graph which is disjoint union of two incomparable graphs G and H, i.e., $G \nrightarrow H$ and $H \nrightarrow G$. Now the subclass $\mathcal{C}_d$ of $Forb_h(F)$ for d larger than the maximum degrees of both G and H, contains both G and H. Thus any bound for $\mathcal{C}_d$ admits a homomorphism from F and therefore $\mathcal{C}_d$ can not be bounded in $Forb_h(F)$.

As a next application we will construct a K_5-free bound for the class of planar graphs, thereby providing another proof for 2.23 without using the four colour theorem. Notice that in order to find a K_5-free bound using Theorem 2.35 we must show that $\chi_{(4,P(5))}(G)$ is bounded for the set of planar graphs. To do this we will define below a certain kind of colouring called "diverse colouring". Then we will show, by introducing a diverse colouring algorithm, that every planar graph admits a diverse colouring using at most k colours, for some fixed k. Finally we will show that the colouring obtained by the diverse-colouring algorithm is also a $(4, P(5))$-colouring.

A theorem of A. Kotzig, which is a consequence of Euler formula for planar graphs, will play an important role in our algorithm. Given a graph G and a vertex x of G let $d(x)$ denote the degree of x in G. Then Kotzig's theorem sates that:

Theorem 2.39 [44] *For a given planar graph G, either G contains a vertex of degree at most 2, or it contains an edge $e = uv$ with $d(u) + d(v) \leq 13$.*

In order to find a $(m, P(k))$-colouring of a graph G, we should find a colouring of G in which every $(k+1)$-chromatic subgraph of G, or equivalently every $(k+1)$-critical subgraph of G, takes at least $m+1$ different colours. In particular to find a K_5-free bound for the class of planar graphs using Theorem 2.35, we should colour every planar graph in such a way that every 5-critical subgraph of it takes at least 6 different colours. This leads us to a study of 5-critical planar graphs.

An alternative form of the four colour theorem is to state that "there is no 5-critical planar graph". In the absence of the four colour theorem the following lemma will help us to achieve our goal of constructing a K_5-free bound for the class of planar graphs. The proof of this lemma is inherited from the Kempe-chain proof of the five colour theorem, but since we have not found this lemma clearly stated in the literature, we will include a proof here.

Lemma 2.40 *If G is a 5-critical planar graph then $\delta(G) \geq 5$.*

Proof. By contradiction, suppose G is a 5-critical planar graph with a vertex x of degree 4 or less. If $d(x) \leq 3$ then any 4 colouring of $G \backslash x$ can be extended to a 4 colouring of G. So we may assume $d(x) = 4$. Consider a 4 colouring of $G \backslash x$, since this is not extendable to a 4 colouring of G then all the four colours must appear on the four neighbours of x.

Consider a planar drawing of G and let 1, 2, 3, and 4 be the four distinct colours of the neighbours of x in a cyclic order. Let x_1 and x_3 be the neighbours of x with colours 1 and 3 respectively. Also let G_{13} be the subgraph of $G \backslash x$ induced by the colours 1 and 3. Then x_1 and x_3 must be in the same connected component of G_{13}, otherwise we will find an extendable 4-colouring of $G \backslash x$ just by exchanging the colours 1 and 3 only in the component of G_{13} containing x_1.

This means that there must be a path connecting x_1 and x_3 which uses only colours 1 and 3. Similarly there must be a path connecting the other two neighbours of x which uses only colours 2 and 4. Obviously these two paths can not intersect in a vertex, but they must intersect somewhere. This contradicts the planarity of G. $\square$

The next step is to define a diverse colouring:

Definition 2.41 For given integers k and l, we say that an l-colouring c of a given graph G is a *k-diverse colouring*, if, for each vertex x of G, at least $min\{d(x), k\}$ different colours appear on the neighbours of x. A k-diverse colouring which uses at most l colours will be called a (k, l)-colouring. $\diamond$

Theorem 2.42 *Given an integer $k \geq 11$, every planar graph admits a $(k, 5k + 8)$-colouring.*

Proof. We will prove this by induction on the number of vertices of G. For graphs on at most $5k+8$ vertices we can colour all the vertices with different colours. Suppose we have found a $(k, 5k + 8)$-colouring for every planar graph on at most n vertices and let G be a planar graph on $n + 1$ vertices. We may assume G is connected, because otherwise $(k, 5k + 8)$-colourings of the components of G all together will give a $(k, 5k + 8)$-colouring of G.

If G has a vertex x with $d(x) = 1$ then any $(k, 5k + 8)$-colouring of $G\backslash x$ can be extended to a $(k, 5k + 8)$-colouring of G. To see this let y be the only neighbour of x. If there are at least k-different colours on the neighbours of y, then any colour different from the colour of y will work. Otherwise $d(y) \leq k - 1$ and we choose a colour which has not appeared on y or any of its neighbours, this is indeed possible because there are more than k colours available.

If G does not have a vertex of degree 1 but it has a vertex x of degree 2 then we identify x with one of its neighbours, remove the loop and the possible multiple edge. We call the new graph G_x. By induction G_x admits a $(k, 5k+8)$-colouring c_x. Colour all the vertices of G except x with the same colour as in the colouring c_x. Notice that neighbours of x have taken two different colours, and in order to extend c_x to a $(k, 5k + 8)$-colouring of G all we need to do is to choose a colour for x different from colours of its neighbours in such a way that the requirement of diversity for the neighbours of x still holds.

For each neighbour y of x, either y already has k different colours on its neighbours or $d(y) \leq k-1$. In the first case the only restriction for the colour of x, coming from y, is to have a colour different from the colour of y, (in order to have a proper colouring). In the second case, i.e., if $d(y) \leq k - 1$, the vertex x must take a colour different from the colours of y and all of its neighbours. In either of the cases, each neighbour of x introduces at most k-colours not admissible for x. Since x has two neighbours there are maximum of $2k$ colours not admissible for x, so c_x can be extended to a $(k, 5k + 8)$-colouring of G.

If neither of the previous two cases happens, then $\delta(G) \geq 3$ and by Theorem 2.39 there is an edge $e = uv$ with $d(u) + d(v) \leq 13$. Without loss of generality assume $d(u) \leq d(v)$. Therefore $d(u) \leq 6$. Identify u and v, remove loops and possible multiple edges and call the new graph G_e. Let v' to be the new vertex in G_e (obtained from identifying u and v), then $d(v') \leq 11$. By induction G_e admits a $(k, 5k+8)$-colouring, we denote this colouring by c_e. Note that all the neighbours of v' have taken different colours (this is because $k \geq 11$).

To find a $(k, 5k + 8)$-colouring of G, colour every vertex $x \notin \{u, v\}$ with $c_e(x)$ and colour v with $c_e(v')$. To complete this colouring all we need is to find an admissible colour for u. Notice that all the neighbours of u have already received different colours. Let $t \neq v$ be a neighbour of u, if $d(t) > k$ then t already has k neighbours with k distinct colours and the only restriction coming from t is that $c(t)$ be different from the colour which we choose for u.

If $d(t) \leq k$ then the colour we would like to choose for u has to be different from colours of t and all of its neighbours. This will remove at most k colours from the list of available colours for u. Similarly there will be also at most $d(v)$ forbidden colours because of the diversity condition for v. In total there will be at most $k(d(u)-1)+d(v)=(k-1)d(u)-k+d(u)+d(v)\leq 5k+7$ forbidden colours for u. Since there are $5k+8$ possible colours, we can find an admissible colour for u. □

In the proof of the last theorem we introduced an inductive algorithm to find a $(k, 5k+8)$-colouring of any planar graph. We will call this algorithm *k-diverse colouring algorithm*. In the next theorem we will show that the colouring obtained from the 11-diverse colouring algorithm satisfies the condition of the Proposition 2.36 and hence provides us with a K_5-free bound for the class of planar graphs.

Theorem 2.43 *Let G be a planar graph and c an $(11, 63)$-colouring of G obtained from 11-diverse colouring algorithm. Then c has the property that every 5-chromatic subgraph of G takes at least 6 different colours.*

Proof. It will be enough to show that every 5-critical subgraph has taken 6 different colours. We prove this by contradiction. Suppose this is not true and algorithm fails at some point. Let G be the smallest graph for which the 11-diverse algorithm fails, i.e., for every graph on at most $|V(G)|-1$ vertices the $(11, 63)$-colouring obtained from 11-diverse colouring algorithm has the required property but the colouring obtained for G by this algorithm uses only five colours on some 5-critical subgraph H of G.

It is easy to see that G does not contain vertices of degree 1 or 2. In fact if $\delta(G)=1$ or 2 then $(11, 63)$-colouring of G has been obtained from $(11, 63)$-colouring of some G_x where x is a vertex of degree 1 or 2. But then every 5-critical subgraph of G is also a subgraph of G_x and therefore takes at least 6 different colours.

So we may assume $\delta(G)\geq 3$. Let u and v be the vertices of G as in the algorithm. Recall that to obtain the colouring c we basically used an $(11, 63)$-diverse colouring of G_e and we found an admissible colour for u. By the minimality of G the 11-diverse colouring of G_e has used at least 6 different colours on any 5-critical subgraph of G_e.

So H could not be a subgraph of G_e, therefore it must contain both u and v. By lemma 2.40 degree of u in H must be at least 5. But all the neighbours of u have been given different colours. By adding the colour of u itself to this collection we will find at least 6 different colours on the vertices of H which is a contradiction. □

Notice that this theorem has only been proved for a 11-diverse colouring obtained from the 11-diverse colouring algorithm. We do not have a proof for a general 11-diverse colouring.

Theorem 2.44 *The class $\mathcal{P}$ is bounded by a K_5-free graph.*

Proof. Applying Theorem 2.35 and Theorem 2.43 we see that the class $\mathcal{P}$ is bounded by $\Pi(63, 5, K_4)$. Proposition 2.36 shows that this graph is K_5-free. □

Some other applications of Theorem 2.35 can be found in the recent papers of P. O. De Mendez and J. Nešetřil, [60] and [61]. In [60] the authors have answered Problem 2.18 affirmatively, by showing that every triangle-free planar graph admits a k-colouring (for some fixed k) in which every odd-cycle takes at least 4 different colours. Also in [61] they have generalized their method to prove the following general theorem.

Theorem 2.45 [61] *Let $\mathcal{C}'$ be the class of all K_k-free graphs in a minor-closed family $\mathcal{C}$ of graphs. Then C' is bounded by a K_k-free graph.*

To see the elegance of this theorem compare it with the equivalent form of the Hadwiger's conjecture: "Any minor-closed family of K_k-free graphs is bounded by K_{k-1}".

2.4 The Π-graphs

In this section we further investigate Π-graphs in order to have a better understanding of them. We find an improved bound for the chromatic number of Π-graphs. We also introduce bounds on the fractional chromatic number of these graphs. Fractional chromatic number has been defined below, but it will be studied in more details in Chapter 4.

Definition 2.46 An n-set colouring of a graph G is an assignment of n-sets to the vertices of G in such a way that any pair of adjacent vertices receive disjoint sets. The minimum number of the total colours required in an n-set colouring of G is denoted by $\chi_n(G)$. The fractional chromatic number of G, denoted by $\chi_f(G)$, is defined to be $liminf\{\frac{\chi_n(G)}{n}\}$. $\diamond$

Fractional chromatic number is also a monotone graph parameter, a proof of this well known fact will be given in Chapter 4. Fractional chromatic number is a difficult parameter to calculate, but if G is a vertex transitive graph, then $\chi_f(G)$ is known to be equal to $\frac{|V(G)|}{\alpha(G)}$, where $\alpha(G)$ is the size of maximum independent set, see [23] for a proof of this.

On the other hand, while the problem of finding core of a graph in general is a difficult problem, there are some methods which helps us to find the core of a vertex transitive graph. For example, it is a known fact that the number of vertices of the core of a vertex transitive graph G, divides the number of vertices of G, see [23]. For these reasons our first attempt in this section is to investigate cases when $\Pi(n, m, U)$ is vertex transitive.

Lemma 2.47 *Let n and m be any two integers. If U is a vertex transitive graph then $\Pi(n, m, U)$ is also vertex transitive.*

Proof. Assume U is a vertex transitive graph. We first prove certain pairs of vertices of $\Pi(n, m, U)$ can be mapped to each other using an automorphism of $\Pi(n, m, U)$. The type of pairs we would like to consider first, are the pairs (A, B) with $A = (1, \varphi)$ and $B = (1, \varphi')$ where φ and φ' differ only in one m-subset S_0 of $[n]$. In other words we may assume $\varphi(S) = \varphi'(S)$ for every S containing 1, except $S_0 = \{1, 2, \cdots m\}$. Let $\varphi(S_0) = x$ and $\varphi'(S_0) = y$, where $x \neq y$.

To prove the existence of an automorphism Θ that maps A to B, notice that since U is a vertex transitive graph there is an automorphism θ of U which maps x to y (i.e., $\theta(x) = y$). Now we may define the automorphism Θ of $\Pi(n, m, U)$ this way:

$$\Theta((x,\psi)) = (x,\psi'), \quad \text{where } \psi'(S) = \begin{cases} \psi(S) & \text{if} \quad S \neq S_0 \\ \theta(\psi(S)) & \text{if} \quad S = S_0. \end{cases}$$

It is easy to check that Θ is a one to one and onto homomorphism of $\Pi(n,m,U)$ to $\Pi(n,m,U)$, therefore it is an automorphism of $\Pi(n,m,U)$. Moreover by the choice of A and B we find that $\Theta(A) = B$.

Now consider a pair $\{C, D\}$ of the vertices of $\Pi(n,m,U)$ where $C = (1,\varphi)$ and $D = (1,\psi)$, with no restriction on φ or ψ. Note that φ and ψ are defined on the same m-subsets of $[n]$. Assume φ and ψ differ in k places, then it will be enough to repeat the previous argument k times to find an automorphism of $\Pi(n,m,U)$ which maps C to D.

To complete our proof it will be enough to show that $(1,\varphi)$ can be mapped (using an automorphism of $\Pi(n,m,U)$) to a vertex of type $(2,\psi)$. But this becomes an obvious fact by considering the automorphism of $\Pi(n,m,U)$ induced by any permutation of $\{1,2,\cdots n\}$ which maps 1 to 2. $\square$

The second lemma concerns the chromatic number of the Π-graphs. The definition of $\Pi(n,m,U)$ implies a natural n-colouring, therefore $\chi(\Pi(n,m,U)) \leq n$. The following easy lemma improves this bound in some cases.

