Ackermann and Primitive-Recursive Bounds with Dickson's Lemma

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Outline

well quasi orderings

generic tools for termination arguments

this talk

beyond termination: complexity upper bounds

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Dickson's Lemma

Definition (wqo)

A wqo is a quasi-order $(S, \leqslant)$ s.t.

$$\forall \mathbf{x} = x_0, x_1, x_2, \dots \in S^\omega,\ \exists i_1 < i_2,\ x_{i_1} \leqslant x_{i_2}\ .$$

Lemma (Dickson's Lemma)

If $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *are two wqo's,*
then $(A \times B, \leqslant_\times)$ *is a wqo,*
where $\leqslant_\times$ *is the product ordering:*

$$\langle a, b\rangle \leqslant_\times \langle a', b'\rangle \stackrel{\text{def}}{\Leftrightarrow} a \leqslant_A a' \wedge b \leqslant_B b'\ .$$

Dickson's Lemma

Definition (wqo)

A wqo is a quasi-order $(S, \leqslant)$ s.t.

$$\forall \mathbf{x} = x_0, x_1, x_2, \cdots \in S^\omega,\ \exists i_1 < i_2,\ x_{i_1} \leqslant x_{i_2}\ .$$

Lemma (Dickson's Lemma)

In this talk: $(\mathbb{N}^k, \leqslant)$ *is a wqo.*

Applications of Dickson's Lemma

Termination and decision of problems on

- well-structured transition systems (Finkel and Schnoebelen, 2001),
- Datalog with constraints (Revesz, 1993),
- Gröbner's bases (Gallo and Mishra, 1994),
- relevance logics (Urquhart, 1999),
- LTL with Presburger constraints (Demri, 2006),
- data logics (Demri and Lazić, 2009; Figueira and Segoufin, 2009),
- ...

An Example

$$
\begin{array}{l}
\textsc{simple}\ (a, b) \\
c \longleftarrow 1 \\
\textbf{while}\ a > 0 \wedge b > 0 \\
\qquad \langle a, b, c \rangle \longleftarrow \langle a-1, b, 2c \rangle \\
\quad \textbf{or} \\
\qquad \langle a, b, c \rangle \longleftarrow \langle 2c, b-1, 1 \rangle \\
\textbf{end}
\end{array}
$$

- in any run $\langle a_0, b_0, c_0 \rangle, \ldots, \langle a_n, b_n, c_n \rangle$, $\langle a_0, b_0 \rangle \not\leq \langle a_n, b_n \rangle$
- Dickson's Lemma: all the runs are finite
- How long can SIMPLE run?

Complexity of SIMPLE

$$\langle 3,3,2^0\rangle, \langle 2,3,2^1\rangle, \langle 1,3,2^2\rangle,$$
$$\langle 2^3,2,1\rangle,\ldots,\langle 1,2,2^{2^3-1}\rangle,$$
$$\langle 2^{2^3},1,1\rangle,\ldots,\langle 1,1,2^{2^{2^3}-1}\rangle,$$
$$\langle 0,1,2^{2^{2^3}}\rangle$$

- $3+2^3+2^{2^3}+1$ steps: non elementary lower bound
- This talk: (matching) upper bound from the use of Dickson's Lemma

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Complexity of SIMPLE

$$\begin{aligned}&\langle 3,3,2^0\rangle, \langle 2,3,2^1\rangle, \langle 1,3,2^2\rangle,\\ &\quad \langle 2^3,2,1\rangle, \ldots, \langle 1,2,2^{2^3-1}\rangle,\\ &\quad \langle 2^{2^3},1,1\rangle, \ldots, \langle 1,1,2^{2^{2^3}-1}\rangle,\\ &\quad \langle 0,1,2^{2^{2^3}}\rangle\end{aligned}$$

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Bad Sequences

- $\mathbf{x} = x_0, x_1, \ldots$ in S^∞ is a *good sequence* if $\exists i_1 < i_2,\ x_{i_1} \leqslant x_{i_2}$,
- a *bad sequence* otherwise,
- $(S, \leqslant)$ wqo: every bad sequence is finite

