The complexity of approximate Nash equilibrium in congestion games with negative delays ⋆

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Abstract. We extend the study of the complexity of computing an ε-approximate Nash equilibrium in symmetric congestion games from the case of positive delay functions to delays of arbitrary sign. Our results show that with this extension the complexity has a richer structure, and it depends on the exact nature of the signs allowed. We first prove that in symmetric games with increasing delay functions and with α-bounded jump the ε-Nash dynamic converges in polynomial time when all delays are negative, similarly to the case of positive delays. We are able to extend this result to monotone delay functions. We then establish a hardness result for symmetric games with increasing delay functions and with α-bounded jump when the delays can be both positive and negative: in that case computing an ε-Nash equilibrium becomes PLS-complete, even if each delay function is of constant sign or of constant absolute value.

1 Introduction

Congestion games were introduced by Rosenthal [19] to model shared resources by selfish players. In these games the strategies of each player correspond to some collection of subsets of a given set of common resources. The cost of a strategy is the sum of the costs of the selected resources, where the cost of a particular resource depends on the number of players having chosen this resource. This dependence is described in the specification of the game by some integer valued delay function for each resource.

Congestion games can describe several interesting routing and resource allocation scenarios in networks. More importantly from a game theoretic perspective, they have some particularly attractive properties. Rosenthal has proven

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that they belong to the class of potential games where, for each player, an improvement (decrease) in his cost is reflected by an improvement in a global function, the potential function. This implies, in particular, that congestion games always have a pure Nash equilibrium. More precisely, a Nash equilibrium can be reached by the so called Nash dynamics, in which an unsatisfied player switches his strategy to a better one, which decreases his cost function. Since the same improvement is mirrored in the potential function, which can not be decreased infinitely, this process indeed has to converge to an equilibrium in a finite number of steps. In an exact potential game the changes in the individual cost functions and the potential function are not only identical in sign, but also in the exact value. Monderer and Shapley [17] have proved that congestion games and exact potential games are equivalent.

The existence of a potential function for congestion games allows us to cast searching for a Nash equilibrium as a local search problem. The states, that is the strategy profiles of the players, are the feasible solutions, and the neighborhood of a state consists of all authorized changes in the strategy of a single player. Then local optima correspond to states where no player can improve individually his cost, that is exactly to Nash equilibria. The potential of a state can be evaluated in polynomial time, and similarly a neighboring state of lower potential can be exhibited, provided that there exists one. This means that the problem of computing a Nash equilibrium in a congestion game belongs to the complexity class PLS, Polynomial Local Search, defined in [13, 18]. The class PLS is a subclass of TFNP [16], the family of NP search problems for which a solution is guaranteed to exist. While PLS is not harder than $\mathsf{NP} \cap \mathsf{coNP}$, it is widely believed to be computationally intractable. It contains several complete problems, such as the weighted 2SAT-FLIP, where one looks for a truth assignment that maximizes the sum of the weights of satisfied clauses among all assignments of Hamming distance one. Fabrikant, Papadimitriou and Talwar [10] have shown that computing a Nash equilibrium in congestion games is also PLS-complete. In addition, they have explicitly constructed games in which the Nash dynamics takes exponential time to converge. It is worth to note that it is also highly unlikely that computing a mixed Nash equilibrium in general games is feasible in polynomial time, even when the number of players is restricted to two [9, 4].

It is therefore natural to look for relaxed versions, and in particular approximations, of Nash equilibria which might be computed in polynomial time. Approximate Nash equilibria of various games have been defined and studied both in the additive [14, 15, 5, 7, 8, 11, 21] and in the multiplicative models of approximation [6, 2]. Here we consider multiplicative ε-approximate Nash equilibria, for $0 < \varepsilon < 1$, that is states where no single player can improve his cost by more than a factor of ε by unilaterally changing his strategy. In this context, the analogous concept of the Nash dynamics is the ε-Nash dynamics, where only ε-moves are permitted, which improve the respective player's cost at least by a factor of ε. Rosenthal's potential function arguments imply again that the ε-Nash dynamics converges to an ε-Nash equilibrium.

In a very interesting positive result, Chien and Sinclair [6] proved that in congestion games with four specific constraints the ε-Nash dynamics indeed does converge fast, in polynomial time. The four constraints require the game to be increasing, positive, symmetric, and with α-bounded jump. The first three constraints are rather standard. A congestion game is increasing (respectively positive) if of all delay functions are non-decreasing (respectively non-negative). It is symmetric if all players have the same strategies. The last constraint puts a limit on the speed of growth of the delay functions. They define an increasing and positive congestion game to be with α-bounded jump, for some $\alpha \geq 1$, if the delay functions can not grow more than a factor α when their argument is increased by one. Their result states that in increasing, positive and symmetric congestion games with α-bounded jump, the ε-Nash dynamics converges in polynomial time in the input length, α and $1/\varepsilon$.

Could it be that the ε-Nash dynamic converges fast in every congestion game? Skopalik and Vöcking have found a very strong evidence for the contrary. In a negative result [20], they proved that for every polynomial time computable $0 < \varepsilon < 1$, computing an ε-approximate Nash equilibrium is PLS-complete, that is just as hard as computing a Nash equilibrium. In fact, they result is even stronger, it shows the PLS-completeness of the problem for increasing positive games.

