Master MPRI — Local cycles and dynamical properties of Boolean networks: Proofs and exercises

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1 Proofs

1 Proposition. *Γ is a bijection between maps from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$ and partial orientations of the hypercube $\mathbb{F}_2^n$.*

Map $f \mapsto$ orientation: edge $x \to x + e_i$ when $f_i(x) \neq x_i$. Orientation $\Gamma \mapsto$ map $f(x) = x + e_I$, where $(x \to x + e_i)_{i \in I}$ = edges leaving x.

2 Remark. *Variant of Leibniz rule: $\partial_i(\varphi\psi) + \partial_i\varphi\partial_i\psi = \varphi\partial_i\psi + \psi\partial_i\varphi$.*

$$\begin{aligned}
\partial_i(\varphi\psi)(x) + \partial_i\varphi(x)\partial_i\psi(x) &= \varphi(x)\psi(x) + \varphi(x+e^i)\psi(x+e^i) \\
&\quad +(\varphi(x) + \varphi(x+e^i))(\psi(x) + \psi(x+e^i)) \\
&= \varphi(x)\psi(x+e^i) + \psi(x)\varphi(x+e^i) \\
\varphi(x)\partial_i\psi(x) + \psi(x)\partial_i\varphi(x) &= \varphi(x)(\psi(x) + \psi(x+e^i)) + \psi(x)(\varphi(x) + \varphi(x+e^i)) \\
&= \varphi(x)\psi(x+e^i) + \psi(x)\varphi(x+e^i).
\end{aligned}$$

3 Lemma. *If f has $\geqslant 2$ attractors, then for some subcube κ, $f\restriction_\kappa$ has $\geqslant 2$ fixed points.*

Let A and B be any two attractors of $\Gamma(f)$ and $(a,b) \in A \times B$ be any pair such that $d(a,b)$ is minimal: then a and b are fixed points of $f\restriction_{[a,b]}$.

4 Proposition (Naldi, Remy, Thieffry, Chaouiya)**.** *Assume $\mathscr{G}(f)$ has no loop on n. Fixed points are preserved by reduction and expansion: x is a fixed point of f' if and only if x' is a fixed point of f. Attractive cycles are preserved by reduction: π maps attractive cycles of f to attractive cycles of f'.*

For any $x \in \mathbb{F}_2^{n-1}$, let $x^\bullet = (x, f_n(x,-)) \in \mathbb{F}_2^n$. Then

$$f(x^\bullet) = (f'(x), f_n(x,-)),$$

because $f_n(x^\bullet) = f_n(x, f_n(x,-)) = f_n(x,-)$. Moreover, $\pi(x^\bullet) = x$, where $\pi : \mathbb{F}_2^n \to \mathbb{F}_2^{n-1}$ is the projection. Now, $f'(x) = x$ if and only if $f_i(x, f_n(x,-)) = f'_i(x) = x_i$ for all $i < n$, if and only if $f_i(x^\bullet) = x_i^\bullet$ for all $i < n$, if and only if $f(x^\bullet) = x^\bullet$, since $f_n(x^\bullet) = f_n(x,-) = x_n^\bullet$.

If $\theta = (\ldots, y, y+e^i, \ldots)$ is an attractive cycle of f, $f(y) = y + e^i$. If $i = n$, $\pi(y) = \pi(y+e^i)$. Otherwise, since θ is attractive, $f_n(y) = y_n$ and $\pi(y)^\bullet = y$. Hence, letting $x = \pi(y)$, we have

$$f'(\pi(y)) = f'(x) = \pi(f(x^\bullet)) = \pi(f(y)) = \pi(y+e^i) = \pi(y) + e^i.$$

Therefore $\pi(\theta)$ is an attractive cycle of $\Gamma(f')$, and for every $x \in \pi(\theta)$, $x^\bullet \in \theta$.

5 Theorem (Remy, Mossé, Chaouiya, Thieffry)**.** *If $\mathscr{G}(f)(x)$ is independent of x and consists in a Hamiltonian positive (resp. negative) cycle, then f has 2 fixed points and no cyclic attractor (resp. f has no fixed point, and a unique attractor which is an attractive cycle).*

If $\mathscr{G}(f)(x) = \mathscr{G}(f)$ is a Hamiltonian positive cycle, we may assume w.l.o.g. that $\mathscr{G}(f) = (1, 2, \ldots, n, 1)$. The sign of the edge from i to $i+1 \mod n$ is $(-1)^{\varepsilon_i}$ with $\varepsilon_i \in \{0,1\}$, so that $f_{k+1}(x) = x_k + \varepsilon_k$, where indices are taken modulo n. Since $\sum_{i=1}^n \varepsilon_i = 0$, the point a defined by

$$a_k = \sum_{i=1}^{k-1} \varepsilon_i \quad \text{for } k = 1, \ldots, n,$$

and its antipode $\overline{a}$ are fixed points of f. Let $K(x)$ be the set of k such that $x_k + x_{k+1} \neq \varepsilon_k$. Then $a + x = e^{K(x)}$ and $d(x,a)$ is the cardinality of K. If x is different from a and $\overline{a}$, $K(x) \neq \varnothing$ and $K(x) \neq \{1, \ldots, n\}$: for any $k \in K$, $f_{k+1}(x) = x_k + \varepsilon_k \neq x_{k+1}$, $\Gamma(f)$ has an edge from x to $x + e^k$, and $K(x+e^k)$ has cardinality smaller than that of $K(x)$. Hence $\Gamma(f)$ has a trajectory from x to a, and in particular, $\Gamma(f)$ has no cyclic attractor.

