The Ordinal-Recursive Complexity of Timed-Arc Petri Nets, Data Nets, and Other Enriched Nets

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Complexity of wqo Algorithms

- well-quasi orders (wqo): tools for proving termination
- generic complexity upper bounds
- enormous complexities
- what about "natural" lower bounds?

Overview

Theorem

Coverability and termination in enriched nets are $F_{\omega^{\omega^{\omega}}}(n+O(1))$*-complete.*

Contents

Enriched Nets

- classes of colored Petri nets:

  timed-arc Petri nets: clocks in $\mathbb{R}$

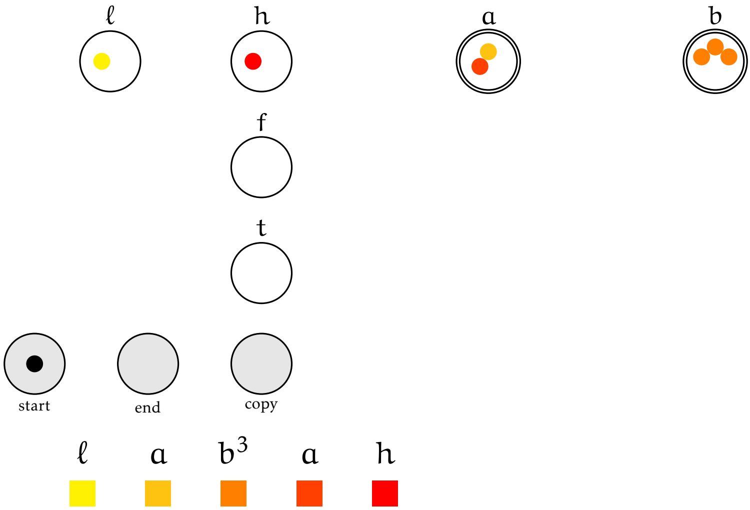
  constrained multiset rewriting systems: natural numbers

  data nets: elements of some linear-ordered dense domain

- equal expressiveness (Abdulla et al., Bonnet et al.)

Petri Data Nets (PDNs)

Example (weak copy)



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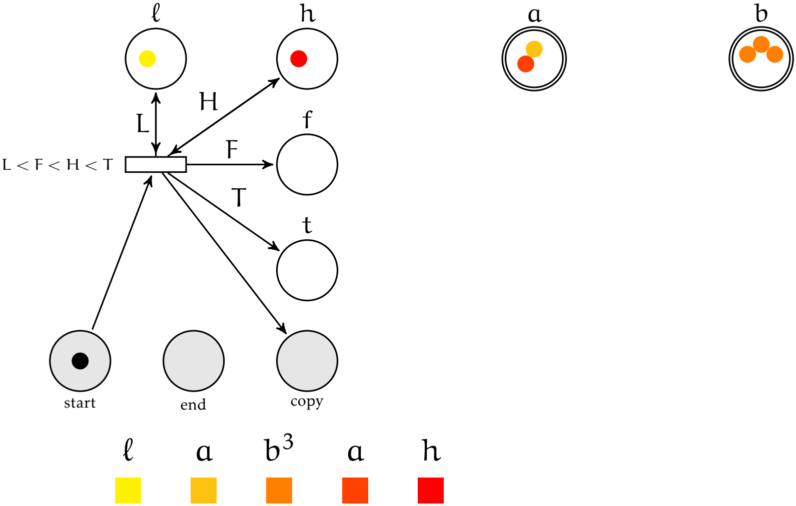

