1 Proposition. $\Gamma$ is a bijection between maps from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ and partial orientations of the hypercube $\mathbb{F}_2^n$.

Map $f \mapsto$ orientation: edge $x \to x + e_i$ when $f_i(x) \neq x_i$. Orientation $\Gamma \mapsto$ map $f(x) = x + e_I$, where $(x \to x + e_i)_{i \in I} =$ edges leaving $x$.

2 Remark. Variant of Leibniz rule: $\partial_i(\varphi \psi) + \partial_i \varphi \partial_i \psi = \varphi \partial_i \psi + \psi \partial_i \varphi$.

$$
\partial_i(\varphi \psi)(x) + \partial_i \varphi(x) \partial_i \psi(x) = \varphi(x) \psi(x) + \varphi(x + e^i) \psi(x + e^i)
+ (\varphi(x) + \varphi(x + e^i)) \psi(x) + \psi(x + e^i)
= \varphi(x) \psi(x + e^i) + \psi(x) \varphi(x + e^i)
\varphi(x) \partial_i \psi(x) + \psi(x) \partial_i \varphi(x) = \varphi(x) (\psi(x) + \psi(x + e^i)) + \psi(x) (\varphi(x) + \varphi(x + e^i))
= \varphi(x) \psi(x + e^i) + \psi(x) \varphi(x + e^i).
$$

3 Lemma. If $f$ has $\geq 2$ attractors, then for some subcube $\kappa$, $f|\kappa$ has $\geq 2$ fixed points.

Let $A$ and $B$ be any two attractors of $\Gamma(f)$ and $(a,b) \in A \times B$ be any pair such that $d(a,b)$ is minimal: then $a$ and $b$ are fixed points of $f|_{[a,b]}$.

4 Proposition (Naldi, Remy, Thieffry, Chaouiya). Assume $\mathcal{G}(f)$ has no loop on $n$. Fixed points are preserved by reduction and expansion: $x$ is a fixed point of $f'$ if and only if $x'$ is a fixed point of $f$. Attractive cycles are preserved by reduction: $\pi$ maps attractive cycles of $f$ to attractive cycles of $f'$.

For any $x \in \mathbb{F}_2^{n-1}$, let $x^* = (x, f_n(x, -)) \in \mathbb{F}_2^n$. Then

$$f(x^*) = (f'(x), f_n(x, -)),$$

because $f_n(x^*) = f_n(x, f_n(x, -)) = f_n(x, -)$. Moreover, $\pi(x^*) = x$, where $\pi : \mathbb{F}_2^n \to \mathbb{F}_2^{n-1}$ is the projection. Now, $f'(x) = x$ if and only if $f_i(x, f_n(x, -)) = f_i(x) = x_i$ for all $i < n$, if and only if $f_i(x^*) = x_i^*$ for all $i < n$, if and only if $f(x^*) = x^*$, since $f_n(x^*) = f_n(x, -) = x_n^*$. 

If \( \theta = (\ldots, y, y + e^i, \ldots) \) is an attractive cycle of \( f \), \( f(y) = y + e^i \). If \( i = n \), \( \pi(y) = \pi(y + e^i) \). Otherwise, since \( \theta \) is attractive, \( f_n(y) = y_n \) and \( \pi(y)^* = y \). Hence, letting \( x = \pi(y) \), we have
\[
f'((\pi(y)) = f'(x) = \pi(f(x^*)) = \pi(f(y)) = \pi(y + e^i) = \pi(y) + e^i.
\]
Therefore \( \pi(\theta) \) is an attractive cycle of \( \Gamma(f') \), and for every \( x \in \pi(\theta) \), \( x^* \in \theta \).

5 **Theorem** (Remy, Mossé, Chaouiya, Thieffry). If \( \mathcal{G}(f)(x) \) is independent of \( x \) and consists in a Hamiltonian positive (resp. negative) cycle, then \( f \) has 2 fixed points and no cyclic attractor (resp. \( f \) has no fixed point, and a unique attractor which is an attractive cycle).

If \( \mathcal{G}(f)(x) = \mathcal{G}(f) \) is a Hamiltonian positive cycle, we may assume w.l.o.g. that \( \mathcal{G}(f) = (1, 2, \ldots, n, 1) \). The sign of the edge from \( i \) to \( i + 1 \mod n \) is \((-1)^{\epsilon_i} \) with \( \epsilon_i \in \{0, 1\} \), so that \( f_{k+1}(x) = x_k + \epsilon_k \), where indices are taken modulo \( n \). Since \( \sum_{i=1}^n \epsilon_i = 0 \), the point \( a \) defined by
\[
a_k = \sum_{i=1}^{k-1} \epsilon_i \quad \text{for} \quad k = 1, \ldots, n,
\]
and its antipode \( \overline{a} \) are fixed points of \( f \). Let \( K(x) \) be the set of \( k \) such that \( x_k + x_{k+1} \neq \epsilon_k \). Then \( a + x = e^{K(x)} \) and \( d(x, a) \) is the cardinality of \( K \). If \( x \) is different from \( a \) and \( \overline{a} \), \( K(x) \neq \emptyset \) and \( K(x) \neq \{1, \ldots, n\} \): for any \( k \in K \), \( f_{k+1}(x) = x_k + \epsilon_k \neq x_{k+1} \), \( \Gamma(f) \) has an edge from \( x \) to \( x + e^k \), and \( K(x + e^k) \) has cardinality smaller than that of \( K(x) \). Hence \( \Gamma(f) \) has a trajectory from \( x \) to \( a \), and in particular, \( \Gamma(f) \) has no cyclic attractor.