Lemma 2.48 *For any two positive integers m and n we have $\chi(\Pi(n,m,U)) \leq \lceil \frac{n}{m} \rceil \chi(U)$.*

Proof. Let c be a $\chi(U)$-colouring of U using the colours $\{1,2,\cdots \chi(U)\}$. The following is then a proper colouring of $\Pi(n,m,U)$ which uses at most $(\lfloor \frac{n-1}{m} \rfloor + 1)\chi(U)$ colours.

$$C((i,\varphi)) = \lfloor \frac{i-1}{m} \rfloor \chi(U) + c(\varphi(S)), \text{ where } S = \{\lfloor \frac{i-1}{m} \rfloor m + 1, \cdots \lfloor \frac{i-1}{m} \rfloor m + m\}.$$

To see this is a proper colouring let $A = (i,\varphi)$ and $B = (j,\phi)$ be two adjacent vertices of $\Pi(n,m,U)$. If $\lfloor \frac{i-1}{m} \rfloor \neq \lfloor \frac{j-1}{m} \rfloor$ then A and B obviously receive different colours.

Otherwise $\lfloor\frac{i-1}{m}\rfloor = \lfloor\frac{j-1}{m}\rfloor$ and the m-subset $S = \{\lfloor\frac{i-1}{m}\rfloor m+1, \cdots \lfloor\frac{i-1}{m}\rfloor m+m\}$ contains both i and j. But then $C(A)-C(B) = c(\varphi(S))-c(\phi(S)) \neq 0$ because $(i,\varphi) \sim (j,\phi)$).

To complete the proof observe that $\lfloor\frac{n-1}{m}\rfloor + 1 = \lceil\frac{n}{m}\rceil$ holds for every pair of positive integers, m and n. $\square$

Example 2.49 As an example we have $\chi(\Pi(4n, 2n, K_n)) \leq 2n$. This bound improves the trivial bound of $4n$. $\diamond$

Our last lemma in this section is about the fractional chromatic number of the Π-graphs. It can be easily seen from the definition that fractional chromatic number is always bounded by the chromatic number, therefore Lemma 2.48 naturally provides an upper bound of $\lceil\frac{n}{m}\rceil\chi(U)$ for the fractional chromatic number $\chi_f(\Pi(n,m,U))$, but since fractional chromatic number admits non integer values we can improve this bound to $\frac{n}{m}\chi(U)$. This is done in the following lemma.

Lemma 2.50 *The fractional chromatic number of* $\Pi(n,m,U)$ *is smaller than or equal to* $\frac{n}{m}\chi(U)$.

Proof. Let c be a $\chi(U)$-colouring of U. We define an assignment A of $\binom{n-1}{m-1}$-sets to the vertices of $\Pi(n,m,U)$ as below:

$$A((i,\varphi)) = \{(S, c(\varphi(S)))|i \in S\}$$

.

We first claim that A is an $\binom{n-1}{m-1}$-set colouring of $\Pi(n,m,U)$. To see this let $a = (i,\varphi)$ and $b = (j,\phi)$ be two adjacent vertices in $\Pi(n,m,U)$. Then $A(a) \cap A(b) = \emptyset$ because for any S which contains both i and j, $\varphi(S)$ and $\phi(S)$ are adjacent vertices in U and therefore $c(\varphi(S)) \neq c(\phi(S))$.

To complete the proof we show that A uses at most $\binom{n}{m}\chi(U)$ colours, but this is clear because there are maximum of $\binom{n}{m}$ choices for S and $\chi(U)$ choices for $c(\varphi(S))$. $\square$

The K_5-free bound for the class of planar graphs we provided in the last section, has a large number of vertices. The following problem, if answered without using the four colour theorem, would provide us with a better K_5-free bound which by Lemma 2.50 would have a fractional chromatic number smaller than or equal to $\frac{24}{5}$. Notice that the best known bound for the fractional chromatic number of planar graphs without using the four colour theorem is that it is strictly smaller than 5.

Problem 2.51 *Without using the four colour theorem, show that every planar graph can be coloured using 6 colours in such a way that every 5-chromatic subgraph receives all 6 different colours.*

2.5 On Hadwiger's Conjecture

Hadwiger's conjecture was introduced in the introduction of this chapter, where we also introduced a reformulation of the conjecture (from [60]), in our terminology. This reformulation splits the conjecture into two different conjectures, Conjecture 2.10 and Conjecture 2.11. Conjecture 2.10 has been introduced in [60] as a weaker form of Hadwiger's conjecture. In this section we show that Conjecture 2.11 is implied by Conjecture 2.10 and therefore prove that Hadwiger's conjecture is equivalent to Conjecture 2.10.

Theorem 2.52 *Suppose every minor-closed family of graphs contains a maximum. Let $\mathcal{C}$ be any minor-closed family of graphs with a maximum element H. Then H must be homomorphically equivalent to a complete graph.*

Proof. We will prove this by contradiction. Assume this is not true for some minor-closed families. Let $\mathcal{G}/K_k$ be the class of all graphs which do not contain K_k as a minor. By Lemma 2.8 any proper minor-closed family is contained in some $\mathcal{G}/K_k$. Let k be the smallest integer such that $\mathcal{G}/K_k$ contains a minor-closed subfamily, $\mathcal{C}$, for which the statement of the theorem does not hold. Note that k must be greater than or equal to 7 (because for the smaller values of k Hadwiger's conjecture has been verified).

Let H be a maximum of $\mathcal{C}$. Then the class formed by H and all of its minors is a finite minor-closed family of graphs for which the statement of the theorem also does not hold. Because of this finiteness we may assume $\mathcal{C}$ is a minimal subfamily of $\mathcal{G}/K_k$ with respect to having a maximum, H, which is not homomorphically equivalent to a complete graph. By the minimality, $\mathcal{C}$ must be formed only by the family of all the minors of H.

By the choice of k, we know K_{k-1} must also be in $\mathcal{C}$, otherwise $\mathcal{C} \subseteq \mathcal{G}/K_{k-1}$ and we are done. Since H is a maximum of $\mathcal{C}$, and K_{k-1} is an element of $\mathcal{C}$ we have $K_{k-1} \to H$. Thus K_{k-1} must be also a subgraph of H. Let K be the subgraph of H which is isomorphic to K_{k-1}.

We first claim that every vertex of K must be adjacent to a vertex of H which is not in K. To see this, suppose there is a vertex x of K which is only adjacent to the $k-2$ vertices of $V(K)\backslash x$. By the minimality of $\mathcal{C}$, the graph H_x obtained from H by deleting the vertex x must be $(k-1)$-colourable. Otherwise H_x with all of its minors form another minor closed family for which the maximum is not homomorphically equivalent to a complete graph. This family is properly contained in $\mathcal{C}$, which contradicts the minimality of $\mathcal{C}$. Since x is adjacent to $k-2$ vertices, any $k-1$ colouring of H_x can be extended to a $k-1$ colouring of H. This implies that H must be homomorphically equivalent to K_{k-1}, which is a contradiction.

Our next claim is that the induced subgraph H' of H on $V(H)\backslash V(K)$ is connected. Again by contradiction assume it has parts H'_1 and H'_2 with no edges from H'_1 to H'_2. Then by a similar argument as before each of the subgraphs induced on $V(H'_1) \cup V(K)$ and $V(H'_2) \cup V(K)$ must be $(k-1)$-colourable. But then just a permutation of colours will produce a $k-1$ colouring of H. Thus H must be homomorphically equivalent to K_{k-1}.

To complete the proof, note that because H' is connected, by contracting all the edges in H' we will obtain a single vertex which must be adjacent to all the vertices of K. Therefore K_k is a minor of H, but this contradicts the choice of $\mathcal{C}$ and H. □

This theorem proves that the validity of Hadwiger's conjecture for all graphs is equivalent to the validity of Conjecture 2.10 for all minor-closed classes. For the sake of completeness we give a proof of this equivalence in the following theorem.

Theorem 2.53 *The following two statements are equivalent:*

(a) Every graph G with $\chi(G) = k$ contains K_k as a minor.

(b) Every proper minor-closed family of graphs contains a maximum with respect to homomorphism order.

Proof. Suppose (a) is true, and let $\mathcal{C}$ be a proper minor-closed family of graphs. Then by Lemma 2.8 the chromatic number of the graphs in $\mathcal{C}$ is bounded. Let k be the maximum chromatic number of the graphs in $\mathcal{C}$ and let G be a graph in $\mathcal{C}$ with the chromatic number equal to k. Then by (a), G contains K_k as a minor, so K_k is in the class, and therefore C contains a maximum.

For the other side assume (b) is true, and let G be a graph with $\chi(G) = k$. Then the class $\mathcal{C}_G$ which is formed from G and all of its minors has a maximum. By Theorem 2.52 such a maximum can be chosen to be a complete graph K_r. But since $G \to K_r$, $r \geq \chi(G)$ and therefore G contains K_k as a minor. □

Note that the statement (b) of the Theorem 2.53 has been proved for an important family of minor closed classes of graphs. Given a surface S, let $\mathcal{C}_S$ be the class of all graphs embedded on S. Then it is known that for every surface S, the class $\mathcal{C}_S$ is bounded by the maximal complete graph in $\mathcal{C}_S$. The most difficult case of this statement is to prove it for the simplest surface, the sphere. This case is equivalent to the four colour theorem.

For any other surface, (that is a surface S with Euler characteristic $\epsilon < 2$), it was proved by P. J. Heawood that any graph embedded on S has chromatic number at most $H(\epsilon) = \lfloor \frac{7+\sqrt{49-24\epsilon}}{2} \rfloor$. Then in a series of work from 1891 to 1974 it was proved by L. Heffter, H. Tietze, G. Ringel and J. W. T. Youngs that any surface of Euler characteristic ϵ, except the Klein bottle, admits an embedding of a complete graph on $H(\epsilon)$ vertices. For the Klein bottle the problem is settled by P. Franklin who proved that any graph embedded on the Klein bottle is 6-colourable, (note that Klein bottle has Euler characteristic 0, and $H(0) = 7$). We refer to [38] and [49] for further details and proofs.

It will also be of interest to find an independent proof for Conjecture 2.11 for a particular minor closed class of graphs. We believe this should not be very difficult to achieve. A simple proof of the following Proposition, due to P. Hell, [34], is an evidence for our claim.

Proposition 2.54 [34] *If the class $\mathcal{P}$ of planar graphs is bounded by a planar graph H, then H must be homomorphically equivalent to K_4.*

A proof of this proposition can be found in the next chapter, (see the second proof of Theorem 3.37).

Note that in the proof of the Theorem 2.52 we only used the weaker assumption that every minor closed family consisting of a graph and all of its minors contains a maximum. Using this reformulation of the Theorem 2.52 it is easy to see that the following is yet another reformulation of Hadwiger's conjecture.

Conjecture 2.55 *Let G be a graph and let H_1 and H_2 be two minors of G, then $\{H_1, H_2\}$ is bounded by a minor of G.*

Chapter 3

Homomorphisms and Planarity

3.1 Introduction

In this chapter we shall consider some of the old colouring problems for planar graphs in the context of graph homomorphisms. Sometimes we will need a specific planar drawing of a planar graph (multi-graph); for this purpose a planar graph with an specific planar drawing will be called a *plane* graph (multi-graph). Given a plane multi-graph G, the *dual* of G is defined to be a multi-graph whose vertex set is the set of faces of G, and two vertices are joined by an edge if they share an edge in G.

One classic result on colouring of planar graphs, is the celebrated theorem of Grötzsch. In the previous chapter we stated Grötzsch's theorem in two ways, and there are still different ways of stating this theorem, each leading to a new subject of study in the theory of colouring of planar graphs. One of the ways of stating Grötzsch theorem is to say that "every planar graph of girth at least 4 admits a homomorphism to C_3". The following conjecture is then a generalization of Grötzsch theorem. This conjecture which is a nice example of an open problem in the theory of colouring of planar graphs, has recently attracted the attention of many graph theorists.

Conjecture 3.1 *Every planar graph of girth at least* $4k$ *admits a homomorphism to* C_{2k+1}.

This conjecture is in fact a restricted version of a conjecture of Jaeger [20], which will be described (in terms of flows) in Section 3.3.

In support of Conjecture 3.1, it was proved by A. Galluccio, L. Goddyn and P. Hell that every planar graph of girth at least $10k-4$ admits a homomorphism to C_{2k+1}, see [21]. It was also conjectured by P. Hell that the condition of "high girth" in their theorem can be replaced by the weaker condition of "high odd grith". Hell's conjecture was proved by W. Klostermeyer and C.Q. Zhang in the following form:

Theorem 3.2 [42] *Every planar graph of odd girth at least* $10k-3$ *admits a homomorphism to* C_{2k+1}.

Their technique in proving this theorem was to show that the crucial cycles in C_{2k+1}-colouring of planar graphs with high odd girth are the facial cycles, then they introduced a lemma so called "Folding lemma". Folding lemma helps to find a homomorphic image of any given plane graph, without reducing the odd girth, while the image is still a plane graph and has the additional important property that every facial cycle is of the length of the odd girth. The following is the full strength of the Folding lemma, but we will only use the weaker form stated below as a corollary.

Folding lemma [42] Let G be a plane graph with odd girth $2g+1$. If $C = v_0v_1\cdots v_{r-1}v_0$ is a facial cycle of G with $r \neq 2g+1$, then there is an $i \in \{0, 1, \cdots r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (indices being taken modulo r) is still of odd girth $2g+1$.

Corollary 3.3 *Let* G *be a plane graph with odd girth at least* $2g+1$. *Then there is a homomorphic image* G' *of* G, *where* G' *is also a plane graph, the odd girth of* G' *is equal to* $2g+1$, *and moreover every facial cycle of* G' *is a* $(2g+1)$*-cycle.*

Proof. If every facial cycle of G is a $(2g+1)$-cycle, then there is nothing to prove, otherwise choose a facial cycle of length different from $2g+1$ and apply the Folding lemma. Repeat this process till there is no facial cycle of length different from g. □

C. Q. Zhang has also proposed a strengthening of Jeager's conjecture, cf. Section 3.3. The following conjecture is Zhang's strengthening of Jaeger's conjecture, restricted to the set of planar graphs and stated in the dual form.

