Canonical Sequent Proofs via Multi-Focusing

Kaustuv Chaudhuri[1], Dale Miller[2], and Alexis Saurin[2]

[1] INRIA Saclay - Île-de-France, kaustuv.chaudhuri@inria.fr
[2] INRIA Saclay - Île-de-France & LIX, École Polytechnique,
{dale.miller,alexis.saurin}@inria.fr

Summary. The sequent calculus admits many proofs of the same conclusion that differ from each other only by trivial permutations of inference rules. Those interested in the identity of proofs and the elimination of "bureaucracy" from proofs usually switch to proof nets or natural deduction formalisms in order to capture equivalence-under-permutations of sequent calculus proofs via more abstract and geometric structures. In this paper we recover canonicity directly in the sequent calculus by restricting Andreoli's focused proofs, a normal form of cut-free sequent calculus proofs, to that of *maximally multi-focused* proofs. We validate this definition by proving a bijection to the well-known proof-nets for MLL and discussing the possibility of a similar correspondence for the still open problem of proof-nets for MALL with units.

1 Introduction

Sequent calculus proofs are much less proof objects than they are traces of the computation of an actual proof. In particular, sequent calculus inference rules are minute and there are many choices in the order of their application that seem equivalent although, formally, they result in different sequent calculus proofs. One way to get a more abstract notion of proof is to declare that two cut-free proofs are *equivalent* if it is possible to permute the inference rules in one to get the other. Such equivalence classes are unsatisfactory for at least two reasons. First, computing permutations of inference rules might require examining and reorganizing arbitrary parts of a proof: attempting to move a given inference rule to the bottom of a proof could cause changes to many parts of the proof. Second, since equivalence clauses are not, themselves, inductive structures, familiar arguments involving inductive reasoning over proof structures cannot be applied easily to equivalence classes. Many people working in proof theory and, particularly, the problem of the *identity of proofs* discard sequent calculus proofs for more abstract proof structures like natural deduction proofs or proof nets. In these later objects, a more geometric structure of proofs requires less sequentialization of inference rule and allows one to work on proofs more abstractly.

We shall argue in this paper that one does not need to discard sequent calculus proofs in order to factor out many of these irrelevant sequentializations in their inference rules. We shall show that there are, in fact, normal forms of sequent calculus proofs that provide unique representatives of equivalence classes of sequent calculus proofs. To be concrete, we shall assume that we are using sequent calculus proofs over multiplicative, additive linear logic (MALL) (including units and literals). Motivating the selection of such representatives goes roughly as follows. A first step is to consider only *focused proofs* [2], with their alternation of negative (invertible) and positive (focused) phases. Focusing proofs systems can be used to distinguish between *micro* rules, *i.e.*, introduction rules in the sequent calculus, from the *macro* rules that correspond to the introduction of *synthetic connectives* [4] that correspond to an entire focusing phase. A first abstraction is then to consider proofs as built up from macro rules introducing synthetic connectives. Unfortunately, this layer of abstraction does not yield canonical representatives of equivalence classes since the selection of foci is still sequentialized even when they can be selected in *parallel*. Such parallelism can be captured by the addition of the *multi-focus* rule that permits focusing on several formulas within one phase. If we then require that such multi-focus inference rules select a "maximal focus" then, as we show in Section 4, we have achieved canonical representatives of equivalence classes of proofs.

Proof nets for MLL and MALL have been used also as abstraction of the class of cut-free proofs under the equivalence of permuting inference rules. We thus show that maximally multi-focused sequent proofs (modulo the weak "iso-polar" equivalence) are in one-to-one correspondence with MLL proof nets [6]: we show how to uniquely associate a maximally multi-focused proof to a MLL proof net. We also discuss MALL (without units) proof nets [9] and for other fragments of linear logic: using maximal multi-focusing proofs should also be applicable in various other richer logics where the nature of proof nets is less well developed or satisfying, such as linear logic with units and exponentials.

The paper is organized as follows: in Sec. 2 we recall the sequent calculus for MALL. In Sec. 3 we present our multi-focal generalization of Andreoli's focusing calculus. In Sec. 4 we define the notion of maximally multi-focus proof and prove the key canonicity result (Theorem 16). In Sec. 5 we establish the bijection between maximally multi-focused proofs and proof-nets for MLL without units.

2 Sequent calculus proofs for MALL

MALL propositions are written using majuscule schema variables ($A, B, \ldots$) with the following grammar:

$$A, B, \ldots ::= a \mid a^\perp \mid A \otimes B \mid \mathbf{1} \mid A \invamp B \mid \perp \mid A \,\&\, B \mid \top \mid A \oplus B \mid \mathbf{0}$$

A *literal* is either an atomic proposition, written using minuscule scheme variables ($a, b, \ldots$), or it is a negated atom ($a^\perp, b^\perp, \ldots$). As usual, MALL propositions are assumed to be in negation-normal form, and the pairs ($\otimes, \invamp$), ($\mathbf{1}, \perp$), ($\&, \oplus$), and ($\top, \mathbf{0}$) are de Morgan duals, *i.e.*, $(A \otimes B)^\perp = A^\perp \invamp B^\perp$, *etc*. The sequent calculus for MALL uses one-sided sequents of the form $\vdash \Gamma$, where the context Γ is a multiset of propositions. Figure 1 contains the standard proof rules for such sequents [6].

$$\dfrac{}{\vdash a, a^{\perp}}\ \mathrm{I} \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \dfrac{}{\vdash \mathbf{1}}\ \mathbf{1} \qquad \dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \invamp B}\ \invamp \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \dfrac{}{\vdash \Gamma, \top}\ \top \qquad \dfrac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2}\ \oplus$$

Fig. 1. Sequent calculus for MALL. In the $\oplus$ rule, $i \in \{1, 2\}$.

