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Research Report LSV-08-33

December 2008

Laboratoire Spécification et Vérification

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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Reversal-bounded counter machines revisited*

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Abstract. We extend the class of reversal-bounded counter machines by authorizing a finite number of alternations between increasing and decreasing mode over a given bound. We prove that extended reversal-bounded counter machines also have effective semi-linear reachability sets. We also prove that the property of being reversal-bounded is undecidable in general even when we fix the bound, whereas this problem becomes decidable when considering Vector Addition System with States.

1 Introduction

The *verification of infinite state systems* has shown in the last years to be an efficient technique to model and verify computer systems. Various models of infinite-state systems have also been proposed as for instance counter systems, lossy channel systems, pushdown automata, timed automata, etc, in order to obtain an automatic verification procedure. Among them, counter systems which consist in finite automata extended with operations on integer variables enjoy a central position for both theoretical results and maturity of tools like FAST [4], LASH [1] and TREX [2].

Reachability problem for counter systems. It has been proved in [18] that Minsky machines, which correspond to counter systems where each counter can be incremented, decremented or tested to zero, have an undecidable reachability problem, even when they manipulate only two counter variables. Because of that, different restrictions over counter systems have been proposed in order to obtain the decidability. For instance, Vector Addition Systems with States (or Petri nets) are a special class of counter systems, in which it is not possible to perform equality tests (equivalent to zero-tests), and for which the reachability problem is decidable [13,17].

Counter systems with semi-linear reachability sets. In many verification problems, it is convenient not only to have an algorithm for the reachability problem, but also to be able to compute effectively the reachability set. In the past, many classes of counter systems with a semi-linear reachability set have been found. Among the VASS (or Petri nets), we distinguish the BPP-nets [6], the cyclic Petri nets [3], the persistent Petri nets [14,16], the regular Petri nets [19], the 2-dimensional VASS [8]. In [9], the class of reversal-bounded counter machines is introduced as follows : each counter can only perform a bounded number of alternations between increasing and decreasing mode. The author shows that reversal-bounded counter machines have a semi-linear reachability set and these results have been extended in [10] authorizing more complex guards and

* Partly supported by project AVERISS (ANR-06-SETIN-001)

restricting the way the alternations are counted. In [15], it has been shown that most of the counter systems with a semi-linear reachability set are in fact flattable, which means that their control graph can be replaced equivalently w.r.t. reachability, by another one with no nested loops. In fact, it has been proved in [7], that counter machines with no nested loops in their control structure have a semi-linear reachability set.

Our contribution. In this paper, we first propose an extension of the definition of reversal-bounded machines saying that a counter machine is k-reversal-b-bounded if each counter does at most k alternations between increasing and decreasing mode above a given bound b. We show that these new reversal-bounded counter machines do also have a semilinear reachability set, which can be effectively computed. We study the decidability of the reversal-boundedness of a given counter machine, proving that the only case, which is decidable, is the one when the two parameters b and k are provided. Finally, we study reversal-bounded VASS, showing that one can decide using the coverability graph whether a VASS is reversal-bounded or not. Doing so, we propose a new recursive class of VASS with semi-linear reachability sets which contains all the bounded VASS. Furthermore, to the best of our knowledge, it is not known whether one can or cannot decide if a VASS has a semi-linear reachability set or if it is flattable.

Due to lack of space, some details can be found in the technical appendix.

2 Preliminaries

2.1 Useful notions

Let $\mathbb{N}$ (resp. $\mathbb{Z}$) denotes the set of nonnegative integers (resp. integers). The usual total order over $\mathbb{Z}$ is written $\leq$. By $\mathbb{N}_\omega$, we denote the set $\mathbb{N} \cup \{\omega\}$ where ω is a new symbol such that $\omega \notin \mathbb{N}$ and for all $k \in \mathbb{N}_\omega$, $k \leq \omega$. We extend the binary operation $+$ and $-$ to $\mathbb{N}_\omega$ as follows : for all $k \in \mathbb{N}$, $k + \omega = \omega$ and $\omega - k = \omega$. For $k, l \in \mathbb{N}_\omega$ with $k \leq l$, we write $[k..l]$ for the interval of integers $\{i \in \mathbb{N} \mid k \leq i \leq l\}$.

Given a set X and $n \in \mathbb{N}$, X^n is the set of n-dim vectors with values in X. For any index $i \in [1..n]$, we denote by $\mathbf{v}(i)$ the i^{th} component of a n-dim vector $\mathbf{v}$. We write $\mathbf{0}$ the vector such that $\mathbf{0}(i) = 0$ for all $i \in [1..n]$. The classical order on $\mathbb{Z}^n$ is also denoted $\leq$ and is defined by $\mathbf{v} \leq \mathbf{w}$ if and only if for all $i \in [1..n]$, we have $\mathbf{v}(i) \leq \mathbf{w}(i)$. We also define the operation $+$ over n-dim vectors of integers in the classical way (ie for $\mathbf{v}$, $\mathbf{v}' \in \mathbb{Z}^n$, $\mathbf{v} + \mathbf{v}'$ is defined by $(\mathbf{v} + \mathbf{v}')(i) = \mathbf{v}(i) + \mathbf{v}'(i)$ for all $i \in [1..n]$).

Let $n \in \mathbb{N}$. A subset $S \subseteq \mathbb{N}^n$ is *linear* if there exist $k+1$ vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{N}^n$ such that $S = \{\mathbf{v} \mid \mathbf{v} = \mathbf{v}_0 + \lambda_1.\mathbf{v}_1 + \ldots + \lambda_k.\mathbf{v}_k$ with $\lambda_i \in \mathbb{N}$ for all $i \in [1..k]\}$. A *semi-linear set* is any finite union of linear sets. We extend the notion of semi-linearity to subsets of $Q \times \mathbb{Z}^n$ where Q is a finite (non-empty) set.

For an alphabet Σ, we denote by Σ^* the set of finite words over Σ and ϵ represents the empty word.

2.2 Counter machines

A *Minsky machine* is a finite control state automaton which manipulates integer variables, called counters. From each control state, the machine can do the following operations : 1) Increment a counter and go to another control state, 2) Test the value of a counter, if it is 0, it passes to a control state, and if not, it decrements the counter and goes to another control state. There is also a control state called the final state (or halting state) from which the machine cannot do anything. The Minsky machine is said to halt when it reaches this control state. We define here a slight extension of Minsky machines.

We call a *n-dim guarded translation* (shortly a translation) any function $t : \mathbb{N}^n \rightarrow \mathbb{N}^n$ such that there exist $\# \in \{=, \leq\}^n$, $\mu \in \mathbb{N}^n$ and $\delta \in \mathbb{Z}^n$ with $0 \leq \mu + \delta$ and $dom(t) = \{\mathbf{v} \in \mathbb{N}^n \mid \mu \# \mathbf{v}\}$ and for all $\mathbf{v} \in dom(t)$, $t(\mathbf{v}) = \mathbf{v} + \delta$. We will sometimes use the encoding $(\#, \mu, \delta)$ to represent a translation. In the following, T_n will denote the set of the n-dim guarded translations. Let $t = (\#, \mu, \delta)$ be a guarded translation in T_n. We define the vector $D_t \in \mathbb{Z}^n$ as follows, $\forall i \in [1..n]$, $D_t(i) = \delta(i)$. We extend this definition to words of guarded translations, recursively as follows, if $\sigma \in T_n^*$ and $t \in T_n$, we have $D_{t\sigma} = D_t + D_\sigma$ and by convention, $D_\epsilon = \mathbf{0}$.

Definition 1. *A n-dim* counter machine *(shortly* counter machine*) is a finite valuated graph $S = \langle Q, E \rangle$ where Q is a finite set of control states and E is a finite relation $E \subseteq Q \times T_n \times Q$.*

The semantics of a counter machine $S = \langle Q, E \rangle$ is given by its associated transition system $TS(S) = \langle Q \times \mathbb{N}^n, \rightarrow \rangle$ where $\rightarrow \subseteq Q \times \mathbb{N}^n \times T_n \times Q \times \mathbb{N}^n$ is a relation defined as follows :

$$(q, \mathbf{v}) \xrightarrow{t} (q', \mathbf{v}') \text{ iff } \exists\ (q, t, q') \in E \text{ such that } \mathbf{v} \in dom(t) \text{ and } \mathbf{v}' = t(\mathbf{v})$$

We write $(q, \mathbf{v}) \rightarrow (q', \mathbf{v}')$ if there exists $t \in T_n$ such that $(q, \mathbf{v}) \xrightarrow{t} (q', \mathbf{v}')$. The relation $\rightarrow^*$ represents the reflexive and transitive closure of $\rightarrow$. Given a configuration $(q, \mathbf{v})$ of $TS(S)$, $\mathtt{Reach}(S, (q, \mathbf{v})) = \{(q', \mathbf{v}') \mid (q, \mathbf{v}) \rightarrow^* (q', \mathbf{v}')\}$. Furthermore, we extend the relation $\rightarrow$ to words in T_n^*. We have then $(q, \mathbf{v}) \xrightarrow{\epsilon} (q, \mathbf{v})$ and if $t \in T_n$ and $\sigma \in T_n^*$, $(q, \mathbf{v}) \xrightarrow{t\sigma} (q'', \mathbf{v}'')$ if $(q, \mathbf{v}) \xrightarrow{t} (q', \mathbf{v}') \xrightarrow{\sigma} (q'', \mathbf{v}'')$.