If $\mathscr{G}(f)(x) = \mathscr{G}(f)$ is a Hamiltonian negative cycle, with the same notations, we have $\sum_{i=1}^n \varepsilon_i = 1$. Thus:

$$1 = \sum_{i=1}^n f_{i+1}(x) + x_i = \sum_{i=1}^n f_i(x) + x_i,$$

so that f has no fixed point. The desired attractive cycle is

$$\theta = (a, a+e^1, a+e^1+e^2, \ldots, \overline{a}, \overline{a}+e^1, \overline{a}+e^1+e^2, \ldots, a).$$

If $x = a + e^I \notin \theta$, then for any $i \in I$ such that $i - 1 \notin I$, where indices are taken modulo n, we have $f_i(x) \neq x_i$ and $\Gamma(f)$ has a trajectory from x to $x + e^i$. Therefore $\Gamma(f)$ has a trajectory from x to some $a + e^1 + \cdots + e^k \in \theta$, with $\{1, \ldots, k\} \subseteq I$.

6 Lemma. *1. An orbit of Boolean networks contains a network with no fixed point if and only if it contains a network with $\geqslant 2$ fixed points.*

2. The set of networks such that each subnetwork $f\restriction_K$ has a unique fixed point is closed under translation.

1. It suffices to observe that if f has no fixed point, $f + \mathrm{id}$ is a non bijective map from a finite set to itself, hence it is not injective: therefore, for some z, there exist distinct points x, y such that $(f+\mathrm{id})(x) = (f+\mathrm{id})(y) = z$, and $f + z$ has two fixed points. On the other hand, if f has two fixed points, $f + \mathrm{id}$ is not bijective, hence not surjective and does not take some value z: then $f + z$ has no fixed point.

2. Let F_n be the set of maps $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that for each subcube κ, $f\restriction_\kappa$ has a unique fixed point. Since the group of translations is generated by translations by basis vectors e_i, it suffices to prove, by induction on n, that for any $f \in F_n$ and $i \in \{1, \dots, n\}$, $f + e_i \in F_n$.

- For $n = 1$, the maps with a unique fixed point are the constant maps, and constant maps are closed under translation.
- If $n > 1$, let $f \in F_n$, $i \in \{1, \dots, n\}$ and $g = f + e_i$. Let κ_0 and κ_1 be the $(n-1)$-dimentional subcubes defined respectively by $x_i = 0$ and $x_i = 1$. By induction hypothesis, $g\restriction_{\kappa_0}$ has a unique fixed point x and $g\restriction_{\kappa_1}$ has a unique fixed point y. On the other hand, since $\mathbb{F}_2^n = \kappa_0 \cup \kappa_1$, f has a unique fixed point, which needs to be either x or y, say it is x. Then $f(y) = y + e_i$ and $g(y) = y$. Moreover, $f(x) = x$, hence $g(x) \neq x$, and we may conclude that y is the unique fixed point of g, hence that $g \in F_n$.

7 Proposition. *f is hereditarily ufp if and only if $f + \mathrm{id}$ is hereditarily bijective.*

If for each subcube κ, $(f + \mathrm{id})\restriction_\kappa$ is bijective, then $(f + \mathrm{id})\restriction_\kappa$ takes exacly once the value 0, and clearly, all the $f\restriction_\kappa$ have a unique fixed point.

On the other hand, in order to prove that for any $n \geqslant 1$ and any $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$, if f is hereditarily ufp, then $f + \mathrm{id}$ is hereditarily bijective, it suffices to prove that for any n and f, if $f\restriction_\kappa$ has a unique fixed point for each subcube κ, then $f + \mathrm{id}$ is bijective: this is because the hypothesis is closed under restriction. Assume this is wrong, so that there exists some $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that $f + \mathrm{id}$ is not bijective while $f\restriction_\kappa$ has a unique fixed point for each subcube κ. Since $f + \mathrm{id}$ is not bijective, the preimage $(f + \mathrm{id})^{-1}(z)$ of some z is not a singleton, hence $(f + z + \mathrm{id})^{-1}(0)$ is not a singleton and $f + z$ does not have a unique fixed point. But by Lemma 6, this contradicts the hypothesis on f, because f and $f + z$ are in the same orbit under translation.

8 Corollary. *If $f + \mathrm{id}$ is hereditarily bijective, then $\Gamma(f)$ is collapsing.*

Under the hypothesis of the corollary, by Proposition 7, for each subcube κ, $f\restriction_\kappa$ has a unique fixed point. Lemma 3 then ensures that this fixed point is the only attractor of $\Gamma(f)$. Let now x be the unique fixed point of f and let Y be the set of points $y \in \mathbb{F}_2^n$ such that $\Gamma(f)$ has no direct trajectory from y to x. Assume for a contradiction that $\Gamma(f)$ is not directly terminating. This implies $Y \neq \varnothing$ and we may choose $y \in Y$ such that $d(x, y)$ is minimal: then x and y are fixed points for $f\restriction_{[x,y]}$, and since $y \neq x$, it follows from Lemma that $(f + \mathrm{id})\restriction_{[x,y]}$ is not bijective.