Script majuscule letters $\mathcal{D}, \mathcal{E}, \ldots$ are used to denote proofs and the expression $\mathcal{D} \vdash \Gamma$ signifies that $\mathcal{D}$ is a proof of $\vdash \Gamma$. It is well-known that the following cut and initial rules are admissible.

$$\dfrac{\vdash \Gamma, A \qquad \Delta, A^{\perp}}{\vdash \Gamma, \Delta} \quad \text{and} \quad \dfrac{}{\vdash A, A^{\perp}}\ ,$$

Local permutations of inference rules forms a natural relation between cut-free proofs [10]. For example, in a proof of the form

$$\dfrac{\mathcal{D} \vdash \Gamma, A \qquad \dfrac{\mathcal{E} \vdash \Delta, B, C \qquad \mathcal{F} \vdash \Delta, B, D}{\vdash \Delta, B, C \,\&\, D}\ \&}{\vdash \Gamma, \Delta, A \otimes B, C \,\&\, D}\ \otimes, \tag{1}$$

the order of the $\otimes$ and & rules may be locally switched to yield the proof

$$\dfrac{\dfrac{\mathcal{D} \vdash \Gamma, A \qquad \mathcal{E} \vdash \Delta, B, C}{\vdash \Gamma, \Delta, A \otimes B, C}\ \otimes \qquad \dfrac{\mathcal{D} \vdash \Gamma, A \qquad \mathcal{F} \vdash \Delta, B, D}{\vdash \Gamma, \Delta, A \otimes B, D}\ \otimes}{\vdash \Gamma, \Delta, A \otimes B, C \,\&\, D}\ \&. \tag{2}$$

This switching causes the proof $\mathcal{D}$ to be duplicated in (2), but does not alter the constituent sub-proofs $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$. We denote a site of a local permutation, *i.e.*, a pair of neighboring inference rules r_1 followed by r_2 as r_1/r_2; for example, (1) ends with a $\&/\otimes$ along the right branch of the final rule.

Consider, instead, the following proof figures.

$$\dfrac{\mathcal{D} \vdash \Gamma, A \qquad \dfrac{}{\vdash \Delta, B, \top}\ \top}{\vdash \Gamma, \Delta, A \otimes B, \top}\ \otimes \qquad\qquad \dfrac{}{\vdash \Gamma, \Delta, A \otimes B, \top}\ \top$$

Moving from left-to-right can be seen as moving the $\top$ inference rule below the $\otimes$ rule: in the process the entire proof $\mathcal{D}$ is deleted. Since we wish to establish an equivalence based on permutations, moving from right-to-left can be seen as "creating" the proof $\mathcal{D}$. While deletion of proofs can be seen as problematic when one is attempting to capture the "essence" of proofs, creation is certainly problematic in this sense. Thus, we introduce the following restriction on permutations to avoid this kind of proof creation within equivalent proofs.

Definition 1 *Two proofs $\mathcal{D}$ and $\mathcal{E}$ of the sequent $\vdash \Gamma$ are* iso-initial, *written $\mathcal{D} \simeq \mathcal{E}$, if each can be rewritten to the other using local permutations* and *the set of initial sequents in both $\mathcal{D}$ and $\mathcal{E}$ are the same. The sets under consideration are of pairs of formula occurrence.*

The additional restriction on the sets of initial sequents allows the deletion and creation of subproofs during permutation only when such proofs are without initial rules. For the $\top$-free fragment of MALL, this restriction is trivial, as all permutations preserve the set of initial sequents. However, because $\top$ can arbitrarily rewrite a branch of a proof, allowing all permutations of it identifies too many proofs. The restriction is further motivated by the observation from proof nets that the axiom links (which correspond to the initial sequents) contain essential dynamics of a proof that that should not be suppressed simply by trivial permutations. Note that because we don't allow all permutations of $\top$, we are decidedly not equating all proofs that are equated in the standard categorical model of MALL (*i.e.*, $\top$ is no longer a terminal object in a suitable $\star$-autonomous category with a Cartesian structure for the additives).

3 Multi-focusing for MALL

The proposition of MALL can be classified, based on their permutative affinities or *polarity*, into the following two classes.

$$
\begin{array}{lrcl}
\textit{(positive)} & P, Q, \ldots & ::= & a \mid A \otimes B \mid \mathbf{1} \mid A \oplus B \mid \mathbf{0} \\
\textit{(negative)} & N, M, \ldots & ::= & a^{\perp} \mid A \,\&\, B \mid \top \mid A \parr B \mid \perp
\end{array}
$$

A logical rule that applies to a positive (resp. negative) proposition will henceforth be called a positive (resp. negative) rule. If r_1 is a positive rule, and r_2 is a negative rule, then r_1/r_2 is an instance of the local permutation class pos/neg; similarly for pos/pos, neg/neg, and neg/pos. All pos/pos and neg/neg permutations are valid. Furthermore, neg/pos permutations are also valid since the negative rules are invertible and, hence, may be applied arbitrarily early (reading bottom-up). From a proof-search perspective, the negative rules are, therefore, *asynchronous* since their application does not depend on the structure of the side contexts. The positive rules, on the other hand, are non-invertible and, therefore, *synchronous*: their application depends on the structure of the remaining context and the sequence of rules that have been applied lower in the proof.

Andreoli [2] presented a *focused* proof system (for all of first-order linear logic) in which proofs have two phases. When reading proofs from the root to the leaves, a *focal* phase begins by granting focus to a positive proposition from the available positive propositions: this focus can be indicated explicitly in the sequents by writing them as as $\vdash \Gamma \Downarrow A$ where A is under focus. Once the focused proposition becomes negative, *i.e.*, the sequent is of the form $\vdash \Gamma \Downarrow N$, the focus is *released* and the search enters the negative (asynchronous) phase where the negative connectives are eagerly decomposed; this phase is indicated in sequents of the form $\vdash \Gamma \Uparrow \Delta$. That this phase separation produces a complete strategy, *i.e.*, that every provable sequent has a focused proof, is non-trivial and requires a careful analysis [2, 12].