If \( \mathcal{G}(f)(x) = \mathcal{G}(f) \) is a Hamiltonian negative cycle, with the same notations, we have \( \sum_{i=1}^n \epsilon_i = 1 \). Thus:
\[
1 = \sum_{i=1}^n f_{i+1}(x) + x_i = \sum_{i=1}^n f_i(x) + x_i,
\]
so that \( f \) has no fixed point. The desired attractive cycle is
\[
\theta = (a, a + e^1, a + e^1 + e^2, \ldots, \overline{a}, \overline{a} + e^1, \overline{a} + e^1 + e^2, \ldots, a).
\]
If \( x = a + e^1 \notin \theta \), then for any \( i \in I \) such that \( i - 1 \notin I \), where indices are taken modulo \( n \), we have \( f_i(x) \neq x_i \) and \( \Gamma(f) \) has a trajectory from \( x \) to \( x + e^i \). Therefore \( \Gamma(f) \) has a trajectory from \( x \) to some \( a + e^1 + \cdots + e^k \in \theta \), with \( \{1, \ldots, k\} \subseteq I \).

6 **Lemma.** 1. An orbit of Boolean networks contains a network with no fixed point if and only if it contains a network with \( \geq 2 \) fixed points.

2. The set of networks such that each subnetwork \( f\mid_k \) has a unique fixed point is closed under translation.

1. It suffices to observe that if \( f \) has no fixed point, \( f + \text{id} \) is a non bijective map from a finite set to itself, hence it is not injective: therefore, for some \( z \), there exist distinct points \( x, y \) such that \( (f + \text{id})(x) = (f + \text{id})(y) = z \), and \( f + z \) has two fixed points. On the other hand, if \( f \) has two fixed points, \( f + \text{id} \) is not bijective, hence not surjective and does not take some value \( z \): then \( f + z \) has no fixed point.
2. Let \( F_n \) be the set of maps \( f : \mathbb{F}_2^n \to \mathbb{F}_2^n \) such that for each subcube \( \kappa \), \( f|_\kappa \) has a unique fixed point. Since the group of translations is generated by translations by basis vectors \( e_i \), it suffices to prove, by induction on \( n \), that for any \( f \in F_n \) and \( i \in \{1, \ldots, n\} \), \( f + e_i \in F_n \).

- For \( n = 1 \), the maps with a unique fixed point are the constant maps, and constant maps are closed under translation.

- If \( n > 1 \), let \( f \in F_n \), \( i \in \{1, \ldots, n\} \) and \( g = f + e_i \). Let \( \kappa_0 \) and \( \kappa_1 \) be the \( (n-1) \)-dimensional subcubes defined respectively by \( x_i = 0 \) and \( x_i = 1 \). By induction hypothesis, \( g|_{\kappa_0} \) has a unique fixed point \( x \) and \( g|_{\kappa_1} \) has a unique fixed point \( y \). On the other hand, since \( \mathbb{F}_2^n = \kappa_0 \cup \kappa_1 \), \( f \) has a unique fixed point, which needs to be either \( x \) or \( y \), say it is \( x \). Then \( f(y) = y + e_i \) and \( g(y) = y \). Moreover, \( f(x) = x \), hence \( g(x) \neq x \), and we may conclude that \( y \) is the unique fixed point of \( g \), hence that \( g \in F_n \).

7 Proposition. \( f \) is hereditarily ufp if and only if \( f + \text{id} \) is hereditarily bijective.

If for each subcube \( \kappa \), \( (f + \text{id})|_\kappa \) is bijective, then \((f + \text{id})|_\kappa\) takes exactly once the value 0, and clearly, all the \((f + \text{id})|_\kappa\) have a unique fixed point.

On the other hand, in order to prove that for any \( n \geq 1 \) and any \( f : \mathbb{F}_2^n \to \mathbb{F}_2^n \), if \( f \) is hereditarily ufp, then \( f + \text{id} \) is hereditarily bijective, it suffices to prove that for any \( n \) and \( f \), if \( f|_\kappa \) has a unique fixed point for each subcube \( \kappa \), then \( f + \text{id} \) is bijective: this is because the hypothesis is closed under restriction. Assume this is wrong, so that there exists some \( f : \mathbb{F}_2^n \to \mathbb{F}_2^n \) such that \( f + \text{id} \) is not bijective while \( f|_\kappa \) has a unique fixed point for each subcube \( \kappa \). Since \( f + \text{id} \) is not bijective, the preimage \((f + \text{id})^{-1}(z)\) of some \( z \) is not a singleton, hence \((f + z + \text{id})^{-1}(0)\) is not a singleton and \( f + z \) does not have a unique fixed point. But by Lemma 6, this contradicts the hypothesis on \( f \), because \( f \) and \( f + z \) are in the same orbit under translation.