Given a counter machine $S = \langle Q, E \rangle$ and an initial configuration $c \in Q \times \mathbb{N}^n$, the pair (S, c) is an intialized counter machine. Since, the notations are explicit, in the following we shall write counter machine for both (S, c) and S.

It is true that any counter machine can be easily encoded into a Minsky machine. For instance to encode a test of the form $x_i = c$, the Minsky machine can decrement c times the counter, test to 0 and increment again c times the counter. Note that this encoding modifies the number of alternations between increasing and decreasing mode for the counters, which is the factor we are interested in when considering reversal-boundedness. That is the reason why we propose this extension of Minsky machine. We do not go further for instance extending the guards, because in [10], it is proved that the reachability problem for reversal-bounded counter machines with linear guards (of the form $x = y$ where x, y are two counters variables) is undecidable.

3 New reversal-bounded counter machines

3.1 Reversal-bounded counter machines

We would like to extend the notion of reversal-bounded to capture and verify a larger class of counter machines. In fact, if we consider the counter machine represented by

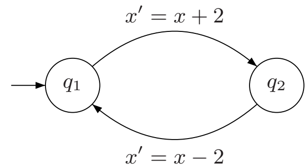

Fig. 1. A simple not reversal-bounded counter machine

the figure 1 with the initial configuration $(q_1, 0)$. Its reachability set is finite equal to $\{(q_1, 0), (q_2, 2)\}$ and consequently semi-linear but the counter machine is not reversal-bounded. We propose here an extension of the notion of reversal-bounded, which allows us to handle such cases and more generally every bounded counter machines.

Given an integer $b \in \mathbb{N}$, we now consider the number of alternations between increasing and decreasing mode when the value of a counter goes above the bound b. Let $S = \langle Q, E \rangle$ be a n-dim counter machine and $TS(S) = \langle Q \times \mathbb{N}^n, \rightarrow \rangle$. From it, we define another transition system $TS_b(S) = \langle Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n, \rightarrow_b \rangle$. Intuitively for a configuration $(q, \mathbf{m}, \mathbf{v}, \mathbf{r}) \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$, the vector $\mathbf{m}$ is used to store the current mode of each counter -increasing ($\uparrow$) or decreasing ($\downarrow$)-, the vector $\mathbf{v}$ contains the values and the vector $\mathbf{r}$ the numbers of alternations performed over b. Formally, we have $(q, \mathbf{m}, \mathbf{v}, \mathbf{r}) \xrightarrow{t}_b (q', \mathbf{m}', \mathbf{v}', \mathbf{r}')$ if and only if the following conditions hold :

1. $(q, \mathbf{v}) \xrightarrow{t} (q', \mathbf{v}')$
2. for each $i \in [1..n]$, the relation expresses by the following array is verified :

$\mathbf{v}(i) - \mathbf{v}'(i)$	$\mathbf{m}(i)$	$\mathbf{m}'(i)$	$\mathbf{v}(i)$	$\mathbf{r}(i)$
> 0	$\downarrow$	$\downarrow$	$-$	$\mathbf{r}(i)$
> 0	$\uparrow$	$\downarrow$	$\leq b$	$\mathbf{r}(i)$
> 0	$\uparrow$	$\downarrow$	$> b$	$\mathbf{r}(i) + 1$
< 0	$\uparrow$	$\uparrow$	$-$	$\mathbf{r}(i)$
< 0	$\downarrow$	$\uparrow$	$\leq b$	$\mathbf{r}(i)$
< 0	$\downarrow$	$\uparrow$	$> b$	$\mathbf{r}(i) + 1$
$= 0$	$\downarrow$	$\downarrow$	$-$	$\mathbf{r}(i)$
$= 0$	$\uparrow$	$\uparrow$	$-$	$\mathbf{r}(i)$

We denote by $\rightarrow_b^*$ the reflexive and transitive closure of $\rightarrow_b$. Given a configuration $(q, \mathbf{m}, \mathbf{v}, \mathbf{r})$ of $TS_b(S)$, $\mathtt{Reach}_b(S, (q, \mathbf{m}, \mathbf{v}, \mathbf{r})) = \{(q', \mathbf{m}', \mathbf{v}', \mathbf{r}') \mid (q, \mathbf{m}, \mathbf{v}, \mathbf{r},) \rightarrow_b^* (q', \mathbf{m}', \mathbf{v}', \mathbf{r}')\}$. We extend this last notation to the configurations of $TS(S)$, saying that if $(q, \mathbf{v}) \in Q \times \mathbb{N}^n$ is a configuration of $TS(S)$, then $\mathtt{Reach}_b(S, (q, \mathbf{v}))$ is equal to the set $\mathtt{Reach}_b(S, (q, \uparrow, \mathbf{v}, \mathbf{0}))$ where $\uparrow$ denotes here the vector with all components equal to $\uparrow$.

Definition 2. *Let $b, k \in \mathbb{N}$. A counter machine (S, c) is k-*reversal-b-bounded *if and only if for all $(q, \boldsymbol{m}, \boldsymbol{v}, \boldsymbol{r}) \in \mathtt{Reach}_b(S, c)$ and for all $i \in [1..n]$, we have $\boldsymbol{r}(i) \leq k$.*

We then say that :

1. A counter machine is *reversal-bounded* if there exist $k, b \in \mathbb{N}$ such that it is k-reversal-b-bounded,

2. For a given $k \in \mathbb{N}$, a counter machine is *k-reversal-bounded*, if there exists $b \in \mathbb{N}$ such that it is k-reversal-b-bounded,
3. For a given $b \in \mathbb{N}$, a counter machine is *reversal-b-bounded*, if there exists $k \in \mathbb{N}$ such that it is k-reversal-b-bounded.

We remark that this definition includes the definition of reversal-bounded given in [9], which corresponds to reversal-0-bounded. In comparison to what is presented in [9], there is a slight difference because we do not have here accepting states and consequently we consider all the possible runs of the counter machine as accepted runs. We will see in section 4 that this difference can change some decidability results. Note that in later works [10], the counter machines are also defined without any accepting state.

3.2 Reachability set

In [9], it has been proved that the reversal-0-bounded counter machines have an effectively computable semi-linear reachability set. We extend here this result to all the reversal-bounded counter machines.

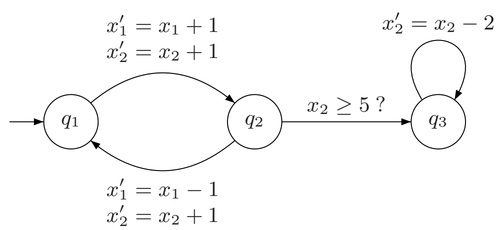

Fig. 2. A 1-reversal-1-bounded counter machine

The idea consists in building from a k-reversal-b-bounded counter machine (S, c) a k-reversal-0-bounded counter machine (S', c') as it is done for the counter machine of the figure 2 (with the initial configuration $(q_1, (0, 0))$) from which we obtain the counter machine represented in the figure 3 (with the initial configuration $((q_1, 0, 0), (0, 0))$). We assume $S = \langle Q, E \rangle$ and $S' = \langle Q', E' \rangle$. First we introduce two symbols $\perp$ and ω_b which are not integers. ω_b represents a counter value strictly greater than b and $\perp$ a counter value for which it is not known whether it is greater or not than b. The location set Q' is then equal to $Q \times B^n$ where $B = \{0, \ldots, b\} \cup \{\omega_b, \perp\}$. Intuitively, the counter machine S' encodes the run of S and when a counter value in S is under the bound b, its value is stored into the control state of S' and the corresponding value of the counter in S' is 0, but when the value goes above b in S then it is restored in the counter in S'. Furthermore (S', c') being k-reversal-0-bounded, we use the results of [9] to compute the reachability set $\mathtt{Reach}(S', c')$ from which we deduce $\mathtt{Reach}(S, c)$.

Theorem 3. *Given a reversal-bounded counter machine, its reachability set is an effectively computable semi-linear set.*

Proof. Let $k, b \in \mathbb{N}$ and (S, c) be an initialized n-dim counter machine k-reversal-b-bounded. Assume $S = \langle Q, E \rangle$. We define two functions $+_B$ and $-_B$ from $B \times \mathbb{N}$ to B, which verify the following rules, for all $d \geq 0$:

– $\omega_b +_B d = \omega_b$,
– $\omega_b -_B d = \bot$,
– for $e \in [1..b]$, $d +_B e = d + e$ if $d + e \leq b$ else $d +_B e = \omega_b$,
– for $e \in [1..b]$ and $d \leq e$, $e -_B d = e - d$.

