Multiply-Recursive Bounds with Higman's Lemma

S. Schmitz Ph. Schnoebelen

LSV, ENS Cachan & CNRS

ICALP 2011, Zürich, July 8, 2011

Outline

well quasi orderings (wqo)

generic tools for termination arguments

this talk

beyond termination: complexity upper bounds

contents

Well Quasi Orderings

Definition (wqo)

A wqo is a quasi-order $(A, \leqslant)$ s.t.

$$\forall \mathbf{x} = x_0, x_1, x_2, \ldots \in A^\omega, \exists i_1 < i_2, x_{i_1} \leqslant x_{i_2} .$$

Example (Basic WQO's)

- $(\mathbb{N}, \leqslant)$,
- $(\{0, 1, \ldots, k\}, \leqslant)$ for any $k \in \mathbb{N}$,
- $(\Gamma_p, =)$ for any finite set Γ_p with p elements.

Algebra of WQO's

Finite Sequences

Lemma (Higman's Lemma)

If $(A, \leqslant)$ *is a wqo, then* $(A^*, \leqslant_*)$ *is a wqo where* $\leqslant_*$ *is the subword embedding ordering:*

$$a_1 \cdots a_m \leqslant_* b_1 \cdots b_n \overset{\text{def}}{\Leftrightarrow} \begin{cases} \exists 1 \leqslant i_1 < \cdots < i_m \leqslant n, \\ \bigwedge_{j=1}^{m} a_j \leqslant_A b_{i_j} \;. \end{cases}$$

Example

$$aba \leqslant_* baaacabbab$$

Algebra of WQO's

Disjoint Sums

Lemma

If $(A_1, \leqslant_{A_1})$ *and* $(A_2, \leqslant_{A_2})$ *are two wqo's, then* $(A_1 + A_2, \leqslant_+)$ *is a wqo,*

where $A_1 + A_2 \stackrel{\text{def}}{=} \{\langle i, a\rangle \mid i \in \{1,2\} \wedge a \in A_i\}$ *and* $\leqslant_+$ *is the sum ordering:*

$$\langle i, a\rangle \leqslant_+ \langle j, b\rangle \stackrel{\text{def}}{\Leftrightarrow} i = j \wedge a \leqslant_{A_i} b\ .$$

Algebra of WQO's

Cartesian Products

Lemma (Dickson's Lemma)

If $(A_1, \leqslant_{A_1})$ *and* $(A_2, \leqslant_{A_2})$ *are two wqo's, then* $(A_1 \times A_2, \leqslant_\times)$ *is a wqo,*

where $\leqslant_\times$ *is the product ordering:*

$$\langle a_1, a_2 \rangle \leqslant_\times \langle b_1, b_2 \rangle \overset{\text{def}}{\Leftrightarrow} a_1 \leqslant_{A_1} b_1 \wedge a_2 \leqslant_{A_2} b_2 \ .$$

WQO's for Termination

Bad Sequences

- $\mathbf{x} = x_0, x_1, \ldots$ in A^∞ is a *good sequence* if $\exists i_1 < i_2,\ x_{i_1} \leqslant x_{i_2}$,
- a *bad sequence* otherwise,
- if $(A, \leqslant)$ is a wqo: every bad sequence is finite

An Example

$$
\begin{array}{l}
\textsc{simple}\ (a, b) \\
c \longleftarrow 1 \\
\textbf{while}\ a > 0 \wedge b > 0 \\
\qquad \langle a, b, c \rangle \longleftarrow \langle a - 1, b, 2c \rangle \\
\quad \textbf{or} \\
\qquad \langle a, b, c \rangle \longleftarrow \langle 2c, b - 1, 1 \rangle \\
\textbf{end}
\end{array}
$$

- in any run, $\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle$ is a bad sequence over $(\mathbb{N}^2, \leqslant_\times)$,
- $(\mathbb{N}^2, \leqslant_\times)$ is a wqo: all the runs are finite
- How long can SIMPLE run?