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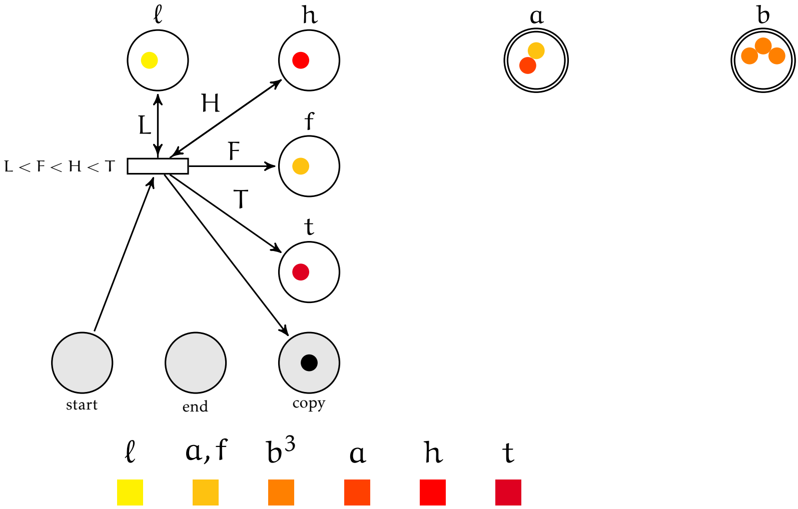

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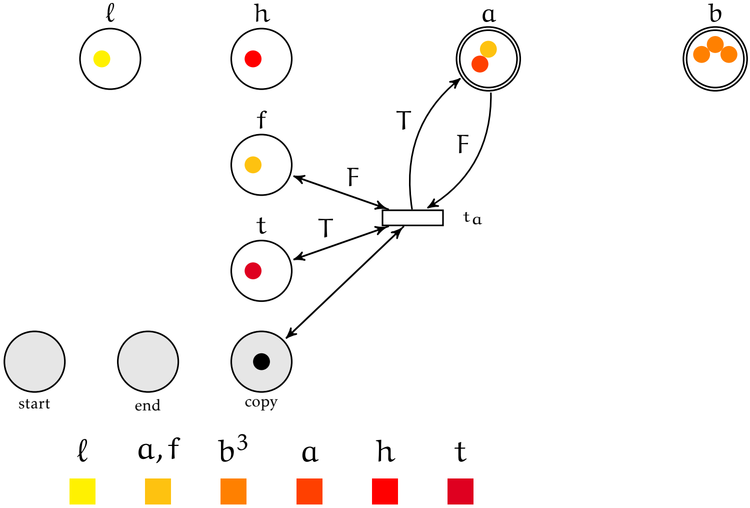

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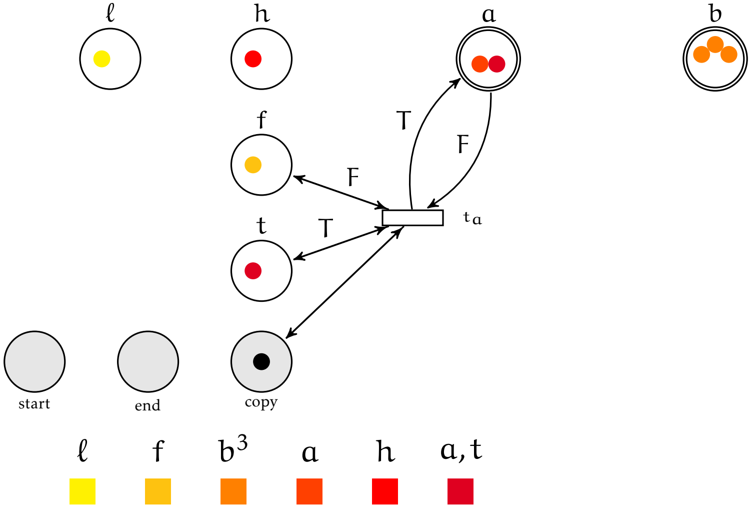