These operations can be extended in the obvious way to vectors.
We then build another counter machine $S' = \langle Q', E' \rangle$ such that $Q' = Q \times B^n$ and E' is defined as follows :

– For each $(q, (\#, \mu, \delta), q') \in E$ and each $\mathbf{u} \in B^n$ such that :
  (i) there is no $i \in [1..n]$ such that $\mathbf{u}(i) = \bot$, and
  (ii) for all $i \in [1..n]$ such that $1 \leq \mathbf{u}(i) \leq b$, $\mu(i)\ \#(i)\ \mathbf{u}(i)$
  Add to E' the transition $((q, \mathbf{u}), (\#', \mu', \delta'), (q', \mathbf{u}'))$ defined by :
  - $\mathbf{u}' = \mathbf{u} +_B \delta$
  - for all $i \in [1..n]$:
    ∗ if $1 \leq \mathbf{u}(i) \leq b$ and $1 \leq \mathbf{u}'(i) \leq b$, then $\#'(i) \in \{\leq\}$ and $\mu'(i) = \delta'(i) = 0$,
    ∗ if $1 \leq \mathbf{u}(i) \leq b$ and $\mathbf{u}'(i) = \omega_b$, then $\#'(i) \in \{\leq\}$ and $\mu'(i) = 0$ and $\delta'(i) = \mathbf{u}(i) + \delta(i)$,
    ∗ if $\mathbf{u}(i) = \omega_b$ then $\#'(i) = \#(i)$, $\mu'(i) = \mu(i)$ and $\delta'(i) = \delta(i)$

– For all $\mathbf{u} \in B^n$, for all $\mathbf{u}' \in (B \setminus \{\bot\})^n$ such that :
  (i) there is $j \in [1..n]$ such that $\mathbf{u}(j) = \bot$, and
  (ii) for all $j \in [1..n]$ such that $\mathbf{u}(i) \neq \bot$, $\mathbf{u}(i) = \mathbf{u}'(i)$
  For all $q \in Q$, add to E' the transitions $((q, \mathbf{u}), (\#', \mu', \delta'), (q, \mathbf{u}'))$ defined as follows, for all $i \in [1..n]$:
  - if $\mathbf{u}(i) \neq \bot$, then $\#'(i) \in \{\leq\}$ and $\mu'(i) = \delta'(i) = 0$,
  - if $\mathbf{u}(i) = \bot$ and $\mathbf{u}'(i) = \omega_b$, then $\#'(i) \in \{\leq\}$, $\mu'(i) = b + 1$ and $\delta'(i) = 0$,
  - if $\mathbf{u}(i) = \bot$ and $1 \leq \mathbf{u}'(i) \leq b$, then $\#'(i) \in \{=\}$, $\mu'(i) = \mathbf{u}'(i)$ and $\delta'(i) = -\mathbf{u}'(i)$.

Given the initial configuration $c = (q, \mathbf{v})$ of S, we build the initial configuration $c' = (q, \mathbf{u}, \mathbf{v}')$ of S' as follows, for all $i \in [1..n]$:

– if $\mathbf{v}(i) \leq b$, $\mathbf{u}(i) = \mathbf{v}(i)$ and $\mathbf{v}'(i) = 0$,
– if $\mathbf{v}(i) > b$, $\mathbf{u}(i) = \omega_b$ and $\mathbf{v}'(i) = \mathbf{v}(i)$.

We define the following relation $\sim_b \subseteq (Q \times \mathbb{N}^n) \times (Q' \times \mathbb{N}^n)$ as follows $(q, \mathbf{v}) \sim_b (q, \mathbf{u}, \mathbf{v}')$ if and only if, for all $1 \leq i \leq n$:

– $\mathbf{u}(i) \neq \bot$,
– if $1 \leq \mathbf{u}(i) \leq b$, $\mathbf{v}(i) = \mathbf{u}(i)$ and $\mathbf{v}'(i) = 0$,
– if $\mathbf{u}(i) = \omega_b$, $\mathbf{v}(i) > b$ and $\mathbf{v}(i) = \mathbf{v}'(i)$.

By construction of (S', c'), we have then the following properties. $(q, \mathbf{v}) \in Q \times \mathbb{N}^n$ belongs to $\mathtt{Reach}(S, c)$ if and only if there exists $(q, \mathbf{u}, \mathbf{v'}) \in Q' \times \mathbb{N}^n$ such that $(q, \mathbf{v}) \sim_b (q, \mathbf{u}, \mathbf{v}')$ and $(q, \mathbf{u}, \mathbf{v}')$ belongs to $\mathtt{Reach}(S', c')$. Furthermore, if (S, c) is k-reversal-b-bounded, by construction (S', c') is k-reversal-0-bounded, in fact in (S', c') none of the counters changes mode under b between increasing and decreasing modes, and all the counters change mode above b as they do in (S, c). From [9], we deduce that $\mathtt{Reach}(S', c')$ is semi-linear and can effectively be computed. Using the first property we have just mentioned, we are able to compute $\mathtt{Reach}(S, c)$ using $\mathtt{Reach}(S', c')$ and a Presburger formula, in fact $\mathtt{Reach}(S, c) = \{(q, \mathbf{v}) \mid \exists (q, \mathbf{u}, \mathbf{v}') \in \mathtt{Reach}(S', c')\ \mathtt{such\ that}\ \forall i \in [1..n], (\mathbf{u}(i) \leq b \Rightarrow \mathbf{v}(i) = \mathbf{u}(i)) \wedge (\mathbf{u}(i) = \omega_b \Rightarrow \mathbf{v}(i) = \mathbf{v}'(i))\}$. $\square$

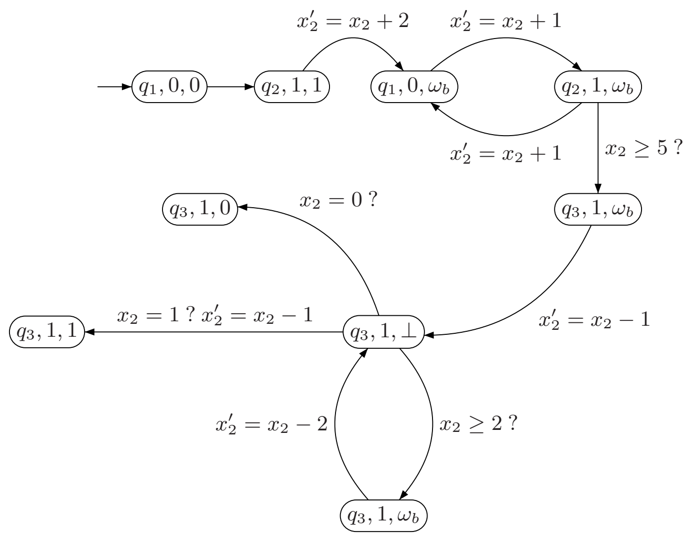

Fig. 3. A 1-reversal-0-bounded counter machine obtained from the counter machine of Fig. 2

4 Deciding reversal-boundedness

In this section, we will study the decidabilty of reversal-boundedness.

4.1 Undecidability

In [9], the author shows that it is not possible to decide whether a counter machine is reversal-0-bounded or not. We prove here that this theorem is still true when considering reversal-boundedness.

Theorem 4. *Verifying if a counter machine is reversal-bounded is undecidable.*

Proof. We reduce the halting problem for 2-counters deterministic Minsky Machines. We consider a deterministic Minsky Machine S with the initial configuration $(q_0, (0,0))$ working over two counter variables x_1 and x_2. "Deterministic" here means that there is a unique possible run starting on $(q_0, (0,0))$. From S, we build a counter machine S' working over three counter variables x_1,x_2 and x_3, such that for each $(q, t, q') \in E$, we add two control states q_1 and q_2 and the transitions (q, t_1, q_1), (q_1, t_2, q_2) and (q_2, t, q') where t_1 and t_2 only change the counter variable x_3 doing $x'_3 = x_3 + 2$ for t_1 and $x'_3 = x_3 - 1$ for t_2. Note that S' starting on $(q_0, (0,0,0))$ is also deterministic. Furthermore $(S', (q_0, (0,0,0)))$ is reversal-bounded if and only if its unique run is finite, which is equivalent to halting. Since S' starting with $(q_0, (0,0,0))$ halts if and only if S starting from $(q_0, (0,0))$ halts and since this last problem is undecidable, we conclude the theorem.□

4.2 Fixing one parameter

We will see here that fixing one of the parameters is not enough to obtain decidability for the reversal-boundedness.

Theorem 5. *Given $b \in \mathbb{N}$, verifying if a counter machine is reversal-b-bounded is undecidable.*

Sketch of Proof, For each b in $\mathbb{N}$, we can reuse the same proof as for the theorem 4, we can show that the 3-counter machine $(S', (q_0, (0,0,0)))$ is reversal-b-bounded if and only if the deterministic Minsky machine (S, c) from which it is built halts. □

Theorem 6. *Given $k \in \mathbb{N}$, verifying if a counter machine is k-reversal-bounded is undecidable.*

Proof. This result can also be proved using the proof of theorem 4. For instance if $k = 0$, deciding if the deterministic 3-counter machine $(S', (q_0, (0,0,0)))$ built in the proof is 0-reversal-bounded is equivalent to know if all its counter are bounded which can only happen if the Minsky machine (S, c) halts. To obtain the result for any $k \in \mathbb{N}$, we can plug $2k$ states such that in the even states, either we increment the counter and we stay in the same state or we go to the next odd states and in the odd states, or we decrement the counter and stay in the same state or we test to zero and go to the next even states. Finally, we connect the last odd state to the initial state q_0 with a test to zero. So for all bound b, there exists a run that do over b at least k alternations between increment and decrement modes and consequently, the counter machine is k-reversal-b-bouned if and only if $(S', (q_0, (0,0,0))))$ is 0-reversal-b-bounded. □

4.3 Fixing the two parameters

We will now prove that if the two parameters b and k are fixed, it is possible to decide if a counter machine is k-reversal-b-bounded. Let $b, k \in \mathbb{N}$ and (S, c) be a counter machine. The idea consists in building a counter machine (S', c') which will be $(k+1)$-reversal-b-bounded and which will reach a special control state q_{err} if and only if (S, c) is not k-reversal-b-bounded. Note that since (S', c') is reversal-bounded, it is possible to decide whether the control state q_{err} is reachable or not. In the control state of (S', c'), we store the mode -increasing (↑) or decreasing (↓)- for each counter and also the number of alternations already performed over b. We also add some control states to test at each step if each counter value is strictly greater (denoted by $b_>$) or smaller than b (denoted by $b_\leq$). The figure 4 gives an example of the counter machine we build to decide if the counter machine from figure 1 with the initial configuration $(q_1, 0)$ is 1-reversal-1-bounded.

