Reachability in Vector Addition Systems is Primitive-Recursive in Fixed Dimension

Sylvain Schmitz
based on joint work with Jérôme Leroux

Highlights 2019

Outline

vector addition systems (VAS)

- central model of computation

reachability problem

- hard conceptually and computationally
- decision via decomposition algorithm

this talk

- new complexity upper bounds

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Vector Addition Systems (with States)

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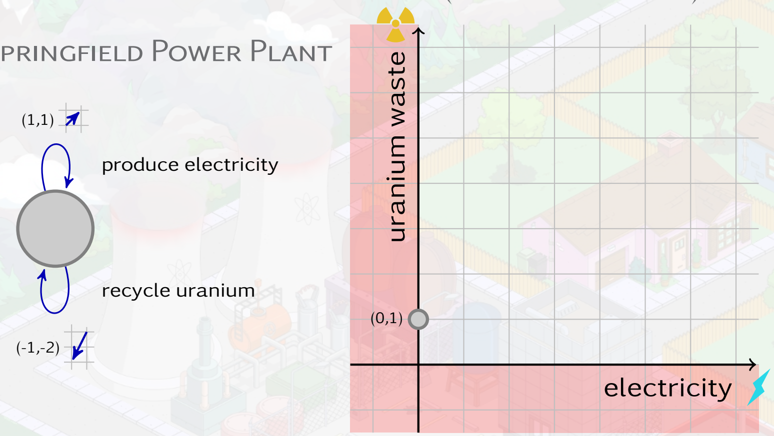
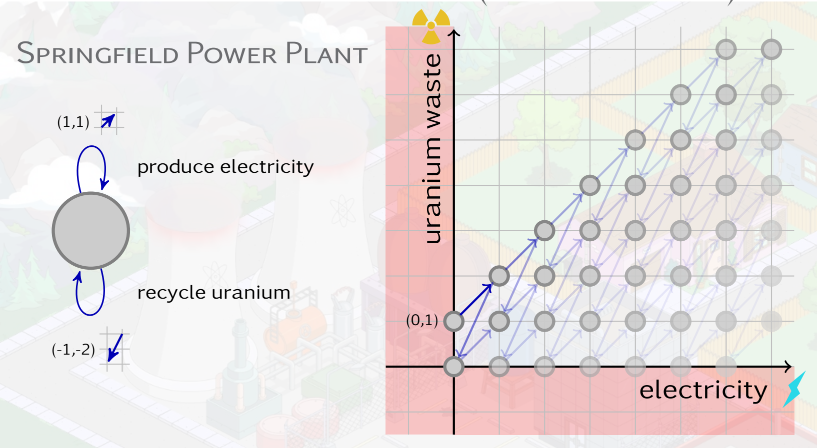
Springfield Power Plant



Vector Addition Systems (with States)



Can we produce unbounded electricity with no left-over uranium waste?

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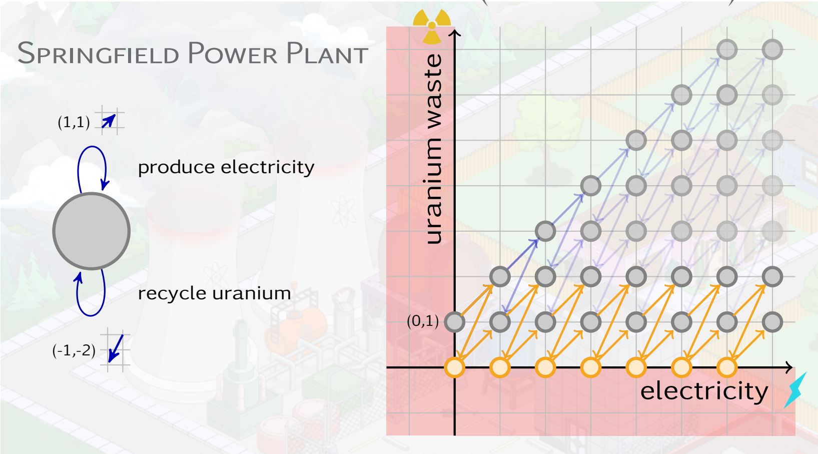

Can we produce unbounded electricity with no left-over uranium waste? Yes, $(\infty,0)$ is reachable

Importance of the Problem

Modelling discrete resources

items, money, molecules, active threads, active data domain, . . .

Central decision problem

Large number of problems interreducible with reachability in vector addition systems

- correctness of population protocols
- satisfiability of logics over data words
- provability of !-Horn linear logic
- . . .

New Upper Bounds

$$F_0(x) = x + 1$$

$$F_1(x) = \overbrace{F_0 \circ \cdots \circ F_0}^{x+1 \text{ times}}(x) = 2x + 1$$

$$F_2(x) = \overbrace{F_1 \circ \cdots \circ F_1}^{x+1 \text{ times}}(x) \approx 2^x$$

$$F_3(x) = \overbrace{F_2 \circ \cdots \circ F_2}^{x+1 \text{ times}}(x) \approx \mathrm{tower}(x)$$

$$\vdots$$

$$F_\omega(x) = F_{x+1}(x) \qquad \approx \mathrm{ackermann}(x)$$

Upper Bound Theorem

VAS Reachability is in $\mathbf{F}_\omega$, and in $\mathbf{F}_{d+4}$ in fixed dimension d

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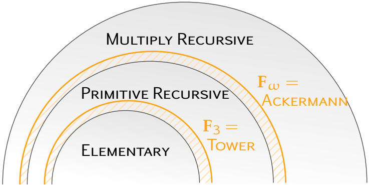

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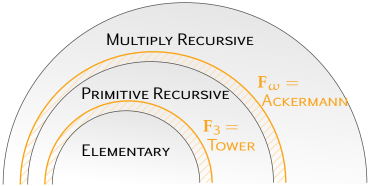


Upper Bound Theorem

VAS Reachability is in $\mathbf{F}_\omega$*, and in* $\mathbf{F}_{d+4}$ *in fixed dimension* d

Decomposition Algorithm

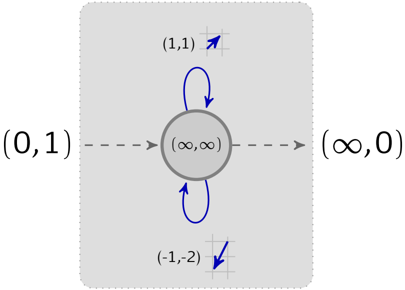

Ernst W. Mayr


S. Rao Kosaraju


Jean-Luc Lambert

"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



"Simple Runs" (Θ Condition)

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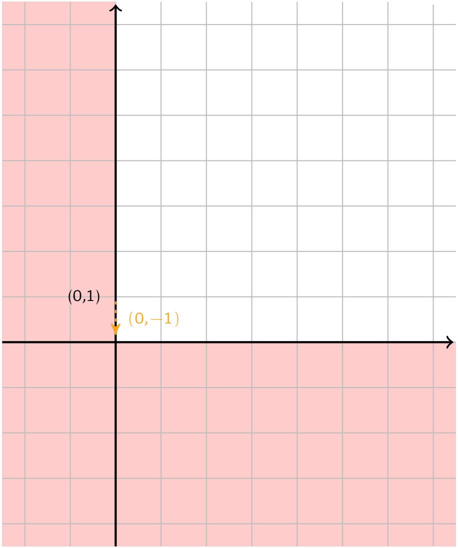

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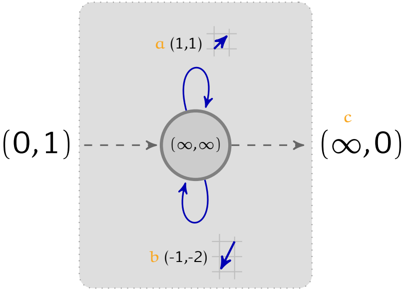


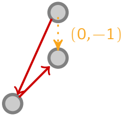
Characteristic System

$$0 + 1 \cdot a - 1 \cdot b = c$$
$$1 + 1 \cdot a - 2 \cdot b = 0$$

Solution for a, b

[1 · , 1 · ]