In this paper we extend these studies to the case when the delays can be also negative, that is some resources might have the special status of improving the cost of the players when they are chosen. We consider negative games where the delay functions may be either increasing or decreasing. We first prove that in negative symmetric games with α-bounded jump, when all delay functions are increasing, the ε-Nash dynamics converges in polynomial time, just as in the case of positive increasing games. We then extend this result to games where all delay functions are monotone, that is either increasing or decreasing. We then prove a hardness result: computing an ε-Nash equilibrium in symmetric and increasing games with α-bounded jump becomes PLS-complete when delay functions of arbitrary sign are allowed. In fact, our result is somewhat stronger: the PLS-completeness holds even when all delay functions are of constant sign or when all the delays are of constant absolute value.

2 Preliminaries and results

2.1 Context

We recall the notions of congestion games, local search problems and approximate Nash equilibrium.

Congestion games For a natural number n, we denote by $[n]$ the set $\{1, \ldots, n\}$. For an integer $n \geq 2$, an *n-player game in normal form* is specified by a set of *(pure) strategies* S_i, and a *cost* function $c_i : S \to \mathbb{Z}$, for each player $i \in [n]$, where

$S = S_1 \times \cdots \times S_n$ is the set of *states*. For $s \in S$, the value $c_i(s)$ is the cost of player i for state s. A game is *symmetric* if $S_1 = \ldots = S_n$.

For a state $s = (s_1, \ldots, , s_n) \in S$, and for a pure strategy $t \in S_i$, we let (s_{-i}, t) to be the state $(s_1, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_n) \in S$. A *pure Nash equilibrium* is a state s such that for all i, and for all pure strategies $t \in S_i$, we have

$$c_i(s) \leq c_i(s_{-i}, t).$$

In general games do not necessarily have a pure Nash-equilibrium.

A specific class of games which always have a pure Nash equilibrium are *congestion games*, where the cost functions are determined by the shared use of resources. More precisely, an n-player congestion game is a 4-tuple $G = (n, E, (d_e)_{e \in E}, (S_i)_{i \in [n]})$, where E is a finite set of *edges* (the common resources), $d_e : [n] \to \mathbb{Z}$ is a *delay function*, for every $e \in E$, and $S_i \subseteq 2^E$ is the set of pure strategies of player i, for $i \in [n]$. Given a state $s = (s_1, \ldots, s_i, \ldots, s_n)$, let the *congestion* of e in s be $f_e(s) = |\{i \in [n] : e \in s_i\}|$. The cost function of user i is defined then as $c_i(s) = \sum_{e \in s_i} d_e(f_e(s))$. Intuitively, each player uses some set of resources, and the cost of each resource e depends on the number of players using it, as described by the delay function. To simplify the notation, we will specify a symmetric congestion game by a 4-tuple $G = (n, E, (d_e)_{e \in E}, Z)$, where by definition the set of pure strategies of every player is $Z \subseteq 2^E$. We will refer to Z as the set of *available* strategies.

A delay function d_e is increasing if for all $t \in [n-1]$, we have $d_e(t) \leq d_e(t+1)$, and it is decreasing if $-d_e$ is increasing. We say that d_e is monotone if its is increasing or decreasing. A congestion game is *increasing* (respectively *decreasing*, *monotone*) if all delay functions are increasing (respectively decreasing, monotone).

That congestion games have indeed a Nash equilibrium can be easily shown by a potential function argument, due to Rosenthal [19], as follows. Let us define the potential function ϕ on the set of states as

$$\phi(s) = \sum_{e \in E} \sum_{t=1}^{f_e(s)} d_e(t).$$

If $s = (s_1, \ldots, s_i, \ldots, s_n)$ and $s' = (s_{-i}, s'_i)$ are two states differing only for player i then $\phi(s) - \phi(s') = c_i(s) - c_i(s')$ since both of these quantities are in fact equal to $\sum_{e \in s_i \setminus s'_i} d_e(f_e(s)) - \sum_{e \in s'_i \setminus s_i} d_e(f_e(s'))$. Therefore, in any state which is not a pure Nash equilibrium, there is always a player that can change unilaterally his strategy so that the induced new state has a smaller potential. In fact the decrease in the cost function and in the potential are identical. This means that a finite sequence of such individual changes, the so-called *Nash dynamics*, necessarily results in a pure Nash equilibrium since the integer valued potential function can not decrease forever. Therefore congestion games can be casted as local search problems, and the computing of a Nash equilibrium can be interpreted as the search of a local optimum.

Local search problems A local search problem is defined by a 4-tuple $\Pi = (\mathcal{I}, F, (v_I)_{I\in\mathcal{I}}, (N_I)_{I\in\mathcal{I}})$, where $\mathcal{I}$ the set of instances, F maps every instance $I \in \mathcal{I}$ to a finite set of feasible solutions $F(I)$, the objective function $v_I : F(I) \to \mathbb{Z}$ gives the value $v_I(S)$ of a feasible solution, and $N_I(S) \subseteq F(I)$ is the neighborhood of $S \in F(I)$. Given an instance I, the goal is to find a feasible solution $S \in F(I)$ such that is also local minimum, that is for all $S' \in N_I(S)$, it satisfies $v_I(S) \leq v_I(S')$. A local search problem is in the class PLS [13, 18] if there exist polynomial algorithms in the instance length to compute: an initial solution S_0; the membership in $F(I)$; the objective value $v_I(S)$; and a feasible solution $S' \in N_I(S)$ such that $v_I(S') < v_I(S)$ whenever S is not a local minimum. Computing a Nash equilibrium of congestion games is then indeed in PLS: Given an instance G, the feasible solutions $F(G)$ are the states S, the value $v_G(s)$ of a state s is its potential $\phi(s)$, and the neighborhood $N_G(s)$ consists of those states which differ in one coordinate from s.