8 Corollary. If \( f + \text{id} \) is hereditarily bijective, then \( \Gamma(f) \) is collapsing.

Under the hypothesis of the corollary, by Proposition 7, for each subcube \( \kappa \), \( f|_\kappa \) has a unique fixed point. Lemma 3 then ensures that this fixed point is the only attractor of \( \Gamma(f) \). Let now \( x \) be the unique fixed point of \( f \) and let \( Y \) be the set of points \( y \in \mathbb{F}_2^n \) such that \( \Gamma(f) \) has no direct trajectory from \( y \) to \( x \). Assume for a contradiction that \( \Gamma(f) \) is not directly terminating. This implies \( Y \neq \emptyset \) and we may choose \( y \in Y \) such that \( d(x, y) \) is minimal: then \( x \) and \( y \) are fixed points for \( f|_{[x, y]} \), and since \( y \neq x \), it follows from Lemma that \((f + \text{id})|_{[x, y]}\) is not bijective.

9 Theorem (R.). If \( f + \text{id} \) is not bijective, then there exist two different points \( x, y \in \mathbb{F}_2^n \) such that \( \mathcal{G}(f)(x) \) and \( \mathcal{G}(f)(y) \) have a cycle.

To prove this theorem, first observe that as a non bijective map from a finite set to itself, \( f + \text{id} \) is not injective: some point \( z \in \mathbb{F}_2^n \) has a preimage of cardinality at least 2 under \( f + \text{id} \). Consider the partially ordered set \( E_z \) of subcubes \( \kappa \), ordered by inclusion, such that \( \pi_\kappa(z) \) has a preimage of cardinality at least 2 under \((f + \text{id})|_\kappa\). By hypothesis, \( E_z \neq \emptyset \). Let \( \kappa \) be a minimal subcube of \( E_z \), and let \( x, y \in \kappa \) be distinct points mapped by \((f + \text{id})|_\kappa\) to \( z|_\kappa \). Since \( \kappa \) is minimal in \( E_z \), \( x \) and \( y \) are antipodes in \( \kappa \), i.e., \( \kappa = [x, y] \). Recall that \( v(x, y) \) denotes the subset \( I \subseteq \{1, \ldots, n\} \) such that \( x + y = e_I \).
• If \( v(x, y) \) is a singleton \( \{i\} \), then \( y = x + e_i \) and \( (f + \text{id})_i(x) = (f + \text{id})_i(y) \), as a consequence \( \partial_i f_i(x) = f_i(x) + f_i(x + e_i) = \partial_i f_i(x + e_i) = 1 \), hence \( \mathcal{G}(f)(x) \) and \( \mathcal{G}(f)(y) \) have an edge from \( i \) to itself.

• If on the other hand \( d(x, y) > 2 \), for any \( i \in v(x, y) \), let \( \lambda_i \) be the subcube \( [x + e_i, y] \), which is smaller than \( \kappa \); then \( (f + \text{id})|_{\lambda_i} (x + e_i) \neq \pi_{\lambda_i}(z) \), since otherwise \( \lambda_i \) would have two different points \( x + e_i \) and \( y \) mapped by \( (f + \text{id})|_{\lambda_i} \) to \( \pi_{\lambda_i}(z) \), and \( \lambda_i \) would belong to \( E_x \), contradicting minimality. Therefore, for any \( i \in v(x, y) \), \( (f + \text{id})|_{\lambda_i} (x + e_i) \neq (f + \text{id})|_{\lambda_i} (x) \), hence there exists \( j \in v(x, y) \), such that \( j \neq i \) and \( \partial_i f_j(x) = f_j(x) + f_j(x + e_i) = 1 \), and \( \mathcal{G}(f)(x) \) has an edge from \( i \) to \( j \). As a consequence, \( \mathcal{G}(f)(x) \) has an infinite path, hence a cycle, and by symmetry, so has \( \mathcal{G}(f)(y) \).

10 Lemma. If \( C \) is a cycle of \( \mathcal{G}(f)(x) \) with vertex set \( I \), then \( C \) is positive (resp. negative) when \( x \) has an even (resp. odd) out-degree in \( \Gamma(f) \), i.e., when \( \sum_{i \in I} (x_i + f_i(x)) = 0 \) (resp. 1). In particular, if \( x \) is a fixed point of \( f \) and \( C \) is any cycle in \( \mathcal{G}(f)(x) \), then \( C \) is positive.

The first assertion follows from the fact that \( C = k_1 \rightarrow \cdots \rightarrow k_p \rightarrow k_1 = k_{p+1} \) is positive if and only if

\[
\sum_{i=1}^{p} (x_{k_i} + f_{k_{i+1}}(x)) = 0 = \sum_{i=1}^{p} x_{k_i} + \sum_{i=1}^{p} f_{k_i}(x) = \sum_{i=1}^{p} (x_{k_i} + f_{k_i}(x)).
\]

11 Theorem (Remy-R.-Thieffry, Comet-Richard). If \( \mathcal{G}(f) \) has no local positive cycle, then \( f \) has a unique attractor.

Assume \( f \) has several attractors. Then by Lemma 3, some projection \( f|_{\kappa} \) has two fixed points. Hence \( f|_{\kappa} + \text{id} \) is not bijective, and by Theorem 9 applied to \( f|_{\kappa} \), there exist two points \( x, y \) such that \( \mathcal{G}(f|_{\kappa})(x) \) and \( \mathcal{G}(f|_{\kappa})(y) \) have cycles. These cycles are also cycles of \( \mathcal{G}(f)(x) \) and \( \mathcal{G}(f)(y) \), and are positive by Lemma 10.