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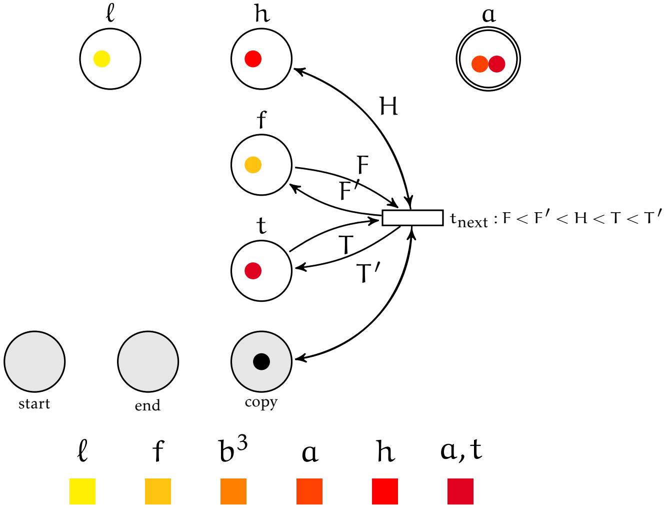

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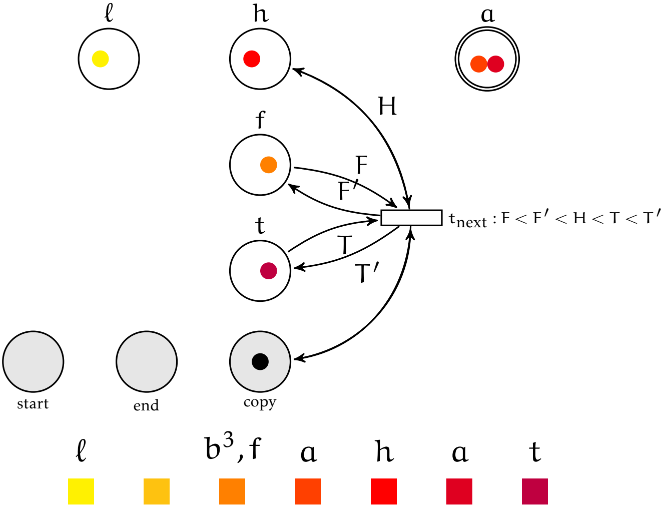

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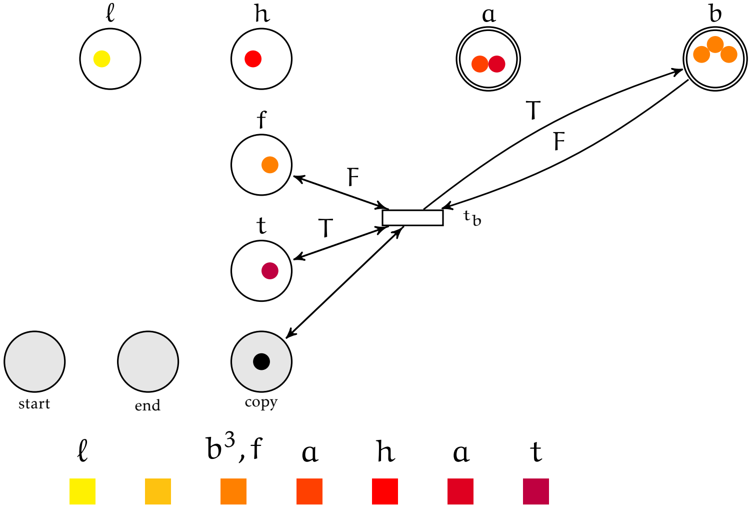