The notion of PLS-reducibility was introduced in [13]. A problem $\Pi = (\mathcal{I}, F, (v_I)_{I\in\mathcal{I}}, (N_I)_{I\in\mathcal{I}})$ is PLS-reducible to $\Pi' = (\mathcal{I}', F', (v'_I)_{I\in\mathcal{I}'}, (N'_I)_{I\in\mathcal{I}'})$ if there exist polynomial time computable functions $f : \mathcal{I} \to \mathcal{I}'$ and $g_I : F(f(I)) \to F(I)$, for $I \in \mathcal{I}$, such that if S' is a local optimum of $f(I)$ then $g_I(S')$ is local optimum of I. Complete problems in PLS are not believed to be solvable by efficient procedures. Therefore, it is highly unlikely that there exists at all a polynomial time algorithm for computing a pure equilibrium in congestion games. Indeed, Fabrikant, Papadimitriou and Talwar [10] have shown that this problem is PLS-complete, even for symmetric games.

Approximate Nash equilibrium Several relaxations of the notion of equilibrium have been considered in the form of approximations. Let $0 < \varepsilon < 1$. In our context ε will be a constant or some polynomial time computable function in the input length. A *multiplicative ε-approximate Nash equilibrium* is a state s such that for all $i \in [n]$, and for all strategies $t \in S_i$, we have

$$c_i(s) - c_i(s_{-i}, t) \leq \varepsilon |c_i(s)|.$$

Given a state s and a strategy $t \in S_i$, we say that (s_{-i}, t) is an *ε-move* for player i if

$$c_i(s) - c_i(s_{-i}, t) > \varepsilon |c_i(s)|.$$

Clearly s is an ε-approximate Nash equilibrium if no player has an ε-move.

The *ε-Nash dynamics* is defined as a sequence of ε-moves, where a player with the **largest absolute gain** makes the change in his strategy, when several players with ε-move are available. Analogously to the exact case, the ε-Nash dynamics converges to an ε-approximate Nash equilibrium. computing an ε-approximate Nash equilibrium is also a problem in PLS. When casting this as a local search, the only difference with the exact equilibrium case is that the neighborhoods are restricted to states which are reachable by an ε-move.

2.2 Related results

In [6] Chien and Sinclair have considered the rate of convergence of the ε-Nash dynamics in symmetric congestion games with three additional restrictions on the delay functions. We say that a delay function d_e is positive if the delays $d_e(t)$ are non-negative integers for all $1 \leq t \leq n$ A congestion game is *positive* if all delay functions are positive. Let $\alpha \geq 1$. A positive and increasing delay function is with α-bounded jump if the delays satisfy $d_e(t+1) \leq \alpha d_e(t)$, for all $t \geq 1$. We can think of α as being a constant, or a polynomial time computable function in the input length of the game. Obviously, a positive delay function with α-bounded jump can never take the value 0. A positive game is *with α-bounded jump* if all delay functions are with α-bounded jump. Chien and Sinclair have shown that in symmetric, positive, increasing games with bounded jump the ε-Nash dynamics converges in polynomial time.

Theorem 1 (Chien and Sinclair [6]).
For every $\alpha \geq 1$ and $0 < \varepsilon < 1$, in n-player symmetric, positive and increasing congestion games with α-bounded jump the ε-Nash dynamics converges from any initial state in $O(n\alpha\varepsilon^{-1}\log(nmD))$ steps, where $m = |E|$, and $D = \max\{d_e(n) : e \in E\}$ is an upper bound on the delay functions.

The hope that the ε-Nash dynamics converges fast in generic congestion games was crushed by Skopalik and Vöcking [20]. In a strongly negative result they proved that computing an ε-approximate Nash equilibrium is PLS-complete, that is just as hard as computing a Nash equilibrium. In fact, they result is even stronger, it shows the PLS-completeness of the problem also in positive increasing games.

Theorem 2 (Skopalik and Vöcking [20]). *For every polynomial time computable $0 < \varepsilon < 1$, computing an ε-approximate Nash equilibrium in a positive and increasing congestion game is* PLS*-complete.*

2.3 Motivations and results

In this paper we mainly study the complexity of computing an ε-approximate Nash equilibrium in congestion games where the delay functions can also have negative values. Our reason to study negative delays is twofold. Firstly, negative delays are motivated by real scenarios worth of investigations. *Profit maximizing games* are defined exactly as congestion games, except that each player tries to maximize its cost. These games are easily seen to be equivalent to congestion games when the delay functions are multiplied by a -1 factor. *Market sharing games* [3, 12], also studied in the context of content distribution in service networks, are specific profit maximizing games, where the delay functions are positive and decreasing as the value of a resource is shared. They are equivalent to congestion games with negative and increasing delay functions. *Market social games*, introduced in section 3.2, generalize market sharing games where the value of some resources, such as Web pages, may increase with the number of players who selected them, whereas some other resources are shared as in market sharing games. They are equivalent to congestion games with negative increasing and decreasing delay functions, that is negative monotone delay functions.