12 Theorem (Remy-R.-Thieffry). If \( \Gamma(f) \) has an attractive cycle, then \( \mathcal{G}(f) \) has a negative cycle.

Attractive cycle \( (x^1, x^2, \ldots, x^p, x^1) \). Sequence \( (k_1, \ldots, k_p) \) defined by \( e^{k_i} = x^i + x^{i+1} \). Subsequence \( (k_\ell, \ldots, k_m) \) maximal without repetition. Then \( k_{m+1} = k_\ell \) and \( \mathcal{G}(f)(x) \) has an edge from \( k_i \) to \( k_{i+1} \) for \( i = \ell, \ldots, m \). Hence \( \mathcal{G}(f) \) has a cycle \( (k_\ell, \ldots, k_m, k_\ell) \). Negative sign because

\[
\sum_{i=\ell}^{m} x^i_{k_i} + f_{k_{i+1}}(x^i) = \sum_{i=\ell}^{m} x^i_{k_i} + x^{i+1}_{k_{i+1}} = x^\ell_{k_\ell} + x^{m+1}_{k_{m+1}} = x^\ell_{k_\ell} + x^m_{k_{m+1}} = 1,
\]

since \( k_{\ell+1}, \ldots, k_m \neq k_\ell \).

13 Lemma. Let \( p \geq 1 \) and \( (x^0, \ldots, x^p) \) be a trajectory of \( \Gamma(f) \). If \( f(x^0) = x^0 + e^i \) and \( f_j(x^p) \neq x^j_p \), then \( \mathcal{G}(f) \) has a path from \( i \) to \( j \) whose sign is positive if and only if \( x^i_i = x^j_j \).
Induction on $p$. If $p = 1$, since $f(x^0) = x^0 + e^i$, $\mathcal{G}(f)$ has an edge from $i$ to $j$, with positive sign if and only if $x^0_i = x^1_j$.

Let $p > 1$ and $q$ be minimal s.t. $q \geq 1$ and

$$f_j(x^q) = f_j(x^{q+1}) = \cdots = f_j(x^p) \neq x^p_j.$$

For some $k$, $x^{q-1} + x^q = e^k$, and $\mathcal{G}(f)(x^{q-1})$ has an edge from $k$ to $j$. Its sign is positive if and only if $x^q_k = f_j(x^{q-1}) = x^p_j$.

If $q = 1$, $k = i$ because $f(x^0) = x^0 + e^i$, and $\mathcal{G}(f)$ has an edge from $i$ to $j$, with positive sign if and only if $x_i^0 = x_j^1$. If $q > 1$, induction with $(x^0, \ldots, x^{q-1})$ gives a path from $i$ to $k$ whose sign is positive if and only if $x_i^0 = x_k^{q-1}$, whence a path from $i$ to $j$ through $k$, whose sign is positive if and only if $x_i^0 = x_j^p$.

14 Theorem (Richard). If $\Gamma(f)$ has a cyclic attractor, then $\mathcal{G}(f)$ has a negative cycle.

Let $\kappa$ be a minimal subcube s.t. $\Gamma(f|_{\kappa})$ has a cyclic attractor; $\kappa$ is an $I$-subcube. Let $\theta$ be a cyclic attractor, $i \in I$, and $\kappa_0, \kappa_1$ be the two subcubes of $\kappa$ defined by $x_i = 0, x_i = 1$. By minimality, $\kappa_1$ has some point $z \in \theta$, and $\Gamma(f|_{\kappa_1})$ has a trajectory from $z$ to some fixed point $x^0$ of $f|_{\kappa_1}$. Since $x^0 \in \theta$, $f(x^0) = x^0 + e^i$. Applying Lemma 13 to a trajectory from $x^0$ to any $y \in \kappa_0$ s.t. $f_i(y) \neq y_i$ gives a path from $i$ to $i$ in $\mathcal{G}(f)$, whose sign is negative because $x_i^0 = 1 \neq 0 = y_i$. To conclude, it remains to note that such a path contains a negative cycle.

15 Theorem (Richard). If $f$ is non-expansive and has no fixed point, then $\mathcal{G}(f)$ has a local negative cycle.

Being non-expansive is a local property: $f$ is non-expansive iff for all $x$, $\mathcal{G}(f)(x)$ has out-degree $\leq 1$. Indeed, if $f$ is non-expansive, $f(x) + f(x + e^j) = 0$ or $e^j$ for some $j$. Conversely, if $\mathcal{G}(f)(x)$ has out-degree $\leq 1$, for any direct path $(x = z_0, \ldots, z_{d(x,y)} = y)$ from $x$ to $y$:

$$d(f(x), f(y)) \leq \sum_{i=1}^{d(x,y)} d(f(z_{i-1}), f(z_i)) \leq \sum_{i=1}^{d(x,y)} 1 = d(x,y).$$

Being non-expansive is a hereditary property: if $f$ is non-expansive, so is any subnetwork. This follows immediately from locality.

