Ackermannian and Primitive-Recursive Bounds with Dickson's Lemma

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Outline

well quasi orderings (wqo)

generic tools for termination arguments

this talk

beyond termination: complexity upper bounds

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Dickson's Lemma
Length of Bad Sequences
Subrecursive Hierarchies

Well Quasi Orderings

Definition (wqo)

A wqo is a quasi-order $(A, \leqslant)$ s.t.

$$\forall \mathbf{x} = x_0, x_1, x_2, \ldots \in A^\omega, \exists i_1 < i_2, \; x_{i_1} \leqslant x_{i_2} \; .$$

Example (Basic WQO's)

- $(\mathbb{N}, \leqslant)$,
- $(\{0, 1, \ldots, k\}, \leqslant)$ for any $k \in \mathbb{N}$,
- $(\Gamma_p, =)$ for any finite set Γ_p with p elements.

Algebra of Polynomial WQO's

Disjoint Sums

Lemma

If $(A_1, \leqslant_{A_1})$ *and* $(A_2, \leqslant_{A_2})$ *are two wqo's, then* $(A_1 + A_2, \leqslant_+)$ *is a wqo,*
where $A_1 + A_2 \stackrel{\text{def}}{=} \{\langle i, a\rangle \mid i \in \{1,2\} \wedge a \in A_i\}$ *and* $\leqslant_+$ *is the sum ordering:*

$$\langle i, a\rangle \leqslant_+ \langle j, b\rangle \stackrel{\text{def}}{\Leftrightarrow} i = j \wedge a \leqslant_{A_i} b \,.$$

Algebra of Polynomial WQO's

Cartesian Products

Lemma (Dickson's Lemma)

If $(A_1, \leqslant_{A_1})$ *and* $(A_2, \leqslant_{A_2})$ *are two wqo's, then* $(A_1 \times A_2, \leqslant_\times)$ *is a wqo,*

where $\leqslant_\times$ *is the product ordering:*

$$\langle a_1, a_2 \rangle \leqslant_\times \langle b_1, b_2 \rangle \stackrel{\text{def}}{\Leftrightarrow} a_1 \leqslant_{A_1} b_1 \wedge a_2 \leqslant_{A_2} b_2 \ .$$

WQO's for Termination

Bad Sequences

- $\mathbf{x} = x_0, x_1, \ldots$ in A^∞ is a *good sequence* if $\exists i_1 < i_2,\ x_{i_1} \leqslant x_{i_2}$,
- a *bad sequence* otherwise,
- if $(A, \leqslant)$ is a wqo: every bad sequence is finite

An Example

$$
\begin{array}{l}
\textsc{simple}\ (a, b) \\
c \longleftarrow 1 \\
\textbf{while } a > 0 \wedge b > 0 \\
\qquad \langle a, b, c \rangle \longleftarrow \langle a - 1, b, 2c \rangle \\
\quad \textbf{or} \\
\qquad \langle a, b, c \rangle \longleftarrow \langle 2c, b - 1, 1 \rangle \\
\textbf{end}
\end{array}
$$

- in any run, $\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle$ is a bad sequence over $(\mathbb{N}^2, \leqslant_\times)$,
- $(\mathbb{N}^2, \leqslant_\times)$ is a wqo: all the runs are finite
- How long can SIMPLE run?

A Computation of SIMPLE$(2,3)$

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
  $\langle a, b, c\rangle \longleftarrow \langle a-1, b, 2c\rangle$
 or
  $\langle a, b, c\rangle \longleftarrow \langle 2c, b-1, 1\rangle$
end

$\langle a, b, c\rangle$	loop iterations
$\langle 2, 3, 2^0\rangle$	0

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$\langle a, b, c\rangle$	loop iterations
$\langle 2, 3, 2^0\rangle$	0
$\langle 1, 3, 2^1\rangle$	1

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end

$\langle a, b, c \rangle$	loop iterations
$\langle 2, 3, 2^0 \rangle$	0
$\langle 1, 3, 2^1 \rangle$	1
$\langle 2^2, 2, 2^0 \rangle$	2

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 or
  $\langle a, b, c \rangle \longleftarrow \langle 2c, b-1, 1 \rangle$
end

$\langle a, b, c \rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 2^2, 2, 2^0 \rangle$	2
$\vdots$	$\vdots$
$\langle 1, 2, 2^{2^2-1} \rangle$	$2 + 2^2 - 1$