Secondly, games with delays of arbitrary sign seem to have a richer mathematical structure than games with only positive delays. Indeed, we show that in symmetric increasing games with α-bounded jump, while computing an ε-approximate Nash equilibrium is easy when delays can only be negative, just as in positive games, the problem becomes significantly harder when they can be of arbitrary sign.

In Section 3 we deal with symmetric monotone games with α-bounded jump, where all delay functions are negative. In **Theorem 3** we show that in negative increasing games the ε-Nash dynamics converges in polynomial time in the input length, α and $1/\varepsilon$. This result is analogous to the result of Chien and Sinclair [6], stated in Theorem 1, but its proof is significantly harder. In positive games the proof essentially shows that while the equilibrium state is not reached, one can always find a player whose cost is polynomially related to the potential, and who either can make an ε-move himself, or whose cost can lower bound the gain of any player with an ε-move. In negative games it is not always possible to find a player with such a high cost, because we can relate the cost functions to the appropriate potential only when it is restricted to edges with non-trivial congestion. To deal with the unaccounted edges, we show that we can find an edge whose initial delay is polynomially related to the remaining part of the potential. We then make a somewhat subtle case analysis which considers also strategies involving these edges. In **Theorem 4** we extend this result to negative monotone games where each delay functions is increasing or decreasing. These positive results applied then to show that in market sharing games and market social games the Nash dynamics converges in polynomial time.

In Section 4 we expand our investigation to symmetric, increasing games with α-bounded jump, and with arbitrary delay functions. We show that in that case the problem of computing an ε-approximate Nash equilibrium becomes PLS-complete. In fact, we can show this even with some specific restrictions on the signs of the delays. In **Theorem 6** we prove the hardness result when all delay functions are of constant sign, and in **Theorem 7** when delays can change sign, but remain of constant absolute value. For the PLS-reductions we use the PLS-complete problem of Skoplik and Vöcking [20] stated in Theorem 2.

Our results describe rather completely how does the complexity of computing an ε-approximate Nash equilibrium in congestion games depend on the sign of the delay functions. Many open questions remain concerning the dependence on other parameters. Theorem 4 is a first step in the study of the dependance on the monotonicity of the delays. While in non symmetric games the Nash dynamics is known to converge in exponential time [20], the complexity in this case remains wide open.

3 Negative games

We start now the study of computing ε-approximate Nash equilibria in congestion games where the delay functions can take negative values. In this section we impose the restriction that the delay functions have only negative values. We further suppose that the games are symmetric, monotone and α-bounded. We show in a result analogous to Theorem 1 that for any polynomial time computable α and ε, the ε-Nash dynamics converges in polynomial time. We then point out that this result applies to symmetric market sharing and social games.

We say that a delay function d_e is negative if the delays $d_e(t)$ are negative integers for all $1 \leq t \leq n$. A congestion game is *negative* if all delay functions are negative. Let $\alpha \geq 1$. A negative and increasing delay function is with α-bounded jump if the delays satisfy $d_e(t+1) \leq d_e(t)/\alpha$, for all $t \geq 1$. A negative and decreasing delay function d_e is with α-bounded jump if $-d_e$ is with α-bounded jump. A negative and monotone game is *with α-bounded jump* if all delay functions are with α-bounded jump.

We show our positive result first for increasing games, then we generalize it to monotone games.

3.1 Increasing delay functions

Theorem 3. *For every $\alpha \geq 1$ and every $\varepsilon > 0$, in an n-player symmetric, negative, increasing congestion game with α-bounded jump the ε-Nash dynamics converges from any initial state in $O((\alpha n^2 + nm)\varepsilon^{-1} \log(nmD))$ steps where $m = |E|$, and $D = \max\{-d_e(1) : e \in E\}$ is an upper bound on magnitude of the delay functions.*

Proof. We will suppose without loss of generality that every edge appears in some strategy, since otherwise the edge can be discarded from E. We first define a positive potential function which will be appropriate to measure the progress of the ε-Nash dynamics. Let ψ be defined over the states as $\psi(s) = -\sum_{e\in E} \sum_{t=f_e(s)+1}^{n} d_e(t)$. The function ψ is clearly positive, and we claim that it is a potential function, that is $\psi(s) - \psi(s') = c_i(s) - c_i(s')$ if the states s and s' differ only in their ith coordinate. This follows immediately from the fact that for every state s, we have $\psi(s) = \phi(s) - k$, where $\phi(s) = \sum_{e\in E} \sum_{t=1}^{f_e(s)} d_e(t)$ is the Rosenthal potential function, and k is the constant $\sum_{e\in E} \sum_{t=1}^{n} d_e(t)$. Observe that $\psi(s)$ is bounded from above by nmD, for every state s.

For an arbitrary initial state $s^{(0)}$, let $s^{(k)}$ be the kth state of the ε-Nash dynamics process. We claim that $\psi(s^{(k+1)}) \leq \psi(s^{(k)})(1 - \varepsilon/4(\alpha n^2 + nm))$, for every k, which clearly implies the theorem. Suppose that $s^{(k)} = s = (s_1, \ldots, s_n)$ is not an ε-equilibrium, and let i be the player which can make the largest gain ε-move. To prove our claim, we will show that there exists a strategy s'_i for player i such that $c_i(s) - c_i(s_{-i}, s'_i) \geq \varepsilon\psi(s)/4(\alpha n^2 + nm)$, and we observe that an ε-move can only be better for player i than playing strategy s'_i.