A Computation of SIMPLE$(2,3)$

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
$\langle a, b, c\rangle \longleftarrow \langle a-1, b, 2c\rangle$
or
$\langle a, b, c\rangle \longleftarrow \langle 2c, b-1, 1\rangle$
end

$\langle a, b, c\rangle$	loop iterations
$\langle 2, 3, 2^0\rangle$	0

A Computation of SIMPLE$(2,3)$

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
    $\langle a, b, c\rangle \longleftarrow \langle a-1, b, 2c\rangle$
  or
    $\langle a, b, c\rangle \longleftarrow \langle 2c, b-1, 1\rangle$
end

$\langle a, b, c\rangle$	loop iterations
$\langle 2, 3, 2^0\rangle$	0
$\langle 1, 3, 2^1\rangle$	1

A Computation of SIMPLE$(2,3)$

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
  $\langle a, b, c\rangle \longleftarrow \langle a-1, b, 2c\rangle$
 or
  $\langle a, b, c\rangle \longleftarrow \langle 2c, b-1, 1\rangle$
end

$\langle a, b, c\rangle$	loop iterations
$\langle 2, 3, 2^0\rangle$	0
$\langle 1, 3, 2^1\rangle$	1
$\langle 2^2, 2, 2^0\rangle$	2

A Computation of SIMPLE$(2,3)$

```
SIMPLE (a, b)
c ← 1
while a > 0 ∧ b > 0
        ⟨a, b, c⟩ ← ⟨a − 1, b, 2c⟩
    or
        ⟨a, b, c⟩ ← ⟨2c, b − 1, 1⟩
end
```

$\langle a,b,c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 2^2, 2, 2^0\rangle$	2
$\vdots$	$\vdots$
$\langle 1, 2, 2^{2^2-1}\rangle$	$2+2^2-1$

A Computation of SIMPLE$(2,3)$

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \land b > 0$
  $\langle a, b, c\rangle \longleftarrow \langle a-1, b, 2c\rangle$
 or
  $\langle a, b, c\rangle \longleftarrow \langle 2c, b-1, 1\rangle$
end

$\langle a, b, c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 1, 2, 2^{2^2-1}\rangle$	$2 + 2^2 - 1$
$\langle 2^{2^2}, 1, 1\rangle$	$2 + 2^2$

A Computation of SIMPLE$(2,3)$

```
SIMPLE (a, b)
c ⟵ 1
while a > 0 ∧ b > 0
      ⟨a, b, c⟩ ⟵ ⟨a − 1, b, 2c⟩
   or
      ⟨a, b, c⟩ ⟵ ⟨2c, b − 1, 1⟩
end
```

$\langle a, b, c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 2^{2^2}, 1, 1\rangle$	$2 + 2^2$
$\vdots$	$\vdots$
$\langle 1, 1, 2^{2^{2^2}-1}\rangle$	$2 + 2^2 + 2^{2^2} - 1$

A Computation of SIMPLE$(2,3)$

```
SIMPLE (a, b)
c ⟵ 1
while a > 0 ∧ b > 0
      ⟨a, b, c⟩ ⟵ ⟨a − 1, b, 2c⟩
   or
      ⟨a, b, c⟩ ⟵ ⟨2c, b − 1, 1⟩
end
```

$\langle a, b, c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 1, 1, 2^{2^{2^2}-1}\rangle$	$2 + 2^2 + 2^{2^2} - 1$
$\langle 0, 1, 2^{2^{2^2}}\rangle$	$2 + 2^2 + 2^{2^2}$

A Computation of SIMPLE(2,3)

```
SIMPLE (a, b)
c ⟵ 1
while a > 0 ∧ b > 0
    ⟨a, b, c⟩ ⟵ ⟨a − 1, b, 2c⟩
  or
    ⟨a, b, c⟩ ⟵ ⟨2c, b − 1, 1⟩
end
```

$\langle a, b, c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 0, 1, 2^{2^{2^2}}\rangle$	$2 + 2^2 + 2^{2^2}$

- Non-elementary complexity
- Derive (matching) upper bounds for termination arguments based on $(\mathbb{N}^2, \leqslant_\times)$ being a wqo

Applications of Higman's Lemma

Decision of problems on

- lossy channel systems (Chambart and Schnoebelen, 2008),
- Post embedding problem $\mathrm{PEP}^{\mathrm{reg}}$ (Chambart and Schnoebelen, 2007),
- 1-clock alternating timed automata (Lasota and Walukiewicz, 2008),
- Metric temporal logic (Ouaknine and Worrell, 2007),
- finite concurrent programs under weak (TSO/PSO) memory models (Atig et al., 2010)
- alternating register automata over ordered domains (Figueira et al., 2010),
- ...