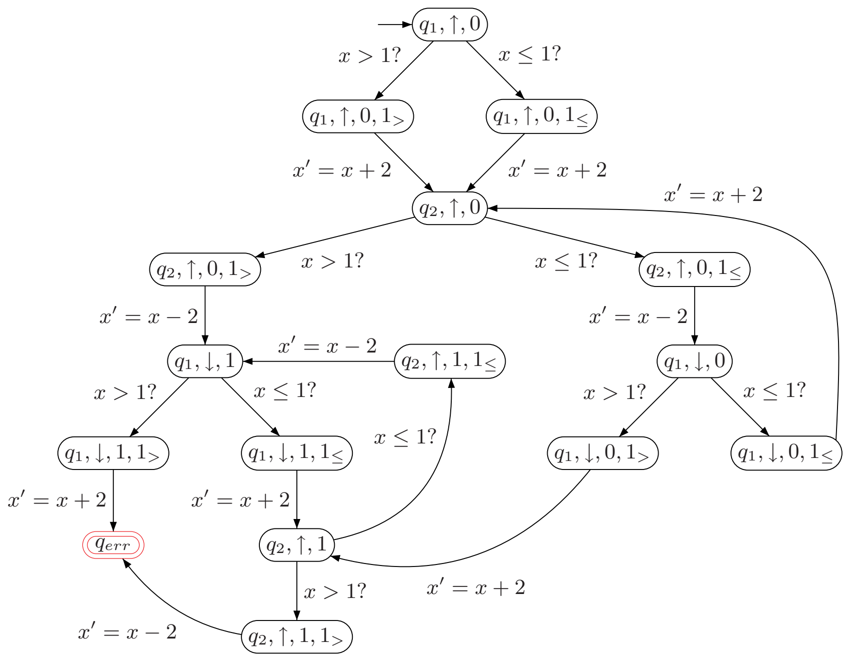

Fig. 4. A 2-reversal-1-bounded counter machine to decide if the counter machine of Fig. 1 is 1-reversal-1-bounded

Theorem 7. *Given $b, k \in \mathbb{N}$, verifying if a counter machine is k-reversal-b-bounded is decidable.*

*Proof.*Let $S = \langle Q, E \rangle$ be a n-dim counter machine and $c = (q_0, \mathbf{v})$ an initial configuration (with $\mathbf{v} \in \mathbb{N}^n$). We will build a counter-machine (S', c') which will be $(k+1)$-reversal-b-bounded and which will have a special location q_{err} such that q_{err} will be reachable in S' from c' if and only if (S, c) is not k-reversal-b-bounded. We define $S' = \langle Q', E' \rangle$ as follows :

– $Q' = Q'' \cup Q_b \cup \{q_{err}\}$ where $Q'' = Q \times \{\uparrow, \downarrow\}^n \times \{1, \dots, k\}^n$. For each counter we store in the control state the current mode (incrementation or decrementation) and the number of alternations (over b) already done, $Q_b = Q'' \times \{b_\leq, b_>\}^n$ is used to know if the different counter values are strictly greater or smaller than b;

– E' is then defined as follows :

1. for each $(q,\mathbf{m},\mathbf{r}) \in Q''$ and $\mathbf{u} \in \{b_{\leq}, b_{>}\}$, we have $((q,\mathbf{m},\mathbf{r}),(\#,\mu,\mathbf{0}),(q,\mathbf{m},\mathbf{r},\mathbf{u})) \in E'$ with for each $i \in [1..n]$:
   - if $\mathbf{u}(i) = b_{\leq}$ then $\#(i) \in \{\geq\}$ (we can in fact simulate this inequality with b transitions doing an equality test) and $\mu(i) = b$
   - if $\mathbf{u}(i) = b_{>}$ then $\#(i) \in \{\leq\}$ and $\mu(i) = b+1$
2. for each $(q,(\#,\mu,\delta),q') \in E$, $((q,\mathbf{m},\mathbf{r},\mathbf{u}),\langle\#,\mu,\delta\rangle,(q,\mathbf{m}',\mathbf{r}'))$ belongs to E' if and only if for each $i \in [1..n]$:
   - if $\mathbf{m}(i) =\uparrow$ and $0 \leq \delta(i)$ then $\mathbf{m}'(i) =\uparrow$ and $\mathbf{r}'(i) = \mathbf{r}(i)$,
   - if $\mathbf{m}(i) =\downarrow$ and $\delta(i) \leq 0$ then $\mathbf{m}'(i) =\downarrow$ and $\mathbf{r}'(i) = \mathbf{r}(i)$,
   - if $\mathbf{m}(i) =\uparrow$ and $\delta(i) < 0$ and $\mathbf{u}(i) = b_{\leq}$ then $\mathbf{m}'(i) =\downarrow$ and $\mathbf{r}'(i) = \mathbf{r}(i)$,
   - if $\mathbf{m}(i) =\downarrow$ and $0 < \delta(i)$ and $\mathbf{u}(i) = b_{\leq}$ then $\mathbf{m}'(i) =\uparrow$ and $\mathbf{r}'(i) = \mathbf{r}(i)$,
   - if $\mathbf{m}(i) =\uparrow$ and $\delta(i) < 0$ and $\mathbf{u}(i) = b_{>}$ and $\mathbf{r}(i) < k$ then $\mathbf{m}'(i) =\downarrow$ and $\mathbf{r}'(i) = \mathbf{r}(i)+1$,
   - if $\mathbf{m}(i) =\downarrow$ and $0 < \delta(i)$ and $\mathbf{u}(i) = b_{>}$ and $\mathbf{r}(i) < k$ then $\mathbf{m}'(i) =\uparrow$ and $\mathbf{r}'(i) = \mathbf{r}(i)+1$,
3. for each $(q,(\#,\mu,\delta),q') \in E$, $((q,\mathbf{m},\mathbf{r},\mathbf{u}),\langle\#,\mu,\delta\rangle,q_{err})$ belongs to E' if and only if there exists $i \in [1..n]$ such that :
   - $\mathbf{m}(i) =\uparrow$ and $\delta(i) < 0$ and $\mathbf{u}(i) = b_{>}$ and $\mathbf{r}(i) = k$, or,
   - $\mathbf{m}(i) =\downarrow$ and $0 < \delta(i)$ and $\mathbf{u}(i) = b_{>}$ and $\mathbf{r}(i) = k$.

We then define c' as follows, $c' = (q_0, \uparrow, \mathbf{0}, \mathbf{0})$. From the way, the initialized counter machine (S', c') is built, we deduce the following property, $(q,\mathbf{m},\mathbf{r},\mathbf{u},\mathbf{v}) \in Q \times \{\uparrow,\downarrow\}^n \times \{1,\ldots,k\}^n \times \{b_{\leq}, b_{>}^n\} \times \mathbb{N}^n$ belongs to $\mathtt{Reach}(S',c')$ if and only if :

(i) $(q,\mathbf{m},\mathbf{v},\mathbf{r}) \in \mathtt{Reach}_b(S,c)$ and
(ii) for all $i \in [1..n]$, $\mathbf{r}(i) \leq k$ and
(iii) for all $i \in [1..n]$, $\mathbf{u}(i) = b_{>} \Leftrightarrow \mathbf{v}(i) > b$ and $\mathbf{u}(i) = b_{\leq} \Leftrightarrow \mathbf{v}(i) \leq b$

Using this property and the way we connect the control state q_{err} in S' and the definition of reversal-boundedness, we deduce that (S,c) is k-reversal-b-bounded if and only if there does not exist $\mathbf{v} \in \mathbb{N}^n$ such that $(q_{err},\mathbf{v}) \in \mathtt{Reach}(S',c')$. By construction (S',c') is $(k+1)$-reversal-b-bounded, we in fact count the exact number of alternations performed over b in the control states and when the counter machines performs the $(k+1)$-th alternations, it moves to the control state q_{err} from which there is no outgoing translation. So using the theorem 3, we can deduce whether the control state q_{err} is reachable or not and hence whether (S,c) is k-reversal-b-bounded or not.$\square$

This result contrasts with the one given in [9], which says that given $k \in \mathbb{N}$, verifying if a counter machine is k-reversal-0-bounded is undecidable. This is due to the fact that in [9], the considered counter machines have accepting control states, whereas our definition is equivalent to have all the control states as accepting. In fact, when we define the reversal-bounded counter machines, we consider all the possible runs and not only the one ending in an accepting state.