Conjecture 3.4 *Every planar graph G of odd girth at least $4k+1$ admits a homomorphism to C_{2k+1}.*

The first case in this conjecture, i.e., the case $k = 1$, is equivalent to the Grötzsch theorem. For $k \geq 2$ the best result in support of this conjecture is due to X. Zhu. Using the folding lemma and the so-called discharging method (see e.g. [49]), Zhu improved the condition of high odd girth in Theorem 3.2 from $10k-3$ to $8k-3$, (see [83]).

On the other hand examples provided by M. Albertson and E. Moore in [1] and M. DeVos [12] show that the conditions of girth at least $4k$ in Conjecture 3.1 and odd girth at least $4k+1$ in Conjecture 3.4, if true, are the best possible.

DeVos' example of a planar graph of odd grith 7 which does not map to C_5 has been depicted in Figure 3.1. For general construction of planar graph of odd girth $4k-1$ which does not map to C_{2k+1}, one can take a $(4k-1)$-cycle with a centeral vertex which is joined to every vertex of the $(4k-1)$-cycle with a disjoint path of length $2k-1$.

To continue further we would like to restate Conjecture 3.4 in our framework of homomorphism and bounds. Recall that $\mathcal{P}_{2g+1}$ represents the class of planar graphs of odd girth at least $2g+1$.

Conjecture 3.4 The class $\mathcal{P}_{4k+1}$ is bounded by C_{2k+1}.

In view of the difference between the odd girth of the proposed bound (i.e., $2k+1$) and the minimum odd girth of the members of $\mathcal{P}_{4k+1}$ (which is $4k+1$) the following conjecture has been posed in [50].

Conjecture 3.5 *The class $\mathcal{P}_{2k+1}$ is bounded by a graph H of odd girth $2k+1$.*

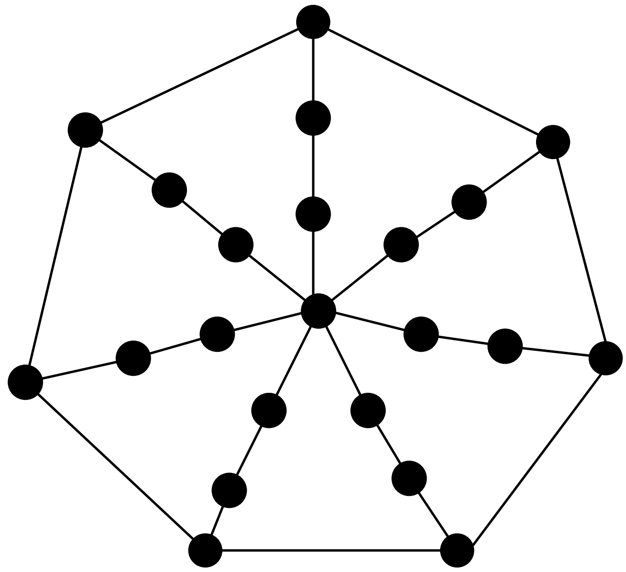

Figure 3.1: DeVos' Example

For $k = 1$, this conjecture claims that the class of planar graphs is bounded. This claim is easy to see by Euler's formula. For $k = 2$, the conjecture is a reformulation of Problem 2.18, and has been answered positively in [60]. It remains open for $k \geq 3$.

Our main propose in this chapter is to study Conjecture 3.5. On the one side, for $k \geq 2$, we will show that if $\mathcal{P}_{2k+1}$ admits a bound H_{2k+1} of odd girth $2k+1$, then H_{2k+1} cannot be planar, concluding that the class $\mathcal{P}_{2k+1}$ does not contain a maximum except when $k = 1$. On the other side we will strengthen this conjecture by proposing some Cayley graphs to be the bounds we are looking for. This will turn out to be a natural generalization of the four colour theorem. Then we will show that this new conjecture is equivalent to an special case of a well known conjecture of P. Seymour in the generalization of an equivalent form of the four colour theorem.

To reach our goals we need to have some introductory sections on the theories of edge-colouring, flows and Cayley graphs. We will also include a section on Tait's original work on the four colour problem, as we believe his work on this problem is the origin of some of these theories and conjectures. The section on edge colouring will also include a new proof of the fact that the Petersen graph is not three-edge colourable.

3.2 Edge colouring of graphs

A *proper edge colouring* of a graph G, or simply an edge colouring of G, is an assignment of colours to the edges of G in such a way that every pair of edges with a common end vertex receive different colours. The minimum number of colours required for an edge colouring of G is called the edge chromatic number of G and is denoted by $\chi'(G)$. One of the most important results in the theory of edge colouring of graphs is the following theorem of Vizing.

Theorem 3.6 [79] *Given a simple graph G with maximum degree Δ we have $\Delta \leq \chi'(G) \leq \Delta + 1$.*

With a strong theorem like this, the only basic problem left in the theory of edge colouring of simple graphs is to decide whether a given graph G is of type I (i.e., $\chi'(G) = \Delta$) or if it is of type II (i.e., $\chi'(G) = \Delta + 1$). However this question turns to be a difficult problem for $\Delta \geq 3$, in fact as it has been proved by I. Holyer in [36], that even deciding whether a given 3-regular graph is of type I or of type II is an NP-complete problem.

Notice that the upper bound of $\Delta + 1$ in Theorem 3.6 only applies to simple graphs; in the case of graphs with multiple edges if μ is the maximum multiplicity then the correct upper bound is $\Delta + \mu$. However the lower bound stays the same for both simple graphs and multi-graphs.

The problem of edge chromatic number is closely related to the theory of matchings and to the 1-factorization problem, specially when it is restricted to the family of regular graphs. In fact a regular graph is of type I if and only if it admits a 1-factorization. One natural obstacle which prevents an r-regular graph G from being of type I is a *small odd cut*. The concepts of "cut" and "odd cut" are defined below.

Definition 3.7 An *edge cut* of a graph, or simply a cut, is a partition of the vertices of G into two sets X and Y. Such an edge cut is normally denoted by (X, Y). The set of all edges between X and Y will be denoted by $[X, Y]$. The *size* of a cut (X, Y) is the number of edges in $[X, Y]$ and is denoted by $|[X, Y]|$. A graph is called k-edge connected if it contains no cut of size smaller than k. An *odd cut* of G is a cut (X, Y) where at least one of the parts, X or Y, contains an odd number of vertices. $\diamond$

Notice that an odd cut has been defined in general, and G does not have to be a regular graph. This definition can be applied even to the multi-graphs. Also note that even though the name "odd cut" suggests that an odd cut should contain an odd number of edges, this is not true in general. However sometimes this is the case. The following easy lemma establishes one of those cases. This lemma will be useful for us later in this chapter.

Lemma 3.8 [70] *Let G be an r-regular multi-graph with r being a positive odd integer. Then a cut (X, Y) is an odd cut if and only if it is of odd size.*

Proof. Let (X, Y) be a cut with an odd number of edges, then the subgraph G_X induced by X has odd number of even degree vertices. But the number of odd degree vertices is always even, therefore there is an odd number of vertices in X, i.e., (X, Y) is an odd cut.

Conversely if (X, Y) is an odd cut then by the definition one of the parts X or Y, in this case both of them, must have an odd number of vertices. Therefore the subgraph G_X of G induced on X, has an odd number of even degree vertices. On the other hand the number of even degree vertices in G_X is congruent to $|[X, Y]|$ modulo 2. So (X, Y) must be of odd size. $\square$

The following folklore lemma shows the importance of the odd cuts.

Lemma 3.9 *Let G be an r-regular multi-graph. If G contains an odd cut of size smaller than r then G is not r-edge colourable.*

Proof. Let G be an r-regular multi-graph and let (X, Y) be an odd cut of G with X having an odd number of vertices. If G is r-edge colourable then every colour class (which is a perfect matching) must meet the edge set $[X, Y]$. Therefore $[X, Y]$ must contains at least r edges with different colours. $\square$

This lemma solves the problem of edge chromatic number of an r-regular graph with an odd cut of size smaller than r. However the problem remains very difficult on the rest of r-regular graphs, so these graphs deserve their own name. Following the notation introduced by P. D. Seymour in [70], we will call them r-graphs.

Definition 3.10 An r-regular graph which does not contain any odd cut of size smaller than r is called an *r-graph*. In a similar vein we define an *r-multi-graph* to be an r-regular multi-graph which does not contain an odd cut of size smaller than r. $\diamond$

Unfortunately small odd cuts are not the only obstacles for a regular graph to be of type I, in other words an r-graph can be either of type I or type II. A classical example of an r-graph of type II is the Petersen graph. This graph which has been a counter example to many conjectures was introduced by J. Petersen as an example of a 3-regular 3-connected graph which is not 3-edge colourable.

The Petersen graph and its properties have been studied during more than a century; there has been even a book written about it. But most known proofs for the fact that it is not 3-edge colourable are based on case studies. The following proof is a simple counting argument based on the symmetries of the Petersen graph:

Proposition 3.11 *The Petersen graph is not 3-edge-colourable.*

Proof. The Petersen graph is usually drawn as an outer 5-cycle, an inner 5-cycle where edges join vertices that are cyclically two apart, and a matching joining corresponding vertices on the two cycles, drawn as depicted in Figure 3.2. Assuming a proper 3-edge-colouring, we obtain a contradiction by showing that each of the three colours must be used twice on the inner cycle, which has only five edges, a contradiction.

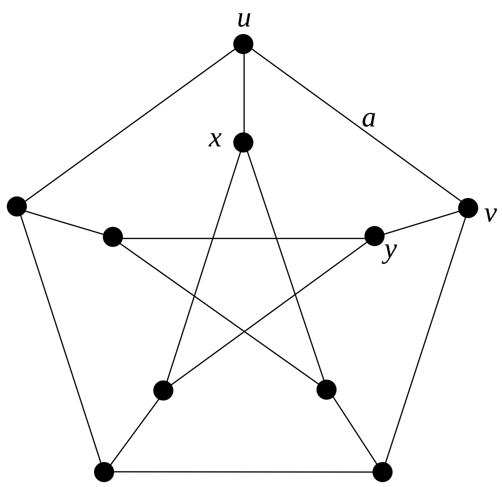

Figure 3.2: The Petersen graph

Since the outer cycle is of odd length, each of the three colours appears on it. Let uv be an edge on the outer cycle with colour a. In a proper 3-edge-colouring of a 3-regular graph, each colour must appear at each vertex. Since a can not appear on ux or vy, where x and y are the neighbours of u and v on the inner cycle, and xy is not an edge, colour a appears on distinct edges of the inner cycle at x and y. □

Constructing simple r-graphs of type II does not seem to be an easy problem for general r. To our knowledge the best work in this direction is due to N. Biggs who proposed the following generalization of Petersen graph (the odd graphs) to be the examples of simple r-graphs.

Definition 3.12 Let S be any $(2k+1)$-set. The odd graph O_k is a graph whose vertices are the k-subsets of S, and the edge set of O_k has been formed by exactly those pairs of vertices A and B for which $A \cap B = \emptyset$. ◇

Notice that O_1 is isomorphic to K_3 and O_2 is isomorphic to the Petersen graph.

It is easy to check that O_k is a $(k+1)$-regular $(k+1)$-edge connected graph, and therefore it is a $(k+1)$-graph. The following conjecture was posed by N. Biggs.

Conjecture 3.13 [6] *For a given positive integer k, the odd graph O_k is a $(k+1)$-graph which is not $(k+1)$-edge colourable.*

3.3 Flows

In this section we will do a short review of the concept of "flows" on graphs. To define a flow we must consider oriented graphs, but as we will see the existence of a flow only depends on the underlying simple graph. In an oriented graph G, an edge which is oriented from x to y will be denoted by (x, y).

Definition 3.14 Let Γ be an abelian group and G be a graph whose edges have been oriented in an arbitrary way. An assignment $f : V(G)^2 \to \Gamma$ is a called a Γ-flow if $f(u, v) = -f(v, u)$, and for every vertex v of G we have:

$$(\maltese) \qquad \sum_{u \in N(v)} f(v, u) = 0,$$

where $N(v)$ is the set of neighbours of v in the underlying simple graph. Moreover, for a subset B of Γ, we say f is a (Γ, B)-flow if for every $e \in E(G)$, $f(e) \in B$. $\diamond$

Notice that f is a function on $V(G)^2$ rather than just the edge set of G. Therefore if f satisfies ($\maltese$) for one orientation of G, then it satisfies ($\maltese$) for every orientation of G. Thus we may speak of flow on a simple graph.

This definition can be generalized to a definition of flows of multi-graphs. Let μ be the maximum multiplicity of a multi-graph G, where every edge is indexed by a number i, $1 \leq i \leq \mu$. Then we define $f : [\mu] \times V(G)^2 \to \Gamma$ to be a Γ-flow if f satisfies ($\maltese$) and if for every oriented edge (x, y) with index i, we have $f(i, x, y) = -f(i, y, x)$.

If we let $\Gamma^* = \Gamma \backslash \{0\}$, then (Γ, Γ^*)-flow is the standard concept of nowhere zero Γ-flow. A k-flow is a $(\mathbb{Z}, [k])$-flow. If $\Gamma = \mathbb{Z}$ and $B = \{d, d+1, \cdots, k-d\}$ then (Γ, B)-flow is known as a (k, d)-flow in some literature.

Given a graph G, let f_1 be a (Γ_1, B_1)-flow on G and f_2 a (Γ_2, B_2)-flow on G. Then $f = f_1 \times f_2$, the product of f_1 and f_2, defined by $f(e) = (f_1(e), f_2(e))$ is a $(\Gamma_1 \times \Gamma_2, B_1 \times B_2)$-flow on G. In particular if f_1 and f_2 are both nowhere zero flows then $f_1 \times f_2$ is also a nowhere zero flow. Products of a set of flows f_i, $i \in I$, are defined analogously.