In this paper, we generalize Andreoli's observation further in the following way: when deciding to focus, we may focus on more than one positive proposition at a time, *i.e.*, our positive sequents are now of the form $\vdash \Gamma \Downarrow \Delta$ (with Δ non-empty).

$$\frac{\vdash \Gamma_1 \Downarrow A, \Delta_1 \qquad \vdash \Gamma_2 \Downarrow B, \Delta_2}{\vdash \Gamma_1, \Gamma_2 \Downarrow A \otimes B, \Delta_1, \Delta_2}\ [\otimes] \qquad \frac{}{\vdash \cdot \Downarrow \mathbf{1}}\ [\mathbf{1}] \qquad \frac{\vdash \Gamma \Downarrow A_i, \Delta}{\vdash \Gamma \Downarrow A_1 \oplus A_2, \Delta}\ [\oplus]$$

$$\frac{\vdash \Gamma \Uparrow A, \Delta \qquad \vdash \Gamma \Uparrow B, \Delta}{\vdash \Gamma \Uparrow A \,\&\, B, \Delta}\ [\&] \qquad \frac{}{\vdash \Gamma \Uparrow \top, \Delta}\ [\top] \qquad \frac{\vdash \Gamma \Uparrow A, B, \Delta}{\vdash \Gamma \Uparrow A \parr B, \Delta}\ [\parr] \qquad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Uparrow \bot, \Delta}\ [\bot]$$

$$\frac{}{\vdash a^\perp \Downarrow a}\ [\mathrm{I}] \qquad \frac{\vdash \Gamma \Downarrow \Delta}{\vdash \Gamma, \Delta \Uparrow \cdot}\ [\mathrm{MF}] \qquad \frac{\vdash \Gamma, A \Uparrow \Delta}{\vdash \Gamma \Uparrow A, \Delta}\ [\mathrm{R}\Uparrow] \qquad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Downarrow \Delta}\ [\mathrm{R}\Downarrow]$$

Fig. 2. Multi-focusing sequent calculus, MF. The contexts on the left of $\Downarrow$ and $\Uparrow$ contain only positive propositions or negated atoms. In the [MF] rule, Δ contains at least one positive proposition. In the [R$\Uparrow$] rule, A is positive or a negated atom. In the [R$\Downarrow$] rule, Δ is all negative. In [$\oplus$], $i \in \{1, 2\}$.

All the propositions under focus are decomposed until only negative formulas remain, at which time, the focus is released and the remaining negative formulas are decomposed in the negative phase. This calculus of *multi-focal* proofs is presented in Figure 2.

Definition 2 *If $\mathcal{D} \vdash \Gamma \Uparrow \Delta$ or $\mathcal{D} \vdash \Gamma \Downarrow \Delta$, then we define $\lfloor \mathcal{D} \rfloor$ to be a proof of $\vdash \Gamma, \Delta$ that replaces every sequent of the form $\vdash \Gamma' \Uparrow \Delta'$ or $\vdash \Gamma' \Downarrow \Delta'$ in $\mathcal{D}$ with $\vdash \Gamma', \Delta'$, elides all instances of* [R$\Uparrow$], [R$\Downarrow$] *and* [MF]*, and renames all other rules to their unbracketed forms (*[$\otimes$] *to* $\otimes$*, etc).*

Theorem 3 (Correctness of multi-focusing)

1. *If $\mathcal{D} \vdash \Gamma \Downarrow \Delta$ or if $\mathcal{D} \vdash \Gamma \Uparrow \Delta$, then $\lfloor \mathcal{D} \rfloor \vdash \Gamma, \Delta$ (soundness).*
2. *If $\vdash \Gamma$, then $\vdash \cdot \Uparrow \Gamma$ (completeness).*

Proof. Soundness is immediate. Completeness follows immediately from the completeness of singly-focused proofs [2, 12]. By a *singly-focused* proof, we mean a proof in MF in which the context Δ in the [MF] rule is always a singleton. Note that the proof in [2] is for full first-order, multiplicative-additive-exponential linear logic, of which our propositional MALL is a simple fragment. □

The following equivalence of proofs is used to define our most primitive equivalence on cut-free proofs.

Definition 4 *Two cut-free proofs $\mathcal{D}$ and $\mathcal{D}'$ of the same sequent are* iso-polar*, written $\mathcal{D} \approx \mathcal{D}'$, if they are equal up to permutation restricted to the pos/pos and neg/neg type.*

Thus, two proofs are iso-polar if they different from each other if they only differ from each other up to permutations "within a phase": that is, the macro rules are the same but possible permutations available to construct that macro rule from micro rules is ignored. This relation seems rather natural since the interchange of the pos/pos and neg/neg inference rules are truly parallel and non-interacting. If one insists, however, that we pick a single representative of $\approx$-classes, we can do so using a technique from Andreoli's original focusing proof system [2]. In particular, consider the structure to the right of both $\Uparrow$ and $\Downarrow$ as *lists* instead of multisets: when a multi-focus is selected, sort the elements in that focus using a fixed but arbitrary ordering and then process the members of right-hand side in the order provided by the lists.

4 Maximality and canonicity

We now revisit the question of permutation, this time asking it of the synthetic connectives. In the unfocused calculus, it is easy to see that the synthetic rule for a negative synthetic connective, which is a sequence of negative rules for the constituents of the synthetic connective, permutes with that of another synthetic negative connective: it is a simple matter of sequencing permutations. Similarly, the positive synthetic rules commute with other positive synthetic rules, and likewise for a neg/pos permutation of synthetic rules. As before, the only disallowed permutations in general are the pos/neg permutations.

Definition 5 *Suppose* $\mathcal{D} = \dfrac{\mathcal{D}' \vdash \Gamma \Downarrow \Delta}{\vdash \Gamma, \Delta \Uparrow \cdot}$ [MF]. *Then,* Δ' *are called the* roots *of* $\mathcal{D}$, *written* $\mathrm{roots}(\mathcal{D})$.

Definition 6 *Two proofs* $\mathcal{D}$ *and* $\mathcal{E}$ *of* $\vdash \Gamma \Updownarrow \Delta$ *are said to be* iso-polar, *written* $\mathcal{D} \approx \mathcal{E}$, *if* $\mathcal{D} \simeq \mathcal{E}$ *and they have the same set of instances of* [MF].

We intend to show that every member of an isoinitial class of proofs of $\vdash \Gamma$ is equivalent to a unique proof (upto iso-polarity) of $\vdash \cdot \Uparrow \Gamma$. In fact, we shall call these representatives of the isonitial equivalence class the *maximally multi-focused proofs*.