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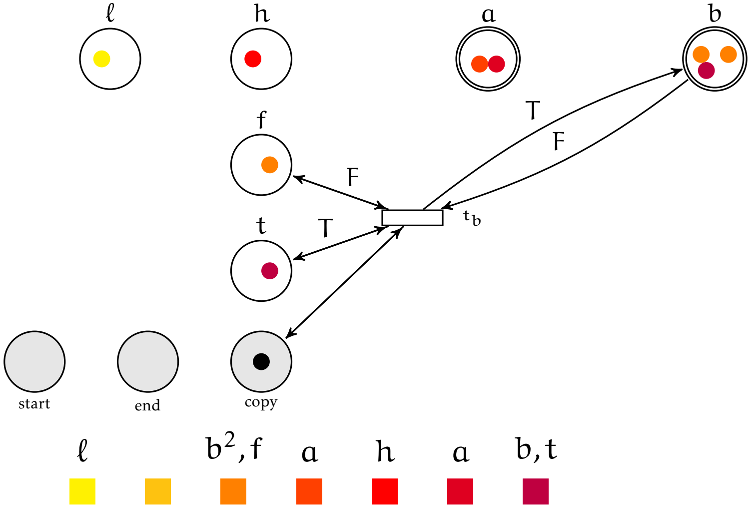

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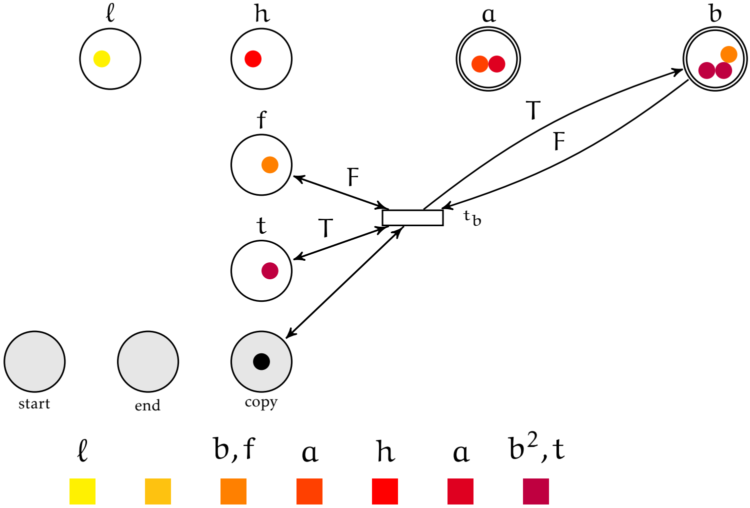

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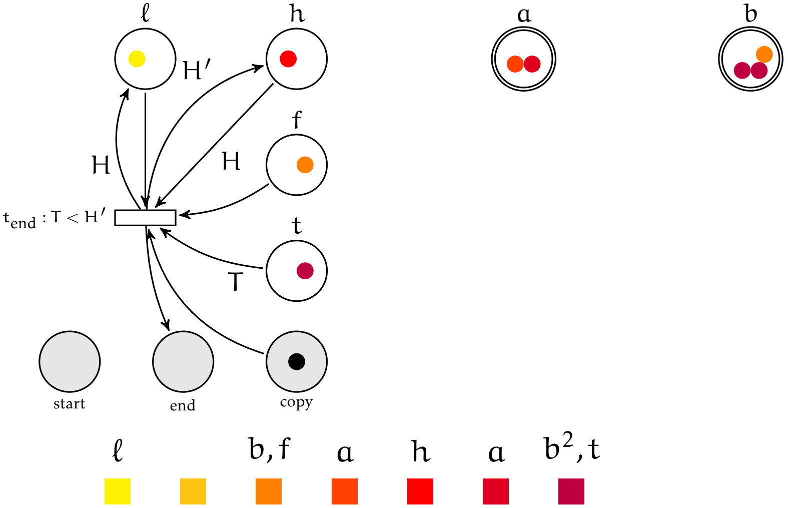

