Proof search in intuitionistic sequent calculus and admissible rules

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Foreword

- The work presented here is an old work I made for my thesis and achieved in 1992 (my thesis and a partial translation are on my web page `http://www.pps.jussieu.fr/~roziere/admiss`)
- Results have since been obtained but by other means, but the approach I followed was purely proof theoretic, so could emphasize other aspects, and could be extended not exactly to the same cases

Summary

In intuitionistic propositional calculus, connections between

- Admissibility = closure under a rule.
  The rule $A_1, \ldots, A_n / C$ is admissible, written $A_1, \ldots, A_n \mathrel{|\!\!\sim} C$,

  iff

  for every substitution s on propositional variables:

  $$\text{if } \vdash s(A_1), \ldots, \vdash s(A_n) \text{ then } \vdash s(C).$$

- Backward derivability = search of possible proofs.

Admissibility = derivability + backward derivability

Emphasizes the role of the restriction on right contraction, in existence of admissible but not derivable rules.

Sequent calculus without cuts

$$\frac{}{\Gamma, \alpha \vdash \alpha}\ (\alpha \text{ variable or } \bot) \qquad \frac{}{\Gamma, \bot \vdash A}$$

$$\frac{\Gamma, A \to B \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Because the lack of contraction rule in the right part:
Every rule, but $(\to_l)$ and $(\vee_r)$, has a reversible formulation.

Two basic examples of admissible rules

$$(\; s(\alpha) = A,\ s(\beta) = B,\ s(\gamma) = C,\ s(\delta) = D \;)$$

$$\dfrac{A \to B \vdash A \qquad A \to B, B \vdash C \vee D}{} \qquad \dfrac{A \to B \vdash C}{} \qquad \dfrac{A \to B \vdash D}{}$$

$$\dotfill$$

$$A \to B \vdash C \vee D$$

$$(\alpha \to \beta) \to (\gamma \vee \delta) \vdash ((\alpha \to \beta) \to \alpha) \vee ((\alpha \to \beta) \to \gamma) \vee ((\alpha \to \beta) \to \delta)$$

redundancy

~~$C \vee D \to B \vdash C \vee D \qquad C \vee D \to B, B \vdash C \vee D$~~ $\qquad \dfrac{C \vee D \to B \vdash C}{} \qquad \dfrac{C \vee D \to B \vdash D}{}$

$$\dotfill$$

$$C \vee D \to B \vdash C \vee D$$

$$((\gamma \vee \delta) \to \beta) \to (\gamma \vee \delta) \vdash [((\gamma \vee \delta) \to \beta) \to \gamma] \vee [((\gamma \vee \delta) \to \beta) \to \delta]$$

Backward derivation = formalization of this procedure.

The backward consequence relation

$$
\dfrac{\dfrac{S_{1,1} \quad \dots \quad S_{1,n}}{\vdots} \qquad \dots \quad \dots \qquad \dfrac{S_{p,1} \quad \dots \quad S_{p,n}}{\vdots}}{S}
$$

redundancy

$$S^{\to} \vdash_{back} (S_{1,1}^{\to} \wedge \dots \wedge S_{1,n}^{\to}) \vee \dots \vee (S_{p,1}^{\to} \wedge \dots \wedge S_{p,n}^{\to})$$

$$(\ (A_1, \dots, A_k \vdash C)^{\to} \ = \ A_1, \dots, A_k \to C \ = \ A_1 \to \dots \to A_k \to C \ )$$

We have to stop when a sequent contains a variable

$$(\Gamma \vdash \alpha)^{\to} \ = \ \Gamma \to \alpha$$ right simple sequents / formulas

$$(\alpha, \Gamma \vdash C)^{\to} \ = \ \alpha, \Gamma \to C$$ left simple sequents / formulas