Given an orientation D of a simple graph G, and a subset X of $V(G)$, let X^+ denotes the set of edges (x, y) with $x \in X$ and $y \notin X$. Similarly we define X^- to be the set of edges (y, x) with $y \notin X$ and $x \in X$. From (✠) it is easy to see that for every subset X of $V(G)$ we have:

$$\sum_{(x,y)\in X^+} f(x, y) = -\sum_{(y,x)\in X^-} f(x, y).$$

This, in particular implies that if there is an edge cut (x, y) of size one with $e = xy$ being the only edge in $[X, Y]$, then for any flow f on G, $f(x, y)$ must be zero. In other words, a graph which contains a cut of size one does not admit a nowhere zero flow. Intuitively speaking, this is dual to the concept of colouring, where a graph with a loop does not admit a proper vertex colouring.

Let G be a plane multi-graph and D be an orientation of G. Let $F = \{F_1, F_2, \cdots F_f\}$ be the set of faces of G. Every edge $e = xy$ of G is incident with two faces, say F_i and F_j, of G. We define $e^* = F_iF_j$ to be the *dual* of e, moreover if F_i is on the right side of (x, y) then we orient F_iF_j from F_i to F_j. Then *dual* of G is now defined to be the multi-graph $G^* = (F, E^*)$, where $E^* = \{e^* | e \in E(G)\}$. The orientation induced by D on the dual graph G^* will be denoted by D^*.

This concept of dual will allow us to show that in the case of planar graphs, flows are the dual concepts of vertex colourings. To see this consider a plane graph G with an orientation D, and let f be a Γ-flow on G. Then we will define below a $|\Gamma|$-colouring of the dual graph G^*. This colouring will be called the *tension* arising from f and will be denoted by T_f.

To define T_f we first define the assignment $g : E(G^*) \to \Gamma$ by $g(e^*) = f(e)$, where e^* is oriented by D^*. Then we choose an arbitrary vertex v of G^* and let $T_f(v) = 0$, for any other vertex u we pick a path $e_0^* e_1^* \cdots e_l^*$ from u to v in the underlying simple graph G^*, and let $T_f(u) = \sum_{i=0}^{l} g(e_i^*)$. It is a well known fact that T_f is well defined, an interested reader can find a proof in any text book on graph theory. It is easy to see that T_f is a proper vertex colouring of G^* if and only if f is a nowhere zero flow.

In fact W. T. Tutte proved that a plane graph G, admits a k-colouring if and only if the dual G^* admits a k-flow. It was also proved by Tutte that a multi graph admits a k-flow if and only if it admits a $(\mathbb{Z}_k, \mathbb{Z}_k^*)$-flow. These two theorems of Tutte, together with the four colour theorem imply the next theorem, but for the sake of completeness we will include a proof of this special case.

Theorem 3.15 [77] *Every 2-edge connected planar multi-graph admits a nowhere zero $\mathbb{Z}_4$-flow.*

Proof. Let G be any 2-edge connected planar multi-graph. Then the dual G^* of G is a planar multi-graph with no loop, and therefore by the four colour theorem it is 4-colourable. So we may use the elements of $\mathbb{Z}_4$ to colour the vertices of G^*. This is in fact a four-colouring of the faces of G, using the four elements of $\mathbb{Z}_4$. Now choose an arbitrary but fixed orientation for all the edges of G. For every edge e define $f(e)$ to be the difference between the colour of the left face and the right face of e. This is a $\mathbb{Z}_4$-flow because in the sum $\sum_{u \in N(v)} f(v, u)$ the colour of every face count twice, once with a positive sign, once with a negative sign. And it is a nowhere zero flow simply because the colouring was proper. $\square$

Unfortunately the duality between vertex colouring and flows does not extend to the class of non planar graphs. In fact it was conjectured by Tutte, that every bridgeless graph admits a 5-flow, and it was proved by P. D. Seymour that every bridgeless graph admits a 6-flow. It seems that a better way of extending the nice properties of flows on planar graphs to general graphs, is using the concept of (Γ, B)-flow with a much more restricted B. The following conjecture of F. Jaeger is a good example of this.

Conjecture 3.16 [37] *Every $4k$-connected graph admits a $(2k+1,k)$-flow.*

Let G be a planar graph and let G^* be the dual of G. Then it is easy to check that G is $4k$-edge connected if any only if G^* is of girth at least $4k$. Assume G admits a $(2k+1,k)$-flow f. It is not hard to see that the tension T_f is a C_{2k+1}-colouring of G^*. Conversely, any C_{2k+1}-colouring of G^* induces a $(2k+1,k)$-flow on G. This proves that Conjecture 3.16, restricted to the class of planar graphs, is equivalent to Conjecture 3.1.

Conjecture 3.16 relates the problem of the existence of a $(2k+1,k)$-flow to the size of smallest cut in G. The following strengthening of the conjecture, introduced by C. Q. Zhang, suggests that in this relation between $(2k+1,k)$-flows and cuts the important cuts are the ones with an odd size.

Conjecture 3.17 [82] *Let G be a graph which does not contain any cut of odd size smaller than $4k+1$. Then G admits a $(2k+1,k)$-flow.*

It can be seen in a similar vein that this conjecture, restricted to the class of planar graphs, is equivalent to Conjecture 3.4.

3.4 Tait's statement

While Kempe's chain method was believed to be a correct proof for the four colour problem, Tait, another mathematician of the time, was trying to apply different methods on the problem. He discovered a certain algebraic structure in the set of 4-colourings of the faces of a planar graph, when the colour set is chosen to be the integer group on base 4, $\mathbb{Z}_4$. He showed that the four colour theorem can be restated as below, however the proof he offered for this statement was not correct.

Theorem 3.18 *[73] The following two statements are equivalent:*

1 *Every planar graph is 4-colourable.*

2 Every planar 3-multi-graph is 3-edge colourable.

Statement 2 of the Theorem 3.18 is known as Tait's statement. A proof of this theorem, with a generalization, will be given later (see Theorem 3.27). To recognize the importance of Tait's statement notice that it is the origin of the theory of edge colouring of graphs. Also the ideas of proving the equivalence in Theorem 3.18 were the first steps on introducing the theory of flows on graphs.

Tait's statement has been, in addition to introducing these theories, a reference for many generalizations of the four colour problem. Tait himself was the first one to generalize his statement. He thought the condition of planarity is not as important as the other conditions, and therefore he claimed that "every 3-graph is 3-edge colourable". But J. Petersen disproved this generalization by introducing a 3-graph which is not 3-edge colourable. This counter example has been named after Petersen. In the previous section we saw a proof of the fact that Petersen graph is not 3-edge colourable.

A comprehensive study of Tait's statement was done by W. T. Tutte who developed the concept of the flows on graphs to a full theory. He observed that the structure of the Petersen's counterexample is essentially needed in any example of a 3-graph which is not 3-edge colourable. Therefore he conjectured that:

Conjecture 3.19 [78] *If a 3-graph G does not admit a 3-edge colouring then it must contain Petersen graph as a minor.*

This conjecture has been proved by N. Robertson, P. D. Seymour and R. Thomas, but only the first step of the proof has so far been published, [67].

The next generalization of Tait's statement we would like to talk about is the following conjecture, which we believe was introduced by P. D. Seymour.

Conjecture 3.20 *Every planar r-multi-graph is r-edge colourable.*

This conjecture is not only about simple graphs, here planar graphs are allowed to have multiple edges but no loops. In fact due to the Euler formula there are no r-regular planar graphs for $r \geq 6$ and therefore the conjecture make sense only for graphs with multiple edges.

This conjecture is a special case of a conjecture of P. Seymour, introduced in [71]. There are also various forms of strengthening of the conjecture. To our knowledge, the most general form of the conjectures, that imply Conjecture 3.20, is due to L. Goddyn, see [22].

The latest progress on Conjecture 3.20 is due to B. Guenin who has proved the following two cases of the conjecture by the discharging method and applying induction on r, therefore using the case $r = 3$ which is equivalent to the four colour theorem.

Theorem 3.21 [28] *Every planar 5-multi-graph (4-multi-graph) is 5-edge colourable (4-edge colourable, respectively).*

This theorem will be used in Section 3.6.

A supporting evidence for Conjecture 3.20 can be obtained from a general theorem of M. De Vos and P. D. Seymour. To introduce their result we need some more definitions and more notation.

A *graft* is a multi-graph G, together with a subset of vertices, T, of even cardinality. A T*-cut* of G is an edge cut which separates T into two sets of odd size. The size of the smallest T-cut is denoted by $\tau(G)$. A T*-join* of G is a subgraph H of G with the property that every vertex of $x \in V(G)$ has an odd degree in H if and only if it is in T. The maximum number of edge disjoint T-join subgraphs of G is denoted by $\nu(G)$. It is easy to check that $\tau(G) \geq \nu(G)$. It was shown by M. De Vos and P. D. Seymour that $\nu(G) \geq \frac{1}{3}\tau(G)$; they have also shown that in the special case when T is the set of all the odd degree vertices of G this inequality can be improved.

Theorem 3.22 [14] *Let G be a graft with $T = \{v|v \in V(G),\ d(v)\ \text{is odd}\}$. Then $\nu(G) \geq \tau(G)$.*

Given a multi-graph G we say a subgraph H of G is an *odd spanning* subgraph if every vertex of G has an odd degree in H. Therefore a perfect matching is an odd subgraph with the minimum number of edges.

If every vertex of G is of odd degree then an important property of an odd spanning subgraph of G is that it intersect every odd cut of G. In this case an odd spanning subgraph is equivalent to a T-join of the graft G with $T = V(G)$. The following then is an special case of Theorem 3.22:

Theorem 3.23 *Every $(2k+1)$-multi-graph contains at least k edge disjoint odd spanning subgraphs.*

Conjecture 3.20 claims that every planar $(2k+1)$-multi-graph can be decomposed into $2k+1$ odd spanning subgraphs. Theorem 3.23 guarantees the existence of at least k such edge disjoint subgraphs.

3.5 Cayley graphs

Let Γ be an additive group and S a subset of Γ closed under taking inverses. Then the Cayley graph $C(\Gamma, S)$ is defined to be a graph whose vertex set is Γ and whose edge set is formed by those pairs of vertices x, y for which $x - y \in S$. Note that since S is closed under taking inverses, $C(\Gamma, S)$ is a simple graph.

Let $k \geq 1$ and $\Gamma = \mathbb{Z}_2^k$ be the k-dimensional group over $\mathbb{Z}_2$. Let S be the set of vectors with exactly two circularly consecutive 1s, i.e., $S = \{s_1 = (1, 1, 0, 0, \cdots 0), s_2 = (0, 1, 1, 0, 0, \cdots 0), \cdots s_k = (1, 0, 0, \cdots 0, 1)\}$. This set is closed under taking inverses because $-s_i = s_i$. The Cayley graph $C(\Gamma, S)$ has two isomorphic connected components. The set of vertices with an even number of 1's induces one component, and the set of vertices with an odd number of 1's induces the other component. We use the notation H_k to denote the graph which is isomorphic to either of the two components of $C(\mathbb{Z}_2^k, S)$.

The graphs H_1, H_2, and H_3 are respectively isomorphic to K_1, K_2, and K_4. In general, H_k is a bipartite graph for all even values of k. The graph H_5 is well known, independently in two different areas. In Ramsey theory, it was introduced by R. E. Greenwood and A. M. Gleason in [24], where the authors used this graph to partition the edge set of K_{16} to three triangle-free subgraphs. Then in [40] J. G. Kalbfleisch and R. G. Stanton proved that the only way to colour the edges of K_{16} using 3 colours without producing a monochromatic triangle is to have each colour class isomorphic to H_5. Thus the graph is called the Greenwood-Gleason graph in some references such as [8].

It is also sometimes called the Clebsch graph because it is one of the few known triangle-free strongly regular graphs, and was introduced by Clebsch for this purpose see [23] for more details. In the next section we will show that H_5 is also a bound for $\mathcal{P}_5$, and that this statement is a direct generalization of the four colour theorem. But we should first introduce some of the properties of these Cayley graphs.

Lemma 3.24 *The graph H_{2k+1} has the following properties.*

(a) It is $2k+1$ regular.

(b) It has edge chromatic number equal to $2k+1$.

(c) It is of odd girth $2k+1$.

Proof. The statements (a) and (b) are obvious, since S has $2k+1$ elements, and each $s \in S$ defines a perfect matching on H_{2k+1}, where every vertex x is matched to $s+x$. (This is a matching since $2s=0$.) The corresponding edge colouring will be called the canonical edge colouring of H_{2k+1}.

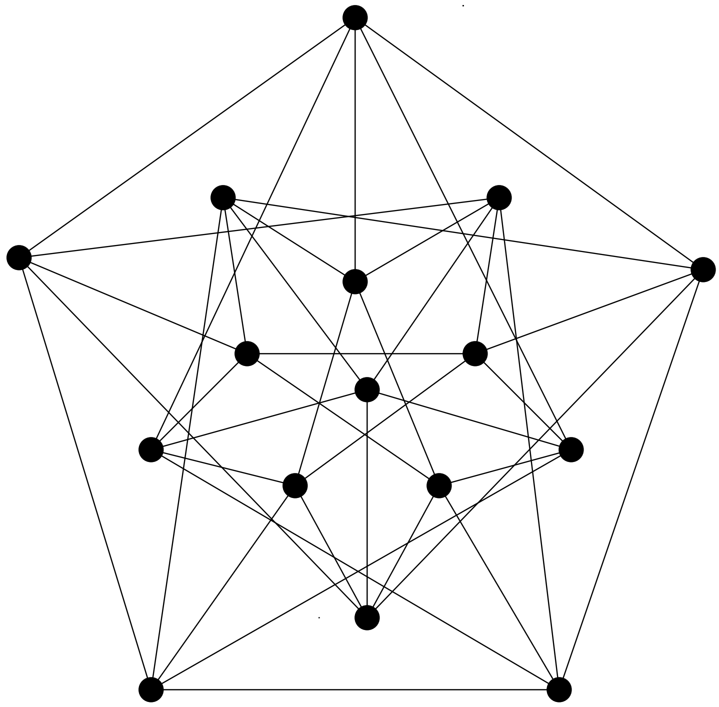

Figure 3.3: Greenwood-Gleason-Clebsch graph

For (c), first note that H_{2k+1} is not a bipartite graph (for $k \geq 1$). For example the set C of vertices defined by $C = \{v_i \,|\, v_i = s_1 + s_2 + \cdots s_i,\ i = 1, \cdots 2k+1\}$ induces an odd cycle of length $2k+1$. To show that H_{2k+1} does not contain any smaller odd cycle, consider the canonical edge colouring of H_{2k+1}, and let C be any cycle in the graph. Note that the sum of the colours of the edges of C is zero.