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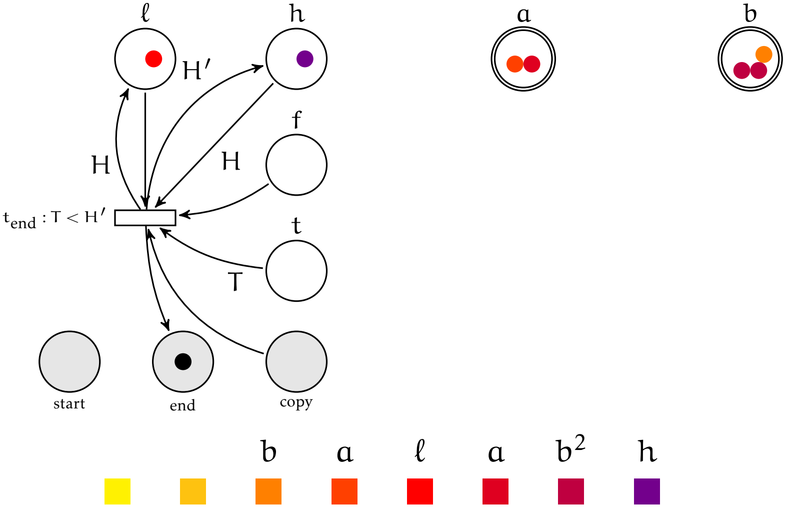
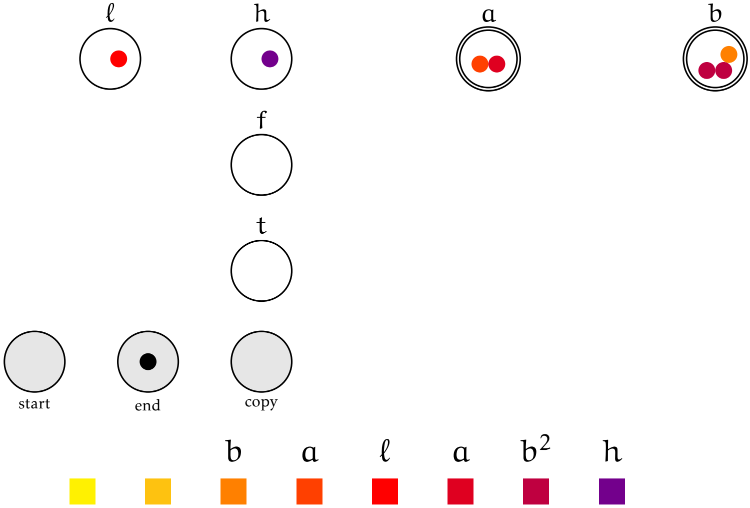


Petri Data Nets (PDNs)

Example (weak copy: $a, b^2 \sqsubseteq a, b^3, a$)



Coverability

input a Petri data net $\mathcal{N}$ and a place p of $\mathcal{N}$

question does there exist a reachable marking with at least one token in p?

Proposition (Lazić et al.)

Petri data net coverability is F_{ω^ω}*-hard.*

Proposition

Petri data net coverability is in $F_{\omega^{\omega^\omega}}(n + O(1))$.

Hardy Hierarchy

Ordinal-indexed hierarchy of functions $(H^\alpha\colon \mathbb{N} \to \mathbb{N})_{\alpha<\varepsilon_0}$:

$$H^0(x) = x \quad H^{\alpha+1}(x) = H^\alpha(x+1) \quad H^\lambda(x) = H^{\lambda_x}(x)$$

where

$$(\gamma + \omega^{\beta+1})_x = \gamma + \omega^\beta \cdot x \quad (\gamma + \omega^\lambda)_x = \gamma + \omega^{\lambda_x}$$

Intuitively: “transfinite iteration of $x \mapsto x+1$”

Hardy Hierarchy

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$$(\gamma + \omega^{\beta+1})_x = \gamma + \omega^{\beta} \cdot x \quad (\gamma + \omega^{\lambda})_x = \gamma + \omega^{\lambda_x}$$