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$\langle a, b, c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 2^{2^2}, 1, 1\rangle$	$2 + 2^2$
$\vdots$	$\vdots$
$\langle 1, 1, 2^{2^{2^2}-1}\rangle$	$2 + 2^2 + 2^{2^2} - 1$

A Computation of SIMPLE$(2,3)$

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$\langle a, b, c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 1, 1, 2^{2^{2^2}-1}\rangle$	$2 + 2^2 + 2^{2^2} - 1$
$\langle 0, 1, 2^{2^{2^2}}\rangle$	$2 + 2^2 + 2^{2^2}$

A Computation of SIMPLE(2, 3)

SIMPLE (a, b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
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$\langle a, b, c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 0, 1, 2^{2^{2^2}}\rangle$	$2 + 2^2 + 2^{2^2}$

- Non-elementary complexity
- Derive (matching) upper bounds for termination arguments based on Dickson's Lemma

Applications of Dickson's Lemma

Termination and decision of problems on

- well-structured transition systems (Finkel and Schnoebelen, 2001),
- Transition invariants (Podelski and Rybalchenko, 2004),
- Datalog with constraints (Revesz, 1993),
- Gröbner's bases (Gallo and Mishra, 1994),
- relevance logics (Urquhart, 1999),
- data logics (Demri and Lazić, 2009; Figueira and Segoufin, 2009),
- ...

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant)$

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant)$
- but: choose any N, and consider the bad sequence $N, N-1, \ldots, 0$ over $\mathbb{N}$
- similarly: $\langle 3,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle, \langle 2,N\rangle, \langle 2,N-1\rangle, \ldots$

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant; |.|_A)$
- associate a *norm function* $|.|_A : A \to \mathbb{N}$ to each wqo $(A, \leqslant)$

Definition (Normed WQO's)

$$|k|_{\mathbb{N}} \stackrel{\text{def}}{=} k \qquad |\langle a, b\rangle|_{A\times B} \stackrel{\text{def}}{=} \max(|a|_A, |b|_B)$$

$$|\langle i, a\rangle|_{A_1+A_2} \stackrel{\text{def}}{=} |a|_{A_i}$$

Controlled Sequences

- bound the length of *controlled* bad sequences over $(A, \leqslant; |.|_A)$
- fix a *control function* $f : \mathbb{N} \to \mathbb{N}$
- $\mathbf{x} = x_0, x_1, \ldots$ over A is (f, t)-*controlled* iff

$$\forall i,\ |x_i|_A < f(i + t)$$

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$$\forall i,\ |x_i|_A < f(i + t)$$

Example (SIMPLE(2,3))

$A = \mathbb{N}^2$, $t = 2 = \lceil \log_2(\max(a, b)) \rceil$, $f(x) = 2^x + 1$

Controlled Sequences

- bound the length of *controlled* bad sequences over $(A, \leqslant ; |.|_A)$
- for fixed A, f, t, there are *finitely* many bad (f, t)-controlled sequences over A
- maximal length function

$$L_{A,f}(t)$$

Technical Overview

1. obtain inequalities for $L_{\mathbb{N}^k,f}$ in terms of "simpler" wqo's
2. define a bounding function M with $L_{\mathbb{N}^k,f}(t) \leqslant M_{\mathbb{N}^k,f}(t)$
3. rank $M_{\mathbb{N}^k,f}$ in a hierarchy of function classes $(\mathscr{F}_n)_n$

Easy Cases

$$L_{\mathbb{N}^0,f}(t) = 1$$
$$L_{\mathbb{N}^1,f}(t) = f(t)$$

the latter sequence being

$$f(t) - 1,\ f(t) - 2,\ \ldots,\ 1,\ 0$$

Inequality for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{ll} \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\langle 0,99\rangle, & (R_{1,0} : x[1] = 0) \\ & (R_{1,1} : x[1] = 1) \\ \langle 3,0\rangle & (R_{2,0} : x[2] = 0) \\ & (R_{2,1} : x[2] = 1) \end{array} \right]$$

Inequality for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{l} \langle 0,100\rangle, \langle 0,99\rangle, \qquad (R_{1,0} : x[1] = 0) \\ \langle 1,5\rangle, \qquad {\color{red}\langle 1,1\rangle}, \qquad\qquad\qquad\qquad (R_{1,1} : x[1] = 1) \\ \qquad \langle 4,0\rangle, \qquad\qquad\qquad\qquad \langle 3,0\rangle\ (R_{2,0} : x[2] = 0) \\ \qquad\qquad {\color{red}\langle 1,1\rangle} \qquad\qquad\qquad\qquad\qquad (R_{2,1} : x[2] = 1) \end{array}\right]$$