Let $\kappa$ be a minimal subcube such that $f|_{\kappa}$ has no fixed point. Then $g = f|_{\kappa}$ is non-expansive, and if $\kappa$ is an $I$-subcube, for each $i \in I$, each of the two subcubes $\kappa_0, \kappa_1$ of $\kappa$ defined by $x_i = 0, x_i = 1$ has a point $x^0 \in \kappa_0, x^1 \in \kappa_1$ which is fixed respectively by $g|_{\kappa_0}, g|_{\kappa_1}$. Thus $x_i^0 \neq x_i^1$ and $g(x^0) + x^0 = e^i = g(x^1) + x^1$. We call a pair $(x^0, x^1)$ with these two properties a mirror pair of $g$.

We now show, by induction on the dimension, that the existence of a mirror pair in a non-expansive network suffices to entail a local negative cycle. In a 1-dimensional network, the result is obvious. Let $f$ be an $n$-dimensional network with a mirror pair. If $d(a,b) < n$ for some mirror pair $(a,b)$, the strict subnetwork $f|_{[a,b]}$ is non-expansive and has the same mirror pair, hence a local negative cycle. So we may assume that any
mirror pair \((a,b)\) is antipodal, i.e. satisfies \(d(a,b) = n\). We may also assume w.l.o.g. that \(f(a) + a = e^i = f(b) + b\), with \(a_0 = 0, b_1 = 1\).

First, we observe that \(f\) has no fixed point. Otherwise, a fixed point \(c\) would satisfy either \(c_1 = 0\) or \(c_1 = 1\). In the first case, \(d(f(c), f(a)) = d(c, a + e^i) = d(c, a) + 1\), and in the second case, \(d(f(c), f(b)) = d(c, b + e^i) = d(c, b) + 1\). Anyway, we get a contradiction.

Let \(a^i = a, a^{i+1} = a + e^i\). Since \(f\) is non-expansive, \(d(f(a^i), f(a^{i+1})) = 1\) and \(f(a^{i+1})\) is not a fixed point, hence \(f(a^{i+2}) = f(a^i) + e^{k_2} = a^2 + e^{k_2}\) for some \(k_2\). This gives rise to sequences

\[
a^1 = a, a^2, a^3, \ldots, a^n \text{ and } k_1 = 1, k_2, k_3, \ldots, k_n
\]

such that \(f(a^i) = a^i + e^{k_i}\) for all \(i\). If for some \(i \neq j, k_i = k_j\), then \((a^i, a^j)\) is a mirror pair such that \(d(a^i, a^j) < n\), contradiction. Thus \(k_1 = 1, k_2, \ldots, k_n\) are all different and \(f(a^n) = b\). Antipodality entails a similar sequence \(a^{n+1} = b, a^{n+2}, \ldots, a^{2n}\) starting from \(b\), with the same sequence \(k_1 = 1, \ldots, k_n\), so that \(f\) has an antipodal attractive cycle

\[
(a^1 = a, a^2, \ldots, a^n, a^{n+1} = b, a^{n+2}, \ldots, a^{2n}, a).
\]

We may assume, up to a permutation of variables, that \(k_i = i\) for all \(i\). Now, for any \(1 \leq i \leq 2n\), \(G(f)(a^i)\) and \(G(f)(a^{i+1})\) have an edge from \(i\) to \(i + 1 \mod n\). Let us prove that, for all \(1 \leq i < p \leq n\), if \(G(f)(a^p)\) has an edge from \(i\) to \(i + 1\), so has \(G(f)(a^{p+1})\). There are two cases.

1. For \(p = i + 1\), we note that

\[
d(a^{i+1}, f(a^i + e^{i+1})) = d(f(a^i), f(a^i + e^{i+1})) \leq d(a^i, a^i + e^{i+1}) = 1.
\]

If \(f_{i+1}(a^i + e^{i+1}) \neq a^i_{i+1}\), then \(f(a^i + e^{i+1}) = a^{i+2} = a^i + e^{i+1}\) and \((a^i + e^{i+1}, a^{i+n \mod 2n})\) is a mirror pair contradicting antipodality. Therefore, \(f_{i+1}(a^i + e^{i+1}) = a^i_{i+1}\) and \(f_{i+1}(a^{i+2}) = a^{i+2}_{i+1} \neq a^i_{i+1}\), so that \(G(f)(a^{i+2})\) has an edge from \(i\) to \(i + 1\).

2. For \(p \geq i + 2\), we have

\[
d(a^{p+1}, f(a^p + e^{i})) = d(f(a^p), f(a^p + e^{i})) \leq d(a^p, a^p + e^{i}) = 1.
\]

Since \(G(f)(a^p)\) has an edge from \(i\) to \(i + 1\) and \(i \neq p, p + 1\), we have \(f_{i+1}(a^p + e^{i}) \neq a^p_{i+1}\), therefore \(f(a^p + e^{i}) = a^{p+1} + e^{i+1}\). On the other hand:

\[
d(f(a^{p+1} + e^{i}), a^{p+1} + e^{i+1}) = d(f(a^{p+1} + e^{i}), f(a^p + e^{i}))
\]

\[
\leq d(a^{p+1} + e^{i}, a^p + e^{i})
\]

\[
= 1.
\]

If \(f_{i+1}(a^{p+1} + e^{i}) = (a^{p+1} + e^{i})_{i+1} = a^{p+1}_{i+1}\), necessarily \(f(a^{p+1} + e^{i}) = a^{p+1}\) and \((a^{p+1} + e^{i}, a^{p+1+n \mod 2n})\) is a mirror pair contradicting antipodality. Therefore \(f_{i+1}(a^{p+1} + e^{i}) \neq a^{p+1}_{i+1} = f_{i+1}(a^p)\) and \(G(f)(a^{p+1})\) has an edge from \(i\) to \(i + 1\).