Solution Path

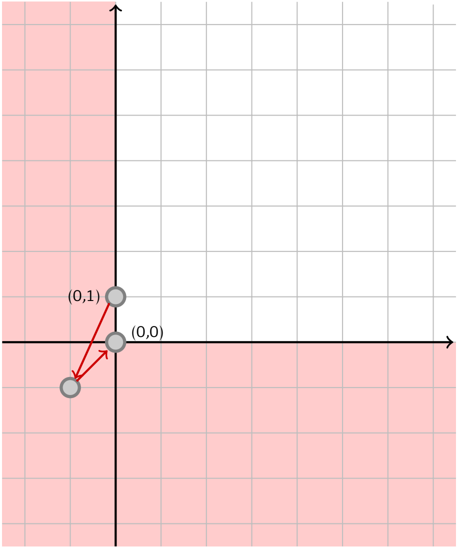


“Simple Runs” (Θ Condition)

[Mayr’81, Kosaraju’82, Lambert’92]

solution path

$\times 1$



"Simple Runs" (Θ Condition)

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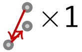

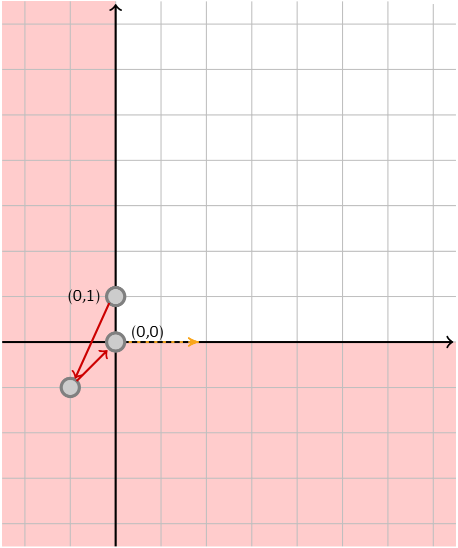

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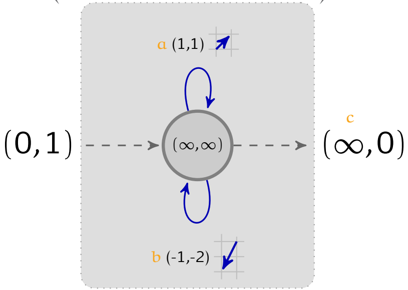


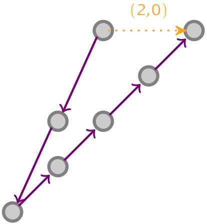
Homogeneous System

$$1 \cdot a - 1 \cdot b = c$$
$$1 \cdot a - 2 \cdot b = 0$$
$$a, b, c > 0$$

Solution for a, b

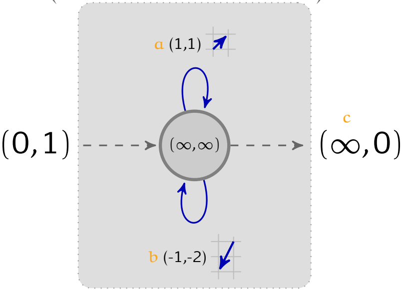
[4 · ↗, 2 · ↙]

Unbounded Path



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



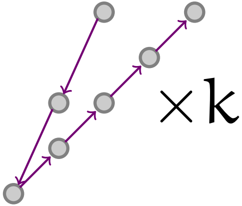
Homogeneous System

$$1 \cdot a - 1 \cdot b = c$$
$$1 \cdot a - 2 \cdot b = 0$$
$$a, b, c > 0$$

Solution for a, b

$[4k \cdot \nearrow, 2k \cdot \swarrow]$

Unbounded Path



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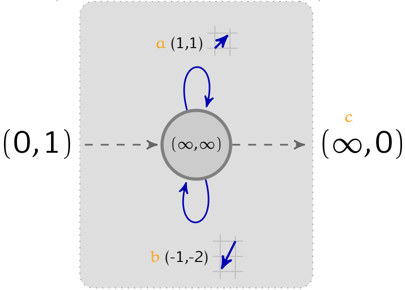


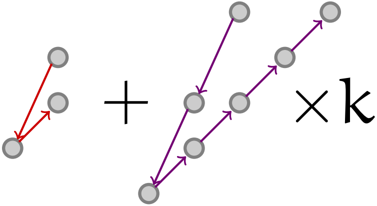
Characteristic System

$$0 + 1 \cdot a - 1 \cdot b = c$$
$$1 + 1 \cdot a - 2 \cdot b = 0$$

Solution for a, b

$$[(1 + 4k) \cdot \nearrow, (1 + 2k) \cdot \swarrow]$$

Path



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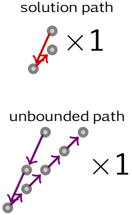

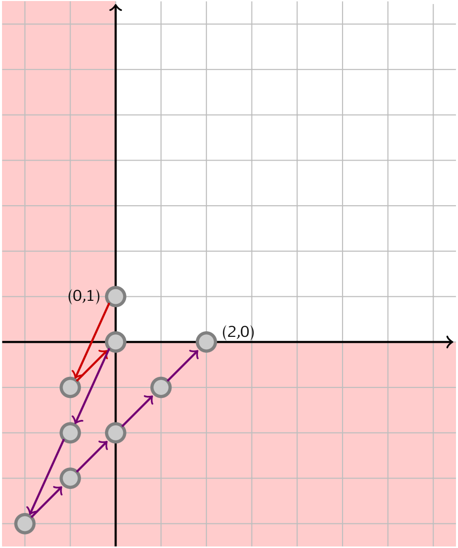

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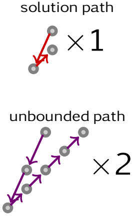

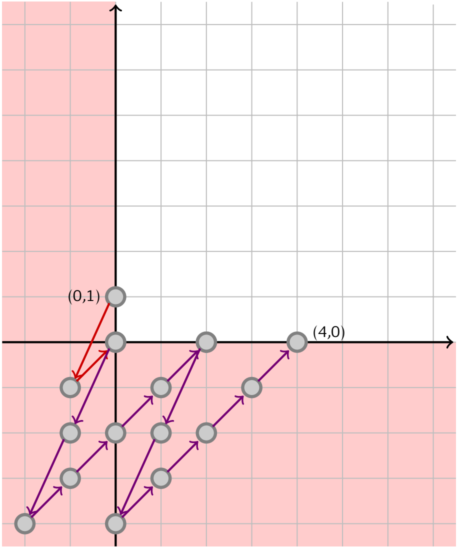

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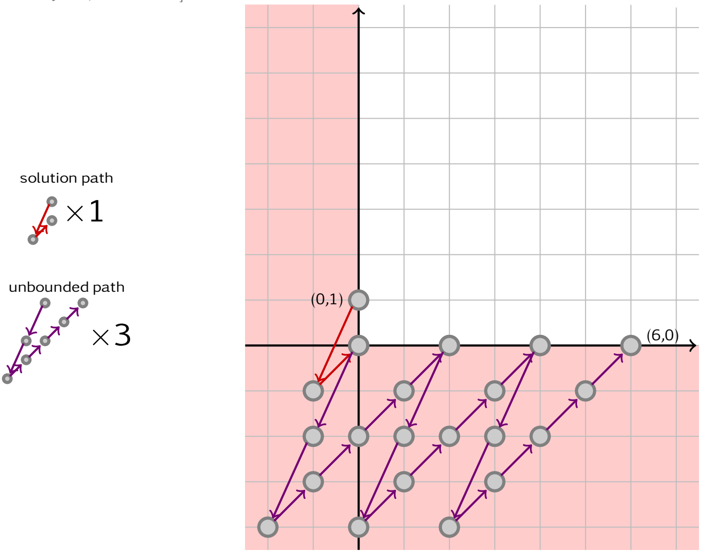

"Simple Runs" (Θ Condition)

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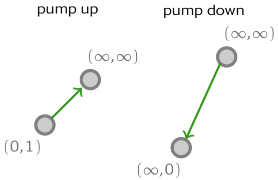
Pumpable Paths