Inequality for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \begin{bmatrix} & & & \langle *,100\rangle,\langle *,99\rangle, & & (R_{1,0} : x[1] = 0) \\ \langle *,5\rangle, & & \langle *,1\rangle, & & & (R_{1,1} : x[1] = 1) \\ & \langle 4,*\rangle, & & & \langle 3,*\rangle & (R_{2,0} : x[2] = 0) \\ & & & & & (R_{2,1} : x[2] = 1) \end{bmatrix}$$

Suffix: a *bad* sequence over

$$\mathbb{N} + \mathbb{N} + \mathbb{N} + \mathbb{N}$$

Inequality for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{lllll} & & \langle 100\rangle,\langle 99\rangle, & & (R_{1,0} : x[1] = 0) \\ \langle 5\rangle, & & \langle 1\rangle, & & (R_{1,1} : x[1] = 1) \\ & \langle 4\rangle, & & \langle 3\rangle & (R_{2,0} : x[2] = 0) \\ & & & & (R_{2,1} : x[2] = 1) \end{array}\right]$$

Suffix: an $(f, t+1)$-*controlled* bad sequence:

$$L_{\mathbb{N}^k,f}(t) \leqslant 1 + L_{\mathbb{N}^{k-1}\cdot k(f(t)-1),f}(t+1)$$

Inequality for $\sum_i \mathbb{N}^{k_i}$

Example

$A = \mathbb{N}^2 + \mathbb{N}^2 + \mathbb{N}^1$:

$$\left[\langle 2,2\rangle, \langle 5\rangle, \langle 1,5\rangle, \langle 4,0\rangle, \langle 3\rangle, \langle 12,1\rangle, \langle 1,1\rangle, \langle 3,5\rangle, \langle 0,100\rangle, \langle 0,99\rangle, \langle 3,0\rangle \right]$$

$$\langle 2,2\rangle \left[\langle 5\rangle, \langle *,5\rangle, \langle 4,*\rangle, \langle 3\rangle, \langle 12,1\rangle, \langle *,1\rangle, \langle 3,5\rangle, \langle *,100\rangle, \langle *,99\rangle, \langle 3,*\rangle \right]$$

Inequality for $\sum_i \mathbb{N}^{k_i}$

Example

$A = \mathbb{N}^2 + \mathbb{N}^2 + \mathbb{N}^1$:

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Inequality for $\sum_i \mathbb{N}^{k_i}$

Example

$$\langle 2,2\rangle \left[\langle 5\rangle, \langle 3\rangle, \langle *,100\rangle, \langle *,99\rangle, \langle *,5\rangle, \langle *,1\rangle, \langle 4,*\rangle, \langle 3,*\rangle, \langle 12,1\rangle, \langle 3,5\rangle \right]$$

$$\partial_x A = \{B + \mathbb{N}^{k-1} \cdot k(x-1) \mid A = B + \mathbb{N}^k\}$$

$$L_{A,f}(t) \leqslant \max_{A' \in \partial_{f(t)} A} \{1 + L_{A',f}(t+1)\}$$

A Bounding Function

$$M_{A,f}(t) \stackrel{\text{def}}{=} \max_{A' \in \partial_{f(t)} A} \{1 + M_{A',f}(t+1)\} \,.$$

- Then for all polynomial A and t

$$L_{A,f}(t) \leqslant M_{A,f}(t)$$

- find the *functional complexity* of M

Fast Growing Hierarchy: $(F_\alpha)_\alpha$

(Löb and Wainer, 1970)

Hierarchy of functions $(F_\alpha)_\alpha$ indexed by ordinals; we only need the *finite* fragment.