The first idea is to try to prove, analogously to the case of positive games, that for some player j, the opposite of its cost $-c_j(s)$ is a polynomial fraction of $\psi(s)$.

Unfortunately this is not necessarily true. The sum $\sum_{j=1}^{n} c_j(s)$ is not necessarily a polynomial fraction of $\psi(s)$ because edges whose congestion is 0 in s do not contribute to the former, but do contribute the latter. Therefore we introduce the function ψ' as ψ restricted to the edges with nontrivial congestion, that is by definition $\psi'(s) = -\sum_{e\in E|f_e(s)\neq 0} \sum_{t=f_e(s)+1}^{n} d_e(t)$. The following Lemma shows that some of the $-c_j(s)$ is at least a polynomial fraction of $\psi'(s)$.

Lemma 1. *There exists a player j such that*

$$-c_j(s) \geq \psi'(s)/n^2.$$

Proof. We claim that

$$-n\sum_{j=1}^{n} c_j(s) \geq \psi'(s),$$

from which the statement clearly follows. To prove the claim we proceed by the following series of (in)equalities:

$$\begin{aligned}
-n\sum_{j=1}^{n} c_j(s) &= -n \sum_{e\in E|f_e(s)\neq 0} f_e(s)\ d_e(f_e(s)) \\
&\geq -n \sum_{e\in E|f_e(s)\neq 0} d_e(f_e(s)) \\
&\geq -\sum_{e\in E|f_e(s)\neq 0} \sum_{t=f_e(s)+1}^{n} d_e(t) \\
&= \psi'(s),
\end{aligned}$$

where the second inequality holds because the delay functions are non-decreasing. □

We fix a value j which satisfies Lemma 1 for the rest of the proof. To upper bound $\psi(s)$, we also have to consider the edges of congestion 0, besides the edges which are accounted for in $\psi'(s)$. We have

$$\psi'(s) - n \sum_{E\in E|f_e(s)=0} d_e(1) \geq \psi(s),$$

again because the delays are non-decreasing. This implies that either $\psi'(s) \geq \psi(s)/2$ or $-n\sum_{e\in E|f_e(s)=0} d_e(1) \geq \psi(s)/2$, and the proof proceeds by distinguishing these two cases.

Case 1: $\psi'(s) \geq \psi(s)/2$. We then reason in two sub-cases by comparing the value of $c_i(s)$ to $\psi'(s)/2\alpha n^2$. If $-c_i(s) \geq \psi'(s)/2\alpha n^2$, then let s_i' be the strategy which makes the biggest gain for player i. Then we have

$$\begin{aligned}
c_i(s) - c_i(s_{-i}, s_i')) &\geq -\varepsilon c_i(s) \\
&\geq \varepsilon\psi(s)/4\alpha n^2,
\end{aligned}$$

where first inequality holds since the move of player i is an ε-move, and the second inequality is true because of the hypotheses. If $-c_i(s) < \psi'(s)/2\alpha n^2$, then let $s'_i = s_j$, the strategy of player j in state s. Observe that s_j is an available strategy for player i since the game is symmetric. Then

$$\begin{aligned} c_i(s) - c_i(s_{-i}, s'_i)) &\geq c_i(s) - c_j(s)/\alpha \\ &\geq \psi'(s)/\alpha n^2 - \psi'(s)/2\alpha n^2 \\ &\geq \psi(s)/4\alpha n^2. \end{aligned}$$

Here the first inequality is true because the game is with α-bounded jump. The second inequality follows from the hypothesis and because $-c_j(s) \geq \psi'(s)/n^2$. Finally, the third inequality holds because $\psi'(s) \geq \psi(s)/2$.

Case 2: $-n \sum_{e \in E | f_e(s)=0} d_e(1) \geq \psi(s)/2$. Then for some edge with $f_e(s) = 0$, we have $-d_e(1) \geq \psi(s)/2nm$. Let's fix such an edge e. We distinguish two sub-cases now by comparing the value of $c_i(s)$ to $d_e(1)/2$. If $c_i(s) \leq d_e(1)/2$ then let s'_i be the strategy which makes the biggest gain for player i. Then, similarly to the first sub-case of Case 1, using the hypotheses and that player i's move is an ε-move, we have

$$\begin{aligned} c_i(s) - c_i(s_{-i}, s'_i)) &\geq -\varepsilon c_i(s) \\ &\geq \varepsilon \psi(s)/4nm. \end{aligned}$$

If $c_i(s) > d_e(1)/2$ then let s'_i be some strategy that contains the edge e. There exists such a strategy since useless edges were discarded from E. Then $f_e(s_{-i}, s'_i)) = 1$ since $f_e(s) = 0$ and s and (s_{-i}, s'_i) differ only for the ith player. This, in turn, implies that $c_i(s_{-i}, s'_i)) \leq d_e(1)$, since the delays are negative. Therefore

$$\begin{aligned} c_i(s) - c_i(s_{-i}, s'_i)) &\geq c_i(s) - d_e(1) \\ &\geq -d_e(1)/2 \\ &\geq \psi(s)/4nm, \end{aligned}$$

where the last two inequalities follow from the hypotheses. □

Market sharing games. In market sharing games [3, 12] n players sell their goods on subsets of m markets $E = \{e_1, \ldots, e_m\}$, and they try to maximize their gains. Each market e has a value $v(e) > 0$. If t sellers choose a market e, they share its value and each earn $v(e)/t$. The gain of player i on a strategy profile $s = (s_1, ..., s_n)$, with $s_i \subseteq E$, is $\sum_{e \in s_i} v(e)/f_e(s)$, where $f_e(s)$ is the number of sellers on the market e. A symmetric market sharing game with available markets strategies $Z \subseteq 2^E$ is a congestion game $(n, E, (d_e)_{e \in E}, Z)$ with delay functions $d_e(t) = -v(e)/t$, which are increasing, negative and with 2-bounded jump.