Now if C is an odd cycle then one of the colours, say s_i, appears an odd number of times. In order for s_i to vanish in the sum, both s_{i+1} and s_{i-1} have to appear an odd number of times. By repeating this argument we will conclude that all s_j's $j = 1, 2, \cdots 2k+1$ must appear on C an odd number of times. In particular, $|C| \geq 2k+1$. □

In the proof of the last lemma we also proved the following:

Corollary 3.25 *In the canonical $(2k+1)$-edge colouring of H_{2k+1}, every $(2k+1)$-cycle takes $2k+1$ different colours.*

3.6 A generalization of the four colour conjecture

The following, which is an strengthening of the Conjecture 3.5, is in fact a direct generalization of the four colour theorem.

Conjecture 3.26 *The class $\mathcal{P}_{2k+1}$, class of planar graphs with odd girth at least $2k+1$, is bounded by the Cayley graph H_{2k+1}.*

Note that for $k = 1$ this statement is exactly the four colour theorem, as H_3 is isomorphic to K_4. For $k = 2$ it claims that the class of triangle-free planar graphs is bounded by the Greenwood-Gleason-Clebsch graph. This latter case will be proven here using the result of Guenin (Theorem 3.21).

The main result of this section is to prove that Conjecture 3.26 is in fact equivalent to the corresponding case of Conjecture 3.20. More precisely we prove the following:

Theorem 3.27 *For a given positive integer k the following two statements are equivalent:*

1. *The class $\mathcal{P}_{2k+1}$ is bounded by H_{2k+1}.*

2. *Ever planar $(2k+1)$-multi-graph is $(2k+1)$-edge colourable.*

Proof. First of all observe that to prove that $\mathcal{P}_{2k+1}$ is bounded by H_{2k+1}, it will be enough to show that every plane graph with odd girth $2k+1$ in which all facial cycles are of length $2k+1$ admits a homomorphism to H_{2k+1}. This is true by the Folding lemma. In fact, by Corollary 3.3 every planar graph of odd girth $2k+1$ or more admits a homomorphic image which is also planar, has odd girth $2k+1$, and moreover all the facial cycles of this image are $2k+1$ cycles. To make the notation easier we will denote this subclass by $\mathcal{P}'_{2k+1}$, so $\mathcal{P}'_{2k+1}$ consists of planar graphs with odd girth exactly $2k+1$, for which there is at least one planar representation such that every facial cycle is a $2k+1$ cycle. When we talk about a member of $\mathcal{P}'_{2k+1}$ we consider it together with a plane representation in which every facial cycle is a $2k+1$ cycle.

The second observation is that being a member of $\mathcal{P}'_{2k+1}$ is dual to being a planar $(2k+1)$-multi-graph. In fact a planar graph G is in $\mathcal{P}'_{2k+1}$ if and only if G^* the dual of G is a planar $(2k+1)$-multi-graph. To see this, first note that for a planar graph being $(2k+1)$-regular is equivalent to have every facial cycle of its dual of length $2k+1$. Secondly by Lemma 3.8 every odd cut of a $(2k+1)$-regular graph has odd number of edges, therefore the condition of no small odd cut is equivalent to the condition of no small odd cycle in the dual.

The third important observation is about the dual of the edge colouring of a planar $(2k+1)$-multi-graphs. It is not hard to see that a planar $(2k+1)$-multi-graph G^*, admits a proper $(2k+1)$-edge colouring if and only if its dual, G, admits an edge colouring (possibly an improper one) in which every facial cycle takes all the $2k+1$ different colours.

With these observations one direction of the theorem is easy to prove. Suppose $\mathcal{P}_{2k+1}$ is bounded by H_{2k+1}, and let G^* be a planar $(2k+1)$-multi-graph. Then the dual G^* of G, is in $\mathcal{P}'_{2k+1}$ and therefore it admits a homomorphism to H_{2k+1}. This homomorphism will induce a $(2k+1)$-edge colouring on G using the canonical $(2k+1)$-edge colouring of H_{2k+1}. This colouring has the property that every facial cycle of G takes all the different colours by Corollary 3.25, and therefore it induces a proper $(2k+1)$-edge colouring on G^*.

For the other direction suppose every planar $(2k+1)$-multi-graph is $(2k+1)$-edge colourable. By the first observation it will be enough to prove that every member of $\mathcal{P}'_{2k+1}$ admits a homomorphism to H_{2k+1}. Let G be a graph in $\mathcal{P}'_{2k+1}$. Then the dual G^* of G is a planar $(2k+1)$-multi-graph. Therefore, by assumption, G^* admits a $(2k+1)$-edge-colouring. Let $\overline{E}_1, \overline{E}_2, \cdots \overline{E}_{2k+1}$ be the colour classes. Then for each i the subgraph induced by $\overline{E}_i \cup \overline{E}_{i+1}$ (indices are being taken modulo $2k+1$) is a union of cycles, and therefore admits a $(\mathbb{Z}_2, \{1\})$-flow. The product of all these flows will be a $(\mathbb{Z}_2^{2k+1}, S)$-flow on G^* where $S = \{(1,1,0,0,\cdots 0),(0,1,1,0,0,\cdots 0),\cdots (1,0,0,\cdots 0,1)\}$. The tension arising from this flow on the dual G of G^* is in fact a homomorphism of G to H_{2k+1}. □

The following theorem is now a consequence of Theorem 3.27 and Theorem 3.21.

Theorem 3.28 *The class of triangle-free planar graphs, $\mathcal{P}_5$, is bounded by H_5.*

The last part of the proof of Theorem 3.27 can be read alternately as a proof of the following interesting connection between edge colourings of $(2k+1)$-regular graphs, and the existence of a special kind of flows. For similar connections between colourings and flows we refer to [13].

Theorem 3.29 *A $2k+1$ regular graph G is $(2k+1)$-edge colourable if and only if it admits a $(\mathbb{Z}_2^{2k+1}, S)$-flow, where S is the set of vectors with exactly two consecutive ones.*

Note that planarity is not required in Theorem 3.29.

Let F_{2g+1} be the Cayley graph $C(\mathbb{Z}_4^{2g+1}, S)$, where S is the set of vectors with $2g-1$ circularly consecutive 0's. In other words, S is the set of vectors with exactly two (circularly consecutive) non zero coordinates. Applying the methods of the proof of Lemma 3.24 we can see that F_{2g+1} is of odd girth $2g+1$:

Lemma 3.30 *Let F_{2g+1} be the Cayley graph defined above, then odd–girth$(F_{2g+1}) \geq 2g+1$.*

Proof. Let u be an edge of F_{2g+1} corresponding to an element of S with the i-th and $(i+1)$-st coordinate (indices are modulo 2g+1) nonzero. Then assign to u the colour c_i.

Now let C be an odd cycle of F_{2g+1}. Then there must be a colour c_i which appears on an odd number of edges of C. Since the sum of vectors corresponding to the edges of C is zero, colours c_{i+1} and c_{i-1} must also appear an odd number of times. Continuing this process we find that every colour must appear an odd number of times, therefore there are at least $2g+1$ different colours and C is of size at least $2g+1$. $\square$

In support of Conjecture 3.26 and Conjecture 3.5 we will prove that $\mathcal{P}_{4g+1}$ is bounded by F_{2g+1}.

Proposition 3.31 *The class $\mathcal{P}_{4g+1}$ is bounded by F_{2g+1}.*

Proof. The proof of this proposition is similar to that of Theorem 3.27, and we will use the same terminology as in the proof of that theorem. So $\mathcal{P}'_{4g+1}$ denotes the class of planar graphs of odd girth $4g+1$, each with a planar representation in which every facial cycle is a $(4g+1)$-cycle. Again using Folding lemma it can be seen that, it suffices to prove that $\mathcal{P}'_{4g+1}$ is bounded by F_{2g+1}.

To prove this we will use Theorem 3.23. Let G be a graph in $\mathcal{P}'_{4g+1}$. Then G^*, dual of G, is a $(4g+1)$-graph and therefore by Theorem 3.23 it contains at least $2g$ edge disjoint odd spanning subgraphs, $T_1, T_2, \cdots T_{2g}$. The subgraph induced on $\cup_{i=1}^{2g} E(T_i)$

is an eulerian graph, therefore the subgraph induced by $E\setminus \cup_{i=1}^{2g} E(T_i)$ is also an odd spanning subgraph which we denote it by T_{2g+1}. Hence $\{T_i\}_{i=1}^{i=2g+1}$ is a decomposition of G into $2g+1$ odd spanning subgraphs.

Let U_i be the subgraph induced on $E(T_i) \cup E(T_{i+1})$, indices being taken modulo $2g+1$. We claim that U_i does not contain an edge cut of size one. By contradiction, suppose (X, Y) is an edge cut of U_i with $|[X, Y]| = 1$. But this is impossible because otherwise the subgraph of U_i induced on X has only one vertex of odd degree.

Now applying Theorem 3.15 we find that every U_i admits a $(\mathbb{Z}_4, \mathbb{Z}^*)$-flow, ϕ_i. Let ϕ be the product of all the flows ϕ_i. This is a $(\mathbb{Z}_4^{2g+1}, S)$-flow. The tension T_ϕ will be a homomorphism of G to F_{2k+1}. $\square$

3.7 The absence of maximum

In this section we will study the properties of a possible bound of odd girth $2g+1$ for the class $\mathcal{P}_{2g+1}$. In particular we will prove that such a bound cannot be planar. We will need the following classical result on transitive planar graphs.

Theorem 3.32 [27] *Let P be a planar graph with a vertex of degree 3 or more. Moreover assume P is vertex transitive, edge transitive and also face transitive. Then P must be isomorphic to one of the platonic graphs, i.e., the cube, or the dodecahedron, or the icosahedron, or the octahedron, or the tetrahedron.*

Next we will introduce the following interesting family of planar graphs.

Definition 3.33 Given positive integer k, let C_{6k-3} be a $(6k-3)$-cycle with vertex set $\{v_1, v_2, \cdots, v_{6k-3}\}$ and v_i being adjacent to v_{i-1} and v_{i+1}, (indices being taken modulo $6k-3$). We define D_{2k+1} to be the graph obtained from C_{6k-3} by adding two new vertices a and b, where a is adjacent to three vertices, v_1, v_{2k} and v_{4k-1} and b is adjacent to v_2, v_{2k+1} and v_{4k}. In the case of $k=1$, since $\{v_1, v_{2k}, v_{4k-1}\} = \{v_2, v_{2k+1}, v_{4k}\}$, we identify a and b to a single vertex. So D_3 is isomorphic to K_4. The graph D_5 has been depicted in 3.4. $\diamond$

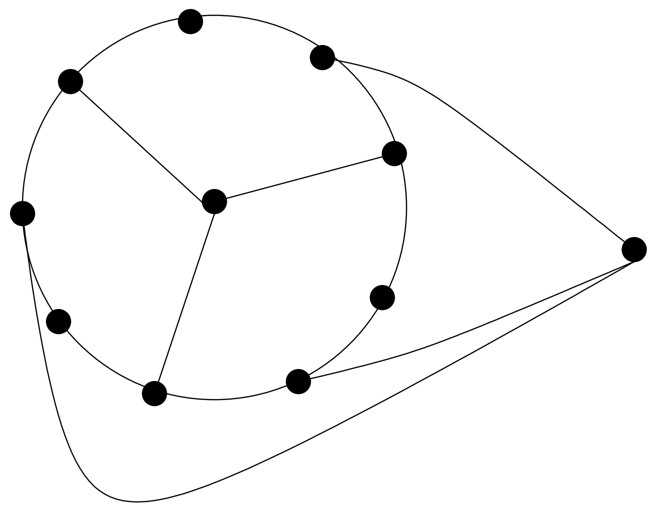

Figure 3.4: D_5

The following lemma, which is easy to see, shows an interesting property of the graph D_{2k+1}.

Lemma 3.34 *Given a positive integer k, any two vertices of the graph D_{2k+1} are joined by a path of odd length $\leq 2k-1$.*

Using this lemma we can prove the following important property of D_{2k+1}.

Lemma 3.35 *Given any positive integer k and a homomorphic image D of D_{2k+1}, if D is of odd girth $2k+1$ then it is isomorphic to D_{2k+1}.*

Proof. Let D be a homomorphic image of D_{2k+1} with *odd–girth*$(D) = 2k+1$. Let f be a homomorphism of D_{2k+1} to D, then f is surjective. To complete the proof we shall show that it is also injective. But if f is not injective then it has identified at least two vertices of D_{2k+1} and thereby f has created an odd cycle of length at most $2k-1$, which is impossible. So f is indeed an isomorphism between D and D_{2k+1}. $\square$

The following corollary is now easy to see.

Corollary 3.36 *For any positive integer k, the graph D_{2k+1} is a core.*

Lemma 3.35 shows that a bound H for $\mathcal{P}_{2k+1}$, if it is of odd grith $2k+1$, cannot be very small. In fact any such bound has to contain the graph D_{2k+1} as a subgraph. It is also not difficult to prove that any such bound for $\mathcal{P}_{2k+1}$, if it exists, can not be a planar graph. This fact will be proved below by two different methods. The first one is an elementary proof based on the Euler formula, the second proof which is somewhat more general argument will be based on Theorem 3.32.

Theorem 3.37 *Given an integer $k \geq 2$, assume $\mathcal{P}_{2k+1}$ is bounded by a graph B_{2k+1} of odd girth $2k+1$. Then B_{2k+1} cannot be a planar graph.*

Proof. By contradiction, suppose $\mathcal{P}_{2k+1}$ admits a planar bound with odd girth $2k+1$ and assume B_{2k+1} is such a bound. Moreover assume B_{2k+1} has the minimum number of vertices among all the planar bounds for $\mathcal{P}_{2k+1}$ with odd girth $2k+1$. This implies that B_{2k+1} is a core, otherwise the core of B_{2k+1} is an smaller bound with all the same properties. It also follows from the Folding lemma that every facial cycle of B_{2k+1} must be a $(2k+1)$-cycle. We first consider the case $k \geq 3$. In this case, by the Euler formula, B_{2k+1} must contain a vertex of degree at most 2.