$$F_0(x) \stackrel{\text{def}}{=} x + 1$$

$$F_{n+1}(x) \stackrel{\text{def}}{=} F_n^{x+1}(x)$$

$$F_1(x) = 2x + 1$$

$$F_2(x) = (x+1) \cdot 2^{x+1} - 1$$

F_3 is non elementary

$F_\omega \stackrel{\text{def}}{=} \lambda x.F_x(x)$ is non primitive-recursive

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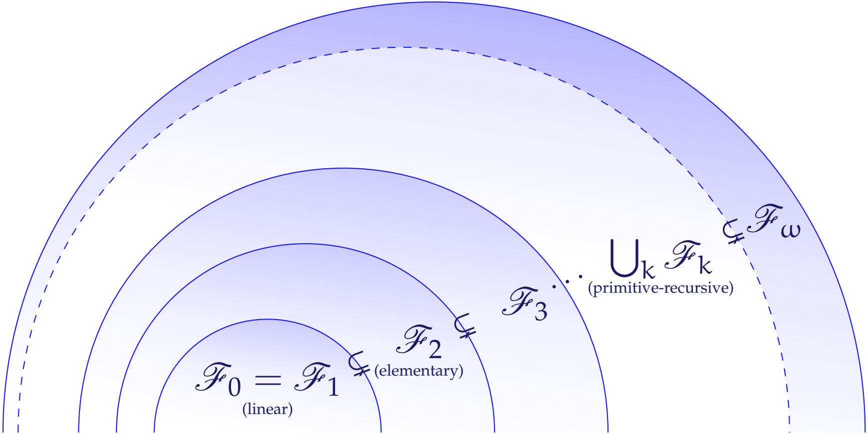
Fast Growing Hierarchy: $(\mathscr{F}_\alpha)_\alpha$

(Löb and Wainer, 1970)

Elementary-recursive closure of the $(F_\alpha)_\alpha$



Complexity Results

Proposition (Upper Bound)

Let $k, r \geqslant 1$ be natural numbers and $\gamma \geqslant 1$. If f is a monotone unary function of $\mathscr{F}_\gamma$ with $f(x) \geqslant \max(1, x)$ for all x, then $M_{\mathbb{N}^k \cdot r}$ is in $\mathscr{F}_{\gamma+k-1}$.

Proposition (Lower Bound)

Let $k, r \geqslant 1$ be natural numbers and $\gamma \geqslant 0$ with $\gamma + k \geqslant 3$. Then $L_{\mathbb{N}^k \cdot r, F_\gamma}$ is bounded below by a function which is **not** in $\mathscr{F}_{\gamma+k-2}$.

Concluding Remarks

- practical applications of wqo's yield upper bounds!
- out-of-the-box upper bounds
- "essentially" matching lower bounds for decision problems on monotone counter systems (lossy counter systems, reset or transfer Petri nets)
- ICALP 2011: Higman's Lemma

References: Upper Bounds for WQO

Dickson's Lemma

McAloon, K., 1984. Petri nets and large finite sets. *Theor. Comput. Sci.*, 32 (1–2):173–183. doi:10.1016/0304-3975(84)90029-X.

Clote, P., 1986. On the finite containment problem for Petri nets. *Theor. Comput. Sci.*, 43:99–105. doi:10.1016/0304-3975(86)90169-6.

Higman's Lemma

Cichoń, E.A. and Tahhan Bittar, E., 1998. Ordinal recursive bounds for Higman's Theorem. *Theor. Comput. Sci.*, 201(1–2):63–84. doi:10.1016/S0304-3975(97)00009-1.

Kruskal's Theorem

Weiermann, A., 1994. Complexity bounds for some finite forms of Kruskal's Theorem. *J. Symb. Comput.*, 18(5):463–488. doi:10.1006/jsco.1994.1059.

References

Fast Growing Hierarchy

Löb, M. and Wainer, S., 1970. Hierarchies of number theoretic functions, I. *Arch. Math. Logic*, 13:39–51. doi:10.1007/BF01967649.

WSTS

Abdulla, P.A., Čerāns, K., Jonsson, B., and Tsay, Y.K., 2000. Algorithmic analysis of programs with well quasi-ordered domains. *Inform. and Comput.*, 160(1–2):109–127. doi:10.1006/inco.1999.2843.

Finkel, A. and Schnoebelen, Ph., 2001. Well-structured transition systems everywhere! *Theor. Comput. Sci.*, 256(1–2):63–92. doi:10.1016/S0304-3975(00)00102-X.

Lower Bounds

Schnoebelen, Ph., 2010. Revisiting Ackermann-hardness for lossy counter machines and reset Petri nets. In *MFCS 2010*, volume 6281 of *LNCS*, pages 616–628. Springer. doi:10.1007/978-3-642-15155-2_54.