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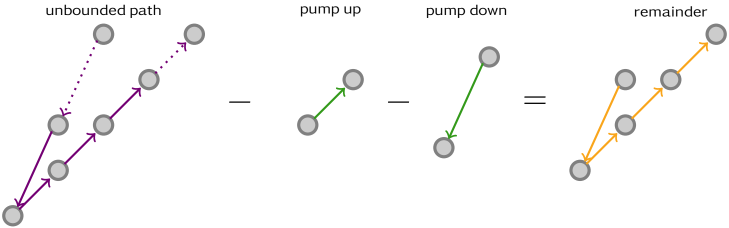
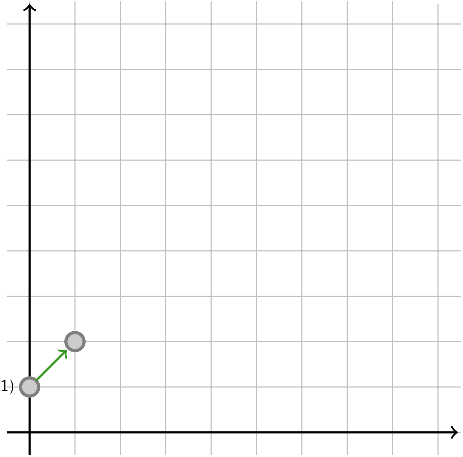


"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

pump up

$\times 1$



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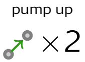

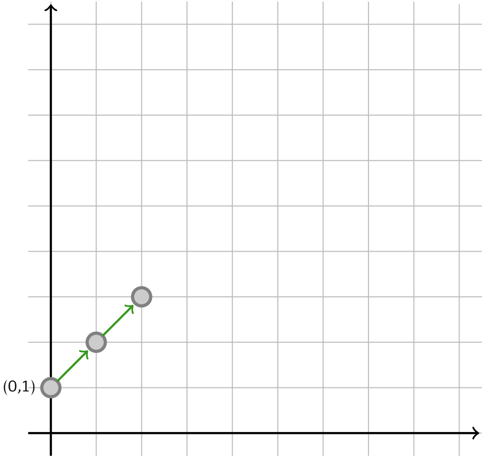

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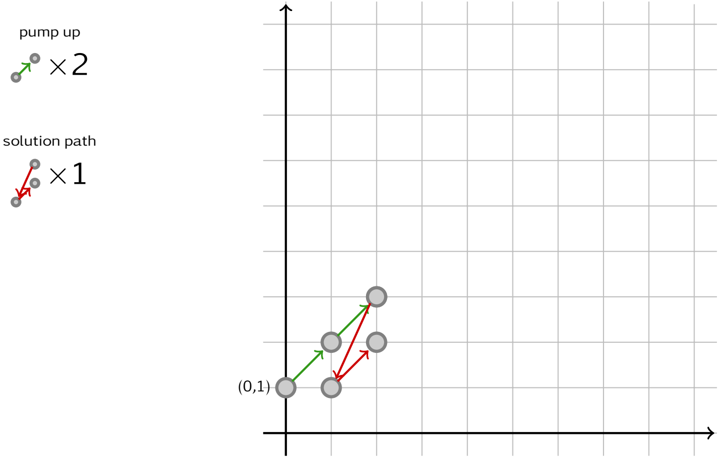

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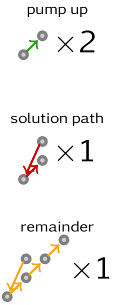

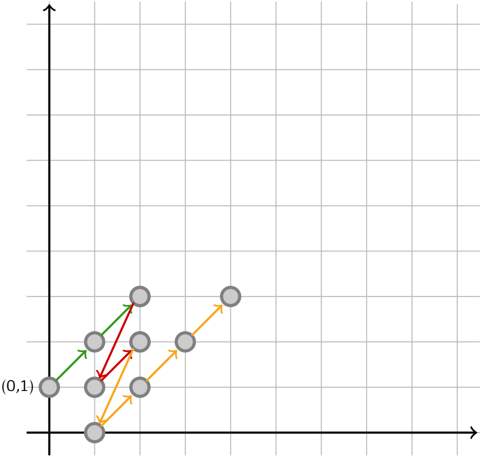

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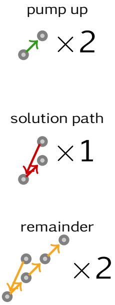

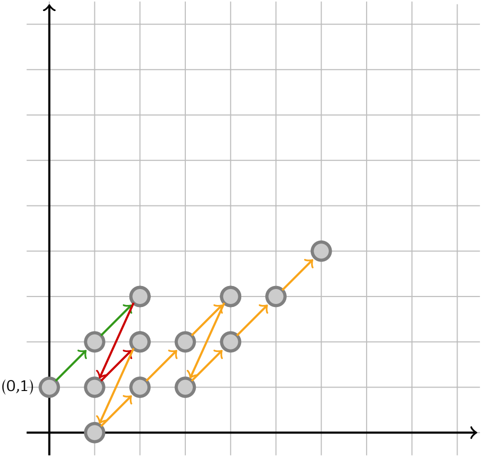

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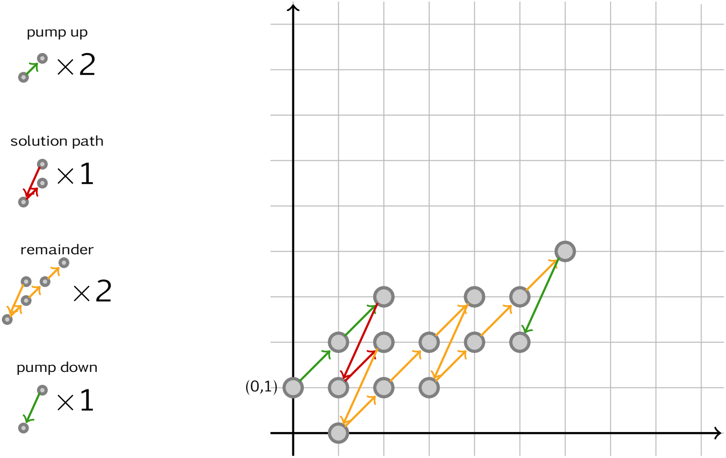

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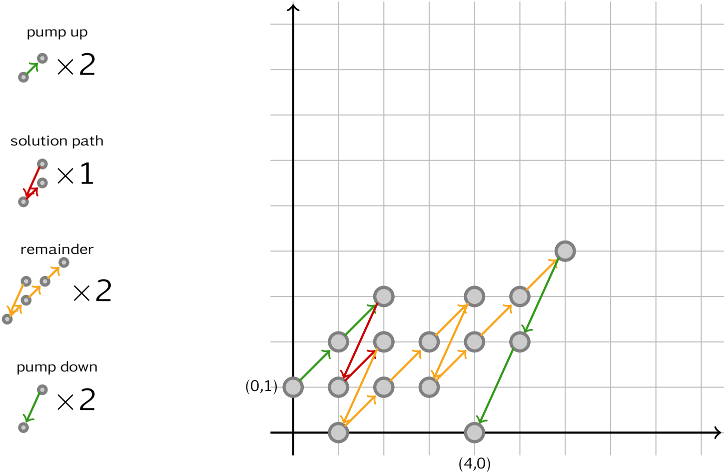

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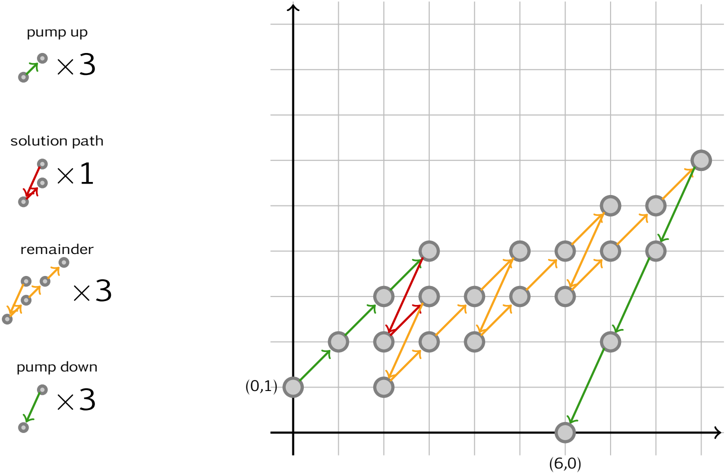
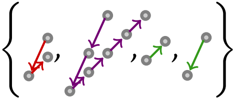


Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"?