4.4 Computing the parameters

When a counter machine is reversal-bounded, it could be useful to characterize the pairs (k, b) for which it is k-reversal-b-bounded, first because it gives us information on the behavior of the counter machine but also because these parameters are involved in the way the reachability set is built as one can see in the proof of theorem 3 and in [9].

Let (S, c) be a counter machine. We define the following set to talk about the parameters of reversal-bounded counter machines :

$$RB(S, c) = \{(k, b) \in \mathbb{N} \times \mathbb{N} \mid (S, c) \text{ is k-reversal-b-bounded}\}$$

Then $RB(S, c) = \emptyset$ if and only if (S, c) is reversal-bounded, hence the non-emptiness problem for $RB(S, c)$ is in general not decidable, but this set is recursive (cf. theorem 7). Furthermore, if there exist (k, b) in $RB(S, c)$ and $(k', b') \in \mathbb{N} \times \mathbb{N}$ such that $(k, b) \leq (k', b')$ then we know, by definition of reversal-boundedness that (S, c) is also k'-reversal-b'-bounded, ie $(k', b') \in RB(S, c)$. Since the order relation $\leq$ on $\mathbb{N} \times \mathbb{N}$ is a well-ordering we can deduce :

Lemma 8. *Let (S, c) be a reversal-bounded counter machine. The set $RB(S, c)$ is upward-closed, it has a finite number of minimal elements, which can effectively be computed.*

*Proof.*Let (S, c) be a reversal-bounded initialized n-dim counter machine. We recall that a set $A \in \mathbb{N}^n$ is upward-closed (according to the order relation $\leq$) if for all $\mathbf{v} \in S$, if there exists $\mathbf{v}' \in \mathbb{N}^n$ such that $\mathbf{v} \leq \mathbf{v}'$, then $\mathbf{v}'$ belongs to A. It is obvious that the set $RB(S, c) \subseteq \mathbb{N}^2$ is upward-closed using the definition of reversal-boundedness.

We will now see how to compute the minimal elements of $RB(S, c)$. Note that in the construction we propose, we use the reachability set of (S, c) which is obtained when knowing at least a pair $(k, b) \in RB(S, c)$ (see the proof of theorem 3). Since we know (S, c) is reversal-bounded, finding an element of $RB(S, c)$ can be done for instance enumerating the pairs $(k, b) \in \mathbb{N}^2$ and testing if (S, c) is k-reversal-b-bounded (possible by theorem 7).

We now assume that (S, c) is k-reversal-b-bounded. In a first step, we will compute b_0 the smallest b' such that (S, c) is reversal-b'-bounded. This constant b_0 can easily be found adding b counters variables to each counter variable, in order to count the number of alternations done over each b' smaller than b. In fact b_0 will then be the smallest b' such that the number of alternations over b' is bounded. This method gives us also the constant k_0 such that (S, c) is k_0-reveral-b_0-bounded and not $(k_0 - 1)$-reversal-b_0-bounded. Note that (k_0, b_0) is already one of the minimal elements of $RB(S, c)$. If (k', b') is a minimal element of $RB(S, c)$ different from (k_0, b_0), we necessarily have $b' > b_0$ and $k' < k_0$ by definition of b_0 and k_0. From the n-dim counter machine (S, c), we build the counter machine (S', c') described in the proof of theorem 7 to decide if (S, c) is k_0-reversal-b_0-bounded. We transform (S', c') into a $(n.k_0)$-dim counter machine (S'', c''). To do that we add, for each counter variable, k_0 counter variables, whose roles are to store for each of the k_0 alternations between increasing and decreasing mode done over b the counter value at the moment of the alternation. We can then use the reachability set and the new counters to decide if the first counter machine is k'-reversal-bounded and this for each $k' < k_0$. In fact, for each $i \in [1..n]$, we look at the reachable configurations $((q, \mathbf{m}, \mathbf{r}), \mathbf{v}) \in Q \times \{\uparrow, \downarrow\} \times [1..k_0]^n \times \mathbb{N}^{k_0 \cdot n}$ in $\texttt{Reach}(S'', c'')$, such that $\mathbf{r}(i) = k_0$, we denote by $R(i)$ the set of such reachable configurations. If (S, c) is $(k_0 - 1)$-reversal-bounded, there should be a $b' \in \mathbb{N}$ such that for all $i \in [1..n]$, for all the configurations $((q, \mathbf{m}, \mathbf{r}), \mathbf{v})$ in $R(i)$, if the values $v(i+1)$, ... $v(i+k_0)$ represent the values of the i-th counter at each of the alternation, then one of these $v(i+k')$ with $k' \in [1..k_0]$ should be smaller than b. Since (S'', c'') is k_0-reversal-b_0-bounded (we have only added counters which always increase), its reachability set is semi-linear and we can build from it (using Presburger formulae for instance) and the previous information the minimal elements of $RB(S, c)$.$\square$

5 Analysis of VASS

In this section, we recall the definition of Vector Addition System with States and show that the notion of reversal-boundedness we newly introduce is well-suited for the verification of these systems.

5.1 VASS and their coverability graphs

Definition 9. *A n-dim counter machine $\langle Q, E \rangle$ is a* Vector Addition System with States *(shortly VASS) if and only if for all transitions $(q, t, q') \in E$, t is a guarded translation $(\#, \mu, \delta)$ such that $\# = (\leq, \ldots, \leq)$,*

Hence in VASS, it is not possible to test if a counter value is equal to a constant but only if it is greater than a constant.

In [12], the authors provide an algorithm to build from a VASS a labeled tree, the *Karp and Miller tree*. We recall here the construction of this tree. We first define a function $\texttt{Acceleration} : \mathbb{N}^n_\omega \times \mathbb{N}^n_\omega \rightarrow \mathbb{N}^n_\omega$ as follows, for $\mathbf{w}, \mathbf{w}' \in \mathbb{N}^n_\omega$ such that $\mathbf{w} \leq \mathbf{w}'$, we have $\mathbf{w}'' = \texttt{Acceleration}(\mathbf{w}, \mathbf{w}')$ if and only if for all $i \in [1..n]$:

- if $\mathbf{w}(i) = \mathbf{w}'(i)$ then $\mathbf{w}''(i) = \mathbf{w}(i)$,
- if $\mathbf{w}(i) < \mathbf{w}'(i)$ then $\mathbf{w}''(i) = \omega$.

The Karp and Miller tree is a labeled tree (P, R, l) where :

- P is a finite set of nodes,
- $l : P \rightarrow Q \times \mathbb{N}^n_\omega$ is a labeling function,
- $R \subseteq P \times T_n \times P$ is the transition relation.

To represent a node p with the label $l(p) = (q, \mathbf{w})$, we will sometimes directly write $p(q, \mathbf{w})$. The algorithm 1 shows how the Karp and Miller tree is obtained from an initialized counter machine.

The main idea of this tree is to cover in a finite way the reachable configurations using the symbol ω, when a counter is not bounded. They have shown that their algorithm always terminates and that it enjoys some good properties. In particular, this tree

Algorithm 1 $T =$ `KMTree`$(\langle Q, E\rangle, c)$

Input : $(\langle Q, E\rangle, c)$ an initialized VASS;
Output : $T = \langle P, R, l\rangle$ the Karp and Miller tree;

$P = \{p_0\}$, $R = \emptyset$, $l(p_0) = c$
$ToBeTreated = \{p_0\}$
while $ToBeTreated \neq \emptyset$ **do**
  Choose $p(q, \mathbf{w}) \in ToBeTreated$
  if there does not exist a predecessor $p'(q, \mathbf{w})$ of p in T **then**
    for each $(q, (\#, \mu, \delta), q') \in E$ **do**
      if $\mu \leq \mathbf{w}$ **then**
        let $\mathbf{w}' = \mathbf{w} + \delta$
        if there exists a predecessor $p'(q', \mathbf{w}'')$ of p in T such that $\mathbf{w}' > \mathbf{w}''$ **then**
          let $\mathbf{w}' =$ `Acceleration`$(\mathbf{w}', \mathbf{w}'')$
        end if
        Add a new node p' to P such that $l(p') = (q', \mathbf{w}')$
        Add $(p, \langle \#, \mu, \delta\rangle, p')$ to R
        Add p' to $ToBeTreated$
      end if
    end for
  end if
  Remove p of $ToBeTreated$
end while

can be used to decide the boundedness of a VASS. In [19], the authors have proposed a further construction based on the Karp and Miller tree in order to test the regularity of the language of the unlabeled traces of a VASS. This last construction is known as the *coverability graph*. To obtain it, the nodes of the Karp and Miller tree with the same labels are grouped together. Formally if (S, c) is a n-dim initialized VASS, we denote by $CG(S, c)$ its coverability graph defined as follows, $CG(S, c) = \langle N, \Delta\rangle$ where :

- $N \subseteq Q \times \mathbb{N}_\omega^n$ is a finite set of nodes,
- $\Delta \subseteq N \times T_n \times N$ is a finite set of edges labeled with guarded transitions.