Controlled Sequences

- bound the length of bad sequences
- but: choose any N, and consider the bad sequence $N, N-1, \ldots, 0$ over $\mathbb{N}$
- similarly: $\langle 3,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle, \langle 2,N\rangle, \langle 2,N-1\rangle, \ldots$

Controlled Sequences

- bound the length of *controlled* bad sequences

- fix a *control function* $f : \mathbb{N} \to \mathbb{N}$
- $\mathbf{x} = x_0, x_1, \ldots$ over $\mathbb{N}^k$ is (f, t)-*controlled* if

$$\forall i = 0, 1, \ldots, \forall 1 \leqslant j \leqslant k, x_i[j] < f(i + t)$$

- for fixed k, f, t, there are *finitely* many (f, t)-controlled sequences over $\mathbb{N}^k$: maximal length

$$L_{k,f}(t)$$

Example

$k = 2, t = 3, f(x) = x + 1$

i	0	1	2	3	4	5	$\cdots$	10	11	12	13	$\cdots$	26	27	28	29	$\cdots$	58	59
$x_i[1]$	3	3	3	3	2	2	$\cdots$	2	2	1	1	$\cdots$	1	1	0	0	$\cdots$	0	0
$x_i[2]$	3	2	1	0	7	6	$\cdots$	1	0	15	14	$\cdots$	1	0	31	30	$\cdots$	1	0
$f(i+t)$	4	5	6	7	8	9	$\cdots$	14	15	16	17	$\cdots$	30	31	32	33	$\cdots$	62	63

Example (SIMPLE)

$k = 2, t = 2 = \lceil \log_2(\max(a, b)) \rceil, f(x) = 2^x + 1$

Technical Overview

1. obtain inequalities for $L_{k,f}$ in terms of "simpler" wqo's
2. define a bounding function M with $L_{k,f}(t) \leqslant M_{k,f}(t)$
3. rank $M_{k,f}$ in a hierarchy of function classes $(\mathscr{F}_k)_k$

Easy Cases

$$L_{0,f}(t) = 1$$
$$L_{1,f}(t) = f(t)$$

the latter sequence being

$$f(t) - 1,\ f(t) - 2,\ \ldots,\ 1,\ 0$$

A More General Problem

- *disjoint sums* $A_1 \oplus A_2$
- wqo for the sum ordering :

$$\begin{aligned} x \leqslant x' \overset{\text{def}}{\Leftrightarrow} & \left(x, x' \in A_1 \wedge x \leqslant_1 x'\right) \\ & \vee \left(x, x' \in A_2 \wedge x \leqslant_2 x'\right) \end{aligned}$$

- multiset notation: $\tau = \{\!\!\{ k_1, k_2, \ldots \}\!\!\}$, $\mathbb{N}^\tau = \bigoplus_i \mathbb{N}^{k_i}$
- shift to $L_{\tau,f}(t)$

Inequality for $\mathbb{N}^k$

A bad sequence $\mathbf{x} = x_0, x_1, \ldots, x_l$ over $\mathbb{N}^k$:

- control: $x_0 \leqslant \langle f(t) - 1, \ldots, f(t) - 1 \rangle$
- badness: $\forall i > 0, \exists j \leqslant k, x_i[j] < x_0[j] \leqslant f(t) - 1$
- each x_i belongs to at least one *region* $R_{j,s}$ depending on its value $s = x_i[j]$ at coordinate j
- $R_{j,s} = \{x \in \mathbb{N}^k \mid x[j] = s\}$
- there are $N_{k,f}(t) \stackrel{\text{def}}{=} k \cdot (f(t) - 1)$ regions in total

Inequality for $\mathbb{N}^k$

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- badness: $\forall i > 0,\ \exists j \leqslant k,\ x_i[j] < x_0[j] \leqslant f(t)-1$
- each x_i belongs to at least one *region* $R_{j,s}$ depending on its value $s = x_i[j]$ at coordinate j
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- there are $N_{k,f}(t) \stackrel{\text{def}}{=} k \cdot (f(t) - 1)$ regions in total