Decomposition Algorithm

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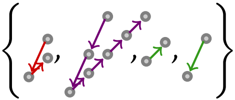
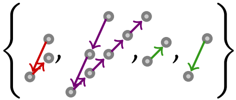
can we build a "simple run"? yes



Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no



decompose

Decomposition Algorithm

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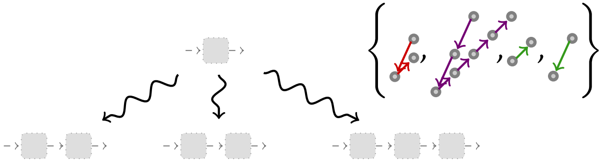

decompose

Decomposition Algorithm

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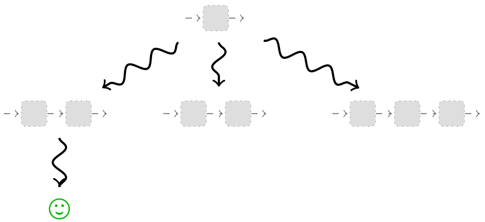

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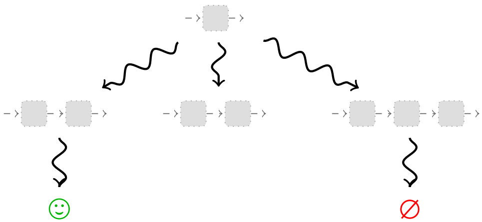

Decomposition Algorithm

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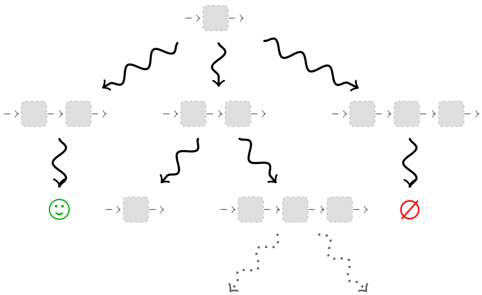
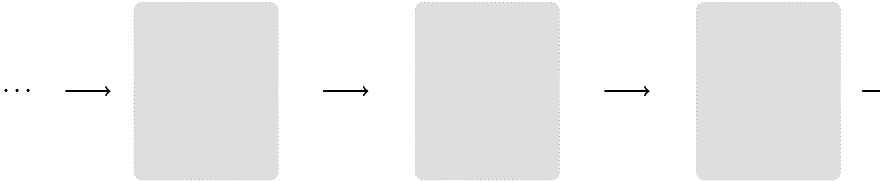


How to Decompose

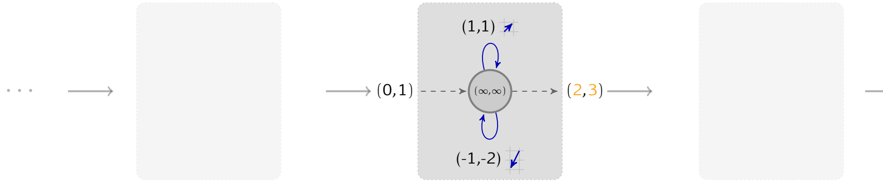
[Mayr'81, Kosaraju'82, Lambert'92]



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

No simple path:



How to Decompose

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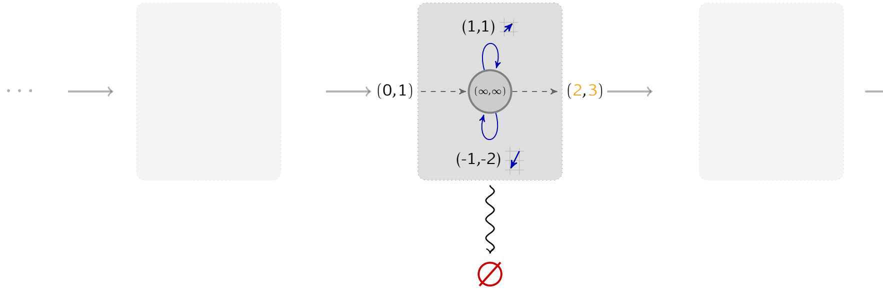
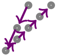
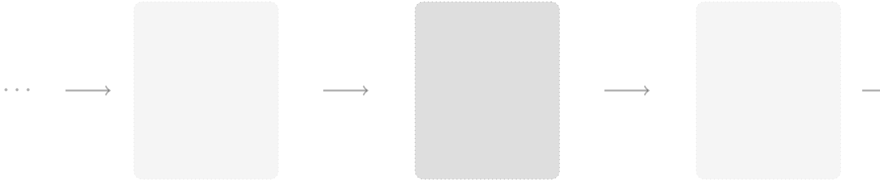


How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

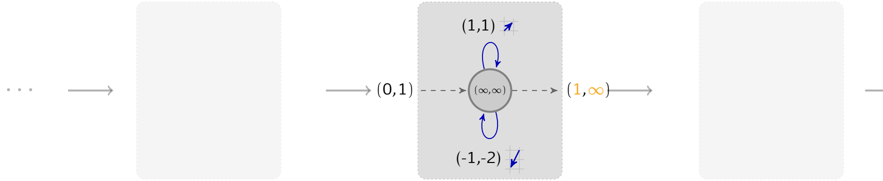
No unbounded path  :



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

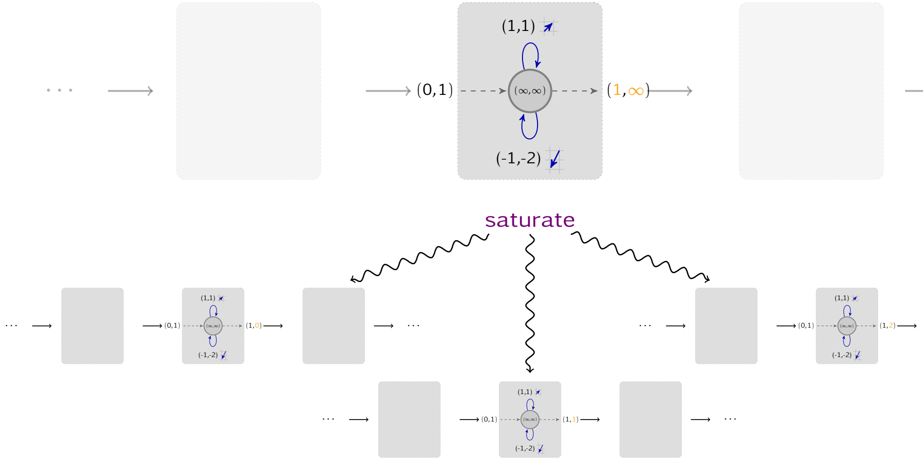
No unbounded path : Case of bounded '∞'



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

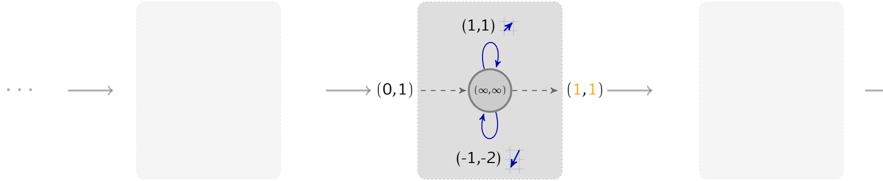
No unbounded path : Case of bounded '∞'



How to Decompose

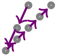
[Mayr'81, Kosaraju'82, Lambert'92]

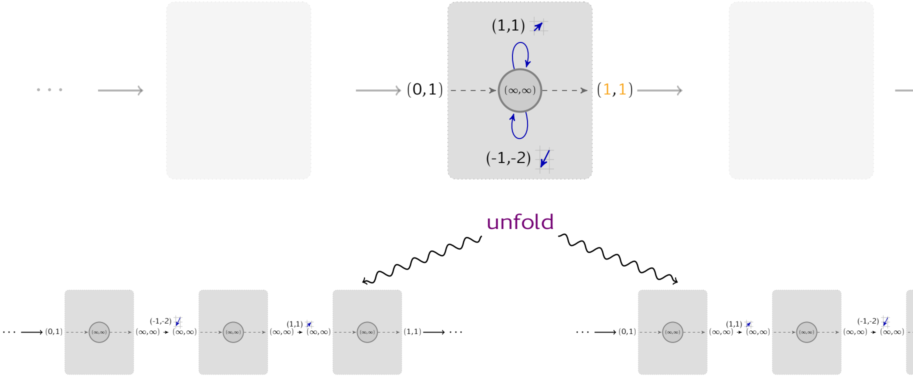
No unbounded path : Case of bounded transitions