We call a *circuit* in the coverability graph a path ending in the starting node and a circuit will be said to be *elementary* if all nodes are different with the exception of the starting and ending nodes. For a vector $\mathbf{w} \in \mathbb{N}_\omega^n$, we denote by `Inf`$(\mathbf{w})$ the set $\{i \in [1..n] \mid \mathbf{w}(i) = \omega\}$ and `Fin`$(\mathbf{w}) = [1..n] \setminus$ `Inf`$(\mathbf{w})$. Using these notions, it has been proved that the coverability graph verifies the following properties.
Let (S, c) be a n-dim initialized VASS with $S = \langle Q, E\rangle$, $TS(S) = \langle Q \times \mathbb{N}^n, \rightarrow\rangle$ its associated transition system and $G = \langle N, \Delta\rangle$ its coverability graph.

Theorem 10. *[12,19]*

1. *If $(q, \mathbf{w})$ is a node in G, then for all $k \in \mathbb{N}$, there exists $(q, \mathbf{v}) \in$ `Reach`(S, c) such that for all $i \in$ `Inf`$(\mathbf{w})$, $k \leq \mathbf{v}(i)$ and for all $i \in$ `Fin`$(\mathbf{w})$, $\mathbf{w}(i) = \mathbf{v}(i)$.*
2. *For $\sigma \in T_n^*$, if $c \xrightarrow{\sigma} (q, \mathbf{v})$ then there is a unique path in G labeled by σ and leading from c to a node $(q, \mathbf{w})$ and for all $i \in$ `Fin`$(\mathbf{w})$, $\mathbf{v}(i) = \mathbf{w}(i)$.*

3. *If $\sigma \in T_n^*$ is a word labeling a circuit in G and $(q, \boldsymbol{w})$ is the initial node of this circuit, then there exist $(q, \boldsymbol{v}) \in \mathtt{Reach}(S, c)$ and $(q', \boldsymbol{v}')$ such that $(q, \boldsymbol{v}) \xrightarrow{\sigma} (q, \boldsymbol{v}')$ and for all $i \in \mathtt{Fin}(\boldsymbol{w})$, $\boldsymbol{w}(i) = \boldsymbol{v}(i) = \boldsymbol{v}'(i)$.*

From this theorem, we deduce the following lemma, we will then use to decide the reversal-boundedness of a VASS :

Lemma 11. *If there exists an elementary circuit $((q_1, \boldsymbol{w}_1) \xrightarrow{t_1} (q_2, \boldsymbol{w}_2) \xrightarrow{t_2} \dots \xrightarrow{t_f} (q_1, \boldsymbol{w}_1))$ in G, then for all $k, l \in \mathbb{N}$, there exist $\boldsymbol{v}_1, \dots, \boldsymbol{v}_l \in \mathbb{N}^n$ such that :*

(i) *$c \rightarrow^* (q_1, \boldsymbol{v}_1) \xrightarrow{\sigma} (q_1, \boldsymbol{v}_2) \xrightarrow{\sigma} \dots \xrightarrow{\sigma} (q_1, \boldsymbol{v}_l)$ in $TS(S)$ with $\sigma = t_1 \dots t_f$, and,*
(ii) *for all $j \in [1..l]$, for all $i \in \mathtt{Inf}(\boldsymbol{w}_1)$, $k \leq \boldsymbol{v}_j(i)$ and for all $i \in \mathtt{Fin}(\boldsymbol{w}_1)$, $\boldsymbol{w}_1(i) = \boldsymbol{v}_j(i)$.*

Proof. We fix $k \in \mathbb{N}$. We define $D_{min} = Min\{D_\sigma(i) \mid i \in [1..n]\}$. Since σ^{l-1} is a circuit, by the point 3 of theorem 10, we deduce there exist $\mathbf{v}'_1, \dots, \mathbf{v}'_l \in \mathbb{N}^n$ such that $c \rightarrow^* (q_1, \mathbf{v}'_1) \xrightarrow{\sigma} (q_1, \mathbf{v}'_2) \xrightarrow{\sigma} \dots \xrightarrow{\sigma} (q_1, \mathbf{v}'_l)$ in $TS(S)$ and such that for all $j \in [1..l]$, $\forall i \in \mathtt{Fin}(\mathbf{u}_1)$, $\mathbf{v}_j(i) = \mathbf{w}_1(i)$. We consider the integer k' defined as follows $k' = Max(k + l.|D_{min}|, Max\{\mathbf{v}_1(i) \mid i \in \mathtt{Inf}(\mathbf{w}_1)\})$ (where $|D_{min}|$ represents the abolute value of D_{min}). From point 1 of theorem 10, we deduce that there exists a $\mathbf{v}_1 \in \mathbb{N}^n$ such that $(q_1, \mathbf{v}_1) \in \mathtt{Reach}(S, c)$ and $\forall i \in \mathtt{Fin}(\mathbf{w}_1)$, $\mathbf{v}_1(i) = \mathbf{w}_1(i)$ and $\forall i \in \mathtt{Inf}(\mathbf{w}_1)$, $k' \leq \mathbf{v}_1(i)$. By definition of k', we deduce that $\mathbf{v}'_1 \leq \mathbf{v}_1$, consequently there exist $\mathbf{v}_2, \dots, \mathbf{v}_l$ such that $(q_1, \mathbf{v}_1) \xrightarrow{\sigma} (q_1, \mathbf{v}_2) \xrightarrow{\sigma} \dots \xrightarrow{\sigma} (q_1.\mathbf{v}_l)$. Furthermore, for all $j \in [1..l]$, $\forall i \in \mathtt{Fin}(\mathbf{w}_1)$, $\mathbf{v}_j(i) = \mathbf{w}_1(i)$ (it is in fact true for $\mathbf{v}_1$, and it can be deduced using the fact that σ is a circuit starting from node $(q_1, \mathbf{w}_1)$). And by property of k', we also deduce that for all $j \in [1..l]$, $\forall i \in \mathtt{Inf}(\mathbf{w}_1)$, $k \leq \mathbf{v}_j(i)$. □

5.2 Deciding if a VASS is reversal-b-bounded

In this section, we show that its possible to decide if a VASS is reversal-b-bounded using a characterization over its coverability graph.

Let $S = \langle Q, E \rangle$ be a n-dim counter machine. We build a $2n$-dim counter machine $\widetilde{S} = \langle Q', E' \rangle$ adding for each counter another counter, whose role is to count the alternation of the first counter between increasing and decreasing mode. Formally, $Q' = Q \times \{\uparrow, \downarrow\}^n$ and T' is built as follows, for each $(q, (\#, \mu, \delta), q') \in E$ and $\mathbf{m}, \mathbf{m}' \in \{\uparrow, \downarrow\}^n$, we have $((q, \mathbf{m}), (\#', \mu', \delta'), (q', \mathbf{m}')) \in E'$ if and only if :

- for all $i \in [1..n]$, $\#'(i) = \#(i)$, $\mu'(i) = \mu_i$ and $\delta'(i) = \delta(i)$;
- for all $i \in [n+1..2n]$, $\#'(i) \in \{\leq\}$ and $\mu'(i) = 0$;
- δ, $\mathbf{m}$, $\mathbf{m}'$ and δ' satisfy for all $i \in [1..n]$ the conditions described in the following array :

$\delta(i)$	$\mathbf{m}(i)$	$\mathbf{m}'(i)$	$\delta'(n+i)$
$= 0$	$\uparrow$	$\uparrow$	0
$= 0$	$\downarrow$	$\downarrow$	0
> 0	$\uparrow$	$\uparrow$	0
> 0	$\downarrow$	$\uparrow$	1
< 0	$\downarrow$	$\downarrow$	0
< 0	$\uparrow$	$\downarrow$	1

By construction, we remark that if S is a VASS then $\widetilde{S}$ is a VASS too. We define then the relation $\sim \in (Q \times \{\uparrow, \downarrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n) \times (Q \times \{\uparrow, \downarrow\}^n \times \mathbb{N}^{2n})$ between the configurations of $TS_0(S)$ and the ones of $TS(\widetilde{S})$ saying that $(q, \mathbf{m}, \mathbf{v}, \mathbf{r}) \sim (q', \mathbf{m}', \mathbf{v}')$ if and only if :

- $q = q'$,
- $\mathbf{m} = \mathbf{m}'$,
- for all $i \in [1..n]$, $\mathbf{v}(i) = \mathbf{v}'(i)$ and $\mathbf{r}(i) = \mathbf{v}'(n+i)$.

The relation $\sim$ is a bisimulation between $TS_0(S)$ and $TS(\widetilde{S})$. Given an initial configuration $c = (q, \mathbf{v})$, we have $(q, \uparrow, \mathbf{v}, \mathbf{0}) \sim (q, \uparrow, (\mathbf{v}, \mathbf{0}))$. Hence, if we denote by $\widetilde{c}$ the triple $(q, \uparrow, (\mathbf{v}, \mathbf{0}))$, we can deduce that the VASS (S, c) is reversal-0-bounded if and only if there exists $k \in \mathbb{N}$ such that for all $(q, \mathbf{m}, \mathbf{v}) \in \mathtt{Reach}(\widetilde{S}, \widetilde{c})$, for all $i \in [1..n]$, $\mathbf{v}(n+i) \leq k$. Using the coverability graph of $(\widetilde{S}, \widetilde{c})$, this last property is decidable for a VASS. Generalizing this method for any $b \in \mathbb{N}$, counting only the alternations that are done above b, we can deduce that :

Theorem 12. *Given $b \in \mathbb{N}$, verifying if a VASS is reversal-b-bounded is decidable.*

5.3 Deciding if a VASS is reversal-bounded

We will now show that the analysis of the coverabilty graph of $(\widetilde{S}, \widetilde{c})$ allows us to decide if a VASS is reversal-bounded (without a fixed bound). Note that this is not a direct consequence of the previous theorem, because it is not possible to enumerate the different bounds b and test if the VASS is reversal-b-bounded, since this method never terminates when the VASS is not reversal-bounded.