9 Theorem (R.)**.** *If $f + \mathrm{id}$ is not bijective, then there exist two different points $x, y \in \mathbb{F}_2^n$ such that $\mathscr{G}(f)(x)$ and $\mathscr{G}(f)(y)$ have a cycle.*

To prove this theorem, first observe that as a non bijective map from a finite set to itself, $f + \mathrm{id}$ is not injective: some point $z \in \mathbb{F}_2^n$ has a preimage of cardinality at least 2 under $f + \mathrm{id}$. Consider the partially ordered set E_z of subcubes κ, ordered by inclusion, such that $\pi_\kappa(z)$ has a preimage of cardinality at least 2 under $(f + \mathrm{id})\restriction_\kappa$. By hypothesis, $E_z \neq \varnothing$. Let κ be a minimal subcube of E_z, and let $x, y \in \kappa$ be distinct points mapped by $(f + \mathrm{id})\restriction_\kappa$ to $z\restriction_\kappa$. Since κ is minimal in E_z, x and y are antipodes in κ, *i.e.*, $\kappa = [x, y]$. Recall that $v(x, y)$ denotes the subset $I \subseteq \{1, \dots, n\}$ such that $x + y = e_I$.

- If $v(x,y)$ is a singleton $\{i\}$, then $y = x + e_i$ and $(f + \mathrm{id})_i(x) = (f + \mathrm{id})_i(y)$, as a consequence $\partial_i f_i(x) = f_i(x) + f_i(x + e_i) = \partial_i f_i(x + e_i) = 1$, hence $\mathscr{G}(f)(x)$ and $\mathscr{G}(f)(y)$ have an edge from i to itself.

- If on the other hand $d(x,y) \geqslant 2$, for any $i \in v(x,y)$, let λ_i be the subcube $[x+e_i, y]$, which is smaller than κ: then $(f + \mathrm{id})\restriction_{\lambda_i}(x + e_i) \neq \pi_{\lambda_i}(z)$, since otherwise λ_i would have two different points $x + e_i$ and y mapped by $(f + \mathrm{id})\restriction_{\lambda_i}$ to $\pi_{\lambda_i}(z)$, and λ_i would belong to E_z, contradicting minimality. Therefore, for any $i \in v(x,y)$, $(f + \mathrm{id})\restriction_{\lambda_i}(x + e_i) \neq (f + \mathrm{id})\restriction_{\lambda_i}(x)$, hence there exists $j \in v(x,y)$, such that $j \neq i$ and $\partial_i f_j(x) = f_j(x) + f_j(x + e_i) = 1$, and $\mathscr{G}(f)(x)$ has an edge from i to j. As a consequence, $\mathscr{G}(f)(x)$ has an infinite path, hence a cycle, and by symmetry, so has $\mathscr{G}(f)(y)$.

10 Lemma. *If C is a cycle of $\mathscr{G}(f)(x)$ with vertex set I, then C is positive (resp. negative) when x has an even (resp. odd) out-degree in $\Gamma(f)$, i.e., when $\sum_{i\in I}(x_i+f_i(x)) = 0$ (resp. 1). In particular, if x is a fixed point of f and C is any cycle in $\mathscr{G}(f)(x)$, then C is positive.*

The first assertion follows from the fact that $C = k_1 \to \cdots \to k_p \to k_1 = k_{p+1}$ is positive if and only if

$$\sum_{i=1}^{p}(x_{k_i} + f_{k_{i+1}}(x)) = 0 = \sum_{i=1}^{p} x_{k_i} + \sum_{i=1}^{p} f_{k_i}(x) = \sum_{i=1}^{p}(x_{k_i} + f_{k_i}(x)).$$

11 Theorem (Remy-R.-Thieffry, Comet-Richard)**.** *If $\mathscr{G}(f)$ has no local positive cycle, then f has a unique attractor.*

Assume f has several attractors. Then by Lemma 3, some projection $f\restriction_\kappa$ has two fixed points. Hence $f\restriction_\kappa$ +id is not bijective, and by Theorem 9 applied to $f\restriction_\kappa$, there exist two points x, y such that $\mathscr{G}(f\restriction_\kappa)(x)$ and $\mathscr{G}(f\restriction_\kappa)(y)$ have cycles. These cycles are also cycles of $\mathscr{G}(f)(x)$ and $\mathscr{G}(f)(y)$, and are positive by Lemma 10.

12 Theorem (Remy-R.-Thieffry)**.** *If $\Gamma(f)$ has an attractive cycle, then $\mathscr{G}(f)$ has a negative cycle.*

Attractive cycle $(x^1, x^2, \ldots, x^p, x^1)$. Sequence $(k_1, \ldots, k_p)$ defined by $e^{k_i} = x^i + x^{i+1}$. Subsequence $(k_\ell, \ldots, k_m)$ maximal without repetition. Then $k_{m+1} = k_\ell$ and $\mathscr{G}(f)(x^i)$ has an edge from k_i to k_{i+1} for $i = \ell, \ldots, m$. Hence $\mathscr{G}(f)$ has a cycle $(k_\ell, \ldots, k_m, k_\ell)$. Negative sign because

$$\sum_{i=\ell}^{m} x^i_{k_i} + f_{k_{i+1}}(x^i) = \sum_{i=\ell}^{m} x^i_{k_i} + x^{i+1}_{k_{i+1}} = x^\ell_{k_\ell} + x^{m+1}_{k_{m+1}} = x^\ell_{k_\ell} + x^{m+1}_{k_\ell} = 1,$$

since $k_{\ell+1}, \ldots, k_m \neq k_\ell$.