Definition 7 *A proof* $\mathcal{D}$ *of* $\vdash \Gamma \Updownarrow \Delta$ *is* maximal *if for every sub-proof* $\mathcal{E} \vdash \Gamma' \Uparrow \cdot$ *of* $\mathcal{D}$, *it is the case for any* $\mathcal{E}' \simeq \mathcal{E} \vdash \Gamma' \Uparrow \cdot$ *that* $\mathrm{roots}(\mathcal{E}') \subseteq \mathrm{roots}(\mathcal{E})$.

We obtain maximal proofs by considering permutations of entire synthetic connectives. Consider two neighbouring bipoles. If the positive phase of the top bipole permutes with the negative phase of the bottom bipole, then in an unfocused form we can perform the permutation and merge the two bipoles by uniting their positive and negative phases, obtaining another (multi-)focused proof. The MF rules are, of course, too rigid to express any but the final points of the permutation. Thus, in this section we shall consider a comparitively more relaxed focusing calculus where a negative phase (of the bottom bipole) can be "carried through" the positive phase (of the top bipole). The bottom negative phase is first (temporarily) pre-empted by the top positive phase; for this, we use sequents of the form $\vdash \Gamma \downarrow \Delta \,;\, \Xi$ where Δ is under focus, and Ξ is a suspended context. Later, when the positive phase has permuted down, the negative phases are awakened into active sequents of the form $\vdash \Gamma \Uparrow \Delta$. The rules of this *pre-emptive multi-focusing* calculus, called PMF, are in Figure 3. A straightforward injection $(-)^{\#}$ from MF to PMF derivations is assumed.

Fact 8 *The following are seen by straightforward induction.*

1. *If* $\vdash_{\mathrm{MF}} \Gamma \Downarrow \Delta$, *then* $\vdash_{\mathrm{PMF}} \Gamma \downarrow \Delta \,;\, \cdot$.

$$\frac{\vdash \Gamma_1 \downarrow A, \Delta_1 \,;\, \Xi_1 \qquad \vdash \Gamma_2 \downarrow B, \Delta_2 \,;\, \Xi_2}{\vdash \Gamma_1, \Gamma_2 \downarrow A \otimes B, \Delta_1, \Delta_2 \,;\, \Xi_1, \Xi_2}\ [\otimes] \qquad \frac{}{\vdash \cdot \downarrow \mathbf{1} \,;\, \cdot}\ [\mathbf{1}] \qquad \frac{\vdash \Gamma \downarrow A_i, \Delta \,;\, \Xi}{\vdash \Gamma \downarrow A_1 \oplus A_2, \Delta \,;\, \Xi}\ [\oplus]$$

$$\frac{}{\vdash a^\perp \downarrow a \,;\, \cdot}\ [\mathrm{I}] \quad \frac{\vdash \Gamma \downarrow \Delta, \Psi \,;\, \Xi}{\vdash \Gamma, \Delta \downarrow \Psi \,;\, \Xi}\ [\mathrm{PMF}]_1 \quad \frac{\vdash \Gamma \downarrow \Delta \,;\, \Xi}{\vdash \Gamma, \Delta \Uparrow \Xi}\ [\mathrm{PMF}]_2$$

$$\frac{\vdash \Gamma, A \Uparrow \Delta}{\vdash \Gamma \Uparrow A, \Delta}\ [\mathrm{R}\Uparrow] \quad \frac{\vdash \Gamma \downarrow \Delta \,;\, N, \Xi}{\vdash \Gamma \downarrow \Delta, N \,;\, \Xi}\ [\mathrm{R}\downarrow] \quad \frac{\vdash \Gamma \Uparrow \Xi}{\vdash \Gamma \downarrow \cdot \,;\, \Xi}\ [\mathrm{R}]$$

Fig. 3. Rules of the pre-emptive multi-focusing calculus, PMF. The rules [&], [⅋], [⊤] and [⊥], and all side conditions from MF (Fig. 2) are carried over.

2. *If* $\vdash_{\mathrm{PMF}} \Gamma \downarrow \Delta \,;\, \Xi$*, then* $\vdash_{\mathrm{MF}} \Gamma \Downarrow \Delta, \Xi$*.*
3. $\vdash_{\mathrm{MF}} \Gamma \Uparrow \Delta$ *if and only if* $\vdash_{\mathrm{PMF}} \Gamma \Uparrow \Delta$*.*

Because both positive and negative phases can be pre-empted using the $[\mathrm{PMF}]_i$ rules, we can explicitly sequence two positive phases by by introducing new instances of $[\mathrm{PMF}]_2$. Note that focus, once granted, cannot be removed until the proposition becomes negative; thus, PMF does not destroy synthetic positive connectives, which are the essential innovation of focusing. After the positive phase of the top bipole has permuted through the negative phase of the bottom bipole, the suspended negative phases are awakened, which might give rise to a number of different sub-derivations (due to &). If **D** is this multiset of sub-derivations, then we indicate that it finishes with the negative phase for Ξ as $\mathbf{D} \,/\, \Xi$.

Definition 9

1. $(\mathbf{D} \,/\, \Xi) \vdash \Gamma \downarrow \Delta \,;\, \Xi$*, where* **D** *is a multiset of derivations, has one of the following forms:*

$$\frac{(\mathbf{D} \,/\, N, \Xi) \vdash \Gamma \downarrow \Delta \,;\, N, \Xi}{\vdash \Gamma \downarrow \Delta, N \,;\, \Xi}\ [\mathrm{R}\downarrow] \quad \frac{(\mathbf{D} \,/\, \Xi) \vdash \Gamma \Uparrow \Xi}{\vdash \Gamma \downarrow \cdot \,;\, \Xi}\ [\mathrm{R}] \quad \frac{(\mathbf{D} \,/\, \Xi) \vdash \Gamma \downarrow \Delta, A_i \,;\, \Xi}{\vdash \Gamma \downarrow \Delta, A_1 \oplus A_2 \,;\, \Xi}\ [\oplus]$$

$$\frac{(\mathbf{D} \,/\, \Xi) \vdash \Gamma_1 \downarrow \Delta_1, A \,;\, \Xi \qquad \mathcal{E} \vdash \Gamma_2 \downarrow \Delta_2, B \,;\, \cdot}{\vdash \Gamma_1, \Gamma_2 \downarrow \Delta_1, \Delta_2, A \otimes B \,;\, \Xi}\ [\otimes]$$