Lemma 13. *A n-dim VASS (S, c) is reversal-b-bounded if and only if for all $i \in [1..n]$, all nodes $(q, \boldsymbol{w})$ belonging to an elementary circuit labeled by $\sigma \in T_n^*$ of $CG(\widetilde{S}, \widetilde{c})$ with $D_\sigma(n+i) > 0$ verify $\boldsymbol{w}(i) \leq b$.*

In other words, this last lemma states that (S, c) is reversal-b-bounded if and only if for all $i \in [1..n]$, there is no elementary circuit in the coverability graph $CG(\widetilde{S}, \widetilde{c})$ which strictly increases the $(n+i)$-th counter and which has a node, whose i-th component is strictly greater than b or equal to ω. In fact, applying the lemma 11, we deduce that if such an elementary circuit exists, we can build a run of the VASS (S, c) which does not respect the definition of reversal-b-boundedness.

Before to prove this lemma, we need some technical lemmas.

Lemma 14. *Let $(S, (q, \boldsymbol{v}))$ be a n-dim counter machine and $b, i \in \mathbb{N}$. If there exists $c_1, c_2, c_3 \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$ and $\sigma_1, \sigma_2, \sigma_3 \in T_n^*$ such that :*

(i) $(q, \uparrow, \boldsymbol{v}, 0) \xrightarrow{\sigma_1}_0 c_1 \xrightarrow{\sigma_2}_0 c_2 \xrightarrow{\sigma_3}_0 c_3$ *in $TS_0(S)$, and,*
(ii) *for all $j \in [1..3]$, $b < \boldsymbol{v}_j(i)$ and $\boldsymbol{r}_1(i) < \boldsymbol{r}_2(i) < \boldsymbol{r}_3(i)$ (with $c_j = (q_j, \boldsymbol{m}_j, \boldsymbol{v}_j, \boldsymbol{r}_j)$)*

then there exist $c'_1, c'_2, c'_3 \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$ such that :

$$(q, \uparrow, \boldsymbol{v}, 0) \xrightarrow{\sigma_1}_b c'_1 \xrightarrow{\sigma_2}_b c'_2 \xrightarrow{\sigma_3}_b c'_3 \text{ in } TS_b(S) \text{ and } \boldsymbol{r}'_1(i) < \boldsymbol{r}'_3(i)$$

Sketch of proof. By definition of $\rightarrow_0$ and $\rightarrow_b$, if we have c_1, c_2 and c_3 such that $(q, \uparrow, \mathbf{v}, 0) \xrightarrow{\sigma_1}_0 c_1 \xrightarrow{\sigma_2}_0 c_2 \xrightarrow{\sigma_3}_0 c_3$, there exist necessarly $c'_1, c'_2, c'_3 \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$ such that $(q, \uparrow, \mathbf{v}, 0) \xrightarrow{\sigma_1}_b c'_1 \xrightarrow{\sigma_2}_b c'_2 \xrightarrow{\sigma_3}_b c'_3$. Remark that we have for all $j \in [1..3]$, $q_j = q'_j$, $\mathbf{m}_j = \mathbf{m}'_j$ and $\mathbf{v}_j = \mathbf{v}'_j$. Furthermore, since $\mathbf{r}_1(i) < \mathbf{r}_2(i) < \mathbf{r}_3(i)$, we deduce that there has been at least two alternations between increasing and decreasing mode while doing the sequence $\sigma_1\sigma_2\sigma_3$, furthermore since $\mathbf{v}_1(i) > b$, $\mathbf{v}_2(i) > b$ and $\mathbf{v}_3(i) > b$, we can deduce that one of this alternation has been done, when the i-th counter value was strictly above b, hence we have $\mathbf{r}'_1(i) < \mathbf{r}'_3(i)$.□

From this lemma, we can directly deduce, the following lemma :

Lemma 15. *Let $(S, (q, \boldsymbol{v}))$ be a counter machine and $k, b \in \mathbb{N}$. If there exist $c_1, c_2, \ldots, c_{2k+1} \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$ and $\sigma_1, \ldots, \sigma_{2k+1} \in T_n^*$ such that :*

$$(q, \uparrow, \boldsymbol{v}, \boldsymbol{0}) \xrightarrow{\sigma_1}_0 c_1 \xrightarrow{\sigma_2}_0 \ldots \xrightarrow{\sigma_{2k+1}} c_{2k+1}$$

and if there exists $i \in [1..n]$, such that for all $j \in [1..2k+1]$, $b < \boldsymbol{v}_j(i)$ and for all $j \in [1..2k]$, $\boldsymbol{r}_j(i) < \boldsymbol{r}_{j+1}(i)$, then $(S, (q, \boldsymbol{v}))$ is not k-reversal-b-bounded.

Sketch of proof. Using the lemma 14, we can in fact build a run in $TS_b(S)$ starting from $(q, \uparrow, \mathbf{v}, \mathbf{0})$, which does not verify the property of being k-reversal-b-bounded, since it does more than k alternations over b between increasing and decreasing mode.□

Lemma 16. *Let $(S, (q, \boldsymbol{v}))$ be a counter machine and $b \in \mathbb{N}$. If (S, c) is not reversal-b-bounded, then there exists $i \in [1..n]$ such that for all $k \in \mathbb{N}$, there exist $c_1, \ldots, c_k \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$ which verify :*

(i) $(q, \uparrow, \boldsymbol{v}, \boldsymbol{0}) \rightarrow_0^* c_1 \rightarrow_0^* \ldots \rightarrow_0^* c_k$ *in $TS_0(S)$, and*
(ii) *for all $j \in [1..k]$, $b < \boldsymbol{v}_j(i)$ and for all $j \in [1..k-1]$, $\boldsymbol{r}_j(i) < \boldsymbol{r}_{j+1}(i)$ (with $\boldsymbol{c}_j = (q_j, \boldsymbol{m}_j, \boldsymbol{v}_j, \boldsymbol{r}_j)$).*

Proof. Let $k \in \mathbb{N}$. Assume $(S, (q, \mathbf{v}))$ is not reversal-b-bounded, in particular $(S, (q, \mathbf{v}))$ is not $(k+1)$-reversal-b-bounded. Hence there exists $(q', \mathbf{m}', \mathbf{v}', \mathbf{r}') \in \texttt{Reach}_b(S, (q, \mathbf{v}))$ and $i \in \mathbb{N}$ such that $\mathbf{r}'(i) \geq k+1$. Since the counters which count the alterations between increasing and decreasing mode only increase at most of one integer value and only when the associated counter is strictly greater than b, we deduce that there exist $c'_1, \ldots, c'_k \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$ such that $(q, \uparrow, \mathbf{v}, \mathbf{0}) \rightarrow_b^* c'_1 \rightarrow_b^* \ldots \rightarrow_b^* c'_k$ in $TS_b(S)$ and such that for all $j \in [1..k]$, $c'_j = (q'_j, \mathbf{m}'_j, \mathbf{v}'_j, \mathbf{r}'_j)$ with $b < \mathbf{v}'_j(i)$ and $\mathbf{r}'_j(i) = j$. Furthermore, by definition of $\rightarrow_b$ and $\rightarrow_0$, we can deduce that there exist $c_1, ..., c_k \in Q \times \{\downarrow, \uparrow\}^n \times \mathbb{N}^n \times \mathbb{N}^n$ which verify the properties given in the lemma. □

From these last lemmas, we are able to proof the lemma 13.

Proof of lemma 13. First, we suppose that the VASS $(S, (q, \mathbf{v}))$ is reversal-b-bounded. So there exists $k \in \mathbb{N}$ such that $(S, (q, \mathbf{v}))$ is k-reversal-b-bounded. The proof is *ad absurdum*. Assume that there is $i \in \mathbb{N}$ and a node $(q', \mathbf{w})$, which belongs to an elementary circuit $((q_1, \mathbf{w}_1) \xrightarrow{t_1} (q_2, \mathbf{w}_2) \xrightarrow{t_2} \ldots (q_f, \mathbf{w}_f) \xrightarrow{t_f} (q_1, \mathbf{w}_1))$ of $CG(\widetilde{S}, \tilde{c})$ with $D_{t_1 \ldots t_f}(n+i) > 0$, such that $b < \mathbf{w}(i)$. First we remark, that since $\mathbf{w}(i) > b$ this means that $\mathbf{w}(i) = b' > b$ or $\mathbf{w}(i) = \infty$. We denote by σ the word $t_1 t_2 \ldots t_f$. By the lemma 11, we deduce that there exist $\mathbf{v}_1, \ldots, \mathbf{v}_{2k+1} \in \mathbb{N}^{2n}$ such that in $TS(\widetilde{S})$, we have :

$$c_0 \rightarrow^* (q', \mathbf{v}_1) \xrightarrow{\sigma} \ldots \xrightarrow{\sigma} (q', \mathbf{v}_{2k+1})$$

with for all $j \in [1..2k+1]$, $b < \mathbf{v}_j(i)$, furthermore since $D_\sigma(n+i) > 0$, we deduce that for all $j \in [1..2k]$, $\mathbf{v}_j(n+i) < \mathbf{v}_{j+1}(n+i)$. Using the fact that the relation $\sim$ is a bisimulation between $TS(\widetilde{S})$ and $TS_0(S)$ and by lemma 15, we deduce that $(S, (q, \mathbf{v}))$ is not k-reversal-b-bounded, which is a contradiction.