Corollary 1. *In symmetric market sharing games the ε-Nash dynamics converges in polynomial time.*

3.2 Monotone delay functions

We extend Theorem 3 to monotone congestion games where the resources can be partitioned into two sets: $E^{\uparrow}$ with increasing delay functions and $E^{\downarrow}$ with decreasing delay functions. Notice that if $E^{\uparrow}$ is empty, then the task of finding a Nash equilibrium becomes trivial. Indeed, if the strategy s^* minimizes $\sum_{e \in s} d_e(n)$ over all available strategies, then the state where all players select s^* is an equilibrium.

Theorem 4. *For every $\alpha \geq 1$ and every $\varepsilon > 0$, in an n-player symmetric, negative, monotone congestion game with α-bounded jump the ε-Nash dynamics converges from any initial state in $O((\alpha n^2 + nm)\varepsilon^{-1} \log(nmD))$ steps where $m = |E|$, and $D = \max\{-d_e(t) : e \in E, t \in [n]\}$ is an upper bound on the magnitude of the delay functions.*

Proof. The proof follows closely the case of increasing delay functions. We define

$$\psi(s) = -\sum_{e \in E^{\uparrow}} \sum_{t=f_e(s)+1}^{n} d_e(t) + \sum_{e \in E^{\downarrow}} \sum_{t=1}^{f_e(s)} d_e(t)$$

which is a potential function since it equals $\phi(s) - k$ where ϕ is the classical Rosenthal potential and k is the constant $\sum_{e \in E^{\uparrow}} \sum_{t=1}^{n} d_e(t)$. The potential ψ can be positive or negative (or even 0), but $|\psi(s)|$ is bounded by nmD.

If $s^{(k)}$ denotes the kth state of the ε-Nash dynamics process, then we claim that

$$|\psi(s^{(k)}) - \psi(s^{(k+1)})| \geq \varepsilon |\psi(s^{(k)})| / 4(\alpha n^2 + nm).$$

This implies that the number of steps of the Nash dynamics where $\psi(s^{(k)})$ and $\psi(s^{(k+1)})$ have the same sign is indeed as it is stated, and there can be only one value of k where they have different signs, or two values of k where one of them is zero.

To prove our claim, we will show that if s is not an ε-Nash equilibrium then for the player i which can make the largest gain ε-move, there exists a strategy s_i' such that $c_i(s) - c_i(s_{-i}, s_i') \geq \varepsilon |\psi(s)| / 4(\alpha n^2 + nm)$. We introduce

$$\psi(s) = -\sum_{e \in E^{\uparrow}, f_e(s) \neq 0} \sum_{t=f_e(s)+1}^{n} d_e(t) + \sum_{e \in E^{\downarrow}} \sum_{t=1}^{f_e(s)} d_e(t),$$

which is again not necessarily positive. The following lemma, whose proof is omitted, is analogous to Lemma 1 where $|\psi'|$ plays the role of ψ'.

Lemma 2. *There exists a player j such that*

$$-c_j(s) \geq |\psi'(s)| / n^2.$$

Since by definition

$$\psi'(s) - \sum_{e \in E^{\uparrow} | f_e(s)=0} \sum_{t=1}^{n} d_e(t) = \psi(s),$$

using the negativity of the delays we get the followings

$$|\psi'(s)| - \sum_{e \in E^{\uparrow} | f_e(s)=0} \sum_{t=1}^{n} d_e(t) \geq |\psi(s)|$$

$$|\psi'(s)| - n \sum_{s \in E^{\uparrow} | f_e(s)=0} d_e(1) \geq |\psi(s)|$$

the later is obtain using the fact that the delays of $E^{\uparrow}$ are increasing. Therefore either $|\psi'(s)| \geq |\psi(s)|/2$ or $-n \sum_{s \in E^{\uparrow} | f_e(s)=0} d_e(1) \geq |\psi(s)|/2$, and the rest of the proof is similar to the case analysis of Theorem 3. $\square$

Market social games. Let us call a symmetric *market social game* a congestion game $(n, E, (d_e)_{e \in E}, Z)$ where the market E is partitioned into $E^{\uparrow}, E^{\downarrow}$. Each market $e \in E$ has a value $v(e) > 0$. The delay functions are defined as $d_e(t) = -v(e)/t$ when $e \in E^{\uparrow}$, and $d_e(t) = -t.v(e)$ when $e \in E^{\downarrow}$. The delays are clearly negative increasing on $E^{\uparrow}$ and negative decreasing on $E^{\downarrow}$. They are also with 2-bounded jump. We interpret $f_e(s)$ as the number of sellers on the market e. These games generalize the market sharing games as some resources are shared between the players, whereas some other resources have a value which increases with the number of players.