To get to a contradiction we will show that B_{2k+1} must have minimum degree at least 3. Let B be the graph obtained from B_{2k+1} by the following method: For every vertex x add a copy of D_{2k+1} and identify x with the vertex a of D_{2k+1}. The graph B obtained this way is obviously planar and has odd girth $2k+1$, therefore B must map to B_{2k+1}. In this mapping, by Lemma 3.35 the image of every D_{2k+1} must be isomorphic to itself. This implies that every vertex x of B_{2k+1} must have at least 3 neighbours, so B_{2k+1} is of minimum degree at least 3, which is a contradiction.

For $k = 2$, the Euler formula only guarantees the existence of vertices of degree 3 or less. In this case the proof follows the same lines. In fact by replacing D_5 with a more sophisticated structure, we prove that B_5 must have minimum degree at least 4.

Let D' be the graph obtained by joining two 5-cycles with an edge. This graph contains a set of four vertices $\{x, y, z, t\}$, each of which is at distance three from the other three. Let D be a graph obtained from D' by adding a new vertex a which is adjacent to four vertices x, y, z and t of D'. In any triangle-free homomorphic image of D, the four neighbours of a must be distinct.

Now for every vertex x of B we add a distinct copy D_x of D and identify x with the vertex a. The graph B' obtained this way is a triangle-free planar graph, so maps to B_5 surjectively (because of the minimality of B_5). Therefore B_5 has minimum degree at least 4. This is a contradiction. $\square$

The second proof which has an algebraic flavor, is somewhat more general and reveals the difference between the nature of the problem for $k = 1$ and $k \geq 2$. This proof has been adopted from the proof of P. Hell for Proposition 2.54, see [34].

Proof. Again we assume B_{2k+1} is planar graph of odd girth $2k + 1$ which bounds $\mathcal{P}_{2k+1}$, moreover we assume B_{2k+1} has minimum number of vertices among all these bounds. Therefore B_{2k+1} must be a core, and that every facial cycle of B_{2k+1} should also be a $(2k + 1)$-cycle, (we are using Folding lemma).

We claim that B_{2k+1} must be a vertex transitive graph. To see this let x and y be two distinct vertices of B_{2k+1}. Form a new graph B from two copies of B_{2k+1}, where the vertex x from one of the copies has been identified with the vertex y from the other copy. Because of the minimality of B_{2k+1}, this new graph B is also planar with odd girth $2k + 1$, so it must admit a homomorphism to B_{2k+1}. This homomorphism is an isomorphism when it is restricted on each copy of B_{2k+1} in B. This gives us an automorphism which transforms x and y to each other.

A similar argument on the edge set will prove that B_{2k+1} must be also edge transitive. In the same vein it must be face transitive too, therefore by Theorem 3.32, the graph B_{2k+1} must be one the five platonic graphs. On the other hand B_{2k+1} must contain D_{2k+1} as a subgraph, and the only possible case is $k = 1$ and $B_{2k+1} \cong K_4$. $\square$

This second proof can also be read independently as a proof for Proposition 2.54, which is a special case of Conjecture 2.11.

3.8 Remarks and open problems

3.8.1 On the size of bounds

By Theorem 3.28 we know that every triangle-free planar graph can be mapped to a triangle-free graph on at most 16 vertices. We believe it is not possible to do any better than 16. In other words we believe the answer for the following question is negative.

Problem 3.38 *Does $\mathcal{P}_5$ admit a triangle-free bound on at most 15 vertices?*

Let $f(5)$ be the smallest integer for which there is a triangle-free bound for $\mathcal{P}_5$ on $f(5)$ vertices. By Theorem 3.28 we know that $f(5) \leq 16$. For the lower bounds, by Lemma 3.35, every triangle-free homomorphic image of D_5 has 11 vertices, therefore $f(5) \geq 11$. Below we will extend D_5 to a graph which proves $f(5) \geq 14$.

Let A' be the graph of Figure 3.5. This graph contains five vertices (named a, b, c, d and e in the figure), such that between every two there is a path of length 3. Let A be the graph obtained from A', by adding a new vertex v which is joined to all the five vertices a, b, c, d, and e. This new graph A is also a triangle-free planar graph. We now construct a new graph D from D_5 as below:

Example 3.39 For every vertex x of D_5, add a distinct copy A_x of A and identify v with x. Let N_x be the five neighbours of x in A_x. Obviously D is a triangle-free planar graph. In the following theorem we prove that D can not be mapped to any triangle-free graph with less than 14 vertices, therefore proving that $f(5) \geq 14$. $\diamond$

Theorem 3.40 *Any triangle-free homomorphic image of D consist of at least* 14 *vertices.*

Proof. Let B be a triangle-free homomorphic image of D, and let c be a B-colouring of D. Since D_5 is a subgraph of D, by Lemma 3.35 the graph B also contains D_5 as

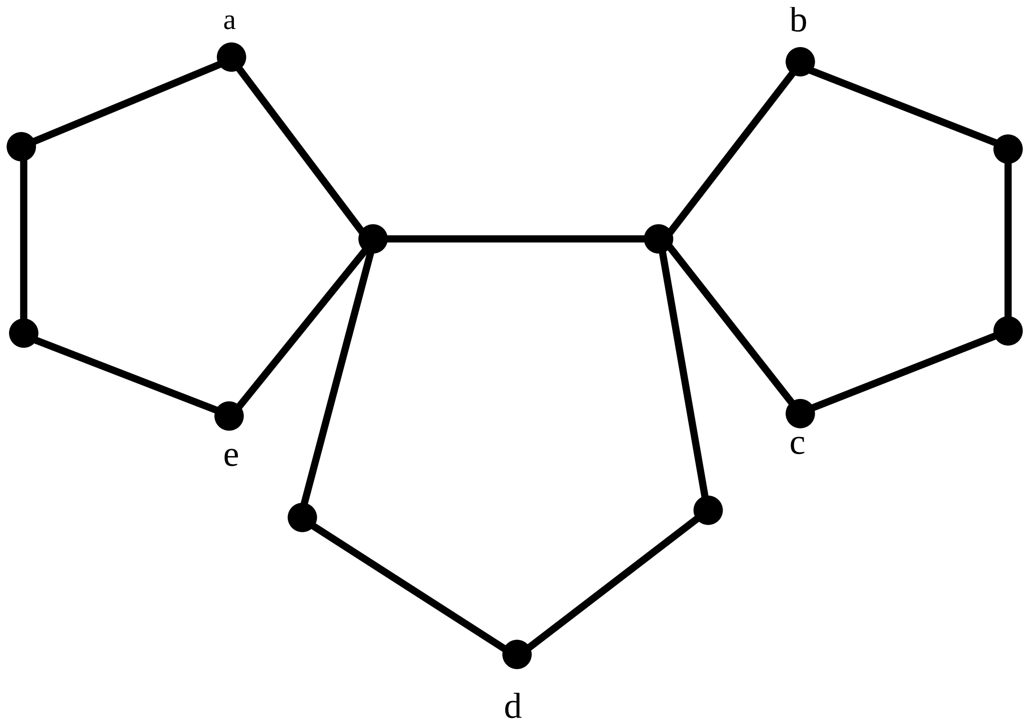

Figure 3.5: The graph A'

a subgraph. To complete the proof we show that B contains three more vertices.

Let x be a vertex of the subgraph D_5 of D. Then c does not map any two distinct vertices of N_x to the same vertex of B (otherwise the image will contain a triangle). It is also not hard to check that c can map at most four of the five vertices in N_x to $c(D_5)$. So for every vertex x in the subgraph D_5 of D, there is a vertex $g(x)$ of D which does not map to $c(D_5)$ under c.

On the other hand if x and y are adjacent in D_5, then $c(g(x))$ must be distinct from $c(g(y))$, otherwise B must contain a triangle. Therefore the assignment $x \to c(g(x))$ is a proper colouring of D_5. Since D_5 is 3-chromatic, B must contain at least 3 more vertices, so B must have at least 14 vertices. □

Problem 3.38 can be naturally generalized to all the odd numbers. In the general case, we would like to find the smallest integer $f(2k+1)$, for which there is a bound of odd girth $2k+1$ with $f(2k+1)$ vertices for the class $\mathcal{P}_{2k+1}$. Conjecture 3.5 is equivalent to say that $f(2k+1)$ exists, and Conjecture 3.26 implies that $f(2k+1) \leq 2^{2k}$. We also conjecture that this is the best possible:

Conjecture 3.41 *For every odd integer $2k+1$, we have $f(2k+1) = 2^{2k}$.*

3.8.2 Powers of planar graphs

Let G be a graph with the adjacency matrix $A(G)$. We define the k-th power G^k of G, to be the graph whose adjacency matrix is the matrix obtained from the k-th power of $A(G)$, by replacing every nonzero element with 1. By this definition, for an odd integer k, the graph G^k is a graph on the same vertex set as G where two vertices x and y are adjacent if there is an odd path of length at most k in G, joining x and y.

We would like to consider the problem of colouring G^k. But notice that when k is an even number then G^k contains a loop unless G does not have any edge. Also for an odd k, the graph G^k is a loopless graph only if *odd–girth*$(G) > k$. So $\mathcal{P}_{2k+1}$ is a natural place to consider the problem of colouring powers of a graph.

Conjecture 3.26 if true would imply that for every planar graph G of odd girth at least $2k+1$ we have $\chi(G^{2k+1}) \leq 2^{2k}$. This observation leads to several interesting problems. The first problem is about the existence of a maximum for the set $A_{2k+1} = \{\chi(G^{2k+1}) | G \in \mathcal{P}_{2k+1}\}$. A positive answer can be viewed as a support for both Conjecture 3.5 and Conjecture 3.26.

Problem 3.42 *Does the set* $A_{2k+1} = \{\chi(G^{2k+1}) | G \in \mathcal{P}_{2k+1}\}$ *have a maximum?*

The graph D_{2k+1}, introduced in Example 3.33, implies that the maximum of A_{2k+1}, if exists, is at least $6k-1$. But we believe the real value of the maximum should be closer to 2^{2k}. This upper bound is being nominated from Conjecture 3.26.

For $k = 2$, the graph D of Example 3.39 provides an example of a graph with $\chi(D^3) \geq 14$, but we do not know if it is possible to construct a triangle-free planar graph G with $\chi(G^3) = 15$ or 16, however note that in this case by Theorem 3.26 $\chi(G^3) = 16$ is the best possible if possible at all.

Let $g(2k+1)$ be the maximum of A_{2k+1}, then $g(2k+1) \leq f(2k+1)$, in other words the existence of $f(2k+1)$ implies the existence of $g(2k+1)$. The following problem is about the inverse of this observation.

Problem 3.43 *Given a positive integer* k*, assume the set* $A_{2k+1} = \{\chi(G^{2k+1})\}$ *has a maximum. Does this imply the existence of a bound of odd grith* $2k+1$ *for the class* $\mathcal{P}_{2k+1}$*?*

3.8.3 Edge colouring and odd graphs

Let H_{2k+1} be a component of $C(\mathbb{Z}_2^{2k+1}, S)$ induced on the set of vectors with even number of 1's. Then $x_0 = (0, 0, \cdots 0)$ is a vertex of H_{2k+1}. Any other vertex can be seen as $y = x_0 + s_{i_1} + s_{i_2} + \cdots s_{i_j}$, where $s_{i_1}, s_{i_2}, \cdots s_{i_j}$, are j distinct vectors of S. In other words, every vertex of H_{2k+1} corresponds to an r-subset of S, with $r \leq k$. Note that since $\sum_{i=1}^{2k+1} s_i = 0$, every subset produces the same vertex as its complement.

Now it is easy to check that in general the diameter of H_{2k+1} is k. To see this we take any vertex x corresponding to a k-subset S_x of S, the distance from x to x_0 is k. In fact, a vertex x is at distance k to x_0 if and only if it corresponds to a subset of size k. On the other hand, two vertices x and y at distance k to x_0, are adjacent if and only if their corresponding k-subsets are disjoint. This proves the following lemma:

Lemma 3.44 *Given the Cayley graph H_{2k+1}, the set of vertices at distance k of x_0 induces a subgraph which is isomorphic to the odd graph O_k.*

The natural appearance of the odd graph O_k, as an induced subgraph of H_{2k+1} together with Theorem 3.29 and Conjecture 3.13 proposes the following generalization of Conjecture 3.19.

Problem 3.45 *Let G be a simple $(2k+1)$-graph with no minor of O_{2k}. Is it always true that G is $(2k+1)$-edge colourable?*

A negative answer to this question perhaps will introduce another obstacle for the existence of a 1-factorization for regular graphs (like the obstacle of having small odd cut). A positive answer would be surprising but at same time very difficult to prove because just for $k = 1$ this is stronger than the four colour theorem. And for $k \geq 2$ we do not even know the edge chromatic number of the odd graph O_{2k} itself.

Chapter 4

The chromatic covering number of graphs

4.1 Fractional chromatic number

We have seen that there is a no-homomorphism lemma for each of the following three graph parameters: the chromatic number, the clique number and the odd girth. Aside from these three, there are several other graph parameters for which a no-homomorphism lemma holds. Fractional chromatic number is one such a parameter. There are different ways to define the fractional chromatic number. One basic definition is the Definition 2.46 of Chapter 3. According to this definition $\chi_f(G)$ is the $\liminf_k \frac{\chi_k(G)}{k}$ where $\chi_k(G)$ is the minimum number of the colours required for a k-set colouring of G. A k-set colouring is an assignment of the k colours to each vertex of G in such a way that no two adjacent vertices have a colour in common.

Let G be a graph and let c be an n-set colouring of G. For any two non adjacent vertices x and y of G, with $c(x) \cap c(y) = \emptyset$, we add a new edge which connects x to y. The graph obtained in this way has the property that every two vertices whose corresponding n-sets are disjoint, are adjacent. This leads to the following definition of a well known family of graphs.

Definition 4.1 Given two positive integers n and k with $n \geq 2k$, we define the *Kneser graph* $K(n, k)$ to be the graph whose vertex set is the set of k-subsets of the n-set, $[n]$, and whose edge set consists of all the pairs of disjoint k-subsets. $\diamond$

Notice that the odd graph O_k, defined in Chapter 3, is the Kneser graphs $K(2k+1, k)$. In particular the Petersen graph is a Kneser graph, namely $K(5, 2)$.