We conclude that \(G(f)(a^{p+1}) = G(f)(b)\) has a Hamiltonian cycle, which has to be negative because \(f(b) + b = e^i\).

16 Proposition (Richard, R.). For an and-net \(f\), a cycle \(C\) of \(G(f)\) is local if and only if it has no delocalizing triple.
It is sufficient to show that, given an edge \((w, s, v)\) of \(\mathcal{G}(f)\) and \(x \in \mathbb{F}_2^6\), \((w, s, v)\) is an edge of \(\mathcal{G}(f)(x)\) if and only if \(f_v\) has no positive input \(u \neq w\) such that \(x_u = 0\), and no negative input \(u \neq w\) such that \(x_u = 1\).

1. On one hand, if \((w, s, v)\) is an edge of \(\mathcal{G}(f)(x)\), then \(f_v(x) \neq f_v(x + e^w)\), so either 
   \[ f_v(x) = 1 \] or 
   \[ f_v(x + e^w) = 1, \]
   and we deduce that \(x_u = (x + e^w)_u = 1\) for every positive input \(u \neq w\) of \(f_v\), and \(x_u = (x + e^w)_u = 0\) for every negative input \(u \neq w\) of \(f_v\).

2. On the other hand, if \(f_v\) has no positive input \(u \neq w\) such that \(x_u = 0\), and no negative input \(u \neq w\) such that \(x_u = 1\), then \(f_v(x) \neq f_v(x + e^w)\) and we deduce that \((w, s, v)\) is an edge of \(\mathcal{G}(f)(x)\).

17 Theorem (R.). If \(f\) is an and-net and has an antipodal attractive cycle, then \(\mathcal{G}(f)\) has a local negative cycle.

We show that an and-net \(f\) has an antipodal attractive cycle if and only if \(\mathcal{G}(f)\) is a (chordless) Hamiltonian negative cycle. The if part is trivial.

Conversely, if an and-net \(f\) has an antipodal attractive cycle \(\theta\), we may assume that \(\theta\) is \((0, \ldots, e^{1 \cdots n-1}, 0, \ldots, e^{1 \cdots n-1}, 0)\) up to translation and a permutation of coordinates, so that \(\mathcal{G}(f)\) has a negative cycle \(C = (1, 2, \ldots, n, 1)\). If \(C\) has a negative chord \((i, j)\), then \(f_j(x) = 0\) as soon as \(x_i = 1\).

- Now, if \(j \leq i\), then \(e^1_{j \cdots j} = 1\). Since \(e^1_{i \cdots j} = 1\) as well, by the above remark, \(f_j(e^1_{i \cdots j}) = 0\), hence \(i + 1\) and \(j\) are degrees of freedom of \(e^1_{i \cdots j}\). Since \((i, j)\) is a chord, \(j \neq i + 1 \mod n\), and \(e^1_{i \cdots j}\) has at least two degrees of freedom.

- Otherwise \(i \leq j - 1\), so \(e^1_{i \cdots j - 1} = 1\), and \(f_j(e^1_{i \cdots j - 1}) = 0\) by the above remark. Since \(e^1_{i \cdots j - 1} = 0\) too, \(j\) is not a degree of freedom of \(e^1_{i \cdots j - 1}\).

In both cases, we have a contradiction with the hypothesis that \(\theta\) is an attractive cycle, and a similar argument applies for a positive chord at \(e^1_{i \cdots j-1}\).

18 Theorem (Richard, R.).

1. If \(f\) is an and-net and every negative cycle of \(\mathcal{G}(f)\) has an internal delocalizing triple, then \(f\) has \(\geq 1\) fixed point.

2. If every odd cycle of a directed graph \(G\) has an internal killing triple, then \(G\) has \(\geq 1\) kernel.

3. The constructions of \(f^*\) and \(G^+\) relate fixed points to kernels, positive and negative cycles to even and odd cycles, and (good) delocalizing triples to killing triples.


19 Proposition. Let \(f\) be the negative and-net associated to the directed graph \(G\). The fixed points of \(f\) are in bijection with the kernels of \(G^\text{op}\).

Let \(K : \mathbb{F}_2^a \to \mathcal{P}([1, \ldots, n])\) be the bijection defined by mapping any point \(x \in \mathbb{F}_2^a\) to the set \(K(x)\) of \(i\) such that \(x_i = 1\). We prove that \(x\) is a fixed point of \(f\) if and only if \(K(x)\) is a kernel of \(G^\text{op}\).
1. Let $x$ be a fixed point of $f$. If $(j, i)$ is an edge of $G^{op}$ and $i \in K(x)$, then $i$ is a negative input of $j$ and $x_i = 1$, so $x_j = f_j(x) = 0$, hence $j \notin K(x)$. Therefore $K(x)$ is an independent set of vertices in $G^{op}$. Moreover, if $j \notin K(x)$, then $0 = x_j = f_j(x)$ and $f_j$ has at least one negative input $i$ such that $x_i = 1$. Therefore $(j, i)$ is an edge of $G^{op}$ and $i \in K(x)$. We conclude that $K(x)$ is an absorbent set of $G^{op}$, hence a kernel.