We will now suppose that for all $i \in [1..n]$, for all nodes $(q, \mathbf{w})$ belonging to an elementary circuit $((q_1, \mathbf{w}_1) \xrightarrow{t_1} (q_2, \mathbf{w}_2) \xrightarrow{t_2} \ldots (q_f, \mathbf{w}_f) \xrightarrow{t_f} (q_1, \mathbf{w}_1))$ of $CG(\widetilde{S}, \tilde{c})$ with $D_{t_1 \ldots t_f}(n+i) > 0$, we have $\mathbf{w}(i) \leq b$. Once again, the proof is *ad absurdum*. Assume that $(S, (q, \mathbf{v}))$ is not reversal-b-bounded. Let N be the number of nodes in $CG(\widetilde{S}, \tilde{c})$. Using lemma 16 and the fact that $\sim$ is a bisimulation between $TS(\widetilde{S})$ and $TS_0(S)$, we deduce that there exist $i \in \mathbb{N}$ and $(q_1, \mathbf{v}_1), \ldots, (q_{N+1}, \mathbf{v}_{N+1}) \in Q \times \mathbb{N}^{2n}$ and $\sigma_1, \ldots \sigma_N$ such that we have in $TS(\widetilde{S})$ $\tilde{c} \rightarrow^* (q_1, \mathbf{v}_1) \xrightarrow{\sigma_1} \ldots \xrightarrow{\sigma_N} (q_{N+1}, \mathbf{v}_{N+1})$ and for all $j \in [1..N+1]$, $b < \mathbf{v}_j(i)$ and for all $j \in [1..N]$, $\mathbf{v}_j(n+i) < \mathbf{v}_{j+1}(n+i)$. First, from this very last point we can immediatly deduce, by definition of D, that for all $1 \leq j \leq N$, $D_{\sigma_j}(n+i) > 0$. Second, from theorem 10, we can say, that there exist $(q_1.\mathbf{w}_1), \ldots, (q_{N+1}, \mathbf{w}_{N+1})$ nodes in $CG(\widetilde{S}, \tilde{c})$ such that we have in $CG(\widetilde{S}, \tilde{c})$, $(q_1, \mathbf{w}_1) \xrightarrow{\sigma_1} \ldots \xrightarrow{\sigma_N} (q_{N+1}, \mathbf{w}_{N+1})$ and for all $j \in [1..N+1]$,$\mathbf{v}_j \leq \mathbf{w}_j$. Since, N is the number of nodes in $CG(\widetilde{S}, \tilde{c})$, we deduce that there exist $(q'_1.\mathbf{w}'_1), \ldots, (q'_f, \mathbf{w}'_f)$ nodes in $CG(\widetilde{S}, \tilde{c})$ and $t_1, \ldots, t_f \in T_n$ such that : $((q'_1, \mathbf{w}'_1) \xrightarrow{t_1} (q'_2, \mathbf{w}'_2) \xrightarrow{t_2} \ldots (q'_f, \mathbf{w}'_f) \xrightarrow{t_f} (q'_1, \mathbf{w}'_1))$ is a circuit in $CG(\widetilde{S}, \tilde{c})$ and $D_{t_1 \ldots t_f}(n+i) > 0$ and there exist $j \in [1..N]$ such that $(q'_j, \mathbf{w}'_j) = (q_j, \mathbf{w}_j)$. We recall that for all the guarded translation t appearing in the $2n$-dim VASS $\widetilde{S}$, for all $i \in [1..n]$, $D_t(n+i) \geq 0$ (by construction of $\widetilde{S}$). We now take the smallest r and the biggest s such that $1 \leq r \leq s \leq f$ and $D_{t_r}(n+i) > 0$ and $D_{t_s}(n+i) > 0$. Since $D_{t_1 \ldots t_f}(n+i) > 0$ and by the previous recall, these integers r and s necessarly exist. We have then in $CG(\widetilde{S}, \tilde{c})$ the circuit $(q'_r, \mathbf{w}'_r) \xrightarrow{t_r} (q', \mathbf{w}') \xrightarrow{\sigma'_1} (q'_1.\mathbf{w}'_1) \xrightarrow{\sigma'_2} (q'_s.\mathbf{w}'_s) \xrightarrow{t_s} (q'', \mathbf{w}'') \xrightarrow{\sigma'_3} (q'_r, \mathbf{w}'_r)$, where $(q', \mathbf{w}'), (q'', \mathbf{w}'') \in \{(q'_1, \mathbf{w}'_1), \ldots, (q'_f, \mathbf{w}'_f)\}$ and $\sigma'_1, \sigma'_2, \sigma'_3 \in T_n^*$. By definition of r and s, we have that $D_{\sigma'_1} = D_{\sigma'_2} = 0$. Since $(q'_r, \mathbf{w}'_r) \xrightarrow{t_r} (q', \mathbf{w}')$ belongs to a circuit, it belongs to an elementary circuit, and since $D_{t_r}(n+i) > 0$, by hypothesis $\mathbf{w}'_r(i) \leq b$ and $\mathbf{w}'(i) \leq b$. For the same reason, we also have $\mathbf{w}'_s(i) \leq b$ and $\mathbf{w}''(i) \leq b$. By definition of $\mathbf{w}'_1$, there exists $j \in [1..N]$ such that $\mathbf{w}'_1 = \mathbf{w}_j$ and as we have pointed out, we have $b < \mathbf{v}_j(i) \leq \mathbf{w}_j(i)$, consequently $b < \mathbf{w}'_1(i)$. If we summarize, we have in $CG(\widetilde{S}, \tilde{c})$, a circuit $(q'_r, \mathbf{w}'_r) \xrightarrow{t_r} (q', \mathbf{w}') \xrightarrow{\sigma'_1} (q'_1.\mathbf{w}'_1) \xrightarrow{\sigma'_2} (q'_s.\mathbf{w}'_s) \xrightarrow{t_s} (q'', \mathbf{w}'') \xrightarrow{\sigma'_3} (q'_r, \mathbf{w}'_r)$ such that :

(i) $\mathbf{w}'(i) \leq b$, $\mathbf{w}'_s(i) \leq b$, $b < \mathbf{w}'_1(i)$ and
(ii) $D_{\sigma'_1} = D_{\sigma'_2} = 0$.

We recall that in S_0 the $n+i$-th counter counts the alternation between the increasing and decreasing mode of the i-th counter. We can deduce that it would be possible to build a run of S_0 from c_0 which passes by a configuration where the i-th counter is smaller than b, then by a configuration where the i-th counter is strictly greater than b and finally by a configuration where the i-th counter is smaller than b, without that the $n+i$-th counter to change, which is a contradiction. We conclude by saying that $(S, (q, \mathbf{v}))$ has to be reversal-b-bounded. $\square$

For a VASS (S, c), the lemma 13 gives us a necessary and sufficient condition over the coverability graph of $(\widetilde{S}, \widetilde{c})$, and this condition can effectively be tested. This allows us to deduce the following decidability result.

Theorem 17. *Verifying if a VASS is reversal-bounded is decidable.*

Unfortunately, the decision algorithm we propose here builds entirely the coverability graph of a VASS, and this building is known to be non-primitive-recursive in space (some details can be found in [11]).

6 Perspectives

In [5], the authors have proved that some liveness problems are decidable for reversal-0-bounded counter machines and others not. For instance, it is decidable to verify if a run of a reversal-bounded counter machine passes infinitely often through a semilinear set of possible configurations; but the same problem becomes undecidable when all the runs are considered. It seems that this result can easily be extended to the class of reversal-bounded counter machines, we have introduced. It would then pave the way to verify more complex properties than reachability over reversal-bounded counter machines. It could also be interesting to look at these liveness problems in the particular case of reversal-bounded VASS.

An other perspective for our work would be to use reversal-bounded counter machines to analyze counter machines which are not necessarily reversal-bounded. In fact, we have seen with the proof of theorem 7, that for any $k, b \in \mathbb{N}$ and from any counter machine, it is possible to build another counter machine, which is k-reversal-b-bounded and whose runs represent an under-approximation of the set of runs of the first one. We could consequently build a tool which given a counter machine would build successively, incrementing the parameters k and b, the corresponding k-reversal-b-bounded counter machines, and would test at each step if the reachability set of the initial counter machine has been built (this can be easily done, since this set is a fixpoint of the reflexive and transitive closure of the transition relation). This algorithm might never terminate, if the reachability set is not semilinear for instance, but it will refine at each step the under-approximation of the reachability set.

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