Inequality for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{llllll} & & & \langle {\color{red}0},100\rangle,\langle {\color{red}0},99\rangle, & & (R_{1,0} : x[1] = 0) \\ \langle {\color{red}1},5\rangle, & & \langle {\color{red}1},1\rangle, & & & (R_{1,1} : x[1] = 1) \\ & \langle 4,{\color{red}0}\rangle, & & & \langle 3,{\color{red}0}\rangle & (R_{2,0} : x[2] = 0) \\ & & & & & (R_{2,1} : x[2] = 1) \end{array}\right]$$

Inequality for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{llllll} & & & \langle 0,100\rangle,\langle 0,99\rangle, & & (R_{1,0} : x[1] = 0) \\ \langle 1,5\rangle, & & \color{red}{\langle 1,1\rangle}, & & & (R_{1,1} : x[1] = 1) \\ & \langle 4,0\rangle, & & & \langle 3,0\rangle & (R_{2,0} : x[2] = 0) \\ & & \color{red}{\langle 1,1\rangle} & & & (R_{2,1} : x[2] = 1) \end{array}\right]$$

Inequality for $\mathbb{N}^k$

Example

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Suffix: a *bad* sequence over

$$\mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N} = \mathbb{N}^{4\times\{1\}} = \mathbb{N}^{N_{k,f}(t)\times\{k-1\}}$$

Inequality for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{llllll} & & & \langle 100\rangle, \langle 99\rangle, & & (R_{1,0} : x[1] = 0) \\ \langle 5\rangle, & & \langle 1\rangle, & & & (R_{1,1} : x[1] = 1) \\ & \langle 4\rangle, & & & \langle 3\rangle & (R_{2,0} : x[2] = 0) \\ & & & & & (R_{2,1} : x[2] = 1) \end{array}\right]$$

Suffix: an $(f, t+1)$-*controlled* bad sequence:

$$L_{\{k\},f}(t) \leqslant 1 + L_{N_{k,f}(t)\times\{k-1\},f}(t+1)$$

Inequality for $\bigoplus_i \mathbb{N}^{k_i}$

Example

$\tau = \{\!|1,2,2|\!\}$:

$$\left[\langle 2,2\rangle, \langle 5\rangle, \langle 1,5\rangle, \langle 4,0\rangle, \langle 3\rangle \langle 12,1\rangle, \langle 1,1\rangle, \langle 3,5\rangle \langle 0,100\rangle, \langle 0,99\rangle, \langle 3,0\rangle\right]$$

$$\langle 2,2\rangle \left[\langle 5\rangle, \langle *,5\rangle, \langle 4,*\rangle, \langle 3\rangle \langle 12,1\rangle, \langle *,1\rangle, \langle 3,5\rangle \langle *,100\rangle, \langle *,99\rangle, \langle 3,*\rangle\right]$$

Inequality for $\bigoplus_i \mathbb{N}^{k_i}$

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Example

$$\langle 2,2\rangle \left[\begin{array}{l} \langle 5\rangle, \quad \langle 3\rangle \quad \langle *,100\rangle, \langle *,99\rangle, \\ \langle *,5\rangle, \quad \langle *,1\rangle, \\ \langle 4,*\rangle, \quad \langle 3,*\rangle \\ \langle 12,1\rangle, \quad \langle 3,5\rangle \end{array}\right]$$

$$\begin{aligned} \tau_{\langle k,t,f\rangle} &\stackrel{\text{def}}{=} \tau - \{k\} + N_{k,f}(t) \times \{k-1\} \\ &= \tau - \{k\} + k(f(t)-1) \times \{k-1\} \end{aligned}$$

$$L_{\tau,f}(t) \leqslant \max_{k\in\tau} \left\{ 1 + L_{\tau_{\langle k,t,f\rangle},f}(t+1) \right\}$$

A Bounding Function

$$M_{\tau,f}(t) \stackrel{\text{def}}{=} \max_{k \in \tau}\{1 + M_{\tau_{\langle k,t,f\rangle},f}(t+1)\}\ .$$