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

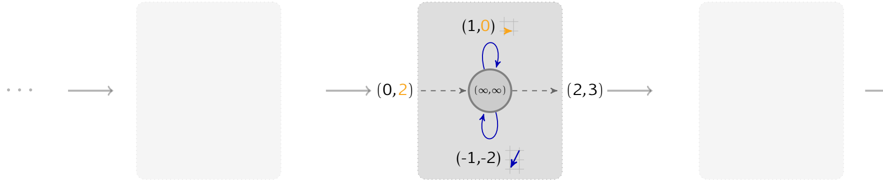
No unbounded path : Case of bounded transitions



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

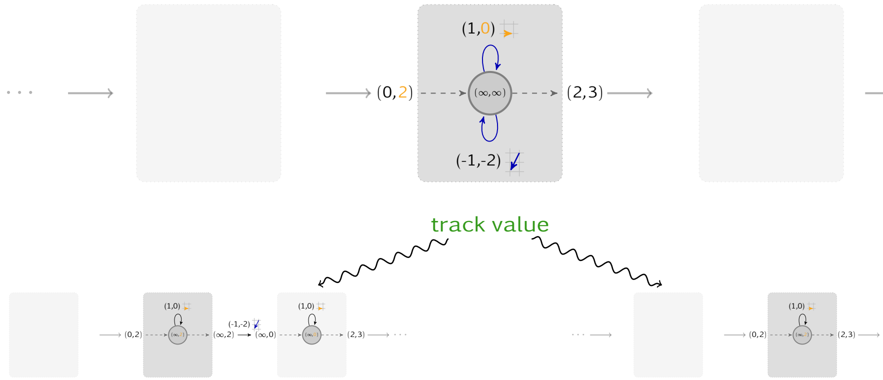
No pumping path ↗ or ↙:



How to Decompose

[Mayr'81, Kosaraju'82, Lambert'92]

No pumping path or :



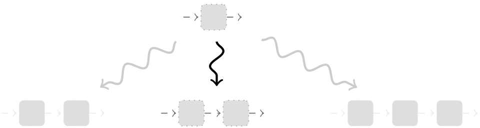
Termination

Ranking Function

α_0

Termination

Ranking Function



α_0

$\vee$

α_1

Termination

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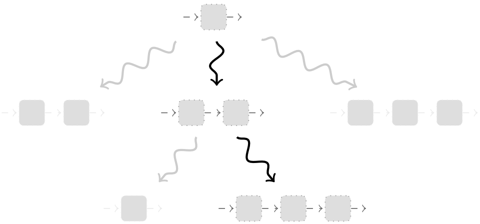


α_0

$\vee$

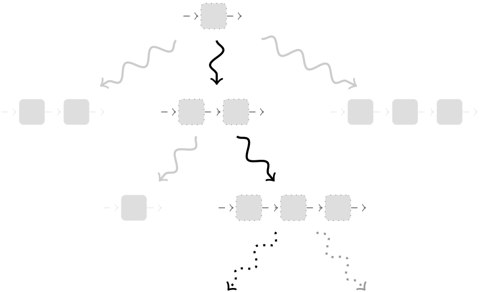
α_1

$\vee$

α_2

Termination

Ranking Function



$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots$$

New Ingredients

Technical Ingredients

[Leroux & S. ′19]

1. new ranking function:

   order type ω^{d+1}

   ω^{ω^3} in [Leroux & S. ′15]

   $\omega^{\omega} \cdot (d+1)$ in [S. ′17]

2. refined analysis of pumpable paths:

   Rackoff-style analysis

   improves complexity from $\mathbf{F}_{2d+2}$ to $\mathbf{F}_{d+4}$

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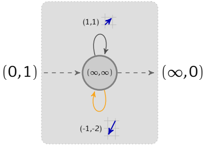
Rank of a Transition

For a transition t in 

$$\{\text{effects of cycles } C \mid t \in C\}$$

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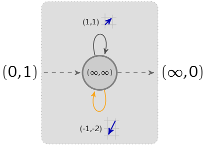
For a transition t in 

$\{ m \cdot$  $+ n \cdot$  $\mid m \geqslant 0, n > 0 \}$

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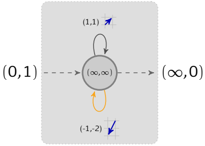
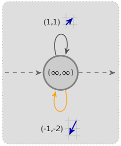
For a transition t in 

$$\mathrm{span}_{\mathbb{Q}}\left(\left\{m\cdot \nearrow + n\cdot \swarrow \mid m \geqslant 0, n > 0\right\}\right) = \mathbb{Q}^2$$

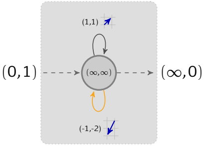
Rank of a Transition

For a transition t in (0,1) 

(∞,0)

$$\dim\left(\operatorname{span}_{\mathbb{Q}}\left(\{m \cdot \nearrow + n \cdot \swarrow \mid m \geqslant 0, n > 0\}\right) = \mathbb{Q}^2\right) = 2$$

Rank of a Transition

For a transition t in 

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here, $\mathrm{rank}(t) = (1,0,0) \qquad \in \mathbb{N}^{d+1}$

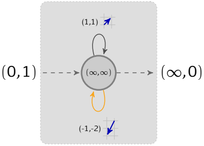
Definition

$$\mathrm{rank}(G) \stackrel{\text{def}}{=} \sum_{t \in G} \mathrm{rank}(t) \qquad \in \mathbb{N}^{d+1}$$

(ordered lexicographically)

Rank of a VAS

For a transition t in



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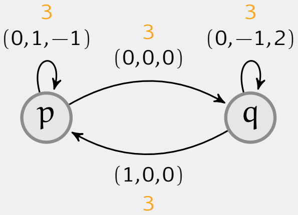
Rank of a VAS

Example



Example

Rank of a VAS

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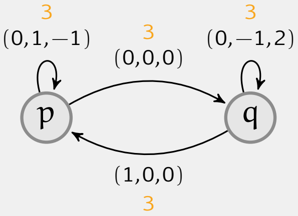

$\mathrm{rank}(G) = (4,0,0,0)$

Example

Rank of a VAS

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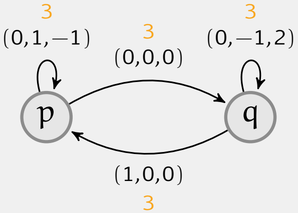

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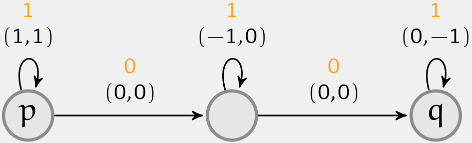

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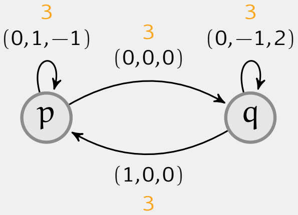

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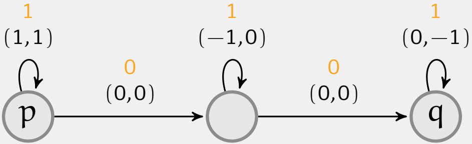
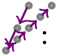


$$\mathrm{rank}(G) = (0,3,2)$$

Decreasing Ranks

Recall decomposition steps:

- no : ∅
- no :
  - bounded '∞': saturate
  - bounded transitions: unfold
- no  or no : track value

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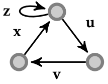
Decreasing Ranks

Proof Idea

Consider a strongly connected VAS G:



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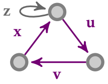
$T \setminus T'$: not in any hom. sol.



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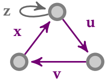

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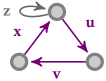

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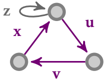

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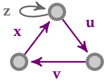

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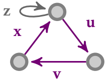

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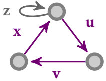

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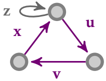

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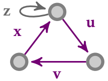
  - $\exists q \in \mathbb{N}$ s.t. $qa, qb, qc \geqslant p\lambda$
  - $[(p + qa - p\lambda)\mathbf{x}, p\mathbf{z}, (p + qb - p\lambda)\mathbf{u}, (p + qc - p\lambda)\mathbf{v}]$ also hom. sol

Decreasing Ranks

Proof Idea

Consider a strongly connected VAS G:

$T \setminus T'$: not in any hom. sol.