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant)$

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant)$
- but: choose any N, and consider the bad sequence $N, N-1, \ldots, 0$ over $\mathbb{N}$
- similarly: $\langle 3,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle, \langle 2,N\rangle, \langle 2,N-1\rangle, \ldots$

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant; |.|_A)$
- associate a *norm function* $|.|_A : A \to \mathbb{N}$ to each wqo $(A, \leqslant)$
- assume $|.|_A$ is *proper* $\overset{\text{def}}{\Leftrightarrow}$ for all n

$$A_{<n} \overset{\text{def}}{=} \{x \in A \mid |x|_A < n\} \text{ is finite}$$

Definition (Normed WQO's)

$$|k|_{\mathbb{N}} \overset{\text{def}}{=} k \quad |a_i|_{\Gamma_p} \overset{\text{def}}{=} 0 \quad |\langle a, b\rangle|_{A\times B} \overset{\text{def}}{=} \max(|a|_A, |b|_B)$$

$$|\langle i, a\rangle|_{A_1+A_2} \overset{\text{def}}{=} |a|_{A_i} \quad |a_1 \cdots a_m|_{A^*} \overset{\text{def}}{=} \max(m, |a_1|_A, \ldots, |a_m|_A)$$

Controlled Sequences

- bound the length of *controlled* bad sequences over $(A, \leqslant; |.|_A)$
- fix a *control function* $g : \mathbb{N} \to \mathbb{N}$ (monotone with $g(x+1) \geqslant g(x) + 1 \geqslant x + 2$)
- $\mathbf{x} = x_0, x_1, \ldots$ over A is (g, n)-*controlled* iff

$$\forall i,\ |x_i|_A < g^i(n)$$

Controlled Sequences

- bound the length of *controlled* bad sequences over $(A, \leqslant; |.|_A)$
- fix a *control function* $g : \mathbb{N} \to \mathbb{N}$
- $\mathbf{x} = x_0, x_1, \ldots$ over A is (g, n)-*controlled* iff

$$\forall i,\ |x_i|_A < g^i(n)$$

Example (SIMPLE(2,3))

$A = \mathbb{N}^2$, $n = 4$, $g(x) = 2x$

Controlled Sequences

- bound the length of *controlled* bad sequences over $(A, \leqslant; |.|_A)$
- for fixed A, g, n, there are *finitely* many bad (g, n)-controlled sequences over A
- maximal length function

$$L_{A,g}(n)$$

Technical Overview

1. *residuals*: inductive definition for $L_{A,g}$
2. *reflections*: approximations to obtain inequalities for $L_{A,g}$ in terms of "simpler" wqo's
3. *ordinal notations*: associate ordinal terms to wqo's in order to work with subrecursive hierarchies

Residuals

Definition (Residual)

The *residual* of a wqo $(A, \leqslant)$ by $a \in A$ is the wqo

$$A/a \stackrel{\text{def}}{=} \{b \in A \mid a \not\leqslant b\}\,.$$

Proposition (Descent Equation)

$$L_{A,g}(n) = \max_{a \in A_{<n}} \left\{1 + L_{A/a,g}(g(n))\right\}$$

Example

$$\Gamma_{p+1}/a \equiv \Gamma_p \qquad\qquad \mathbb{N}/k \equiv \{0, \ldots, k-1\}$$
$$L_{\Gamma_p,g}(n) = p \qquad\qquad L_{\mathbb{N},g}(n) = n\,.$$

Reflections

Definition (Normed Reflection)

A *normed reflection* is a mapping $h : A \to B$ between two normed wqo's satisfying

$$\forall a, b \in A, h(a) \leqslant_B h(b) \text{ implies } a \leqslant_A b$$
$$\forall a \in A, |h(a)|_B \leqslant |a|_A$$

Notation

$A \hookrightarrow B$: there exists a normed reflection $h : A \to B$

Example

$$\{0, \ldots, k-1\} \hookrightarrow \Gamma_k \qquad \Gamma_p \hookrightarrow \Gamma_{p+1}$$

Reflections

Proposition (Monotony of Length Function)

$$A \hookrightarrow B \text{ implies } \forall n, L_{A,g}(n) \leqslant L_{B,g}(n)\ .$$

Proposition (Reflections for Residuals)

$$\begin{aligned}
(A+B)/\langle 1,a\rangle &= (A/a)+B\\
(A+B)/\langle 2,b\rangle &= A+(B/b)\\
(A\times B)/\langle a,b\rangle &\hookrightarrow [(A/a)\times B]+[A\times(B/b)]\\
\Gamma^*_{p+1}/a_1\cdots a_m &\hookrightarrow \Gamma_m\times(\Gamma^*_p)^m
\end{aligned}$$