Corollary 2. *In symmetric market social games the ε-Nash dynamics converges in polynomial time.*

4 Games without sign restriction

In this section we deal with congestion games with no restriction on the sign of the delay functions. Our overall result is that in that case computing an ε-approximate Nash equilibrium is PLS-hard, even when the remaining restrictions of Chien and Sinclair are kept, that is when the game is symmetric, increasing and with α-bounded jump, for $\alpha \geq 1$. Observe that the smaller α the stronger is the hardness result, therefore we deal only with constant α. Our first step is to observe that a simple consequence of Theorem 2 is that computing an ε-approximate Nash equilibrium in positive and increasing games remains PLS-complete even if we additionally suppose that the game is symmetric. Our reductions will use the hardness of this latter problem. The proof of this statement is a PLS-reduction of the search of an ε-approximate Nash equilibrium in positive and increasing games to the same problem in symmetric, positive, increasing games. This reduction is basically identical to the analogous reduction for pure Nash equilibria, due to Fabrikant, Papdimitriou and Talwar [10]. We include here the proof just for the sake of completeness.

Theorem 5. *For every polynomial time computable $0 < \varepsilon < 1$, computing an ε-approximate Nash equilibrium in a symmetric, positive, increasing congestion game is* PLS*-complete.*

Proof. Given a congestion game G with edge set E and strategy sets $S_1, \ldots, S_n$, we map it to the symmetric game G' defined as follows. The edge set of G' is $E \cup \{e_1, \ldots, e_n\}$ where the e_i's are new edges. The set of available common strategies is $\bigcup_{i=1}^{n} S'_i$ where $S'_i = \{s \cup \{e_i\} : s \in S_i\}$. Set $D = \sum_{e \in E} d_e(n)$. The delays of the edges in E don't change, and for every $i \in [n]$, the delay of e_i is defined as

$$d_{e_i}(t) = \begin{cases} 0 & \text{if } t = 1, \\ D/(1-\varepsilon) & \text{if } t \geq 2. \end{cases}$$

Observe that D is an upper bound on the cost of the players in any state of G. Let $s' = (s'_1, ..., s'_n)$ be an ε-approximate Nash equilibrium in G'. For every i, there is necessarily a unique j_i such that $s'_{j_i} \in S_i$. Indeed, if the strategies of several players would belong to the same S_i then any of these players could pick a strategy in which the congestion of the new edge would be 1, and therefore its cost would drop by at least a factor of ε. It is then immediate that the state $s = (s_1, \ldots, s_n)$, where by definition $s_i = s'_{j_i} \setminus \{e_i\}$, is an ε-approximate Nash equilibrium in G. □

We need to discuss now the right notion of α-bounded jump when the jump occurs from a negative to a positive value in the delay function. One possibility could be to require $d_e(t+1) \leq -\alpha d_e(t)$ when $d_e(t) < 0$ and $d_e(t+1) \geq 0$, but there are also other plausible definitions. In fact, we will avoid to give a general definition because it turns out that this is not necessary for our hardness results. Indeed, we will be able to establish a hardness result for congestion games where there is no jump at all around 0, that is for delay functions of constant sign (still some of the delay functions can be negative while some others positive). We say that a congestion game is *non-alternating*, if every delay function is positive or negative. Let $\alpha > 1$ be a constant. A non-alternating congestion game is *with α-bounded jump* if all delay functions are with α-bounded jump.

What happens when $\alpha = 1$? We could also consider non-alternating games with 1-bounded jump, but they are not interesting. If the delays are increasing these are games with constant delay functions for which a pure Nash equilibrium can be determined trivially. Indeed, the cost functions of the individual players are independent from the strategies of the other players, and therefore any choice of a least expensive strategy, for each player, forms a Nash equilibrium.

Nonetheless, if we authorize a jump around 0, then even if the jump changes only the sign without changing the absolute value (which corresponds intuitively to the case $\alpha = 1$ in that situation), the game becomes already hard. We say that a delay function d_e is a *flip function*, if there exists a positive integer c such that for some $1 \leq k \leq n$, the function satisfies:

$$d_e(t) = \begin{cases} -c & \text{if} \quad t < k, \\ c & \text{if} \quad t \geq k. \end{cases}$$

Flip functions are either constant positive functions, or they are simple step functions, which are constant negative up to some point, where an alternation occurs which keeps the absolute value. After the alternation the function remains constant positive. A congestion game is a *flip* game if all delay functions are flip functions. The next two theorems state our hardness results respectively for non-alternating games with α-bounded jump and for flip games.

Theorem 6. *For every constant $\alpha > 1$, and for every polynomial time computable $0 < \varepsilon < 1$, computing an ε-approximate Nash equilibrium in n-player symmetric, non-alternating, increasing congestion games with α-bounded jump is* PLS*-hard.*

Proof. As stated in Theorem 5 computing an ε-approximate Nash equilibrium in a symmetric, positive, increasing congestion game is PLS-complete [20]. We present a PLS-reduction from this problem to the problem of computing an ε-approximate Nash equilibrium in a symmetric, non-alternating, positive game with α-bounded jump.