- Then for all τ and t

$$L_{\tau,f}(t) \leqslant M_{\tau,f}(t)$$

- find the *functional complexity* of M

Fast Growing Hierarchy: $(F_\alpha)_\alpha$

(Löb and Wainer, 1970)

Hierarchy of functions $(F_\alpha)_\alpha$ indexed by ordinals; we only need the *finite* fragment.

$$F_0(x) \stackrel{\text{def}}{=} x + 1$$

$$F_{n+1}(x) \stackrel{\text{def}}{=} F_n^{x+1}(x)$$

$$F_1(x) = 2x + 1$$

$$F_2(x) = (x + 1) \cdot 2^{x+1} - 1$$

F_3 is non elementary

$F_\omega \stackrel{\text{def}}{=} \lambda x.F_x(x)$ is non primitive-recursive

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Fast Growing Hierarchy: $(\mathscr{F}_\alpha)_\alpha$

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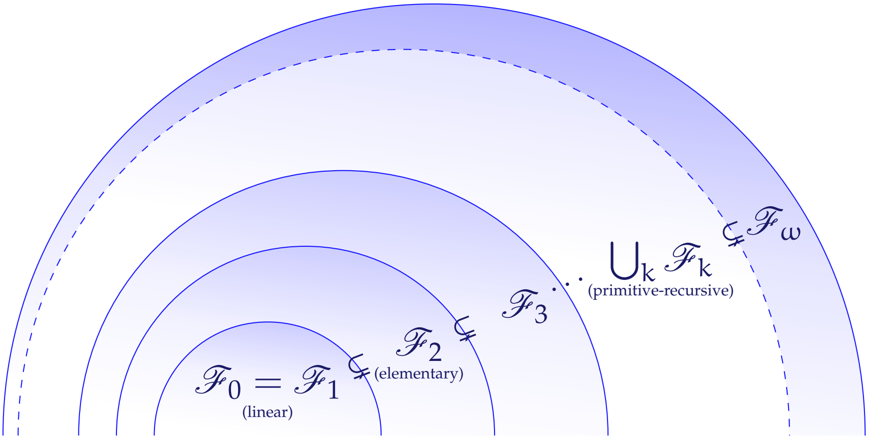
Elementary-recursive closure of the $(F_\alpha)_\alpha$



Maximizing Strategy

Lemma

For all $\tau \neq \emptyset$, $t \geqslant 0$, $M_{\tau,f}(t) = 1 + M_{\tau_{\langle \min \tau, t, f\rangle}, f}(t+1)$.

Example

$t = 1$, $f(x+t) = x + t$, $\tau = \{\!\{2, 1\}\!\}$: compare

$$\langle 0,0\rangle, \langle 1\rangle, \langle 0\rangle$$

$$\langle 0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \langle 0,3\rangle, \langle 0,2\rangle, \langle 0,1\rangle, \langle 0,0\rangle$$

A Variant of $(F_k)_k$

$$G_{1,f}(x) \stackrel{\text{def}}{=} f(x) + x$$
$$G_{k+1,f}(x) \stackrel{\text{def}}{=} G_{k,f}^{N_{k+1,f}(x)}(x+1)$$

Lemma

For all $k \geqslant 1$, $r \geqslant 1$, $x \geqslant 0$, $M_{r\times\{k\},f}(x) = G_{k,f}^{r}(x) - x$

Lemma

Let $\gamma \geqslant 1$ *and* f *be unary monotone in* $\mathscr{F}_\gamma$ *with* $f(x) \geqslant \max(1, x)$ *for all* x*. Then for all* $k \geqslant 1$, $G_{k,f}$ *belongs to* $\mathscr{F}_{\gamma+k-1}$.

Complexity Results

Proposition (Upper Bound)

Let $k, r \geqslant 1$ be natural numbers and $\gamma \geqslant 1$. If f is a monotone unary function of $\mathscr{F}_\gamma$ with $f(x) \geqslant \max(1, x)$ for all x, then $M_{r\times\{k\},f}$ is in $\mathscr{F}_{\gamma+k-1}$.