13 Lemma. *Let $p \geqslant 1$ and $(x^0, \ldots, x^p)$ be a trajectory of $\Gamma(f)$. If $f(x^0) = x^0 + e^i$ and $f_j(x^p) \neq x^p_j$, then $\mathscr{G}(f)$ has a path from i to j whose sign is positive if and only if $x^0_i = x^p_j$.*

Induction on p. If $p = 1$, since $f(x^0) = x^0 + e^i$, $\mathscr{G}(f)$ has an edge from i to j, with positive sign if and only if $x_i^0 = x_j^1$.

Let $p > 1$ and q be minimal s.t. $q \geqslant 1$ and

$$f_j(x^q) = f_j(x^{q+1}) = \cdots = f_j(x^p) \neq x_j^p.$$

For some k, $x^{q-1} + x^q = e^k$, and $\mathscr{G}(f)(x^{q-1})$ has an edge from k to j. Its sign is positive if and only if

$$x_k^{q-1} = f_j(x^{q-1}) = x_j^p.$$

If $q = 1$, $k = i$ because $f(x^0) = x^0 + e^i$, and $\mathscr{G}(f)$ has an edge from i to j, with positive sign if and only if $x_i^0 = x_j^p$. If $q > 1$, induction with $(x^0, \ldots, x^{q-1})$ gives a path from i to k whose sign is positive if and only if $x_i^0 = x_k^{q-1}$, whence a path from i to j through k, whose sign is positive if and only if $x_i^0 = x_j^p$.

14 Theorem (Richard)**.** *If $\Gamma(f)$ has a cyclic attractor, then $\mathscr{G}(f)$ has a negative cycle.*

Let κ be a minimal subcube s.t. $\Gamma(f\restriction_\kappa)$ has a cyclic attractor; κ is an I-subcube. Let θ be a cyclic attractor, $i \in I$, and κ_0, κ_1 be the two subcubes of κ defined by $x_i = 0, x_i = 1$. By minimality, κ_1 has some point $z \in \theta$, and $\Gamma(f\restriction_{\kappa_1})$ has a trajectory from z to some fixed point x^0 of $f\restriction_{\kappa_1}$. Since $x^0 \in \theta$, $f(x^0) = x^0 + e^i$. Applying Lemma 13 to a trajectory from x^0 to any $y \in \kappa_0$ s.t. $f_i(y) \neq y_i$ gives a path from i to i in $\mathscr{G}(f)$, whose sign is negative because $x_i^0 = 1 \neq 0 = y_i$. To conclude, it remains to note that such a path contains a negative cycle.

15 Theorem (Richard)**.** *If f is non-expansive and has no fixed point, then $\mathscr{G}(f)$ has a local negative cycle.*

Being non-expansive is a local property: f is non-expansive iff for all x, $\mathscr{G}(f)(x)$ has out-degree $\leqslant 1$. Indeed, if f is non-expansive, $f(x) + f(x + e^i) = 0$ or e^j for some j. Conversely, if $\mathscr{G}(f)(x)$ has out-degree $\leqslant 1$, for any direct path $(x = z_0, \ldots, z_{d(x,y)} = y)$ from x to y:

$$d(f(x), f(y)) \leqslant \sum_{i=1}^{d(x,y)} d(f(z_{i-1}), f(z_i)) \leqslant \sum_{i=1}^{d(x,y)} 1 = d(x, y).$$

Being non-expansive is a hereditary property: if f is non-expansive, so is any subnetwork. This follows immediately from locality.

Let κ be a minimal subcube such that $f\restriction_\kappa$ has no fixed point. Then $g = f\restriction_\kappa$ is non-expansive, and if κ is an I-subcube, for each $i \in I$, each of the two subcubes κ_0, κ_1 of κ defined by $x_i = 0, x_i = 1$ has a point $x^0 \in \kappa_0, x^1 \in \kappa_1$ which is fixed respectively by $g\restriction_{\kappa_0}, g\restriction_{\kappa_1}$. Thus $x_i^0 \neq x_i^1$ and $g(x^0) + x^0 = e^i = g(x^1) + x^1$. We call a pair (x^0, x^1) with these two properties a *mirror pair* of g.

We now show, by induction on the dimension, that the existence of a mirror pair in a non-expansive network suffices to entail a local negative cycle. In a 1-dimensional network, the result is obvious. Let f be an n-dimensional network with a mirror pair. If $d(a, b) < n$ for some mirror pair (a, b), the strict subnetwork $f\restriction_{[a,b]}$ is non-expansive and has the same mirror pair, hence a local negative cycle. So we may assume that any mirror pair (a, b) is *antipodal*, *i.e.* satisfies $d(a, b) = n$. We may also assume w.l.o.g. that $f(a) + a = e^1 = f(b) + b$, with $a_1 = 0, b_1 = 1$.