(And the symmetric case for [⊗]*.)*

2. $(\mathbf{D} \,/\, \Xi) \vdash \Gamma \Uparrow \Delta, \Xi$ *where* **D** *is a multiset of derivations, has one of the following forms:*

$$\frac{(\mathbf{D}_1 \,/\, \Xi', A) \vdash \Gamma \Uparrow \Delta, \Xi', A \qquad (\mathbf{D}_2 \,/\, \Xi', B) \vdash \Gamma \Uparrow \Delta, \Xi', B}{\vdash \Gamma \Uparrow \Delta, \Xi', A \,\&\, B}\ [\&] \qquad \ldots \text{and } \mathbf{D} = \mathbf{D}_1, \mathbf{D}_2$$

$$\frac{}{\vdash \Gamma \Uparrow \Delta, \Xi', \top}\ [\top] \qquad \ldots \text{and } \mathbf{D} = \cdot$$

$$\frac{(\mathbf{D} \,/\, \Xi', A, B) \vdash \Gamma \Uparrow \Delta, \Xi', A, B}{\vdash \Gamma \Uparrow \Delta, \Xi', A \parr B}\ [\parr] \quad \frac{(\mathbf{D} \,/\, \Xi') \vdash \Gamma \Uparrow \Delta, \Xi'}{\vdash \Gamma \Uparrow \Delta, \Xi', \bot}\ [\bot] \quad \frac{(\mathbf{D} \,/\, \Xi', P) \vdash \Gamma, P \Uparrow \Delta, \Xi'}{\vdash \Gamma \Uparrow \Delta, \Xi', P}\ [\mathrm{R}\Uparrow]$$

Additionally, $(\mathcal{D} \,/\, \cdot) = \mathcal{D}$*.*

We define the merge operation in terms of a rewrite $\longrightarrow$ between PMF proofs such that in each case of the rewrite at least one root of a $[\mathrm{PMF}]_1$ is permuted lower in the derivation. Eventually, this will bring two instances of $[\mathrm{PMF}]_i$ next to each other, at which point they are merged. All negative rules encountered during the rewrite are immediately suspended, causing them to permute above the positive phase rooted at the [PMF] being permuted. To obtain confluence globally, we must first split the roots to obtain the subset that can merge with the roots of the bottom bipole; otherwise, we might merge bipoles in the wrong order and block possible merges.

Definition 10 *The rewrite $\longrightarrow$ between* PMF *proofs has the following rules.*

$$\dfrac{\mathcal{D} \vdash \Gamma \downarrow \Delta, \Delta' ; \Xi}{\vdash \Gamma, \Delta, \Delta' \Uparrow \Xi}\,[\text{PMF}]_1 \quad\longrightarrow\quad \dfrac{\dfrac{\mathcal{D} \vdash \Gamma \downarrow \Delta, \Delta' ; \Xi}{\vdash \Gamma, \Delta \downarrow \Delta' ; \Xi}\,[\text{PMF}]_2}{\vdash \Gamma, \Delta, \Delta' \Uparrow \Xi}\,[\text{PMF}]_1$$

$$\dfrac{\dfrac{(\mathbf{D} \,/\, \Xi) \vdash \Gamma, P \downarrow \Delta ; \Xi}{\vdash \Gamma, P, \Delta \Uparrow \Xi}\,[\text{PMF}]_1}{\vdash \Gamma, \Delta \Uparrow \Xi, P}\,[\text{R}\Uparrow] \quad\longrightarrow\quad \dfrac{(\mathbf{D} \,/\, \Xi, P) \vdash \Gamma \downarrow \Delta ; P, \Xi}{\vdash \Gamma, \Delta \Uparrow \Xi, P}\,[\text{PMF}]_1$$

$$\dfrac{\dfrac{(\mathbf{D}_1 \,/\, \Xi, C) \vdash \Gamma \downarrow \Delta ; \Xi, C}{\vdash \Gamma, \Delta \Uparrow \Xi, C}\,[\text{PMF}]_1 \qquad \dfrac{(\mathbf{D}_2 \,/\, \Xi, D) \vdash \Gamma \downarrow \Delta ; \Xi, D}{\vdash \Gamma, \Delta \Uparrow \Xi, D}\,[\text{PMF}]_1}{\vdash \Gamma, \Delta \Uparrow \Xi, C \,\&\, D}\,[\&]$$

$$\longrightarrow\quad \dfrac{(\mathbf{D}_1, \mathbf{D}_2 \,/\, \Xi, C \,\&\, D) \vdash \Gamma \downarrow \Delta ; \Xi, C \,\&\, D}{\vdash \Gamma, \Delta \Uparrow \Xi, C \,\&\, D}\,[\text{PMF}]_1$$

$$\dfrac{\dfrac{(\mathbf{D} \,/\, \Xi, C, D) \vdash \Gamma \downarrow \Delta ; \Xi, C, D}{\vdash \Gamma, \Delta \Uparrow \Xi, C, D}\,[\text{PMF}]_1}{\vdash \Gamma, \Delta \Uparrow \Xi, C \parr D}\,[\parr] \quad\longrightarrow\quad \dfrac{(\mathbf{D} \,/\, \Xi, C \parr D) \vdash \Gamma \downarrow \Delta ; \Xi, C \parr D}{\vdash \Gamma, \Delta \Uparrow \Xi, C \parr D}\,[\text{PMF}]_1$$

$$\dfrac{\dfrac{(\mathbf{D} \,/\, \Xi) \vdash \Gamma \downarrow \Delta ; \Xi}{\vdash \Gamma, \Delta \Uparrow \Xi}\,[\text{PMF}]_1}{\vdash \Gamma, \Delta \Uparrow \Xi, \bot}\,[\bot] \quad\longrightarrow\quad \dfrac{(\mathbf{D} \,/\, \Xi, \bot) \vdash \Gamma \downarrow \Delta ; \Xi, \bot}{\vdash \Gamma, \Delta \Uparrow \Xi, \bot}\,[\text{PMF}]_1$$