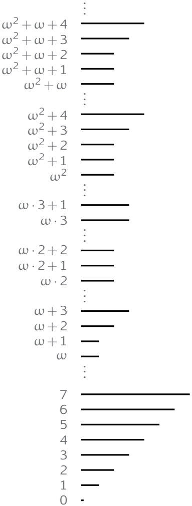
T': in an homogeneous solution $[a\mathbf{x}, b\mathbf{u}, c\mathbf{v}]$

Claim: if $T' \subsetneq T$, then $\mathrm{rank}(G') < \mathrm{rank}(G)$

- let $\mathbf{V}$, resp. $\mathbf{V}'$, be the vector space spanned by the cycles of T, resp. T'
- we want to show $\dim(\mathbf{V}') < \dim(\mathbf{V})$
- as $\mathbf{V}' \subseteq \mathbf{V}$, it suffices to show that $\mathbf{V}' = \mathbf{V}$ implies $T' = T$
  - pick cycle using every transition: effect $\mathbf{x} + \mathbf{z} + \mathbf{u} + \mathbf{v} \in \mathbf{V}$
  - $\mathbf{V} = \mathbf{V}'$ thus $\exists \lambda \in \mathbb{Q}$ s.t. $\mathbf{x} + \mathbf{z} + \mathbf{u} + \mathbf{v} = \lambda(\mathbf{x} + \mathbf{u} + \mathbf{v})$
  - pick $p \in \mathbb{N}_{>0}$ s.t. $p\lambda \in \mathbb{Z}$; $p(\mathbf{x} + \mathbf{z} + \mathbf{u} + \mathbf{v}) - p\lambda(\mathbf{x} + \mathbf{u} + \mathbf{v}) = \mathbf{0}$
  - $\exists q \in \mathbb{N}$ s.t. $qa, qb, qc \geqslant p\lambda$
  - $[(p + qa - p\lambda)\mathbf{x}, p\mathbf{z}, (p + qb - p\lambda)\mathbf{u}, (p + qc - p\lambda)\mathbf{v}]$ also hom. sol

Complexity Upper Bounds

Ordinals



- Cantor normal form for ordinals $\alpha < \varepsilon_0$:

$$\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_k} \cdot c_k$$

$$\alpha > \alpha_1 > \cdots > \alpha_k \text{ in CNF}, \qquad 0 < c_1, \ldots, c_k < \omega$$

- norm of ordinals $\alpha < \varepsilon_0$: "maximal constant"

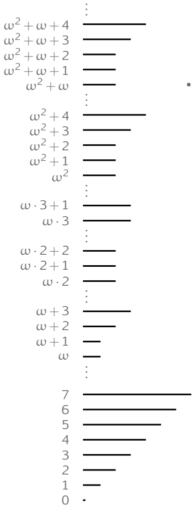
$$N\alpha \overset{\text{def}}{=} \max_{1 \leqslant i \leqslant k} (\max(N\alpha_i, c_i))$$

Example

$$N7 = 7 \qquad N(\omega \cdot 3 + 1) = 3$$

$$N(\omega^2 + \omega) = 2 \qquad N(\omega^2 + \omega + 4) = 4$$

Descending Ordinal Sequences

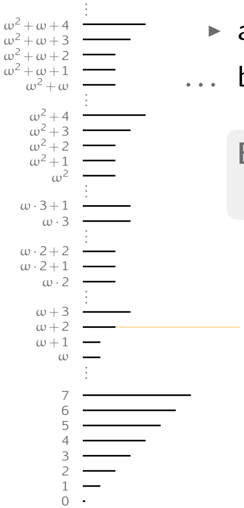


- always finite

... but can be of arbitrary length

Example

Descending Ordinal Sequences



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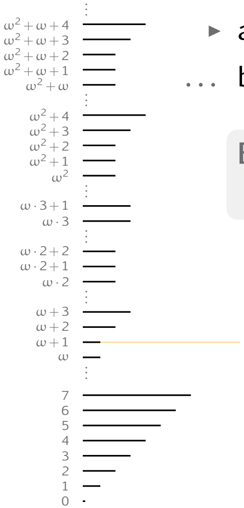
Example

$$\omega+2$$

Descending Ordinal Sequences



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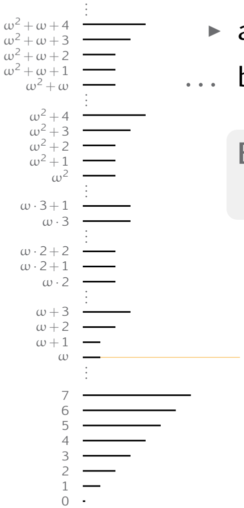
Example

$$\omega + 2 > \omega + 1$$

Descending Ordinal Sequences



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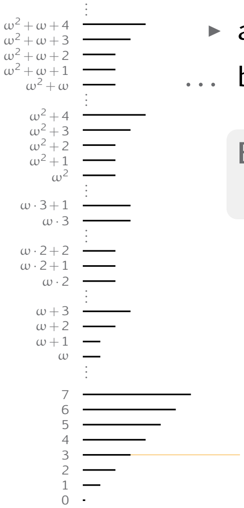
Example

$$\omega+2 > \omega+1 > \omega$$

Descending Ordinal Sequences



- always finite

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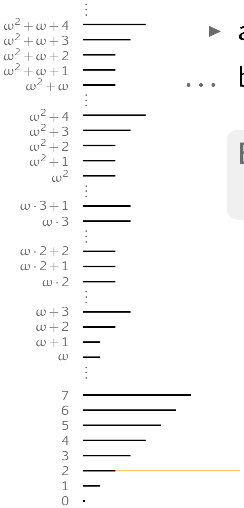
Example

$$\omega+2>\omega+1>\omega>3$$

Descending Ordinal Sequences



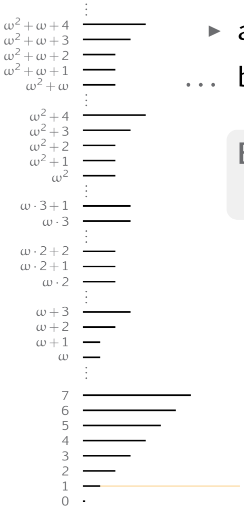
- always finite

... but can be of arbitrary length

Example

$$\omega+2>\omega+1>\omega>3>2$$

Descending Ordinal Sequences



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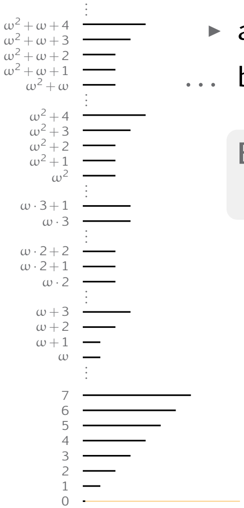
Example

$$\omega+2 > \omega+1 > \omega > 3 > 2 > 1$$

Descending Ordinal Sequences



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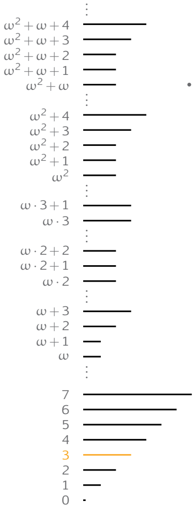

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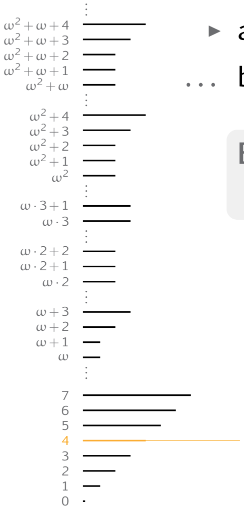