$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$

"Jullien's Decomposition"

Example

$$aba, \overbrace{aaa,\ bbb,\ aabb,\ baa,\ abb,\ bb}^{\in \Gamma_2^*/aba}$$

$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$

"Jullien's Decomposition"

Example

$$aba, \overbrace{aaa,\ bbb,\ aabb,\ baa,\ abb,\ bb}^{\in \Gamma_2^*/aba}$$

$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$

"Jullien's Decomposition"

Example

$$aba, \overbrace{\begin{bmatrix} & bbb, & & & bb \\ aaa, & & & baa, & \\ & & aabb, & & abb, \end{bmatrix}}^{\in \Gamma_3 \times \Gamma_2^*}$$

$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$

"Jullien's Decomposition"

Example

$$aba, \overbrace{\left[\langle \varepsilon, aa, \varepsilon\rangle, \langle bbb, \varepsilon, \varepsilon\rangle, \langle \varepsilon, a, b\rangle, \langle b, a, \varepsilon\rangle, \langle \varepsilon, \varepsilon, b\rangle, \langle bb, \varepsilon, \varepsilon\rangle \right]}^{\in \Gamma_3 \times (\Gamma_1^*)^3}$$

Ordinal Terms

- *maximal order type* $o : WQO \to \mathrm{CNF}(\omega^{\omega^{\omega}})$ (de Jongh and Parikh, 1977; Hasegawa, 1994)
- well-founded relations ∂_n over $\mathrm{CNF}(\omega^{\omega^{\omega}})$ implement reflection of residuals of size $< n$

Example

$$
\begin{array}{ccc}
\Gamma_2^* & \xrightarrow{\bigcup_{|x|<4}[\cdot/x \hookrightarrow \cdot]} & \Gamma_3 \times (\Gamma_1^*)^3 \\
\Big\downarrow o & & \Big\downarrow o \\
\omega^{\omega} & \xrightarrow[\partial_4]{} & \omega^3 \cdot 3
\end{array}
$$

Main Inequality

$$L_{o^{-1}(\alpha),g}(n) \leqslant \max_{\alpha' \in \partial_n \alpha} \left\{ 1 + L_{o^{-1}(\alpha'),g}(g(n)) \right\} .$$

A Bounding Function

$$M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha} \{1 + M_{\alpha',g}(g(n))\} .$$

- Then for all α and n

$$L_{A,g}(n) \leqslant M_{o(A),g}(n)$$

- find the *functional complexity* of M

Subrecursive Hierarchies

Hierarchies of functions (and function classes) indexed by *ordinal terms*.

Fundamental Sequences

Subrecursive hierarchies are defined through an assignment of *fundamental sequences* $(\lambda_x)_{x<\omega}$ for limit ordinal terms λ, s.t. $\lambda_x < \lambda$ and $\lambda = \sup_x \lambda_x$: e.g.

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^{\beta} \cdot (x+1)$$

$$(\gamma + \omega^{\lambda})_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x} \; ,$$

Example

$$\omega_x = x + 1 \qquad\qquad (\omega^{\omega^{p+1}})_x = \omega^{\omega^{p}\cdot(x+1)}$$

Fast Growing Hierarchy: $(F_\alpha)_\alpha$

(Löb and Wainer, 1970)

$$F_0(x) \stackrel{\text{def}}{=} x + 1, \quad F_{\alpha+1}(x) \stackrel{\text{def}}{=} F_\alpha^{x+1}(x), \quad F_\lambda \stackrel{\text{def}}{=} F_{\lambda_x}(x)\,.$$