$$\dfrac{\dfrac{(\mathbf{D} \,/\, N, \Xi) \vdash \Gamma \downarrow \Delta, \Psi ; N, \Xi}{\vdash \Gamma, \Delta \downarrow \Psi ; N, \Xi}\,[\text{PMF}]_1}{\vdash \Gamma, \Delta \downarrow \Psi, N ; \Xi}\,[\text{R}\downarrow] \quad\longrightarrow\quad \dfrac{\dfrac{(\mathbf{D} \,/\, N, \Xi) \vdash \Gamma \downarrow \Delta, \Psi ; N, \Xi}{\vdash \Gamma \downarrow \Delta, \Psi, N ; \Xi}\,[\text{R}\downarrow]}{\vdash \Gamma, \Delta \downarrow \Psi, N ; \Xi}\,[\text{PMF}]_1$$

$$\dfrac{\dfrac{(\mathbf{D} \,/\, \Xi) \vdash \Gamma \downarrow \Delta ; \Xi}{\vdash \Gamma, \Delta \Uparrow \Xi}\,[\text{PMF}]_1}{\vdash \Gamma, \Delta \downarrow \cdot ; \Xi}\,[\text{R}] \quad\longrightarrow\quad \dfrac{(\mathbf{D} \,/\, \Xi) \vdash \Gamma \downarrow \Delta ; \Xi}{\vdash \Gamma, \Delta \downarrow \cdot ; \Xi}\,[\text{PMF}]_2$$

$$\dfrac{\dfrac{\mathcal{D} \vdash \Gamma_1 \downarrow \Psi, \Delta_1, A ; \Xi_1}{\vdash \Gamma_1, \Psi \downarrow \Delta_1, A ; \Xi_1}\,[\text{PMF}]_2 \qquad \mathcal{E} \vdash \Gamma_1 \downarrow \Delta_2, B ; \Xi_2}{\vdash \Gamma_1, \Gamma_2, \Psi \downarrow \Delta_1, \Delta_2, A \otimes B ; \Xi_1, \Xi_2}\,[\otimes]$$

$$\longrightarrow\quad \dfrac{\dfrac{\mathcal{D} \vdash \Gamma_1 \downarrow \Psi, \Delta_1, A ; \Xi_1 \qquad \mathcal{E} \vdash \Gamma_1 \downarrow \Delta_2, B ; \Xi_2}{\vdash \Gamma_1, \Gamma_2 \downarrow \Psi, \Delta_1, \Delta_2, A \otimes B ; \Xi_1, \Xi_2}\,[\otimes]}{\vdash \Gamma_1, \Gamma_2, \Psi \downarrow \Delta_1, \Delta_2, A \otimes B ; \Xi_1, \Xi_2}\,[\text{PMF}]_2$$

$$\dfrac{\dfrac{\mathcal{D} \vdash \Gamma \downarrow \Psi, \Delta, A ; \Xi}{\vdash \Gamma, \Psi \downarrow \Delta, A ; \Xi}\,[\text{PMF}]_2}{\vdash \Gamma, \Psi \downarrow \Delta, A \oplus B ; \Xi}\,[\oplus] \quad\longrightarrow\quad \dfrac{\dfrac{\mathcal{D} \vdash \Gamma \downarrow \Psi, \Delta, A ; \Xi}{\vdash \Gamma \downarrow \Psi, \Delta, A \otimes B ; \Xi}\,[\oplus]}{\vdash \Gamma, \Psi \downarrow \Delta, A \oplus B ; \Xi}\,[\text{PMF}]_2$$

$$\dfrac{\dfrac{\mathcal{D} \vdash \Gamma \downarrow \Psi_1, \Psi_2, \Delta ; \Xi}{\vdash \Gamma, \Psi_1 \downarrow \Delta, \Psi_2 ; \Xi}\,[\text{PMF}]_2}{\vdash \Gamma, \Psi_1, \Psi_2 \downarrow \Delta ; \Xi}\,[\text{PMF}]_2 \quad\longrightarrow\quad \dfrac{\mathcal{D} \vdash \Gamma \downarrow \Psi_1, \Psi_2, \Delta ; \Xi}{\vdash \Gamma, \Psi_1, \Psi_2 \downarrow \Delta ; \Xi}\,[\text{PMF}]_2$$

$$\dfrac{\dfrac{\mathcal{D} \vdash \Gamma \downarrow \Psi_1, \Psi_2, \Delta ; \Xi}{\vdash \Gamma, \Psi_1 \downarrow \Psi_2 ; \Xi}\,[\text{PMF}]_2}{\vdash \Gamma, \Psi_1, \Psi_2 \Uparrow \Xi}\,[\text{PMF}]_1 \quad\longrightarrow\quad \dfrac{\mathcal{D} \vdash \Gamma \downarrow \Psi_1, \Psi_2 ; \Xi}{\vdash \Gamma, \Psi_1, \Psi_2 \Uparrow \Xi}\,[\text{PMF}]_1$$

The symmetric cases for $[\mathrm{PMF}]_1$ / $[\otimes]$ *and* $[\mathrm{PMF}]_1$ / $[\oplus]$ *are elided.*

Definition 11 *If* $\mathcal{D}, \mathcal{E} \vdash_{\mathrm{MF}} \Gamma \Updownarrow \Delta$*, and* $\mathcal{D}^{\#} \longrightarrow^{*} \mathcal{E}^{\#}$*, then* $\mathcal{D} \longrightarrow \mathcal{E}$*.*

Lemma 12 *If* $\mathcal{D} \vdash \Gamma \Updownarrow \Delta$ *is maximal and* $\mathcal{D} \longrightarrow \mathcal{E}$*, then* $\mathcal{D} \approx \mathcal{E}$*.*

Proof (Sketch). Note that in every case of the rewrite $\longrightarrow$ on PMF derivations, an instance of $[\mathrm{PMF}]_1$ is brought closer to the root of the derivation. Therefore, the rewrite $\longrightarrow$ on MF proofs can only enlarge the lowermost roots in $\mathcal{D}$. But, $\mathcal{D}$ is already maximal. So $\mathcal{E}$ has the same instances of [MF] as $\mathcal{D}$, *i.e.*, $\mathcal{D} \approx \mathcal{E}$. □