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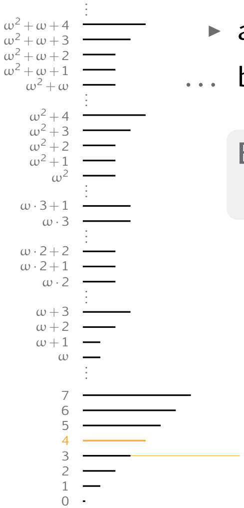
Example

$$\omega+2 > \omega+1 > \omega > 4$$

Descending Ordinal Sequences



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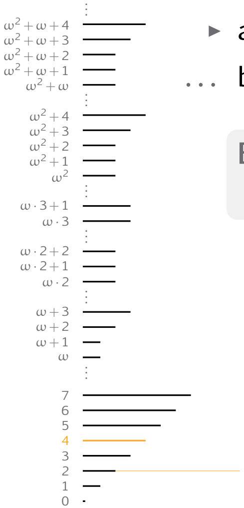
Example

$$\omega+2>\omega+1>\omega>4>3$$

Descending Ordinal Sequences



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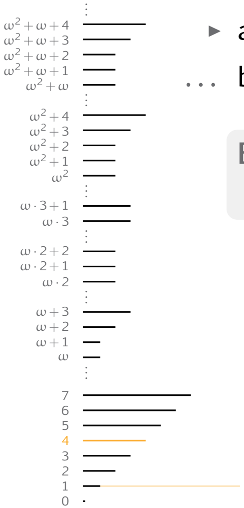

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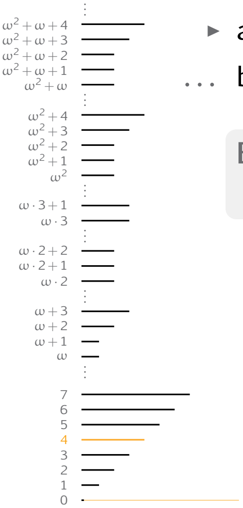
$$\omega+2>\omega+1>\omega>4>3>2>1$$

Descending Ordinal Sequences



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Example

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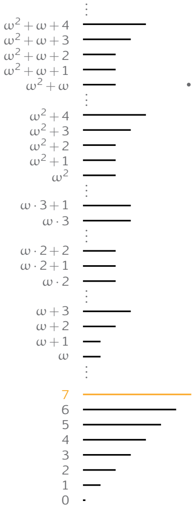

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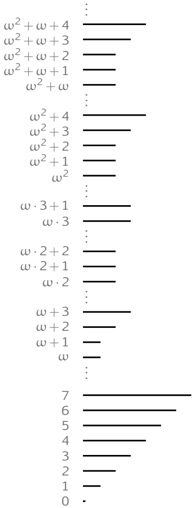
Example

$$\omega+2>\omega+1>\omega>7>6>5>4>3>2>1>0$$

Descending Ordinal Sequences



- $\alpha_0 > \alpha_1 > \ldots$ is controlled by $g\colon \mathbb{N} \to \mathbb{N}$ (monotone inflationary) and $n_0 \in \mathbb{N}$ if

$$\forall i \,.\, N\alpha_i \leqslant g^i(n_0)$$

Example $(g(x) = x + 1, n_0 = 2)$

Proposition
Descending sequences of ordinals in $\alpha < \varepsilon_0$ controlled by g and n_0 have a maximal length, noted $L_{g,\alpha}(n_0)$.

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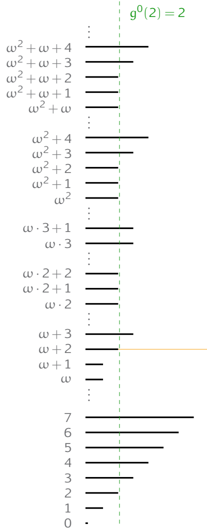

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$\omega + 2$

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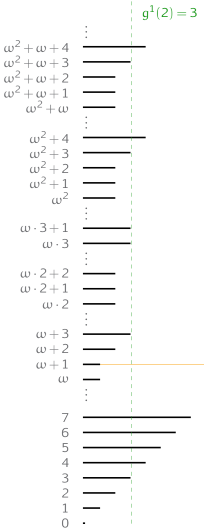

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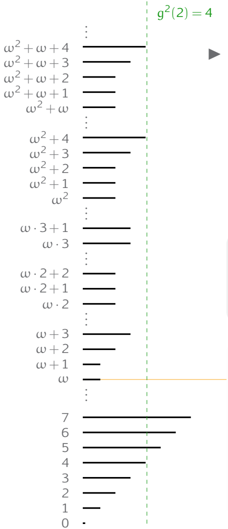

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$$\omega+2 > \omega+1 > \omega$$

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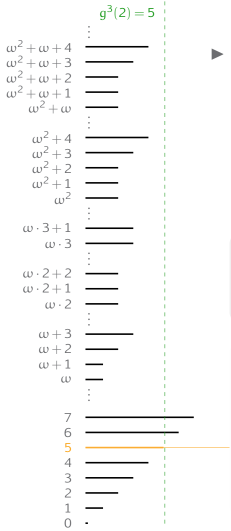

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Example $(g(x) = x+1, n_0 = 2)$

$$\omega+2 > \omega+1 > \omega > 5$$

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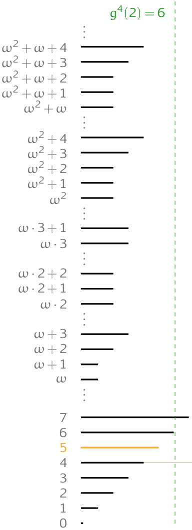
Descending Ordinal Sequences



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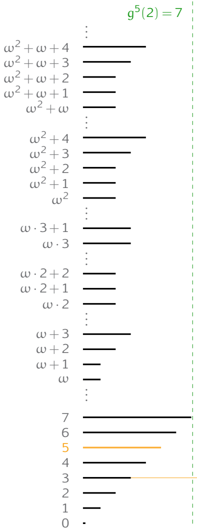

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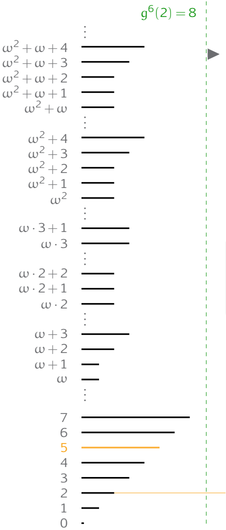

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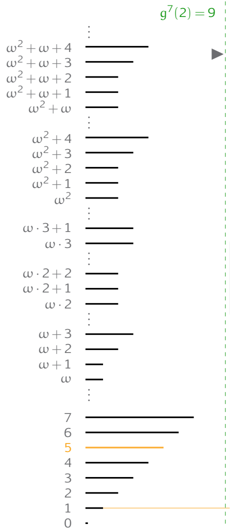

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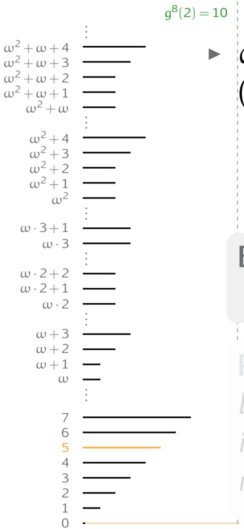

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Proposition

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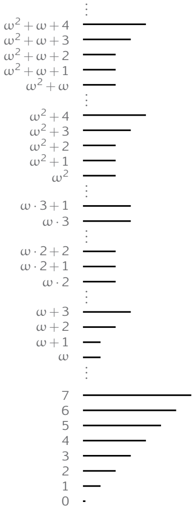
DESCENDING ORDINAL SEQUENCES



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EXAMPLE $(g(x) = x+1, n_0 = 2)$

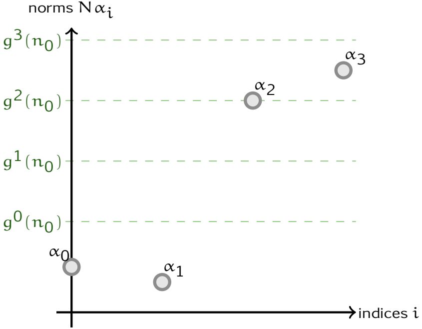
$$\omega+2 > \omega+1 > \omega > 5 > 4 > 3 > 2 > 1 > 0$$

PROPOSITION

Descending sequences of ordinals in $\alpha < \varepsilon_0$ controlled by g and n_0 have a maximal length, noted $L_{g,\alpha}(n_0)$.