Example

$$F_1(x) = 2x + 1$$

$$F_2(x) = (x+1) \cdot 2^{x+1} - 1$$

F_3 is non elementary

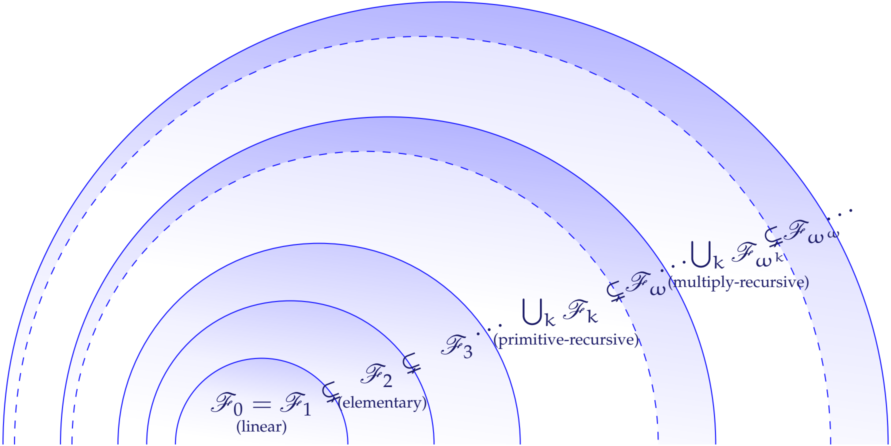
F_ω is non primitive-recursive

F_{ω^ω} is non multiply-recursive

Extended Grzegorczyk Hierarchy: $(\mathscr{F}_\alpha)_\alpha$

(Löb and Wainer, 1970)

Elementary-recursive closure of the $(F_\alpha)_\alpha$



Comparison with $(\mathscr{F}_\alpha)_\alpha$

Theorem (Simple Version)

Let $p \geqslant 2$ *and* g *primitive-recursive. Then* $L_{\Gamma_p^*, g}$ *is bounded by a function in* $\mathscr{F}_{\omega^{p-1}}$.

Concluding Remarks

- practical applications of wqo's yield upper bounds!
- out-of-the-box upper bounds
- "essentially" matching lower bounds

F_{ω^ω}-Complete Problems

Decision of problems on

- lossy channel systems (Chambart and Schnoebelen, 2008),
- Post embedding problem $\mathrm{PEP}^{\mathrm{reg}}$ (Chambart and Schnoebelen, 2007),
- 1-clock alternating timed automata (Lasota and Walukiewicz, 2008),
- Metric temporal logic (Ouaknine and Worrell, 2007),
- finite concurrent programs under weak (TSO/PSO) memory models (Atig et al., 2010)
- alternating register automata over ordered domains (Figueira et al., 2010),
- ...

References: Upper Bounds for WQO

Dickson's Lemma

McAloon, K., 1984. Petri nets and large finite sets. *Theor. Comput. Sci.*, 32 (1–2):173–183. doi:10.1016/0304-3975(84)90029-X.

Clote, P., 1986. On the finite containment problem for Petri nets. *Theor. Comput. Sci.*, 43:99–105. doi:10.1016/0304-3975(86)90169-6.

Figueira, D., Figueira, S., Schmitz, S., and Schnoebelen, Ph., 2011. Ackermannian and primitive-recursive bounds with Dickson's Lemma. In *LICS 2011*. IEEE Press. arXiv:1007.2989[cs.LO].

Higman's Lemma

Cichoń, E.A. and Tahhan Bittar, E., 1998. Ordinal recursive bounds for Higman's Theorem. *Theor. Comput. Sci.*, 201(1–2):63–84. doi:10.1016/S0304-3975(97)00009-1.

Kruskal's Theorem

Weiermann, A., 1994. Complexity bounds for some finite forms of Kruskal's Theorem. *J. Symb. Comput.*, 18(5):463–488. doi:10.1006/jsco.1994.1059.

References

Fast Growing Hierarchy

Löb, M. and Wainer, S., 1970. Hierarchies of number theoretic functions, I. *Arch. Math. Logic*, 13:39–51. doi:10.1007/BF01967649.

WSTS

Abdulla, P.A., Čerāns, K., Jonsson, B., and Tsay, Y.K., 2000. Algorithmic analysis of programs with well quasi-ordered domains. *Inform. and Comput.*, 160(1–2):109–127. doi:10.1006/inco.1999.2843.

Finkel, A. and Schnoebelen, Ph., 2001. Well-structured transition systems everywhere! *Theor. Comput. Sci.*, 256(1–2):63–92. doi:10.1016/S0304-3975(00)00102-X.

Lower Bounds

Chambart, P. and Schnoebelen, Ph., 2008. The ordinal recursive complexity of lossy channel systems. In *LICS 2008*, pages 205–216. IEEE. doi:10.1109/LICS.2008.47.