Lemma 13 *If* $\mathcal{D} \simeq \mathcal{E} \vdash \Gamma \Updownarrow \Delta$ *and* $\mathcal{E}$ *is maximal, then* $\mathcal{D} \longrightarrow \mathcal{E}$*.*

Proof (Sketch). We have to show that all ways of permuting a root downwards in a proof can be generated by $\longrightarrow$. But this is easily seen because the $\longrightarrow$ is allowed to divide the roots and permute only the necessary fragment downwards. For a representative example, consider the following sub-derivation of $\mathcal{D}^{\#}$:

$$\mathcal{F} = \dfrac{\dfrac{\dfrac{\mathcal{F}' \vdash \Gamma, P \downarrow \Delta, Q \,;\, \cdot}{\vdash \Gamma, P, Q, \Delta \Uparrow \cdot}\,[\mathrm{PMF}]_1}{\vdash \Gamma, \Delta \Uparrow P, Q}\,[\mathrm{R}\Uparrow]^2}{\vdash \Gamma, \Delta \Uparrow P \parr Q}\,[\parr]$$

Of the roots Δ, Q, only Δ can possibly permute below $P \parr Q$, because Q is one of its sub-formulas. According to the rewrite rules, we first remove Q from the roots of the [PMF] rule by inserting another [PMF]. The permutation can now proceed (for some $\mathbf{F} \,/\, P, Q \simeq \mathcal{F}''$):

$$\dfrac{\dfrac{\dfrac{\mathcal{F}'' = \dfrac{\mathcal{F}' \vdash \Gamma, P \downarrow \Delta, Q \,;\, \cdot}{\vdash \Gamma, P, Q \downarrow \Delta \,;\, \cdot}\,[\mathrm{PMF}]_2}{\vdash \Gamma, P, Q, \Delta \Uparrow \cdot}\,[\mathrm{PMF}]_1}{\vdash \Gamma, \Delta \Uparrow P, Q}\,[\mathrm{R}\Uparrow]^2}{\vdash \Gamma, \Delta \Uparrow P \parr Q}\,[\parr] \longrightarrow \dfrac{\dfrac{(\mathbf{F} \,/\, P, Q) \vdash \Gamma \downarrow \Delta \,;\, P, Q}{\vdash \Gamma, \Delta \Uparrow P, Q}\,[\mathrm{PMF}]_1}{\vdash \Gamma, \Delta \Uparrow P \parr Q}\,[\parr]$$

$$\longrightarrow \dfrac{(\mathbf{F} \,/\, P \parr Q) \vdash \Gamma \downarrow \Delta \,;\, P \parr Q}{\vdash \Gamma, \Delta \Uparrow P \parr Q}\,[\mathrm{PMF}]_1$$

The instance of $[\mathrm{PMF}]_1$ that permutes down is thus free of the disallowed root Q. □

Corollary 14 *If* $\mathcal{D} \simeq \mathcal{E} \vdash \Gamma \Uparrow \cdot$ *are both maximal, then* $\mathcal{D} \approx \mathcal{E}$*.*

Proof. Immediately from lemmas 12 and 13. □

Definition 15 *If* $\mathcal{D} \vdash \Gamma$ *and* $\mathcal{E} \vdash \cdot \Uparrow \Gamma$ *are such that* $\mathcal{E}$ *is maximal and* $\mathcal{D} \simeq \lfloor \mathcal{E} \rfloor$*, then we say that* $\mathcal{E}$ *is* maximal for $\mathcal{D}$*.*

Theorem 16 (Canonicity) *Maximal proofs for iso-initial derivations are isopolar.*

Proof. A simple consequence of Cor. 14.

5 Multi-focusing and proof nets

In this section, we deal with a restricted fragment of MALL proofs, the unit-free multiplicative fragment, MLL$^-$, for which proof nets have been proposed as a natural candidate for the canonical representation of proofs [8]. We shall show that the maximally multi-focused proofs that we established as canonical in the previous section are indeed in bijection with proof nets. We recall in Figure 4 the syntax for proof structures. A proof net of conclusion Γ is a proof structure that is the desequentialization of a sequent proof of $\vdash \Gamma$ [6, 8].

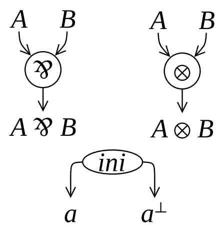

Fig. 4. Unit-free MLL proof nets

Theorem 17 *Let Γ be a multiset of occurrences of formulas of MLL$^-$. Two maximally multifocused proofs of $\vdash \cdot \Uparrow \Gamma$ are iso-polar iff they have the same proof net.*

We actually show that for every proof net there is a unique maximally multi-focused proof (up to iso-polarity) associated with it. This fact can be seen as a trivial consequence of the previous section, since all sequentialization of an MLL$^-$ proof net are iso-initial sequent proofs. However, our approach is of interest as it unveils a simple process to build a maximal multi-focused sequent proof starting with an MLL$^-$ proof net.

We first recall two definitions from [1] which develops a focused sequentialization algorithm for MLL$^-$ proof nets:

Definition 18 ($\mathtt{split}(\pi)$, $\mathtt{foc}(\pi)$, from [1]) *Let π be an MLL$^-$ proof net.*

1. *$\mathtt{split}(\pi)$ is the set of positive conclusions P of π such that removing the concluding $\otimes$-link of P disconnects π in two proof nets π_1 and π_2.*
2. *$\mathtt{foc}(\pi)$ is the set of conclusions F of π such that F is a positive atom and π is just an ini link; or $F \in \mathtt{split}(\pi)$ and its premisses A and B are conclusions of the two sub-nets π_1 and π_2 where A (resp. B) is negative or $A \in \mathtt{foc}(\pi_1)$ (resp. $B \in \mathtt{foc}(\pi_2)$).*

Proof (of Theorem 17). Let π be a MLL$^-$ proof net of conclusions Γ. We outline a sequentialization algorithm producing a maximally multi-focused proof of conclusion $\vdash \cdot \Uparrow \Gamma$ (if Γ contains some negative non-atomic formula) or $\vdash \Gamma \setminus \mathtt{foc}(\pi) \Downarrow \mathtt{foc}(\pi)$ (otherwise) which is maximally multi-focused. We reason by induction on the size of π.