2. Assume on the other hand that $K(x)$ is a kernel of $G^{op}$, and let $j$ be any vertex. If $j \in K(x)$, then any vertex $i$ dominated by $j$ in $G^{op}$ is not in $K(x)$. In other words, $f_j$ has no input $i$ in $K(x)$, and we deduce that $f_j(x) = 1 = x_j$. On the other hand, if $j \notin K(x)$, then $j$ dominates some vertex $i \in K(x)$ in $G^{op}$. Hence $f_j$ has an input $i$ such that $x_i = 1$, and we deduce that $f_j(x) = 0 = x_j$. Therefore $x$ is a fixed point of $f$.

**20 Theorem** (Remy-R.-Thieffry, Comet-Richard). If $\gamma$ is a discrete network such that $\mathcal{G}(\gamma)$ has no local positive cycle, then $\gamma$ has $\leq 1$ fixed point.

Assume that $\gamma$ has two fixed points, and let $\kappa$ be a minimal subspace such that $\gamma|_{\kappa}$ has two fixed points. Then $\kappa = [x, y]$ for two fixed points $x, y$ of $\gamma|_{\kappa}$. Let $y'$ be the unique point of $\kappa$ satisfying $|x_i - y'_i| = 1$ for all $i$ such that $x_i \neq y'_i$, and $\beta$ be the increasing bijection $[x, y'] \to \mathbb{F}_2^d$. We show, by induction on the diameter $k$ of $\kappa$, that $\mathcal{G}(\gamma)(x, y')$ has a positive cycle.

If $k = 1$, then $y = y' = x + e^i$ or $x - e^i$ for some $i$, and $\mathcal{G}(\gamma)(x, y')$ has a positive loop on $i$.

If $k > 1$, let $f$ be the map associated to the Boolean network $\beta(\gamma|_{[x, y']})$. Let $i \in I$ and $x'$ be the unique point of $\kappa$ such that $d(x, x') = 1$ and $x_i \neq x'_i$. By minimality, $y$ is the unique fixed point of the subnetwork $\gamma|_{[x', y]}$. If $\gamma|_{[x', y]}$ has an edge leaving $x'$ in the direction $i$, then $\gamma|_{[x', x]}$ has two fixed points, in contradiction with minimality. Therefore, there exists $j \neq i$ such that $f_j(\beta(x) + e^i) = f_j(\beta(x') \neq f_j(\beta(x))$, and $\mathcal{G}(f)(\beta(x))$ has an edge from $i$ to $j$. Since this holds for any $i \in I$, $\mathcal{G}(f)(\beta(x))$ has a cycle, which is positive because $\beta(x)$ is a fixed point of $f$.

**21 Theorem** (Richard). If $\gamma$ is a discrete network such that $\mathcal{G}(\gamma)$ has no local cycle, then $\gamma$ has a unique fixed point.

Assume that $\gamma$ has no fixed point, and let $\kappa = \prod_{i \in I} \{m_i, \ldots, M_i\}$ be a minimal subspace such that $\gamma|_{\kappa}$ has no fixed point. Let $i \in I$. There exist $a, b \in \kappa$ such that $a_i = m_i, b_i = M_i$ and the only edges of $\mu = \gamma|_{\kappa}$ leaving $a$ or $b$ are in direction $i$. Define the discrete network $N_i(\mu)$ to be the same as $\mu$, except for edges in direction $i$: if $d(x, y) = 1$ and $x_i \neq y_i$, $N_i(\mu)$ has an edge from $x$ to $y$ when $\mu$ has not. Then for any $x, y$, the local graphs $\mathcal{G}(N_i(\mu))(x, y)$ and $\mathcal{G}(\mu)(x, y)$ have the same edges (maybe with different signs). Since $a, b$ are two fixed points of $N_i(\mu)$, $\mathcal{G}(N_i(\mu))$ has a local (positive) cycle, and $\mathcal{G}(\mu)$ has a local cycle.
2 Exercises

1. Compute the Jacobian matrix $J(f)(x)$ of
   \[
   f(x) = \begin{pmatrix}
   (x_3 \lor x_4) - x_2 \\
   x_3 \land x_4 \\
   x_4 - x_1 \land x_2 \\
   x_1 x_2 - x_3 
   \end{pmatrix},
   \]
   and the subnetworks $f|_{\kappa_0}, f|_{\kappa_1}$, where $\kappa_0, \kappa_1$ are the two $\{1, 2, 3\}$-subcubes defined by $x_4 = 0, x_4 = 1$. Compute the reduced network $f'$ of $f$ on 4.

2. Show that the following local inverse theorem holds: if $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ and $x \in \mathbb{F}_2^n$ are such that $J(f)(x)$ is invertible, then the restriction of $f$ to the unit ball $B(x, 1)$, defined by $d(x, y) \leq 1$, is injective.

3. Show that arbitrary cyclic attractors are generally not preserved by reduction.

4. Prove the theorems of Robert and Bahi-Michel: if $\mathcal{G}(f)$ has no cycle, then $f$ has a unique fixed point and the iteration of $f$ terminates in $\leq n$ steps, and $\Gamma(f)$ is collapsing.

5. Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a Boolean network. Show that if $x \in \mathbb{F}_2^n$ has odd out-degree in $\Gamma(f)$ and the Jacobian matrix $J(f)(x)$ is invertible, then $\mathcal{G}(f)(x)$ has a negative cycle.