First, we observe that f has no fixed point. Otherwise, a fixed point c would satisfy either $c_1 = 0$ or $c_1 = 1$. In the first case, $d(f(c), f(a)) = d(c, a + e^1) = d(c, a) + 1$, and in the second case, $d(f(c), f(b)) = d(c, b+e^1) = d(c, b)+1$. Anyway, we get a contradiction.

Let $a^1 = a, a^2 = a + e^1$. Since f is non-expansive, $d(f(a^1), f(a^2)) \leqslant 1$ and $f(a^2)$ is not a fixed point, hence $f(a^2) = f(a^1) + e^{k_2} = a^2 + e^{k_2}$ for some k_2. This gives rise to sequences

$$a^1 = a, a^2, a^3, \dots, a^n \text{ and } k_1 = 1, k_2, k_3, \dots, k_n$$

such that $f(a^i) = a^i + e^{k_i}$ for all i. If for some $i \neq j$, $k_i = k_j$, then (a^i, a^j) is a mirror pair such that $d(a^i, a^j) < n$, contradiction. Thus $k_1 = 1, k_2, \dots, k_n$ are all different and $f(a^n) = b$. Antipodality entails a similar sequence $a^{n+1} = b, a^{n+2}, \dots, a^{2n}$ starting from b, with the same sequence $k_1 = 1, \dots, k_n$, so that f has an antipodal attractive cycle

$$(a^1 = a, a^2, \dots, a^n, a^{n+1} = b, a^{n+2}, \dots, a^{2n}, a).$$

We may assume, up to a permutation of variables, that $k_i = i$ for all i. Now, for any $1 \leqslant i \leqslant 2n$, $\mathscr{G}(f)(a^i)$ and $\mathscr{G}(f)(a^{i+1})$ have an edge from i to $i + 1 \mod n$. Let us prove that, for all $1 \leqslant i < p \leqslant n$, if $\mathscr{G}(f)(a^p)$ has an edge from i to $i + 1$, so has $\mathscr{G}(f)(a^{p+1})$. There are two cases.

1. For $p = i + 1$, we note that

$$d(a^{i+1}, f(a^i + e^{i+1})) = d(f(a^i), f(a^i + e^{i+1})) \leqslant d(a^i, a^i + e^{i+1}) = 1.$$

   If $f_{i+1}(a^i+e^{i+1}) \neq a^i_{i+1}$, then $f(a^i+e^{i+1}) = a^{i+2} = a^i+e^{i,i+1}$ and $(a^i+e^{i+1}, a^{i+n \mod 2n})$ is a mirror pair contradicting antipodality. Therefore, $f_{i+1}(a^i + e^{i+1}) = a^i_{i+1}$ and $f_{i+1}(a^{i+2}) = a^{i+2}_{i+1} \neq a^i_{i+1}$, so that $\mathscr{G}(f)(a^{i+2})$ has an edge from i to $i + 1$.

2. For $p \geqslant i + 2$, we have

$$d(a^{p+1}, f(a^p + e^i)) = d(f(a^p), f(a^p + e^i)) \leqslant d(a^p, a^p + e^i) = 1.$$

   Since $\mathscr{G}(f)(a^p)$ has an edge from i to $i+1$ and $i \neq p, p+1$, wa have $f_{i+1}(a^p + e^i) \neq a^p_{i+1} = a^{p+1}_{i+1}$, therefore $f(a^p + e^i) = a^{p+1} + e^{i+1}$. On the other hand:

$$\begin{aligned} d(f(a^{p+1} + e^i), a^{p+1} + e^{i+1}) &= d(f(a^{p+1} + e^i), f(a^p + e^i)) \\ &\leqslant d(a^{p+1} + e^i, a^p + e^i) \\ &= 1. \end{aligned}$$

   If $f_{i+1}(a^{p+1}+e^i) = (a^{p+1}+e^i)_{i+1} = a^{p+1}_{i+1}$, necessarily $f(a^{p+1}+e^i) = a^{p+1}$ and $(a^{p+1} + e^i, a^{i+n \mod 2n})$ is a mirror pair contradicting antipodality. Therefore $f_{i+1}(a^{p+1} + e^i) \neq a^{p+1}_{i+1} = f_{i+1}(a^{p+1})$ and $\mathscr{G}(f)(a^{p+1})$ has an edge from i to $i + 1$.

We conclude that $\mathscr{G}(f)(a^{n+1}) = \mathscr{G}(f)(b)$ has a Hamiltonian cycle, which has to be negative because $f(b) + b = e^1$.

16 Proposition (Richard, R.). *For an and-net f, a cycle C of $\mathscr{G}(f)$ is local if and only if it has no delocalizing triple.*

It is sufficient to show that, given an edge (w, s, v) of $\mathscr{G}(f)$ and $x \in \mathbb{F}_2^n$, (w, s, v) is an edge of $\mathscr{G}(f)(x)$ if and only if f_v has no positive input $u \neq w$ such that $x_u = 0$, and no negative input $u \neq w$ such that $x_u = 1$.

1. On one hand, if (w, s, v) is an edge of $\mathscr{G}(f)(x)$, then $f_v(x) \neq f_v(x + e^w)$, so either $f_v(x) = 1$ or $f_v(x + e^w) = 1$, and we deduce that $x_u = (x + e^w)_u = 1$ for every positive input $u \neq w$ of f_v, and $x_u = (x + e^w)_u = 0$ for every negative input $u \neq w$ of f_v.