References: Applications

Chambart, P. and Schnoebelen, Ph., 2007. Post embedding problem is not primitive recursive, with applications to channel systems. In Arvind, V. and Prasad, S., editors, *FSTTCS 2007*, volume 4855 of *LNCS*, pages 265–276. Springer. doi:10.1007/978-3-540-77050-3_22.

Ouaknine, J.O. and Worrell, J.B., 2007. On the decidability and complexity of Metric Temporal Logic over finite words. *Logic. Meth. in Comput. Sci.*, 3 (1):8. doi:10.2168/LMCS-3(1:8)2007.

Lasota, S. and Walukiewicz, I., 2008. Alternating timed automata. *ACM Trans. Comput. Logic*, 9(2):10. doi:10.1145/1342991.1342994.

Atig, M.F., Bouajjani, A., Burckhardt, S., and Musuvathi, M., 2010. On the verification problem for weak memory models. In *POPL 2010*, pages 7–18. ACM Press. doi:10.1145/1706299.1706303.

Figueira, D., Hofman, P., and Lasota, S., 2010. Relating timed and register automata. In Fröschle, S. and Valencia, F., editors, *EXPRESS 2010*, volume 41 of *EPTCS*, pages 61–75. doi:10.4204/EPTCS.41.5.

$\mathbb{N}^2/\langle 2,2\rangle \hookrightarrow \mathbb{N}/2 \times \mathbb{N} + \mathbb{N}/2 \times \mathbb{N}$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \begin{bmatrix} \langle 1,5\rangle, & & \langle 1,1\rangle,\langle 0,100\rangle,\langle 0,99\rangle, & & (\{0,1\}\times\mathbb{N}) \\ & \langle 4,0\rangle, & & \langle 3,0\rangle & (\{0,1\}\times\mathbb{N}) \end{bmatrix}$$

$\mathbb{N}^2/\langle 2,2\rangle \hookrightarrow \mathbb{N}/2 \times \mathbb{N} + \mathbb{N}/2 \times \mathbb{N}$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle,\begin{bmatrix}\langle 1,5\rangle, & & {\color{red}\langle 1,1\rangle},\langle 0,100\rangle,\langle 0,99\rangle, & & (\{0,1\}\times\mathbb{N})\\ & \langle 4,0\rangle,{\color{red}\langle 1,1\rangle}, & & \langle 3,0\rangle & (\{0,1\}\times\mathbb{N})\end{bmatrix}$$

Ordinal Terms for WQO's

Quick Reminder

Definition (Ordinal Terms)

$$\alpha ::= 0 \mid \omega^{\alpha} \mid \alpha + \alpha$$

Definition (Cantor Normal Form)

$$\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$$

with $\alpha > \beta_1 \geqslant \cdots \geqslant \beta_m \geqslant 0$ with each β_i in CNF itself.

Ordinal Terms for WQO's

Operations on CNF

Definition (Natural Sum)

$$\sum_{i=1}^{m} \omega^{\beta_i} \oplus \sum_{j=1}^{n} \omega^{\beta'_j} \stackrel{\text{def}}{=} \sum_{k=1}^{m+n} \omega^{\gamma_k},$$

where $\gamma_1 \geqslant \cdots \geqslant \gamma_{m+n}$ is a rearrangement of $\beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_n$.

Definition (Natural Product)

$$\sum_{i=1}^{m} \omega^{\beta_i} \otimes \sum_{j=1}^{n} \omega^{\beta'_j} \stackrel{\text{def}}{=} \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \omega^{\beta_i \oplus \beta'_j}.$$

Ordinal Terms for WQO's

Maximal Order Type (de Jongh and Parikh, 1977; Hasegawa, 1994)

A bijection between normed WQO's with $+$, $\times$, and * over finite sets, and $\mathrm{CNF}(\omega^{\omega^\omega})$:

$$o(\Gamma_p) \stackrel{\text{def}}{=} p\,, \qquad o(\Gamma_0^*) \stackrel{\text{def}}{=} \omega^0\,, \qquad o(\Gamma_{p+1}^*) \stackrel{\text{def}}{=} \omega^{\omega^p}\,,$$

$$o(A+B) \stackrel{\text{def}}{=} o(A) \oplus o(B)\,, \quad o(A \times B) \stackrel{\text{def}}{=} o(A) \otimes o(B)\,.$$

Example

$$o((\Gamma_{p+2}^*)^{k+1} \times \Gamma_{q+1}) = \omega^{\omega^{p+1}\cdot(k+1)} \cdot (q+1)\,.$$

Ordinal Terms for WQO's

- translate reflections of residuals into *derivative* ordinal terms
- $\forall n$, we define a well-founded relation ∂_n over $\mathrm{CNF}(\omega^{\omega^\omega})$ s.t. if $a \in A_{<n}$, then $\exists \alpha' \in \partial_n o(A)$ s.t. $A/a \hookrightarrow o^{-1}(\alpha')$.