All simple sequents in a backward derivation are leaves

Completeness

- The rule A/C is obtained by backward and forward derivation, written $A \vdash_{b,f} C$, when it is obtained by a (finite) sequence of backward derivations and usual derivation

$$\vdash_{b,f} = (\vdash_{back} + \vdash)^*$$

- Soundness

$$A \vdash_{b,f} C \implies A \mid\!\sim C$$

- Completeness

$$A \mid\!\sim C \implies A \vdash_{b,f} C$$

Infinite base of rules for admissibilty

As a corollary of completeness, all admissible rules can be obtained by composing derivable rules and some of the rules (ad_n) (Visser rules) :

$$\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to (\gamma \vee \delta) \vdash \begin{cases} \bigvee_{j=1}^{n} (\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to \alpha_j) \\ \vee \\ (\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to \gamma) \\ \vee \\ (\{\alpha_i \to \beta_i\}_{1\leq i\leq n} \to \delta) \end{cases} \qquad (ad_n)$$

Not completly straightforward because of redundancies.

Eliminating "pruning" of redundancies: an example

We have seen

$$((\gamma \vee \delta) \to \beta) \to (\gamma \vee \delta) \vdash [((\gamma \vee \delta) \to \beta) \to \gamma] \vee [((\gamma \vee \delta) \to \beta) \to \delta] \, .$$

It can be reduce by $(\gamma \vee \delta) \to \beta \equiv (\gamma \to \beta) \wedge (\delta \to \beta)$ to

$$(\gamma \to \beta), (\delta \to \beta) \to (\gamma \vee \delta) \vdash \begin{cases} [(\gamma \to \beta), (\delta \to \beta) \to \gamma] \\ \vee \\ [(\gamma \to \beta), (\delta \to \beta) \to \delta] \end{cases}$$

instance of (ad_2)

The only rule leading to possible redundancies is $(\to_I)$.
This rule can be rewritten in order to avoid it.

Eliminating "pruning" of redundancies

$$\frac{\Gamma, A \to B \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}$$

can be replaced by:

$$\frac{\Gamma, E \to B, F \to B \vdash C}{\Gamma, (E \vee B) \to B \vdash C} \qquad \frac{\Gamma, E \to F \to B \vdash A}{\Gamma, (E \wedge F) \to B \vdash C}$$

$$\frac{\Gamma, E, F \to B \vdash F \qquad \Gamma, B \vdash C}{\Gamma, (E \to F) \to B \vdash C} \qquad \frac{\Gamma, \alpha, B \vdash C}{\Gamma, \alpha, \alpha \to B \vdash C}$$

(old trick that apparently go back to Vorob'ev (1958))

For admissibility we use only the 3 first and keep instance of usual left rule for A atomic.

Completeness proof (sketch)

The skeleton is an usual one:

- Forward and backward derivation plays the syntactic part;
- Substitutions play the semantic part.

Two steps :

- Construct all saturated sets containing a given set of formulas;
- Associate to each saturated set a particular substitution.

We have to deal with finite sets of formulas, in order to construct substitutions. Then we need :

- Restriction of saturation to a convenient finite set of formulas (corresponding to sequent of subformulas);

As all is finite we can :

- Construct a sufficient but finite collection of saturated sets containing a given finite set of formulas.

Extending subformulas for saturation

We define saturation on formulas obtained from sequents of subformulas (sequent that appears in a backward derivation of the original formula).

- $\mathscr{F}^{\to}(\Gamma)$: formulas $A_1, \ldots, A_n \to C$
  where $A_1, \ldots, A_n$ are distinct negative subformulas of Γ
  C is a positive subformula of Γ
- $\mathscr{F}^{\to,\wedge,\vee}(\Gamma)$: disjunctions of distinct conjunctions of distinct formulas in $\mathscr{F}^{\to}(A)$;

Proposition.

- $\mathscr{F}^{\to}(\Gamma)$ and $\mathscr{F}^{\to,\wedge,\vee}(\Gamma)$ are finite.
- If $B \in \mathscr{F}^{\to}(\Gamma)$, then every formula of $\mathscr{F}^{\to}(B)$ is equivalent to a formula of $\mathscr{F}^{\to}(B) \cap \mathscr{F}^{\to}(\Gamma)$. Hence :

$$\mathscr{F}^{\to}(\mathscr{F}^{\to}(\Gamma))/_{\equiv} = \mathscr{F}^{\to}(\Gamma)/_{\equiv} \quad \mathscr{F}^{\to,\wedge,\vee}(\mathscr{F}^{\to}(A))/_{\equiv} = \mathscr{F}^{\to,\wedge,\vee}(A)/_{\equiv}$$

Saturation property

Definition.