2. On the other hand, if f_v has no positive input $u \neq w$ such that $x_u = 0$, and no negative input $u \neq w$ such that $x_u = 1$, then $f_v(x) \neq f_v(x + e^w)$ and we deduce that (w, s, v) is an edge of $\mathscr{G}(f)(x)$.

17 Theorem (R.). *If f is an and-net and has an antipodal attractive cycle, then $\mathscr{G}(f)$ has a local negative cycle.*

We show that an and-net f has an antipodal attractive cycle if and only if $\mathscr{G}(f)$ is a (chordless) Hamiltonian negative cycle. The if part is trivial.

Conversely, if an and-net f has an antipodal attractive cycle θ, we may assume that θ is $(0, \ldots, e^{1,\ldots,n-1}, \overline{0}, \ldots, \overline{e^{1,\ldots,n-1}}, 0)$ up to translation and a permutation of coordinates, so that $\mathscr{G}(f)$ has a negative cycle $C = (1, 2, \ldots, n, 1)$. If C has a negative chord (i, j), then $f_j(x) = 0$ as soon as $x_i = 1$.

- Now, if $j \leqslant i$, then $e_j^{1,\ldots,i} = 1$. Since $e_i^{1,\ldots,i} = 1$ as well, by the above remark, $f_j(e^{1,\ldots,i}) = 0$, hence $i + 1$ and j are degrees of freedom of $e^{1,\ldots,i}$. Since (i, j) is a chord, $j \neq i + 1 \bmod n$, and $e^{1,\ldots,i}$ has at least two degrees of freedom.

- Otherwise $i \leqslant j - 1$, so $e_i^{1,\ldots,j-1} = 1$, and $f_j(e^{1,\ldots,j-1}) = 0$ by the above remark. Since $e_j^{1,\ldots,j-1} = 0$ too, j is not a degree of freedom of $e^{1,\ldots,j-1}$.

In both cases, we have a contradiction with the hypothesis that θ is an atractive cycle, and a similar argument applies for a positive chord at $\overline{e^{1,\ldots,i}}$.

18 Theorem (Richard, R.). *1. If f is an and-net and every negative cycle of $\mathscr{G}(f)$ has an internal delocalizing triple, then f has $\geqslant 1$ fixed point.*

2. If every odd cycle of a durected graph G has an internal killing triple, then G has $\geqslant 1$ kernel.

3. The constructions of f^* and $G^\pm$ relate fixed points to kernels, positive and negative cycles to even and odd cycles, and (good) delocalizing triples to killing triples.

See *From kernels in directed graphs to fixed points and negative cycles in Boolean networks*, A. Richard, P. Ruet (Discrete Applied Mathematics 161: 1106-1117, 2013).

19 Proposition. *Let f be the negative and-net associated to the directed graph G. The fixed points of f are in bijection with the kernels of G^{op}.*

Let $K : \mathbb{F}_2^n \to \mathscr{P}(\{1, \ldots, n\})$ be the bijection defined by mapping any point $x \in \mathbb{F}_2^n$ to the set $K(x)$ of i such that $x_i = 1$. We prove that x is a fixed point of f if and only if $K(x)$ is a kernel of G^{op}.

1. Let x be a fixed point of f. If (j,i) is an edge of $g^{\rm op}$) and $i \in K(x)$, then i is a negative input of j and $x_i = 1$, so $x_j = f_j(x) = 0$, hence $j \notin K(x)$. Therefore $K(x)$ is an independent set of vertices in $G^{\rm op}$. Moreover, if $j \notin K(x)$, then $0 = x_j = f_j(x)$ and f_j has at least one negative input i such that $x_i = 1$. Therefore (j,i) is an edge of $G^{\rm op}$ and $i \in K(x)$. We conclude that $K(x)$ is an absorbent set of $G^{\rm op}$, hence a kernel.

2. Assume on the other hand that $K(x)$ is a kernel of $G^{\rm op}$, and let j be any vertex. If $j \in K(x)$, then any vertex i dominated by j in $G^{\rm op}$ is not in $K(x)$. In other words, f_j has no input i in $K(x)$, and we deduce that $f_j(x) = 1 = x_j$. On the other hand, if $j \notin K(x)$, then j dominates some vertex $i \in K(x)$ in $G^{\rm op}$. Hence f_j has an input i such that $x_i = 1$, and we deduce that $f_j(x) = 0 = x_j$. Therefore x is a fixed point of f.

20 Theorem (Remy-R.-Thieffry, Comet-Richard)**.** *If γ is a discrete network such that $\mathscr{G}(\gamma)$ has no local positive cycle, then γ has $\leqslant 1$ fixed point.*

Assume that γ has two fixed points, and let κ be a minimal subspace such that $\gamma\restriction_\kappa$ has two fixed points. Then $\kappa = [x,y]$ for two fixed points x, y of $\gamma\restriction_\kappa$. Let y' be the unique point of κ satisfying $|x_i - y'_i| = 1$ for all i such that $x_i \neq y_i$, and β be the increasing bijection $[x,y'] \to \mathbb{F}_2^I$. We show, by induction on the diameter k of κ, that $\mathscr{G}(\gamma)(x,y')$ has a positive cycle.