Example

$$\partial_n 0 = \emptyset, \qquad \partial_n 1 = \{0\}, \qquad \partial_n \omega = \{n-1\},$$

$$\partial_n(\omega^{\omega^{p+1}\cdot(k+1)} \cdot (q+1)) = \{\omega^{\omega^{p+1}\cdot(k+1)} \cdot q + \omega^{[\omega^{p+1}\cdot k + \omega^p \cdot (n-1)]} \cdot (k+1)(n-1)\}.$$

Fundamental Sequences

Subrecursive hierarchies are defined through an assignment of *fundamental sequences* $(\lambda_x)_{x<\omega}$ for limit ordinal terms λ, s.t. $\lambda_x < \lambda$ and $\lambda = \sup_x \lambda_x$: e.g.

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^{\beta} \cdot (x+1)$$
$$(\gamma + \omega^{\lambda})_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x} ,$$

Example

$$\omega_x = x + 1 \qquad (\omega^{\omega^{p+1}})_x = \omega^{\omega^{p} \cdot (x+1)}$$

Well-Structured Transition Systems

- transition systems $(Q, \rightarrow, q_0)$ with a wqo $\leqslant$ on Q compatible with transitions:

$$\forall p, q, p' \in Q, (p \xrightarrow{a} q \wedge p \leqslant p') \Rightarrow \exists q', (q \leqslant q' \wedge p' \xrightarrow{a} q')$$

- a generic framework for decidability results: safety, termination, EF model checking, ...
- many classes of concrete systems are WSTS:
  - over $(\mathbb{N}^k, \leqslant_\times)$: vector addition systems, resets/transfer Petri nets, increasing counter systems, ...
  - over $(\Gamma_p^*, \leqslant_*)$: lossy channel systems, ...
  - beyond: data nets, ...

Example: (Non) Termination

- given $(Q, \to, q_0)$, decide whether there exists an infinite run $q_0 \to q_1 \to \cdots$
- holds iff there exists $q_i \leqslant q_j$ with $q_0 \to^* q_i \to^+ q_j$
- thanks to wqo, termination is both r.e. and co-r.e.
- what is the complexity?

Example: (Non) Termination

- given $(Q, \to, q_0)$, decide whether there exists an infinite run $q_0 \to q_1 \to \cdots$
- holds iff there exists $q_i \leqslant q_j$ with $q_0 \to^* q_i \to^+ q_j$
- thanks to wqo, termination is both r.e. and co-r.e.
- **what is the complexity?**

Example: Lossy Channel Systems

- $\langle Q, M, C, \delta \rangle$
- Q finite set of q states, M size-m message alphabet, C set of c channels, $\delta \subseteq Q \times (\{\mathrm{nop}\} \uplus (\{!, ?\} \times M))^C \times Q$ set of transitions,
- configurations in $A = Q \times (M^*)^C$,
- WSTS for $(s, w_1, \ldots, w_{|C|}) \to (s', w'_1, \ldots, w'_{|C|})$ iff $\exists (s, e_1, \ldots, e_{|C|}, s') \in \delta$ s.t. $\forall i$

$$\begin{cases} w'_i \leqslant_* w_i & \text{if } e_i = \mathrm{nop} \\ \exists w \in M^*, w \leqslant_* w_i \wedge w'_i \leqslant_* aw & \text{if } e_i = !a \\ \exists w \in M^*, wa \leqslant_* w_i \wedge w'_i \leqslant_* w & \text{if } e_i = ?a \end{cases}$$

Example: Lossy Channel Systems

- $o(A) = \omega^{\omega^{m-1} \cdot c} \cdot q$
- linear control by $g(x) = x + 1$ in $\mathscr{F}_1$
- non-terminating run from $s_{\text{init}} = (s_0, w_1, \ldots, w_{|C|})$ iff there exists a run of length $L_{A,g}(|s_{\text{init}}|)$
- non termination in $\mathscr{F}_{\omega^{m-1} \cdot c}$