The following lemma is now easy to see.

Lemma 4.2 *If a graph G admits a homomorphism to the Kneser graph $K(n, k)$, then $\chi_f(G) \leq \frac{n}{k}$.*

Proof. Note that a omomorphism of G to $K(n, k)$ will induce a k-set colouring using at most n colours. Therefore $\chi_k(G) \leq n$ and $\chi_f(G) = \liminf_k \frac{\chi_k(G)}{k} \leq \frac{n}{k}$. $\square$

Definition 2.46 is not the only standard way of defining the fractional chromatic number of graphs. The following equivalent definition helps us to establish some basic properties of the fractional chromatic number.

Definition 4.3 Let G be a graph and $\mathcal{I}$ the set of all the independent subsets of $V(G)$. A fractional weight of G is a function f from $\mathcal{I}$ to $\mathbb{R}^+ \cup \{0\}$ for which the inequality

$$\sum_{x \in I} f(I) \geq 1 \tag{4.1}$$

holds for every vertex x of G. Here the sum is taken over all the independent subsets I which contain x. The fractional chromatic number is then defined to be the infimum of $\sum_{I \in \mathcal{I}} f(I)$, where the infimum is taken over all the fractional weights of G. $\diamond$

It is not hard to see that the infimum of the Definition 4.3 is equal to that of the Definition 2.46. We do not give a proof here, instead we refer the interested reader to [68].

The inequalities of (4.1) in Definition 4.3 form a feasible linear program with integer coefficients. This linear programming is feasible because there is a trivial solution defined by $f(I) = 1$ for every independent set $I \in \mathcal{I}$. Therefore there exist an optimal solution $X_f(G)$. Moreover such an optimal solution must be a rational number. This proves the following lemma.

Lemma 4.4 *For any given graph G we have $\chi_f(G) = \min_k\{\frac{\chi_k(G)}{k}\} = \min_f \sum_{I\in\mathcal{I}} f(\mathcal{I})$, where the minimum in the sum is being taken over all the fractional weight functions.*

Using this lemma, the Lemma 4.2 can be improved as below:

Lemma 4.5 *For a given graph G the fractional chromatic number of G, $\chi_f(G)$, is equal to the smallest ratio $\frac{n}{k}$ for which there exists a homomorphism of G to the Kneser graph $K(n, k)$.*

Proof. By Lemma 4.4 the fractional chromatic number of G is equal to $\frac{\chi_k(G)}{k}$ for some k. The k-set colouring of G using $\chi_k(G)$ colours provides a homomorphism of G to $K(\chi_k(G), k)$. □

Notice that in the above lemma, n and k need not be relatively prime. For example the Kneser graph $K(2n, 2k)$ has the same fractional chromatic number as the Kneser graph $K(n, k)$, but if $n > 2k$ then there is no homomorphism of $K(2n, 2k)$ to $K(n, k)$. This can be seen from their chromatic numbers, see Theorem 4.6.

By Lemma 4.5, the role Kneser graphs play for the fractional chromatic number is similar to the role that the complete graphs play for the ordinary chromatic number. For this reason, and because Kneser graphs will be used later in this chapter, we would like to say more about them.

The name Kneser is associated with these graphs because of a conjecture of M. Kneser on their chromatic number. In order to colour a Kneser graph, notice that $K(n, k)$ has no edge for $n < 2k$, and therefore is 1-colourable. For $n = 2k$, $K(n, k)$ is a perfect matching and thus 2-colourable.

For $n > 2k$ a natural colouring can be constructed by induction on n. Let x be an element of the n-set used to construct the Kneser graph $K(n,k)$. Then the subset K_x of all the vertices containing x, forms an independent set. Colour the vertices in K_x with the same colour. The remaining vertices (i.e., those which do not contain x), form a Kneser graph of $K(n-1,k)$. Do the same procedure for the new Kneser graph (using a new colour), and continue this process of colouring till the remaining graph is the Kneser graph $K(2k,k)$. Now, use two new colours to colour this last graph. What we obtain, is a $(n-2k+2)$-colouring of $K(n,k)$.

In [43] Kneser conjectured that this is the optimal colouring. This conjecture was proved by L. Lovász in [46].

Theorem 4.6 [46] *For given positive integers, n and k, with $n \geq 2k$ we have $\chi(K(n,k)) = n-2k+2$.*

Proof of this theorem is one of the deep results in graph theory. Lovász used algebraic topology and in particular Borsuk-Ulam theorem in his proof. Since then, various authors have tried to improve this result by simplifying the proof, generalizing the theorem, and etc. Among all of these works a remarkable one is due to A. Schrijver [69]. His nice argument for the theorem also characterizes the set of minimal subgraphs of $K(n,k)$ which has the chromatic number as the Kneser graphs, i.e., the set of $(n-2k+2)$-critical subgraphs of $K(n,k)$.

These minimal subgraphs which we call them Schrijver graphs are introduced below:

Definition 4.7 Let a, b be two integers such that $b \geq 2a$. We define $S(a,b)$ to be the graph whose vertices are the a-independent sets of a b-cycle, where two of these independent sets are joined by an edge if they are disjoint. $\diamond$

Obviously the Schrijver graph $S(a,b)$ is a subgraph of the Kneser graph $K(a,b)$. Schrijver proved that they are the $(n-2k+2)$-critical subgraphs of $K(n,k)$. We will only need to know their chromatic number. This is stated in the next theorem.

Theorem 4.8 [69] *Let a, b be positive integers such that $b \geq 2a$. Then $\chi(S(a,b)) = b - 2a + 2$.*

To complete our discussion on the fractional chromatic number, notice that a complete graph K_n is also the Kneser graph $K(n, 1)$. Therefore, by Lemma 4.2, the fractional chromatic number is bounded above by the ordinary chromatic number. But the chromatic number does not provide any lower bound for the fractional chromatic number. This can be seen by considering the following family of the Kneser graphs. Let $n > 2k$ and let $\mathcal{K} = \{K(an, ak) | a \in \mathbb{N}\}$ be a family of Kneser graphs. The fractional chromatic number of the members of $\mathcal{K}$ is bounded above by $\frac{n}{k}$ while the chromatic number tends to infinity when a tends to infinity, (by Theorem 4.6).

Recall that the fractional chromatic number is the minimum of $\sum f(I)$, where f runs over the set of all functions from $\mathcal{I} \to \mathbb{R}^+ \cup \{0\}$, satisfying the inequalities of 4.1. The chromatic number can be defined in a similar vein, it is the minimum of $\sum f(I)$, where f runs over the set of all functions from $\mathcal{I} \to \{0, 1\}$, satisfying the same inequalities. Hence there is more freedom for the choice of values of f in the definition of the fractional chromatic number than that of the ordinary chromatic number. This is responsible for the fact that ordinary chromatic number can be arbitrarily larger than the fractional chromatic number. In the next section we will define a function with a similar flavour as the fractional chromatic number but more dependent on the structure of the graph.

We finish our discussion of the fractional chromatic number with the following no-homomorphism lemma. This lemma can be compared to the other no-homomorphism lemmas, Lemma 2.1 and Lemma 2.2.

Lemma 4.9 *Let G and H be two graphs for which there is a homomorphism of G to H. Then we have $\chi_f(G) \leq \chi_f(H)$.*

Proof. Suppose $\chi_f(H) = \frac{n}{k}$, then by Lemma 4.5 for some positive integer a there exists a homomorphism of H to the Kneser graph $K(an, ak)$. But since $G \to H$, the graph G also admits a homomorphism to $K(an, ak)$. Therefore by Lemma 4.2 we have $\chi_f(G) \leq \frac{n}{k}$. $\square$

Our aim in this chapter is to study another graph parameter, namely the chromatic covering number of graphs, which has a similar flavor as the fractional chromatic number. This parameter has arisen naturally in the study of random lifts of graphs, see [3]. Here we study this parameter for its own interesting properties, specially for its homomorphism properties.

In the next section we will introduce the chromatic covering number and give lower bounds and upper bounds for this parameter in terms of the chromatic number. In Section 4.3 we construct a family of Kneser like graphs which play the role of complete graphs for the chromatic covering number. Using these constructions we show that the upper bounds for the chromatic covering number, obtained from the chromatic number, are tight. In the last section we introduce some more properties of the chromatic covering number, together with some related problems.

4.2 Chromatic covering number

We start this section by the following definition of chromatic covering and the chromatic covering number.

Definition 4.10 Let G be a graph and $G_1, \ldots, G_k$ induced subgraphs of G. If for every vertex u of G we have $\sum_{u \in V(G_i)} \frac{1}{\chi(G_i)} \geq 1$, then $\{G_1, \ldots G_k\}$ is called a *chromatic covering* of G. The *chromatic covering number* $F_\chi(G)$ of G is the smallest value k such that G admits a chromatic covering with at most k induced subgraphs. $\diamond$

For example the chromatic covering number of the complete graph K_n is equal to n. This can easily be seen from the following two lemmas.

Lemma 4.11 *[3] The chromatic covering number of a graph G is less than or equal to the chromatic number of G.*

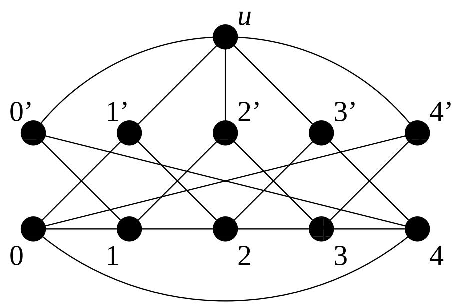

Figure 4.1: The Grötzsch graph

Proof. Let c be a $\chi(G)$-colouring of G, and let G_i be the subgraph induced on the vertices of colour i, (so G_i has no edge and therefor $\chi(G_i) = 1$). Then $\{G_1, G_2, \cdots G_{x(G)}\}$ is a chromatic covering of G. □

Lemma 4.12 *[3] The chromatic covering number of a graph G is bigger than or equal to the fractional chromatic number.*

Proof. Let $\{G_1, G_2, \dots G_k\}$ be a chromatic covering of G where $k = F_\chi(G)$. For $i = 1, 2, \dots k$, let $I_{i,1}, \dots, I_{i,\chi(G)}$ be the colour classes in a proper $\chi(G)$-colouring of G_i. Now we define a weight function on the set of the independent subsets of the graph G as below:

$$\mu(I) = \begin{cases} \frac{1}{\chi(G_i)}, & \text{if} \quad I = I_{i,j} \quad \text{for some} \quad i \text{ and } j \\ 0 & otherwise. \end{cases}$$

The weight function μ satisfies the inequalities (4.1) of the Definition 4.3 because $\sum_{u \in I} \mu(I) = \sum_{u \in V(G_i)} \frac{1}{\chi(G_i)} \geq 1$. Therefore μ is a fractional weight of G with a total weight of $k = F_\chi(G)$. Hence $\chi_f(G) \leq F_\chi(G)$. □

The next example shows how to find the chromatic covering number of the Grötzsch graph.

Example 4.13 Grötzsch's graph G above provides a good illustration of the dynamics of chromatic coverings. It is well known that this graph is 4-chromatic, yet it contains relatively large bipartite subgraphs. In particular the graph G_1 obtained from G by removing the vertices $0, 0', u$ is bipartite, as is the subgraph G_2 obtained from G by removing $1, 2, 4$. Noting that the subgraph G_3 of G induced by $\{0, 0', u, 1, 2, 4\}$ is again bipartite, we conclude that $\{G_1, G_2, G_3\}$ is a collection of bipartite induced subgraphs of G such that every vertex of G is in two of these subgraphs. Thus $F_\chi(G) \leq 3$.

On the other hand it is known (see [45]) that the fractional chromatic number of Grötzsch's graph is $\frac{29}{10}$. Now by Lemma 4.12 we find that $F_\chi(G) = 3$. $\diamond$

Even though chromatic covering number is similar to the fractional chromatic number, they behave differently in some other aspects. As we saw in the previous section, the chromatic number is not bounded above by any function of the fractional chromatic number. In contrary, it has been proven in [3] that for any graph G we have $\chi(G) \leq 2\left(F_\chi(G)\right)^2$. It was asked by J. Matoušek, [48], whether this bound can be improved. The next theorem answers this question, then in the next chapter we show that the bound of this theorem is the best possible.

Theorem 4.14 *For every graph G, $\chi(G) \leq \left\lfloor \left(\frac{F_\chi(G)+1}{2}\right)^2 \right\rfloor$.*

Proof. Let $\{G_1, \ldots G_k\}$ be a chromatic covering of G, where $k = F_\chi(G)$. Moreover, by permuting the indices, if required, we may assume that $\chi(G_1) \leq \chi(G_2) \leq \ldots \leq \chi(G_k)$. Since $\bigcup_{i=1}^k V(G_i) = V(G)$, there exists a smallest index ℓ such that $\bigcup_{i=1}^\ell V(G_i) = V(G)$. By the choice of ℓ there must exist a vertex $u \in V(G_\ell) \setminus (V(G_1) \cup \ldots \cup V(G_{\ell-1}))$. By the covering condition for the vertex u we have

$$1 \leq \sum_{u \in V(G_i)} \frac{1}{\chi(G_i)}, \tag{4.2}$$

but since $u \notin G_i$ for $i < l$ we see that

$$\sum_{u \in V(G_i)} \frac{1}{\chi(G_i)} \leq \sum_{i \geq \ell} \frac{1}{\chi(G_i)}. \tag{4.3}$$

Finally, because $\chi(G_\ell) \leq \chi(G_i)$ for $i \geq \ell$,

$$\sum_{i \geq \ell} \frac{1}{\chi(G_i)} \leq \frac{k - \ell + 1}{\chi(G_\ell)}. \tag{4.4}$$

These three inequalities combined together imply that $\chi(G_\ell) \leq k - \ell + 1$. On the other hand, since $V(G) = \bigcup_{i=1}^{\ell} V(G_i)$ and that G_i's are induced we have

$$\chi(G) \leq \sum_{i \leq \ell} \chi(G_i) \leq \sum_{i \leq \ell} \chi(G_\ell) \leq \ell(k - \ell + 1) \leq \left\lfloor \left(\frac{k+1}{2} \right)^2 \right\rfloor. \tag{4.5}$$

□

Notice that in order to have $\chi(G) = \left\lfloor \left(\frac{F_\chi(G)+1}{2} \right)^2 \right\rfloor$ each of the inequalities in (4.2) ... (4.5) must be an equality. In this case from (4.4) and the second inequality of (4.5) we find that all G_i's must have the same chromatic number. Therefore the equality in (4.2) implies that every vertex must appear in the same number of G_i's (in $\chi(G_i)$ of them). Finally considering the relation between ℓ and k from the last inequality of 4.5 we will have two separate cases based on the parity of k.