If $k = 1$, then $y = y' = x + e^i$ or $x - e^i$ for some i, and $\mathscr{G}(\gamma)(x,y')$ has a positive loop on i.

If $k > 1$, let f be the map associated to the Boolean network $\beta(\gamma\restriction_{[x,y']})$. Let $i \in I$ and x' be the unique point of κ such that $d(x,x') = 1$ and $x_i \neq x'_i$. By minimality, y is the unique fixed point of the subnetwork $\gamma\restriction_{[x',y]}$. If $\gamma\restriction_{[x',y]}$ has an edge leaving x' in the direction i, then $\gamma\restriction_{[x,x']}$ has two fixed points, in contradiction with minimality. Therefore, there exists $j \neq i$ such that $f_j(\beta(x) + e^i) = f_j(\beta(x')) \neq f_j(\beta(x))$, and $\mathscr{G}(f)(\beta(x))$ has an edge from i to j. Since this holds for any $i \in I$, $\mathscr{G}(f)(\beta(x))$ has a cycle, which is positive because $\beta(x)$ is a fixed point of f.

21 Theorem (Richard)**.** *If γ is a discrete network such that $\mathscr{G}(\gamma)$ has no local cycle, then γ has a unique fixed point.*

Assume that γ has no fixed point, and let $\kappa = \prod_{i\in I}\{m_i, \ldots, M_i\}$ be a minimal subspace such that $\gamma\restriction_\kappa$ has no fixed point. Let $i \in I$. There exist $a, b \in \kappa$ such that $a_i = m_i, b_i = M_i$ and the only edges of $\mu = \gamma\restriction_\kappa$ leaving a or b are in direction i. Define the discrete network $N_i(\mu)$ to be the same as μ, except for edges in direction i: if $d(x,y) = 1$ and $x_i \neq y_i$, $N_i(\mu)$ has an edge from x to y when μ has not. Then for any x, y, the local graphs $\mathscr{G}(N_i(\mu))(x,y)$ and $\mathscr{G}(\mu)(x,y)$ have the same edges (maybe with different signs). Since a, b are two fixed points of $N_i(\mu)$, $\mathscr{G}(N_i(\mu))$ has a local (positive) cycle, and $\mathscr{G}(\mu)$ has a local cycle.

22 Lemma (R.)**.** *Let f be a negative and-net and S be a set of cycles of $\mathscr{G}(f)$. If f has an S-quasi-delocalizing function, then f can be expanded to an and-net g such that every cycle of $\mathscr{G}(g)$ above a cycle of S is delocalized.*

Let χ be an S-quasi-delocalizing function of f. We proceed in two steps. We first define an and-net f' by replacing in $\mathscr{G}(f)$ each edge $(i, j) \in \mathrm{Im}(\chi_2)$ by two edges $(i, i'), (i', j)$, where i' is a new vertex, (i, i') is positive and (i', j) is negative. Since $\mathrm{Im}(\chi_1) \cap \mathrm{Im}(\chi_2) = \varnothing$, $\mathrm{Im}(\chi_1)$ is a set of negative egdes of f'.

We then define g by adding to f', for each $(i, k) \in \mathrm{Im}(\chi_1)$, three edges $(i, i''), (i'', i')$ and (i'', k), where i'' is a new vertex, $(i, i''), (i'', i')$ are positive and (i'', k) is negative.

Now, f' reduces to f and g reduces to f', so these two steps are expansions, as required. Finally, a cycle of $\mathscr{G}(g)$ which is above

$$C = (i, j, \ldots, k, \ldots, i) \in S,$$

where $\chi(C) = ((i, k), (i, j))$, is

$$\text{either } (i, i', j, \ldots, k, \ldots, i) \text{ or } (i, i'', i', j, \ldots, k, \ldots, i).$$

The first one is delocalized by the triple (i'', i', k), the second one by (i, i', k).

23 Corollary (R.)**.** *Let f be a negative and-net and S be the set of negative cycles of $\mathscr{G}(f)$. If f has an S-quasi-delocalizing function, then f can be expanded to an and-net without local negative cycle.*

Let g be the and-net given by expansion of f in Lemma 22. Each elementary negative cycle of $\mathscr{G}(g)$ is above some negative cycle of $\mathscr{G}(f)$, thus above a cycle of S: it is therefore delocalized.

2 Exercises

1. Compute the Jacobian matrix $\mathscr{J}(f)(x)$ of

$$f(x) = \begin{pmatrix} (x_3 \vee x_4)\neg x_2 \\ x_3 \neg x_4 \\ x_4 \neg x_1 \neg x_2 \\ x_1 x_2 \neg x_3 \end{pmatrix},$$

   and the subnetworks $f\restriction_{\kappa_0}, f\restriction_{\kappa_1}$, where κ_0, κ_1 are the two $\{1, 2, 3\}$-subcubes defined by $x_4 = 0, x_4 = 1$. Compute the reduced network f' of f on 4.

2. Show that the following local inverse theorem holds: if $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ and $x \in \mathbb{F}_2^n$ are such that $\mathscr{J}(f)(x)$ is invertible, then the restriction of f to the unit ball $B(x, 1)$, defined by $d(x, y) \leqslant 1$, is injective.