- Γ is Θ-saturated :
  $\forall C, D \in \mathscr{F}^{\rightarrow,\wedge,\vee}(\Theta),\ \Gamma \vdash_{b,f} C \vee D \ \Rightarrow\ \Gamma \vdash C$ or $\Gamma \vdash D$.
- Γ is saturated if and only if Γ is Γ-saturated.

Fact. If $\Gamma \subset \mathscr{F}^{\rightarrow}(\Theta)$ and Γ is Θ-saturated, then Γ is saturated.

Lemma. For every formula A, there exists $\Gamma_1, \ldots, \Gamma_n$ saturated such that

$$A \vdash_{b,f} (\bigwedge \Gamma_1) \vee \ldots \vee (\bigwedge \Gamma_n)$$

$$(\bigwedge \Gamma_1) \vee \ldots \vee (\bigwedge \Gamma_n) \vdash A$$

In order to show that this notion of saturation is sufficient, the key point is that :

Γ is a saturated set, iff Γ is projective.

Projective unifier and admissibility

A finite set of formulas Γ is projective if
there exists a projective unifier s for Γ, that is

- $\forall C \in \Gamma, \ \vdash s(C)$
- $\forall \alpha, \ \Gamma \vdash \alpha \leftrightarrow s(\alpha)$ and then
  $\forall C, \ \Gamma \vdash C \leftrightarrow s(C)$ and $\Gamma \to C \equiv \Gamma \to s(C)$

$\Downarrow$ usual Disjunction Property

$\Uparrow$ equivalent to the main step of completness proof

Γ has the disjunction property for admissibility
i.e.
$\forall C, D, \ (\Gamma \mathrel{|\!\!\sim} C \vee D \ \text{ iff } \ \Gamma \vdash C \text{ or } \Gamma \vdash D)$

$\Downarrow$ (take $C = D$)

Γ has the same admissible and derivable consequences: $\forall C, \ \Gamma \mathrel{|\!\!\sim} C \ \text{ iff } \ \Gamma \vdash C$

Projective unifier and saturated set

Proposition. The three following propositions are equivalent.

1. Γ is a saturated set.
2. There exists a projective unifier for Γ, or $\Gamma \vdash \bot$.
3. Γ has the disjunction property for admissibility.

(3)⇒(1) by soundness of "$\vdash_{b,f}$" for "$\mid\!\sim$".
(2)⇒(3) is easy and has been seen
It is then sufficient to prove (1)⇒(2)
We can restrict to set of simple formulas.
The construction of the projective unifier for Γ in two steps

- A first substitution "eliminate" left simple formulas $\alpha \to G$
- It is then composed with the suitable substitution for right simple formulas $\Gamma \to \alpha$

Simple formulas

unifier	formula	
A simple example		
$s(\alpha_i) = \top,\ s(\beta_i) = \bot$	$\bigwedge_i \alpha_i \wedge \bigwedge_i \neg \beta_i$	
The two key examples		
$s(\alpha_i) = F \to \alpha_i,\ i \in I$	$F = \bigwedge_{i \in I} (\Gamma_i \to \alpha_i)$	right simple formulas
$s(\alpha_i) = \alpha_i \wedge F,\ i \in I$	$F = \bigwedge_{i \in I} (\alpha_i \to G_i)$	left simple formulas

The two key examples correspond to homogeneous sets of simple sequents

$$\Gamma \vdash \alpha \ \text{ or } \ \Gamma, \alpha \vdash C$$

Note that, by Glivenko Theorem, the case where a formula is not classically satisfiable is trivial

$$\Gamma \vdash_c \bot \ \text{ iff } \ \Gamma \vdash \bot \ \text{ iff } \ \Gamma \mathrel{|\!\!\sim} \bot$$