For k odd, say $k = 2p - 1$, we have $\left\lfloor \left(\frac{k+1}{2} \right)^2 \right\rfloor = p^2$, and equality holds in the Theorem 4.14 only if $\{G_1, \ldots, G_{2p-1}\}$ are p-chromatic subgraphs such that every vertex u of G is in p of these subgraphs.

For $k = 2p$ we have $\left\lfloor \left(\frac{k+1}{2} \right)^2 \right\rfloor = p^2 + p$, and equality holds in the Theorem 4.14 only if either $\{G_1, \ldots, G_{2p}\}$ are p-chromatic subgraphs such that every vertex u of G is in p of these subgraphs, or $\{G_1, \ldots, G_{2p}\}$ are $(p+1)$-chromatic subgraphs such that every vertex u of G is in $p + 1$ of these subgraphs.

These considerations will help us to characterize likely candidates to reach the upper bound in the next section. But before closing this section we should prove the following no-homomorphism lemma which for us is the most interesting property of the chromatic covering number.

Lemma 4.15 *Let G and H be two graphs. If $G \to H$ then $F_\chi(G) \leq F_\chi(H)$.*

Proof. The inequality $F_\chi(G) \leq F_\chi(H)$ when G is a subgraph of H is easy to see. Now let f be a homomorphism of G to H, and let $H' = f(G)$ be the image of G in H. Let $\{H'_1, H'_2 \cdots H'_k\}$ be a chromatic covering of H'. Then $\{f^{-1}(H'_1), f^{-1}(H'_2) \cdots f^{-1}(H_k)\}$ forms a chromatic covering of G. So the chromatic covering of G is smaller than or equal to chromatic covering number of H', but $F_\chi(H') \leq F_\chi(H)$. □

4.3 Kneser-like graphs

The condition under which the upper bound of the Theorem 4.14 is tight leads to the following construction which was first introduced by C. Tardif in [74].

Definition 4.16 Let n, r and s be positive integers such that $r \leq s$. We define the graph $K_n^{r,s}$ as follows: The vertices of $K_n^{r,s}$ are the subsets $A = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ of $\{1, \ldots, s\} \times \{1, \ldots, n\}$ such that $i_1, \ldots, i_r$ are all distinct. Two of these subsets are joined by an edge in $K_n^{r,s}$ if they are disjoint. ◇

In [76], $K_n^{r,s}$ is called a *fractional multiple* of the complete graph K_n. It can also be represented as follows: The vertices of $K_n^{r,s}$ represent independent r-sets in a disjoint union of s copies of K_n, and two of these are joined by an edge in $K_n^{r,s}$ if they are disjoint.

The following lemma shows that for certain kind of chromatic covering the fractional multiple graphs will play the role of a complete graph, when the chromatic covering is compared to the ordinary colouring.

Lemma 4.17 *A graph G admits a homomorphism to $K_n^{r,s}$ if and only if G can be covered by s n-colourable subgraphs $G_1, \ldots, G_s$ such that every vertex of G is in r of these subgraphs.*

Proof. Suppose that $\{G_1, \ldots, G_s\}$ is such a covering of G. For $i = 1, \ldots, n$, fix an n-colouring $f_i : G_i \mapsto \{1, \ldots, n\}$ of G_i. We first define the mapping $\phi : G \mapsto K_n^{r,s}$ by

$$\phi(u) = \{(i, f_i(u)) : u \in V(G_i)\}.$$

Now we will show that ϕ is in fact a homomorphism of G to the fractional multiple $K_n^{r,s}$. To see this note that for every edge uv of G, if $u, v \in G_i$ then we have $f_i(u) \neq f_i(v)$. Therefore, $\phi(u)$ is disjoint from (that is, adjacent to) $\phi(v)$. So ϕ is a homomorphism.

Conversely, if $\phi : G \mapsto K_n^{r,s}$ is a homomorphism, then for $i = 1, \ldots, s$, let G_i to be the subgraph of G induced by

$$V(G_i) = \{u \in V(G) : \phi(u) \cap \{(i,1), \ldots, (i,n)\} \neq \emptyset\}.$$

It is obvious that G_i is n-colourable. On the other hand if $\phi(u) = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ then the vertex u belongs to only G_{i_j} for $j = 1, 2, \ldots r$ and therefore every vertex u of G belongs to r of the induced subgraphs G_is. □

In the previous section we showed that if a graph G satisfies $F_\chi(G) = 2p - 1$ and $\chi(G) = p^2$, then G can be covered by $2p - 1$ p-chromatic subgraphs such that every vertex of G is in p of these subgraphs. By Lemma 4.17, such a graph G would then admit a homomorphism to $K_p^{p,2p-1}$. Therefore by Lemma 4.15, the upper bound of p^2 on the chromatic number of the graphs with chromatic covering number at most $2p - 1$ is tight if and only if $\chi(K_p^{p,2p-1}) = p^2$.

A similar argument holds in the case of even chromatic covering numbers, except that this time there are two candidates which may satisfy the upper bound: a graph G which satisfies $F_\chi(G) = 2p$ and $\chi(G) = p^2 + p$ may be covered by $2p$ p-chromatic subgraphs such that every vertex of G is in p of these subgraphs, or by $2p$ $(p + 1)$-chromatic subgraphs such that every vertex of G is in $p + 1$ of these subgraphs. By Lemma 4.17, such a graph admits a homomorphism into $K_p^{p,2p}$ or $K_{p+1}^{p+1,2p}$. Therefore the upper bound of $p^2 + p$ on the chromatic number of the graphs with chromatic covering number at most $2p$ is tight if and only if $\chi(K_p^{p,2p}) = p^2 + p$ or $\chi(K_{p+1}^{p+1,2p}) = p^2 + p$.

So to prove the tightness of the upper bound of the Theorem 4.14 we must find the chromatic number of certain type of the fractional multiple graphs. The following easy lemma provides an upper bound on the chromatic number of $K_n^{r,s}$.

Lemma 4.18 *Let n, r and s be positive integers such that $r \leq s$. Then $K_n^{r,s}$ is $(n(s-r+1))$-colourable, i.e., $\chi(K_n^{r,s}) \leq n(s-r+1)$.*

Proof. To prove this we first label the elements of $X = \{1,\dots,s-r+1\} \times \{1,\dots,n\}$ with $n(s-r+1)$ colours. Then for every vertex A of $K_n^{r,s}$ choose a pair $x_A = (i,j)$ which is both in X and A. Note that this is possible because A intersects $\{1,\dots,s-r+1\} \times \{1,\dots,n\}$. Now colour every vertex A of $K_n^{r,s}$ with the colour of x_A in X. In this way, since adjacent vertices have no element in common, they receive different colours. Thus, $\chi(K_n^{r,s}) \leq n(s-r+1)$. $\square$

The next step will be to show that the upper bound of the previous lemma is tight, i.e., there are no colouring of $K_n^{r,s}$ using less than $n(s-r+1)$ colours. This problem has the flavor of Kneser's conjecture. The next theorem will answer this problem for the even values of n. We will use Schrijver's strengthening on the chromatic number of Kneser graphs (Theorem 4.8) in our proof of this theorem.

Theorem 4.19 *Let n, r, s be integers such that n is even and $r \leq s$. Then $\chi(K_n^{r,s}) = n(s-r+1)$.*

Proof. By Lemma 4.18 we have $\chi(K_n^{r,s}) \leq n(s-r+1)$. To show that the bound is tight, we first define a homomorphism ϕ from Schrijver graph $S(a,b)$ to $K_n^{r,s}$, where $a = \frac{n}{2}(r-1)+1$ and $b = ns$.

Suppose that C_b is a b-cycle where the vertices are labelled consecutively by

$$(1,1),(1,2),\dots,(1,n),(2,1),\dots,(2,n),\dots,(s,1),\dots,(s,n).$$

Then for any a-independent set I of the cycle, there exist at least r values $i_1,\dots,i_r$ such that I intersects $\{(i_k,1),\dots,(i_k,n)\}$ for $k = 1,\dots,r$. We can then select j_k such that $(i_k,j_k) \in I$ for $k = 1,\dots,r$, and put

$$\phi(I) = \{(i_1,j_1),\dots,(i_r,j_r)\}.$$

By this definition if I and J are two independent a-subsets of C_b then $\phi(I)$ and $\phi(J)$ are two independent r-subsets of $\{1, 2, \cdots\} \times \{1, 2, \cdots n\}$, therefore ϕ is a homomorphism of $S(a, b)$ to $K_n^{r,s}$.

Now by Lemma 2.1 $\chi(K_n^{r,s}) \geq \chi(S(a, b))$. But by Schrijver's theorem (Theorem 4.8) $\chi(S(a, b) = b - 2a + 2 = n(s - r + 1)$, therefore $\chi(K_n^{r,s}) = n(r - s + 1)$. □

Corollary 4.20 *Let k be an integer not congruent to 1 modulo 4. Then there exists a graph with chromatic covering number k and chromatic number $\left\lfloor \left(\frac{k+1}{2}\right)^2 \right\rfloor$.*

Proof. If k is even, say $k = 2q$, then one of the two graphs $K_q^{q,2q}$ and $K_{q+1}^{q+1,2q}$ is guaranteed to fit the bound for chromatics number by Theorem 4.19. For $k = 4p - 1$, the only candidate is $K_{2p}^{2p,4p-1}$, and this graph is indeed $4p^2$-chromatic by Theorem 4.19. Lemma 4.17 shows that each of these graphs indeed has a chromatic k-covering.
□

We will end this chapter by some comments and open problems on the fractional chromatic number and the chromatic covering number.

4.4 Concluding comments

4.4.1 Odd cases

We have shown that the bound in Theorem 4.14 is best possible in most cases. We do not doubt that the bound should be tight in all cases; this only depends on the identity $\chi(K_{2p+1}^{2p+1,4p+1}) = (2p + 1)^2$ being valid for all $p \geq 1$. This in turn would be implied by the following conjecture in completion of Theorem 4.19.

Conjecture 4.21 *Let n, r, s be integers such that n is odd and $r \leq s$. Then $\chi(K_n^{r,s}) = n(s - r + 1)$.*

4.4.2 Some observations

We have shown that chromatic covering is also a monotone graph parameter, i.e., if $G \to H$ then $F_\chi(G) \leq F_\chi(H)$. The relation between chromatic coverings and homomorphisms extends as follows:

> For every sequence $(a_1, a_2, \ldots, a_k)$, there exists a graph $K(a_1, a_2, \ldots, a_k)$ with the property that a graph G admits a homomorphism to $K(a_1, a_2, \ldots, a_k)$ if and only if G admits a chromatic covering $\{G_1, \ldots, G_k\}$ such that $\chi(G_i) \leq a_i$ for $i = 1, \ldots, k$.

The proof essentially follows the lines of that of Lemma 4.17, given a suitable definition of $K(a_1, \ldots, a_k)$. Therefore among the graphs $K(a_1, \ldots, a_k)$ with $a_i \leq k$ for $i = 1, \ldots, k$, we find a finite family of graphs which are maximal (in the sense of homomorphisms) with respect to the property of having a chromatic covering number at most k.

It can be shown that $K_2^{2,3}$ is the only maximal graph with chromatic covering number 3, but the situation changes in the case of larger chromatic covering numbers. In fact it can be shown that neither of the graphs $K_2^{2,4}$ and $K_3^{3,4}$ admits a homomorphism to the other. Now consider the graph $M = K_2^{2,4} \cup K_3^{3,4}$. Then M does not admit a homomorphism to $K_2^{2,4}$ or to $K_3^{3,4}$; however since M is 6-chromatic, this implies that $F_\chi(M) > 4$. Thus, the identity $F_\chi(G \cup H) = \max\{F_\chi(G), F_\chi(H)\}$ does not hold in general.

It seems that the identity $F_\chi(G \times H) = \min\{F_\chi(G), F_\chi(H)\}$ (where $\times$ is the categorical product) should not hold either. Indeed, $K_{10}^{4,10} \times K_{15}^{9,15}$ has a natural chromatic covering induced by ten 10-chromatic subgraphs of $K_{10}^{4,10}$ and fifteen 15-chromatic subgraphs of $K_{15}^{9,15}$. Thus $F_\chi(K_{10}^{4,10} \times K_{15}^{9,15}) \leq 25$, but there are no obvious chromatic coverings of $K_{10}^{4,10}$ or $K_{15}^{9,15}$ with 25 subgraphs (though we have no proof that it cannot be done).

It is interesting to compare these observations with results and problems concerning chromatic numbers: The inequalities $\chi(G \cup H) \geq \max\{\chi(G), \chi(H)\}$ and $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$ both follow from the fact that if there is a homomorphism from G to H, then $\chi(G) \leq \chi(H)$. It is easy to verify that equality always holds in the first case, while the question as to whether equality always holds in the second case is a notorious open problem. The situation is a bit more symmetric with chromatic covering numbers, in the sense that both inequalities can be strict.

4.4.3 Degeneracy covering numbers

The degeneracy covering and degeneracy covering number is defined in a similar way. Given a graph G, we say a collection $G_1, G_2, \cdots G_k$ is a *k-degeneracy covering* of G is for every vertex x of G the following holds:

$$\sum_{u \in V(G_i)} \frac{1}{\deg(G_i) + 1} \geq 1.$$

The *degeneracy covering number* is the smallest k for which G admits a k-degeneracy covering.

By this definition and by Theorem 2.6, every k-degeneracy covering is also a k-chromatic covering, therefore the degeneracy covering number is bounded by the chromatic covering number. In fact the chromatic covering number was defined in [3] in order to get bounds on the degeneracy covering number.

On the other hand it is not an easy problem to construct graphs with a difference between their chromatic covering number and colouring covering number. The problem of finding such a graph in a certain case was posed by C. Tardif and the author in a graph homomorphism workshop in Vancouver, 2000, and was answered affirmatively by A. V. Pyatkin in [64].

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