References: Applications

Demri, S., 2006. Linear-time temporal logics with Presburger constraints: An overview. *J. Appl. Non-Classical Log.*, 16(3–4):311–347. doi:10.3166/jancl.16.311-347.

Demri, S. and Lazić, R., 2009. LTL with the freeze quantifier and register automata. *ACM Trans. Comput. Logic*, 10(3). doi:10.1145/1507244.1507246.

Figueira, D. and Segoufin, L., 2009. Future-looking logics on data words and trees. In *MFCS 2009*, volume 5734 of *LNCS*, pages 331–343. Springer. doi:10.1007/978-3-642-03816-7_29.

Gallo, G. and Mishra, B., 1994. A solution to Kronecker's Problem. *Appl. Algebr. Eng. Comm.*, 5(6):343–370.

Podelski, A. and Rybalchenko, A., 2004. Transition invariants. In *LICS 2004*, pages 32–41. IEEE. doi:10.1109/LICS.2004.1319598.

Revesz, P.Z., 1993. A closed-form evaluation for Datalog queries with integer (gap)-order constraints. *Theor. Comput. Sci.*, 116(1):117–149. doi:10.1016/0304-3975(93)90222-F.

Urquhart, A., 1999. The complexity of decision procedures in relevance logic II. *J. Symb. Log.*, 64(4):1774–1802. doi:10.2307/2586811.

Program Termination Proofs

(Podelski and Rybalchenko, 2004)

Monolithic Termination Argument

- prove that the program's transition relation R is *well-founded*
- *ranking function* ρ from program configurations $\mathbf{x} = x_0, x_1, \ldots$ into a wqo s.t. $R \subseteq \{(x_i, x_j) \mid \rho(x_i) \not\leq \rho(x_j)\}$
- for SIMPLE: $\rho(a, b, c) = \omega \cdot b + a$

Program Termination Proofs

(Podelski and Rybalchenko, 2004)

Disjunctive Termination Argument

- find well-founded relations $T_1, \ldots, T_k$ on program configurations
- prove $R^+ \subseteq T_1 \cup \cdots \cup T_k$
- for SIMPLE:

$$T_1 = \{(\langle a, b, c\rangle, \langle a', b', c'\rangle) \mid a > 0 \wedge a' < a\}$$
$$T_2 = \{(\langle a, b, c\rangle, \langle a', b', c'\rangle) \mid b > 0 \wedge b' < b\}$$

- at the heart of the TERMINATOR tool

Termination by Dickson's Lemma

- each T_j shown well-founded thanks to a ranking function ρ_j into a wqo $(S_j, \leqslant_j)$
- map any sequence of program configurations

$$\mathbf{x} = x_0, x_1, \ldots$$

  to a sequence of tuples

$$\mathbf{y} = \langle \rho_1(x_0), \ldots, \rho_k(x_0) \rangle, \langle \rho_1(x_1), \ldots, \rho_k(x_1) \rangle, \ldots$$

  in $S_1 \times \cdots \times S_k$
- $\mathbf{y}$ is *bad*: if $i_1 < i_2$, there exists j s.t.

$$(x_{i_1}, x_{i_2}) \in R^+ \cap T_j$$

  but

$$\rho_j(x_{i_1}) \not\leqslant \rho_j(x_{i_2})$$

Termination by Dickson's Lemma

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- map any sequence of program configurations

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  in $S_1 \times \cdots \times S_k$
- $\mathbf{y}$ is *bad*: if $i_1 < i_2$, there exists j s.t.

$$(x_{i_1}, x_{i_2}) \in R^+ \cap T_j$$

  but

$$\rho_j(x_{i_1}) \not\leqslant \rho_j(x_{i_2})$$

Bounds on Program Complexity

Make some assumptions:

- complexity bound g on atomic program operations
  - for instance polynomial
- complexity bound ρ on ranking functions into $\mathbb{N}$
  - for instance polynomial
- y controlled by $g^i \circ \rho$ in some $\mathscr{F}_\gamma$
  - in this case an exponential function in $\mathscr{F}_2$
- time complexity in $\mathscr{F}_{\gamma+k-1}$
  - in this case $\mathscr{F}_{k+1}$
- matches the lower bound (expand SIMPLE to dimension k instead of 2)

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  - in this case an exponential function in $\mathscr{F}_2$
- time complexity in $\mathscr{F}_{\gamma+k-1}$
  - in this case $\mathscr{F}_{k+1}$
- matches the lower bound (expand SIMPLE to dimension k instead of 2)