Construction of the substitution

First step. Because of composition, it is useful, for left simple formulas to block some later substitutions, with the constant $\top$:

$$s(\alpha) = \alpha \wedge A[\top/\alpha]$$

Let $A^{-\alpha} = A[\top/\alpha]$, and $G = \wedge\Gamma$.
The substitutions s_i, σ_i $i \in \{1, \ldots, n\}$ are defined by induction on i

- $s_0 = \sigma_0 = Id$,
- $s_{i+1} = [\alpha_{i+1} \wedge \sigma_i(G)^{-\alpha_{i+1}}/\alpha_{i+1}]$; $\sigma_i = s_i \circ \cdots \circ s_1 \circ s_0$.

If $Var_\Gamma = \{\alpha_1, \ldots, \alpha_n\}$, then $\sigma_n(G)$ is equivalent to a set of simple right formulas.
Idea of the proof : take a maximal backward derivation tree of $\sigma_n(G)$, then choose, by saturation, a derivation with leaf sequents that are consequences of G.
Difficulty : subformulas of $\sigma_n(G)$ are not directly in $\mathscr{F}^{\rightarrow,\wedge,\vee}(G)$.
Second step. As $\sigma_n(G)$ is equivalent to a set of right simple formulas, we can use the substitution still defined :

$$s(\alpha_i) = \sigma_n(G) \rightarrow \alpha_i$$

Subformulas of $\sigma_n(G)$

Substitution verify :

$$G \vdash G \leftrightarrow \sigma_i(G) \quad \text{hence } G \vdash \sigma_i(G)$$

A subformula B of $\sigma_n(G)$ is a variable α_i or a substituate of a subformula B^0 of G by $\sigma_{i_1,\ldots,i_l;n}$ for some $1 \leq i_1 < \cdots < i_l$, with:

- $\sigma_{i_1,\ldots,i_l;0} = \sigma_0(C) = id$
- if $q+1 \not\in \{i_1,\ldots,i_l\}$, then $\sigma_{i_1,\ldots,i_l;q+1} = s_{q+1} \circ \sigma_{i_1,\ldots,i_l;q}$
- if $q+1 \in \{i_1,\ldots,i_l\}$, then $\sigma_{i_1,\ldots,i_l;q+1} = \sigma_{i_1,\ldots,i_l;q}[\top/\alpha_{q+1}]$

Then

$$\alpha_{i_1},\ldots,\alpha_{i_l},\sigma_{i_1,\ldots,i_l;n}(G) \vdash \sigma_n(G)\,.$$

Saturation can be used to find a conjunction of simple sequents S_k corresponding to a derivation of $\sigma_n(G)$, such that :

$$G \equiv \bigwedge_k (S_k^{\rightarrow})^0 \vdash \bigwedge_k S_k^{\rightarrow} \vdash \sigma_n(G)$$

Elimination of left simple formulas

Always using analysis on subformulas in $\sigma_n(G)$ we obtained that under this hypothesis :

$$G \equiv \bigwedge_k (S_k^{\rightarrow})^0 \vdash \bigwedge_k S_k^{\rightarrow} \vdash \sigma_n(G)$$

among S_k's, all left simple sequents are consequences of the right simple sequents.

The problem to solve is that a substitution $[\alpha \wedge A/\alpha]$ applied to a right simple sequent $\Gamma \vdash \alpha$ leads to two sequents (in the backward derivation) :

$$\Gamma \vdash \alpha \quad \text{and} \quad \Gamma \vdash A$$

The formula A is a $\sigma_{i_1,\ldots,i_l;p}(G)$.

The point is that all these formulas are consequences of G and the variables α_{i_j}, but remaining sequents $\Gamma \vdash \alpha_{i_j}$ give these variables.

Conclusion

Other consequences

- Finitary unification type
- Rybakov result on admissibilty

Conclusion

- Purely proof theoretic analysis
- Non inversible rules play the key role
- Proof that we can construct a “good” substitution for a saturated set is very intricated (but hopefully could be simplified)