Case: Γ contains at least one negative formula. We remove all negative cells (that is, the ⅋ cells) of π up to reaching a positive cell or an initial cell. The resulting proof structure is a proof net π' and its conclusions Γ' are positive. By induction hypothesis, we can sequentialize it into a maximally multi-focused proof $\mathcal{D}$ of conclusion $\vdash \Gamma' \setminus \mathtt{foc}(\pi') \Downarrow \mathtt{foc}(\pi')$ by sequentializing in an arbitrary order[3] the negative rules

[3] the different possibilities give rise to iso-polar proofs.

that have been removed in the previous step, we obtain a proof Π of the form given in the adjoining figure.

$$\dfrac{\mathcal{D} \vdash \Gamma' \setminus \mathtt{foc}(\pi') \Downarrow \mathtt{foc}(\pi')}{\vdash \Gamma' \Uparrow \cdot}\,[\mathrm{MF}] \quad \vdots\,[\parr] \quad \vdash \cdot \Uparrow \Gamma$$

Case: Γ contains only positive formulas. We consider the formulas in $\mathtt{foc}(\pi) \neq \emptyset$ and remove the top-most positive connectives of every $F \in \mathtt{foc}(\pi)$. The resulting proof structure is not a proof net since it is not connected; however, each of its connected components is. Let them be $\pi_1, \ldots, \pi_n$. For $1 \leq i \leq n$, π_i has conclusions Γ_i which has at least one negative formula or which is reduced to an axiom link. In the first case, one can inductively sequentialize it into of maximally multi-focused proof $\mathcal{D}_i$. In order to conclude, we only need to show that one can obtain a proof of $\vdash \cdot \Uparrow \Gamma$ from the $\mathcal{D}_i$ and the positive cells of the formulas of $\mathtt{foc}(\pi)$, which follows from the fact that the formulas in $\mathtt{foc}(\pi)$ are hereditarily splitting: applying these formulas in any order (as long as the sub-formula priority is maintained), gives rise to a way to sequentialize π.

We finally need to check that the proof obtained with this process is indeed maximal, but this is done very easily: let F be a formula that could potentially enlarge the set of foci and let us consider a proof $\mathcal{D}_F$ that witnesses this fact ($\mathcal{D}_F$ focuses on F). By desequentializing $\mathcal{D}_F$, we get a proof net π and since $\mathcal{D}_F$ is a sequentialization of π that focuses on F which is positive, then F is hereditarily splitting, that is $F \in \mathtt{foc}(\pi)$, so $\mathtt{foc}(\pi)$ is maximal. □

We showed in this section that there is a bijection between MLL$^-$ proof nets and classes of isopolar maximally multi-focused proofs. MLL$^-$ proof nets are certainly the most concise canonical structures for this fragment. There are candidates to extend MLL$^-$ proof nets to broader fragments (MLL with units [11], MALL [9] or MELL) but they are not as satisfactory as for MLL$^-$. The problem of proof-nets for MALL with units is still open. Yet, these fragments have standard sequent calculi with well understood focusing systems. We expect that an analysis of the maximally multi-focused sequent proofs would yield a better understanding of proof net-like structures for such fragments.

6 Conclusion

The contributions of this paper are three-fold: (*i*) we provide a non-trivial result about multi-focused proofs; (*ii*) we show that maximally multi-focused proofs are representatives of their $\simeq$-equivalence class, and (*iii*) that when we restrict our attentions to MLL without units, proof nets are in one-to-one correspondence with maximal multi-focused proofs.

Andreoli studied focusing in proof nets [1, 3] and introduced a notion of “multifocus” [3]. In his paper, multifocus has a different meaning since it refers to a part of the context which is needed in order to perform of focusing rule. Andreoli also investigates a way to do proof net construction but only captures a restricted fragment of MLL$^-$. Faggian and Curien [5] introduced L-nets as a generalization of the designs of Ludics [7]: L-nets can be seen as designs with a flexible degree of sequentiality, between sequent proofs and proof nets. This may be close to multi-focused proofs which allow both singly focused sequent proofs and maximally multi-focused proofs.

We plan to develop this use of multi-focused proofs in a couple of directions: First, investigating the links between L-nets and multi-focused proofs; next, treatment of exponentials in the context of maximal multi-focus proofs; finally, the use of multi-focused derivations to guide or parallelize proof-search.

References

1. J.-M. Andreoli and R. Maieli. Focusing and proof nets in linear and noncommutative logic. In *International Conference on Logic for Programming and Automated Reasoning (LPAR)*, volume 1581 of *LNAI*. Springer, 1999.
2. Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. of Logic and Computation*, 2(3):297–347, 1992.
3. Jean-Marc Andreoli. Focussing proof-net construction as a middleware paradigm. In Andrei Voronkov, editor, *Automated Deduction - CADE-18*, number 2392 in LNAI, pages 501–516. Springer, 2002.
4. Pierre-Louis Curien. Introduction to linear logic and ludics, part I. *Advances in Mathematics (China)*, 34(5):513–544, January 2005.
5. Pierre-Louis Curien and Claudia Faggian. L-nets, strategies and proof-nets. In *CSL 2006: Computer Science Logic*, 2006.
6. Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
7. Jean-Yves Girard. Locus solum. *Mathematical Structures in Computer Science*, 11(3):301–506, June 2001.
8. Jean-Yves Girard. *Le Point Aveugle: Cours de logique: Tome 1, Vers la perfection*. Hermann, 2006.
9. Dominic Hughes and Rob Van Glabbeek. Proof nets for unit-free multiplicative-additive linear logic. *ACM Trans. on Computational Logic*, 6:784–842, 2005.
10. Stephen Cole Kleene. Permutabilities of inferences in Gentzen's calculi LK and LJ. *Memoirs of the American Mathematical Society*, 10:1–26, 1952.
11. François Lamarche and Lutz Straßburger. From proof nets to the free *-autonomous category. *Logical Methods in Computer Science*, 2(4:3):1–44, 2006.
12. Dale Miller and Alexis Saurin. From proofs to focused proofs: a modular proof of focalization in linear logic. In J. Duparc and T. A. Henzinger, editors, *CSL 2007: Computer Science Logic*, volume 4646 of *LNCS*, pages 405–419. Springer-Verlag, 2007.