Descent Equation

(g, n_0)-controlled descending sequence $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$ over an ordinal α:



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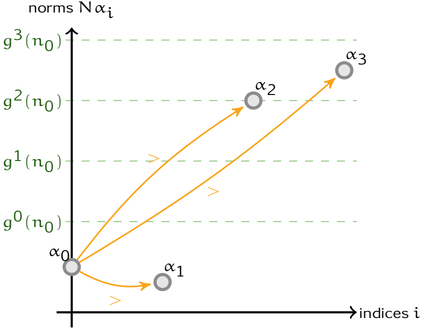

over the suffix $\alpha_1, \alpha_2, \alpha_3, \ldots, \forall i > 0$,

$$\alpha_0 > \alpha_i$$

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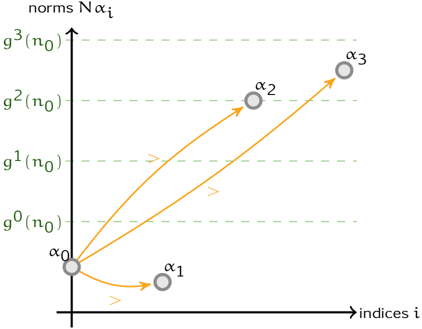

over the suffix $\alpha_1, \alpha_2, \alpha_3, \ldots, \forall i > 0$,

$\alpha_i \in \alpha_0$

Descent Equation

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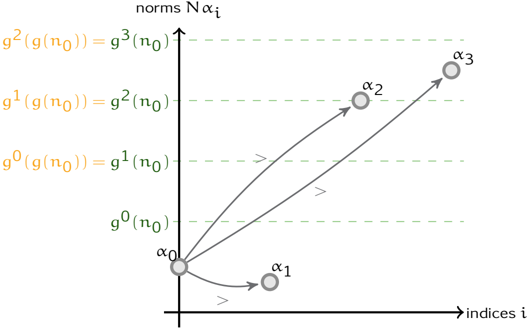

over the suffix $\alpha_1, \alpha_2, \alpha_3, \ldots$, $\forall i > 0$,

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$$N\alpha_i \leqslant g^{i-1}(g(n_0))$$

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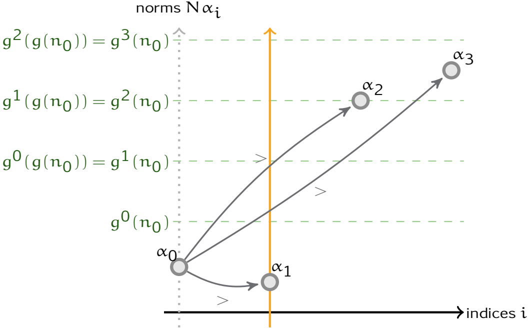

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$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, N\alpha_0 \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

Descent Equation

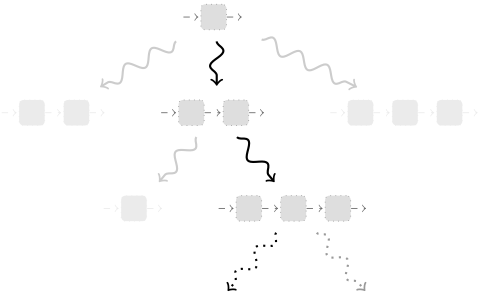
$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, N\alpha_0 \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

Consequence of (S. '14, '16)

For g elementary, $L_{g,\omega^{d+1}}(n_0) \leqslant F_{d+4}(e(n_0))$ *for some elementary function* e.

The Length of Decomposition Branches

[Leroux & S. '19]



$$\omega^{d+1} > \alpha_0 > \alpha_1 > \alpha_2 > \vdots$$

Corollary

The decomposition tree is of size at most $F_{d+4}(e(n))$ for some elementary function e.

The Length of Decomposition Branches

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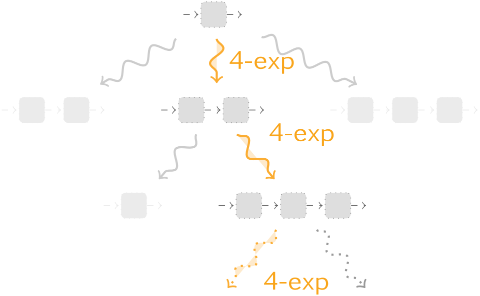

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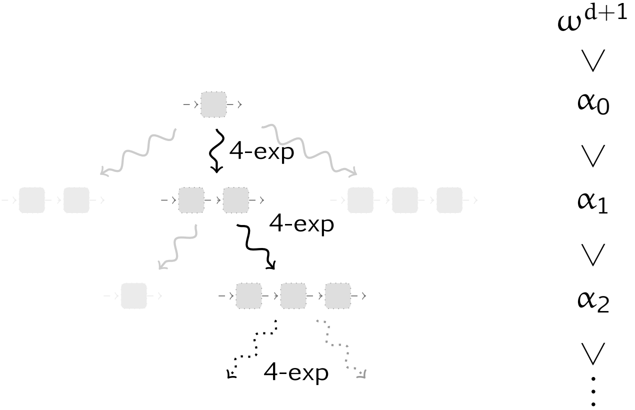


Corollary
The decomposition tree is of size at most $F_{d+4}(e(n))$ for some elementary function e.

New Upper Bounds

$$F_0(x) = x+1$$

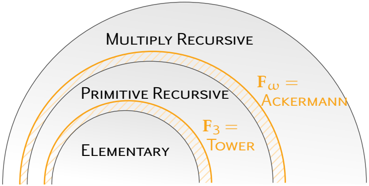
$$F_1(x) = \overbrace{F_0 \circ \cdots \circ F_0}^{x+1 \text{ times}}(x) = 2x+1$$

$$F_2(x) = \overbrace{F_1 \circ \cdots \circ F_1}^{x+1 \text{ times}}(x) \approx 2^x$$

$$F_3(x) = \overbrace{F_2 \circ \cdots \circ F_2}^{x+1 \text{ times}}(x) \approx \text{tower}(x)$$

$$\vdots$$

$$F_\omega(x) = F_{x+1}(x) \approx \text{ackermann}(x)$$



Upper Bound Theorem

VAS Reachability is in $\mathbf{F}_\omega$*, and in* $\mathbf{F}_{d+4}$ *in fixed dimension* d

New Upper Bounds

$$F_0(x) = x + 1$$

$$F_1(x) = \overbrace{F_0 \circ \cdots \circ F_0}^{x+1 \text{ times}}(x) = 2x + 1$$

$$F_2(x) = \overbrace{F_1 \circ \cdots \circ F_1}^{x+1 \text{ times}}(x) \approx 2^x$$

$$F_3(x) = \overbrace{F_2 \circ \cdots \circ F_2}^{x+1 \text{ times}}(x) \approx \text{tower}(x)$$

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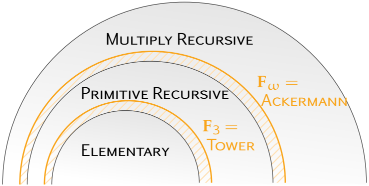

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Theorem

VAS Reachability reduces to bounded VAS Reachability

A Related Problem

labelled VAS transitions carry labels from some alphabet

$L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ the language of labels in runs from **source** to **target**

$\downarrow L$ the set of scattered subwords of the words in the language L

Example (scattered subword ordering)

$$\mathtt{aba} \leqslant_* \mathtt{baaacabbab}$$

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Downwards Language Inclusion Problem

input: *two labelled VAS* $\mathcal{V}$ *and* $\mathcal{V}'$ *and configurations* $\mathbf{source}, \mathbf{target}, \mathbf{source}', \mathbf{target}'$

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Given a labelled VAS $\mathcal{V}$ *and configurations* $\mathbf{source}$ *and* $\mathbf{target}$ *and its decomposition, one can construct a finite automaton for* $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ *in polynomial time.*

Corollary

The Downwards Language Inclusion is in Ackermann.

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Theorem (Zetzsche'16)

The Downwards Language Inclusion is Ackermann*-hard.*

Perspectives

1. complexity gap for VAS reachability
   - Tower-hard [Czerwinski et al.'19]
   - decomposition algorithm: requires $\mathbf{F}_\omega$ = Ackermann time, because downward language inclusion is $\mathbf{F}_\omega$-hard [Zetzsche'16]
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   - decidable in VAS with hierarchical zero tests [Reinhardt'08]
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     - unordered data Petri nets
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