Let $G = (n, E, (d_e)_{e \in E}, Z)$ a symmetric, positive, increasing congestion game, and let $\alpha > 1$ be a constant. In our reduction we map G to the symmetric game $G' = (n, E', (d_{e'})_{e' \in E'}, Z')$ that we define now. For each $e \in E$, we set $E_e = \{e_1, e_2^+, e_2^-, \ldots, e_n^+, e_n^-\}$, and for every $z \subseteq E$, we define $z' = \bigcup_{e \in z} E_e$ (and therefore $E' = \bigcup_{e \in E} E_e$). The set of available strategies is defined as $Z' = \{z' : z \in Z\}$. Finally the delay functions are defined as follows. The delay d_{e_1} is simply the constant function $d_e(1)$. For $k \geq 2$, we set

$$d_{e_k^+}(t) = \begin{cases} (d_e(k) - d_e(k-1))\frac{\alpha}{\alpha^2-1} & \text{if } t < k, \\ (d_e(k) - d_e(k-1))\frac{\alpha^2}{\alpha^2-1} & \text{if } t \geq k, \end{cases}$$

and

$$d_{e_k^-}(t) = \begin{cases} -(d_e(k) - d_e(k-1))\frac{\alpha}{\alpha^2-1} & \text{if } t < k, \\ -(d_e(k) - d_e(k-1))\frac{1}{\alpha^2-1} & \text{if } t \geq k. \end{cases}$$

The game G' is clearly non-alternating, increasing and with α-bounded jump.

Observe that there is a bijection between the states of G and G'. Indeed, the states of G' are of the form $s' = (s'_1, \ldots, s'_n)$, where $s = (s_1, \ldots, s_n) \in Z^n$ is a state of G. For the reduction we will simply show that if s' is an ε-approximate Nash equilibrium in G' then s is an ε-approximate Nash equilibrium in G (our construction satisfies also the reverse implication). In fact, we show a stronger statement about cost functions: for every state s, and for every player i, the cost of player i for s in G is the same as the cost of player i for s' in G'.

The edges e_k^+ and e_k^- are such that the sum of their delay functions emulates the jump $d_e(k) - d_e(k-1)$ when $t \geq k$. Therefore the sum of the delays corresponding to edges in E_e is just d_e which is expressed in the following lemma.

Lemma 3. *For every edge $e \in E$, and for every $1 \leq t \leq n$,*

$$\sum_{e' \in E_e} d_{e'}(t) = d_e(t).$$

Proof. It is immediate from the definition that the delays satisfy for every $2 \leq k \leq n$, and $1 \leq t \leq n$,

$$d_{e_k^+}(t) + d_{e_k^-}(t) = \begin{cases} 0 & \text{if } t < k, \\ d_e(k) - d_e(k-1) & \text{si } t \geq k. \end{cases}$$

Therefore

$$\begin{aligned} \sum_{e' \in E_e} d_{e'}(t) &= d_e(1) + \sum_{k=2}^{t} (d_{e_k^+}(t) + d_{e_k^-}(t)) \\ &= d_e(1) + \sum_{k=2}^{t} (d_e(k) - d_e(k-1)) \\ &= d_e(t). \end{aligned}$$

□

We now claim the following strong relationship between the cost functions in the two games.

Lemma 4. *For all state $s = (s_1, \ldots, s_n)$ in G, and for every player i, we have*

$$c_i(s') = c_i(s),$$

where $s' = (s'_1, \ldots, s'_n)$.

Proof. It is easy to verify the following sequence of equalities:

$$\begin{aligned} c_i(s') &= \sum_{e' \in s'_i} d_{e'}(f_{e'}(s')) \\ &= \sum_{e \in s_i} \sum_{e' \in E_e} d_{e'}(f_{e'}(s')) \\ &= \sum_{e \in s_i} \sum_{e' \in E_e} d_{e'}(f_e(s)) \\ &= \sum_{e \in s_i} d_e(f_e(s)) \\ &= c_i(s). \end{aligned}$$

Indeed the first and last equalities are just the definitions of the cost functions, and the second one is true by the definition of s'_i. The third equality holds because for every edge $e \in E$, every $e' \in E_e$, and every player i, the strategy s_i contains e if and only if s' contains e', and therefore $f_{e'}(s') = f_e(s)$. The fourth equality follows from Lemma 3.

□

By Lemma 4 we can trivially deduce an ε-approximate Nash equilibrium for G, given an ε-approximate Nash equilibrium for G'. This concludes the proof of the theorem. □

Theorem 7. *For every polynomial time computable* $0 < \varepsilon < 1$*, computing an* ε*-approximate Nash equilibrium in* n*-player symmetric, flip congestion games is* PLS*-hard.*

Proof. The proof is very similar to the proof of the previous theorem. In the reduction the definition of the game G' is as in Theorem 6 except for the delay functions, for $2 \leq k \leq n$, which are defined in the first case as

$$d_{e_k^+}(t) = (d_e(k) - d_e(k-1))/2 \qquad \text{for every } t$$

and

$$d_{e_k^-}(t) = \begin{cases} -(d_e(k) - d_e(k-1))/2 & \text{if } t < k, \\ (d_e(k) - d_e(k-1))/2 & \text{if } t \geq k. \end{cases}$$

These are indeed flip functions, and Lemma 3 holds again. This implies, similarly to Theorem 6, that the reduction is correct. □

5 Conclusion

We studied symmetric congestion games with delay functions of arbitrary sign. We proved that for monotone negative delay functions with the α-bounded jump property, the ε-Nash dynamics converges in polynomial time. In the general case when the delay functions can be positive or negative, finding en ε-Nash equilibrium becomes PLS-complete. We leave the following open questions:

Are negative games as hard as positive games without bounded jump assumption extending results from [20] to negative games? Can we extend theorem 4 on the convergence of monotone negative games to monotone positive games? More generally, what is the exact situation of the complexity of approximate Nash equilibrium in congestion games with non monotone delay functions?

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