3. Show that arbitrary cyclic attractors are generally not preserved by reduction.

4. Prove the theorems of Robert and Bahi-Michel: if $\mathscr{G}(f)$ has no cycle, then f has a unique fixed point and the iteration of f terminates in $\leqslant n$ steps, and $\Gamma(f)$ is collapsing.

5. Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a Boolean network. Show that if $x \in \mathbb{F}_2^n$ has odd out-degree in $\Gamma(f)$ and the Jacobian matrix $\mathscr{J}(f)(x)$ is invertible, then $\mathscr{G}(f)(x)$ has a negative cycle.

6. Show that if If $f+\mathrm{id}$ is bijective, then $\mathscr{G}(f)$ has a local positive loop (cycle of length 1) if and only if it has a local negative loop.

7. Let $f : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ be defined by

$$f(x) = \begin{pmatrix} x_4(x_2+1)(x_3+1) \\ x_1(x_3+1)(x_4+1) \\ x_2(x_4+1)(x_1+1) \\ x_3(x_1+1)(x_2+1). \end{pmatrix}.$$

Show that f is hereditarily ufp, but that $\mathscr{G}(f)$ has a unique local cycle, which is positive. It is therefore not true that if a Boolean network f is hereditarily ufp, then $\mathscr{G}(f)$ has a local positive cycle if and only if it has a local negative cycle.

8. Is the set of Boolean networks which have a unique fixed point closed under translation? Give either a proof or a counterexample.

9. Show that a Boolean network f may be directly terminating but not hereditarily ufp. Show that the fact $f+\mathrm{id}$ is bijective does not suffice to conclude that $\Gamma(f)$ is weakly terminating.

10. Show that a Boolean network may be hereditarily upf and have a local cycle.

11. A pair $(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$ is called a *mirror pair* of $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ when $(f+\mathrm{id})\restriction_{[x,y]}(x) = (f+\mathrm{id})\restriction_{[x,y]}(y)$, *i.e.*, when x and y have the same degrees of freedom for the map $f\restriction_{[x,y]}$. Prove that $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is hereditarily ufp if and only if f has no mirror pair.

12. A point $x \in \mathbb{F}_2^n$ is said to be *even* (resp. *odd*) when $\sum_{i=1}^n x_i = 0$ (resp. 1). The sum here is again addition in the field $\mathbb{F}_2$. A network $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is *even* (resp. *odd*) when the image $\mathrm{Im}(f+\mathrm{id})$ of $f+\mathrm{id}$ is the set of even (resp. odd) points of $\mathbb{F}_2^n$. Letting $\overline{x}$ denote the *antipode* $x+e_1+\cdots+e_n$ of $x \in \mathbb{F}_2^n$, a network f is said to be a *mirror network* when for any $x \in \mathbb{F}_2^n$, $f(\overline{x}) = \overline{f(x)}$, *i.e.* $(x, \overline{x})$ is a mirror pair.

   (a) Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a Boolean network. Assume that for any $i \in \{1, \ldots, n\}$ and $\{1, \ldots, n\} \setminus \{i\}$-subcube κ, $(f+\mathrm{id})\restriction_\kappa$ is bijective, and that $f+\mathrm{id}$ is not bijective. Let $F = f+\mathrm{id}$. Show that for any $a \in \mathbb{F}_2^n$ and $i \in \{1, \ldots, n\}$, $F^{-1}(\{a, a+e_i\})$ has cardinality 2.

   (b) Prove that, under the same assumptions, f is an even or odd mirror network.

   (c) Show that $f+\mathrm{id}$ is hereditarily bijective if and only if f has no even or odd mirror subnetwork.

13. Show that if the graph associated to a square matrix M with entries in $\mathbb{F}_2$ has no cycle, then M is nilpotent. Show that the converse is wrong.

14. An $n \times n$ matrix $M = (M_{i,j})_{i,j\in\{1,\ldots,n\}}$ with entries in $\mathbb{F}_2$ is said to be *hereditarily invertible* (resp. *hereditarily nilpotent*) when so are all square submatrices $M_I = (M_{i,j})_{i,j\in I}$, for $I \subseteq \{1, \ldots, n\}$. Show that the following are equivalent:

(a) the graph whose adjacency matrix is M has no cycle;

(b) M is hereditarily nilpotent;

(c) $\mathscr{I} + M$ is hereditarily invertible, where $\mathscr{I}$ denotes the identity matrix.

Prove that if $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is such that $\mathscr{J}(f)(x)$ is hereditarily invertible for each $x \in \mathbb{F}_2^n$, then f is hereditarily bijective.

15. Given a permutation $\sigma \in \mathfrak{S}_{2^n}$, let $F : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}$ be the map defined on the subcube $0[\{1, \dots, n\}]$ by:

$$F(x, 0) = \begin{cases} (\sigma(x), 0) & \text{if } \sigma(x) \text{ is even,} \\ (\sigma(x), 1) & \text{otherwise,} \end{cases}$$

and by $F(x, 1) = F(\overline{x}, 0)$. Show that $f = F + \mathrm{id}$ is an even mirror network, and that any even mirror network can be constructed as above.

16. Let f be a non-expansive Boolean network. Show that if f has a cyclic attractor, then $\mathscr{G}(f)$ has a local negative cycle. (Hint: take a minimal subcube such that the assumption holds.)