6. Show that if $f + \text{id}$ is bijective, then $\mathcal{G}(f)$ has a local positive loop (cycle of length 1) if and only if it has a local negative loop.

7. Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be defined by
   \[
   f(x) = \begin{pmatrix}
   x_4(x_2 + 1)(x_3 + 1) \\
   x_1(x_3 + 1)(x_4 + 1) \\
   x_2(x_4 + 1)(x_1 + 1) \\
   x_3(x_1 + 1)(x_2 + 1)
   \end{pmatrix}.
   \]
   Show that $f$ is hereditarily ufp, but that $\mathcal{G}(f)$ has a unique local cycle, which is positive. It is therefore not true that if a Boolean network $f$ is hereditarily ufp, then $\mathcal{G}(f)$ has a local positive cycle if and only if it has a local negative cycle.

8. Is the set of Boolean networks which have a unique fixed point closed under translation? Give either a proof or a counterexample.

9. Show that a Boolean network $f$ may be directly terminating but not hereditarily ufp. Show that the fact $f + \text{id}$ is bijective does not suffice to conclude that $\Gamma(f)$ is weakly terminating.

10. Show that a Boolean network may be hereditarily ufp and have a local cycle.
11. A pair \((x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n\) is called a mirror pair of \(f : \mathbb{F}_2^n \to \mathbb{F}_2^n\) when \((f + \text{id})|_{[x, y]}(x) = (f + \text{id})|_{[x, y]}(y)\), i.e., when \(x\) and \(y\) have the same degrees of freedom for the map \(f|_{[x, y]}\). Prove that \(f : \mathbb{F}_2^n \to \mathbb{F}_2^n\) is hereditarily ufp if and only if \(f\) has no mirror pair.

12. A point \(x \in \mathbb{F}_2^n\) is said to be even (resp. odd) when \(\sum_{i=1}^n x_i = 0\) (resp. 1). The sum here is again addition in the field \(\mathbb{F}_2\). A network \(f : \mathbb{F}_2^n \to \mathbb{F}_2^n\) is even (resp. odd) when the image \(\text{Im}(f + \text{id})\) of \(f + \text{id}\) is the set of even (resp. odd) points of \(\mathbb{F}_2^n\). Letting \(\overline{x}\) denote the antipode \(x + e_1 + \cdots + e_n\) of \(x \in \mathbb{F}_2^n\), a network \(f\) is said to be a mirror network when for any \(x \in \mathbb{F}_2^n\), \(f(x) = f(\overline{x})\), i.e. \((x, \overline{x})\) is a mirror pair.

(a) Let \(f : \mathbb{F}_2^n \to \mathbb{F}_2^n\) be a Boolean network. Assume that for any \(i \in \{1, \ldots, n\}\) and \(\{1, \ldots, n\} \setminus \{i\}\)-subcube \(\kappa\), \((f + \text{id})|_{\kappa}\) is bijective, and that \(f + \text{id}\) is not bijective. Let \(F = f + \text{id}\). Show that for any \(a \in \mathbb{F}_2^n\) and \(i \in \{1, \ldots, n\}\), \(F^{-1}(\{a, a + e_i\})\) has cardinality 2.

(b) Prove that, under the same assumptions, \(f\) is an even or odd mirror network.

(c) Show that \(f + \text{id}\) is hereditarily bijective if and only if \(f\) has no even or odd mirror subnetwork.

13. Show that if the graph associated to a square matrix \(M\) with entries in \(\mathbb{F}_2\) has no cycle, then \(M\) is nilpotent. Show that the converse is wrong.

14. An \(n \times n\) matrix \(M = (M_{i,j})_{i,j \in \{1, \ldots, n\}}\) with entries in \(\mathbb{F}_2\) is said to be hereditarily invertible (resp. hereditarily nilpotent) when so are all square submatrices \(M_I = (M_{i,j})_{i,j \in I}\), for \(I \subseteq \{1, \ldots, n\}\). Show that the following are equivalent:

(a) the graph whose adjacency matrix is \(M\) has no cycle;

(b) \(M\) is hereditarily nilpotent;

(c) \(\mathcal{I} + M\) is hereditarily invertible, where \(\mathcal{I}\) denotes the identity matrix.

Prove that if \(f : \mathbb{F}_2^n \to \mathbb{F}_2^n\) is such that \(\mathcal{I}(f)(x)\) is hereditarily invertible for each \(x \in \mathbb{F}_2^n\), then \(f\) is hereditarily bijective.

15. Given a permutation \(\sigma \in \mathcal{S}_{2^n}\), let \(F : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}\) be the map defined on the subcube \(0 \{1, \ldots, n\}\) by:

\[
F(x, 0) = \begin{cases} 
(\sigma(x), 0) & \text{if } \sigma(x) \text{ is even,} \\
(\sigma(x), 1) & \text{otherwise,}
\end{cases}
\]

and by \(F(x, 1) = F(\overline{x}, 0)\). Show that \(f = F + \text{id}\) is an even mirror network, and that any even mirror network can be constructed as above.

16. Let \(f\) be a non-expansive Boolean network. Show that if \(f\) has a cyclic attractor, then \(\mathcal{D}(f)\) has a local negative cycle. (Hint: take a minimal subcube such that the assumption holds.)