Hardy Hierarchy: $(h^\alpha)_\alpha$

Fix $h : \mathbb{N} \to \mathbb{N}$:

$$h^0(x) \stackrel{\text{def}}{=} x, \quad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^\alpha(h(x)), \quad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda_x}(x)\,.$$

Example

For $h(x) = x + 1$:

$$H^\omega(x) = H^{x+1}(x) = 2x + 1 \quad H^{\omega\cdot 2}(x) = H^{\omega+x+1}(x) = 4x + 3$$

Lemma

For all $r < \omega$, α, *and* x,

$$h^{\omega^\alpha \cdot r}(x) = f_\alpha^r(x)\,.$$

Length Hierarchy: $(h_\alpha)_\alpha$

Fix $h : \mathbb{N} \to \mathbb{N}$:

$$h_0(x) \stackrel{\text{def}}{=} 0,\ h_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + h_\alpha(h(x)),\ h_\lambda(x) \stackrel{\text{def}}{=} h_{\lambda_x}(x)\,.$$

Lemma

For all α, x

$$h_\alpha(x) \leqslant h^\alpha(x) - x$$

Lemma

Define the predecessor at x of $\alpha > 0$ as

$$P_x(\alpha+1) \stackrel{\text{def}}{=} \alpha\,, \qquad P_x(\lambda) \stackrel{\text{def}}{=} P_x(\lambda_x)$$

Then $$h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x))\,.$$

Monotonicity Matters

Lemma

For h *monotone with* $h(x) \geqslant x$ *and any* α*,*

$$x < y \text{ implies } h^\alpha(x) \leqslant h^\alpha(y) \; .$$

But: for $x < n$,
$H^\omega(x) = 2x + 1 < x + n + 1 = H^{n+1}(x)$, i.e.

$$\alpha < \beta \text{ does } \textit{not} \text{ imply } h^\alpha(x) \leqslant h^\beta(x)$$

Monotonicity Matters

Lemma

For h *monotone with* $h(x) \geqslant x$ *and any* α*,*

$$x < y \textit{ implies } h^{\alpha}(x) \leqslant h^{\alpha}(y) \; .$$

But: for $x < n$,
$H^{\omega}(x) = 2x + 1 < x + n + 1 = H^{n+1}(x)$, i.e.

$$\alpha < \beta \text{ does } \textit{not} \text{ imply } h^{\alpha}(x) \leqslant h^{\beta}(x)$$

Monotonicity Matters

Definition (Pointwise Ordering)

For all x, $\prec_x$ is the smallest transitive relation s.t.

$$\alpha \prec_x \alpha + 1 \qquad\qquad \lambda_x \prec_x \lambda\ .$$

Lemma

For h monotone with $h(x) \geqslant x$ and any x,

$$\alpha \prec_x \beta \textit{ implies } h^\alpha(x) \leqslant h^\beta(x)\ .$$

Comparison with $(h_\alpha)_\alpha$

Contrast $M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha}\{1 + M_{\alpha',g}(g(n))\}$ with $h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x))$:

Proposition

For all α in $\mathrm{CNF}(\omega^{\omega^\omega})$, there is a constant k s.t. for all $n > 0$, $M_{\alpha,g}(n) \leqslant h_\alpha(kn)$ where $h(x) \stackrel{\text{def}}{=} x \cdot g(x)$.

Example (Higman's Lemma)

For bad (g, n)-controlled sequences in Γ_p^*:

$$L_{\Gamma_p^*,g}(n) \leqslant h_{\omega^{\omega^{p-1}}}((p-1)n) \quad \text{where } h(x) \stackrel{\text{def}}{=} x \cdot g(x) .$$

Comparison with $(h_\alpha)_\alpha$

Contrast $M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha}\{1 + M_{\alpha',g}(g(n))\}$ with $h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x))$:

Proposition

For all α in $\mathrm{CNF}(\omega^{\omega^\omega})$, there is a constant k s.t. for all $n > 0$, $M_{\alpha,g}(n) \leqslant h_\alpha(kn)$ where $h(x) \stackrel{\text{def}}{=} x \cdot g(x)$.

Example (Higman's Lemma)

For bad (g, n)-controlled sequences in Γ_p^*:

$$L_{\Gamma_p^*,g}(n) \leqslant h_{\omega^{\omega^{p-1}}}((p-1)n) \quad \text{where } h(x) \stackrel{\text{def}}{=} x \cdot g(x) \; .$$