Proposition (Lower Bound)

Let $k, r \geqslant 1$ be natural numbers and $\gamma \geqslant 0$ with $\gamma + k \geqslant 3$. Then $L_{r\times\{k\},F_\gamma}$ is bounded below by a function which is **not** in $\mathscr{F}_{\gamma+k-2}$.

Applications of our Bounds

Termination and decision of problems on

- well-structured transition systems (Finkel and Schnoebelen, 2001),
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Program Termination Proofs

(Podelski and Rybalchenko, 2004)

Monolithic Termination Argument

- prove that the program's transition relation R is *well-founded*
- *ranking function* ρ from program configurations $\mathbf{x} = x_0, x_1, \ldots$ into a wqo s.t. $R \subseteq \{(x_i, x_j) \mid \rho(x_i) \not\leq \rho(x_j)\}$
- for SIMPLE: $\rho(a, b, c) = \omega^b + a$

Program Termination Proofs

(Podelski and Rybalchenko, 2004)

Disjunctive Termination Argument

- find well-founded relations $T_1, \ldots, T_k$ on program configurations
- prove $R^+ \subseteq T_1 \cup \cdots \cup T_k$
- for SIMPLE:

$$T_1 = \{(\langle a, b, c\rangle, \langle a', b', c'\rangle) \mid a > 0 \wedge a' < a\}$$
$$T_2 = \{(\langle a, b, c\rangle, \langle a', b', c'\rangle) \mid b > 0 \wedge b' < b\}$$

- at the heart of the TERMINATOR tool

Termination by Dickson's Lemma

- each T_j shown well-founded thanks to a ranking function ρ_j into a wqo $(S_j, \leqslant_j)$
- map any sequence of program configurations

$$\mathbf{x} = x_0, x_1, \ldots$$

to a sequence of tuples

$$\mathbf{y} = \langle \rho_1(x_0), \ldots, \rho_k(x_0) \rangle, \langle \rho_1(x_1), \ldots, \rho_k(x_1) \rangle, \ldots$$

in $S_1 \times \cdots \times S_k$

- $\mathbf{y}$ is *bad*: if $i_1 < i_2$, there exists j s.t.

$$(x_{i_1}, x_{i_2}) \in R^+ \cap T_j$$

but

$$\rho_j(x_{i_1}) \not\leqslant \rho_j(x_{i_2})$$

Termination by Dickson's Lemma

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$$\rho_j(x_{i_1}) \not\leqslant \rho_j(x_{i_2})$$

Bounds on Program Complexity

Make some assumptions:

- complexity bound g on atomic program operations
  - for instance polynomial
- complexity bound ρ on ranking functions into $\mathbb{N}$
  - for instance polynomial
- y controlled by $g^i \circ \rho$ in some $\mathscr{F}_\gamma$
  - in this case an exponential function in $\mathscr{F}_2$
- time complexity in $\mathscr{F}_{\gamma+k-1}$
  - in this case $\mathscr{F}_{k+1}$
- matches the lower bound (expand SIMPLE to dimension k instead of 2)

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Concluding Remarks

- practical applications of wqo's yield upper bounds!
- out-of-the-box upper bounds
- "essentially" matching lower bounds for decision problems on monotone counter systems (lossy counter systems, reset or transfer Petri nets)
- the future: Higman's Lemma

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References: Upper Bounds for WQO

Dickson's Lemma

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Lower Bounds

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Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\text{lex}})$, of length $\ell_{k,f}(t)$.

Example

$k = 2$, $t = 1$, $f(x) = x + 3$:

i	0	1	2	3	4	5	$\cdots$	10	11	12	13	$\cdots$	26	27	28	29	$\cdots$	58	59
$x_i[1]$	3	3	3	3	2	2	$\cdots$	2	2	1	1	$\cdots$	1	1	0	0	$\cdots$	0	0
$x_i[2]$	3	2	1	0	7	6	$\cdots$	1	0	15	14	$\cdots$	1	0	31	30	$\cdots$	1	0
$f(i+t)$	4	5	6	7	8	9	$\cdots$	14	15	16	17	$\cdots$	30	31	32	33	$\cdots$	62	63

$$\begin{aligned}
5 &= 1 + 4 &&= 1 + \ell_{1,f}(1) \\
13 &= 5 + 8 &&= 5 + \ell_{1,f}(5) \\
29 &= 16 + 13 &&= 13 + \ell_{1,f}(13)
\end{aligned}$$

Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{lex})$, of length $\ell_{k,f}(t)$.
In general, on the $k+1$th coordinate:

$$\underbrace{f(t)-1\ f(t)-1\ \cdots\ f(t)-1}_{\ell_{k,f}(t)\text{ times}}\quad \underbrace{f(t)-2,\ f(t)-2,\ \cdots,\ f(t)-2}_{\ell_{k,f}(o_{k,f}(t))\text{ times}}$$

$$\cdots\qquad \underbrace{0,\ 0,\ \cdots,\ 0}_{\ell_{k,f}\left(o_{k,f}^{f(t)-1}(t)\right)\text{ times}}$$

$$o_{k,f}(t) \stackrel{\text{def}}{=} t + \ell_{k,f}(t)$$

$$\ell_{k+1,f}(t) = \sum_{j=1}^{f(t)} \ell_{k,f}\left(o_{k,f}^{j-1}(t)\right)$$

Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\text{lex}})$, of length $\ell_{k,f}(t)$. One can have $\ell_{k,f}(t) < L(\{k\}, t)$: let $f(x) = 2x$ and $t = 1$,

$$\langle 1,1\rangle,\langle 1,0\rangle,\langle 0,5\rangle,\langle 0,4\rangle,\langle 0,3\rangle,\langle 0,2\rangle,\langle 0,1\rangle, \langle 0,0\rangle$$
$$\langle 1,1\rangle,\langle 0,3\rangle,\langle 0,2\rangle,\langle 0,1\rangle,\langle 9,0\rangle,\langle 8,0\rangle,\langle 7,0\rangle,\langle 6,0\rangle,\langle 5,0\rangle,\ldots,\langle 0,0\rangle$$

$$\ell_{2,f}(1) = 8 \qquad\qquad L_{\{2\},f}(1) \geqslant 14$$

Well-structured transition systems

- transition systems $(Q, \to, q_0)$ with a wqo $\leqslant$ on Q compatible with transitions:

$$\forall p, q, p' \in Q, (p \xrightarrow{a} q \wedge p \leqslant p') \Rightarrow \exists q', (q \leqslant q' \wedge p' \xrightarrow{a} q')$$

- a generic framework for decidability results: safety, termination, EF model checking, ...
- many classes of concrete systems are WSTS:
  - over $(\mathbb{N}^k, \leqslant)$: vector addition systems, resets/transfer Petri nets, increasing counter systems, ...
  - over $(\Sigma^*, \sqsubseteq)$: lossy channel systems, ...
  - beyond: data nets, ...

Example: (Non) Termination

- given $(Q, \rightarrow, q_0)$, decide whether there exists an infinite run $q_0 \rightarrow q_1 \rightarrow \cdots$
- holds iff there exists $q_i \leqslant q_j$ with $q_0 \rightarrow^* q_i \rightarrow^+ q_j$
- thanks to wqo, termination is both r.e. and co-r.e.
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- **what is the complexity?**

Affine Counter Systems

- $\mathcal{C} = \langle Q, k, \delta, m_0 \rangle$
- transitions (q, g, q') where $g(x) = Ax + B$ an affine function, $A \in \mathbb{N}^{k \times k}$, $B \in \mathbb{Z}^k$
- $m_0 \in \mathbb{N}^k$
- generalize reset/transfer Petri nets, broadcast protocols,. . .

Termination for ACS

Given $\langle \mathcal{C} \rangle$ a k-ACS, does every run of $\mathcal{C}$ terminate?

- exponential control in $\mathscr{F}_2$
- $t < |m_0| < |\mathcal{C}|$
- upper bound: $\mathscr{F}_{k+1}$
- lower bound: $\mathscr{F}_{k-O(1)}$ (Schnoebelen, 2010)
- if k is not fixed, non-primitive recursive, with an upper bound in $\mathscr{F}_\omega$