Example

$$H^1(x) = x + 1 \quad H^{\omega}(x) = 2x \quad H^{\omega^2}(x) = 2^x x$$

H^{ω^3} non elementary

H^{ω^ω} non primitive-recursive

$H^{\omega^{\omega^\omega}}$ non multiply-recursive

$F_{\omega^{\omega^\omega}} = H^{\omega^{\omega^{\omega^\omega}}}$ this talk

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Example

Let $\Omega = \omega^{\omega^{\omega^{\omega}}}$ and $k > 0$ then

$$\Omega_k = \omega^{\omega^{\omega^k}} \qquad (\Omega_k)_k = \omega^{\omega^{\omega^{k-1} \cdot k}}$$

$$F_{\omega^{\omega^\omega}}(k) = H^{\Omega}(k) = H^{(\Omega_k)_k}(k)$$

Hardy Computations

$$H^0(x) = x \quad H^{\alpha+1}(x) = H^{\alpha}(x+1) \quad H^{\lambda}(x) = H^{\lambda_x}(x)$$

$$(\gamma + \omega^{\beta+1})_x = \gamma + \omega^{\beta} \cdot x \quad (\gamma + \omega^{\lambda})_x = \gamma + \omega^{\lambda_x}$$

Rewrite system over $\Omega_k \times \mathbb{N}$:

$$\pi + 1, n \to \pi, n+1$$
$$\pi + \omega^{\alpha+1}, n \to \pi + \omega^{\alpha} \cdot n, n$$
$$\pi + \omega^{\alpha+\omega^{\beta+1}}, n \to \pi + \omega^{\alpha+\omega^{\beta} \cdot n}, n$$
$$\pi + \omega^{\alpha+\omega^{\beta+\omega^{m+1}}}, n \to \pi + \omega^{\alpha+\omega^{\beta+\omega^{m} \cdot n}}, n$$

Hardy Computations

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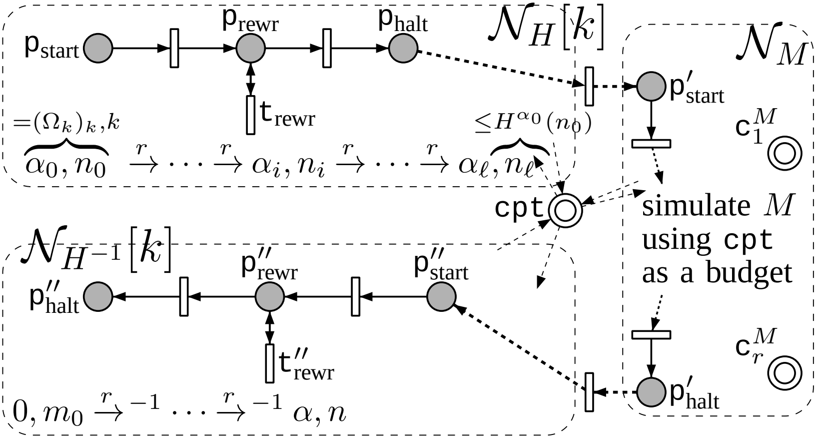
A Hardy computation:

$$\pi_0, n_0 \to \pi_1, n_1 \to \cdots \to 0, H^{\pi_0}(n_0)$$

Hardness of Coverability in PDNs

- Reduction: from the halting problem of a Minsky machine M with counters bounded by $H^{\Omega}(|M| + O(1)) = H^{\Omega}(k)$.
- weak computation of $H^{\Omega}(k)$: some smaller value could be reached
- also perform a weak computation of the inverse $(H^{\Omega})^{-1}$

Hardness of Coverability in PDNs



Cumulative Ordinal Encoding

- ordinals $< \omega^k$ as vectors v in $\mathbb{N}^k$:

$$\beta(v) = \omega^{k-1} \cdot v[k-1] + \cdots + \omega^0 \cdot v[0]$$

- ordinals $< \omega^{\omega^k}$ as vector sequences $\mathbf{V} = v_1 \cdots v_p$:

$$\alpha(\mathbf{V}) = \omega^{\beta(v_1)} + \cdots + \omega^{\beta(v_p)}$$

- ordinals $< \Omega_k$ as #-separated vector sequences $x = \mathbf{V}_1 \# \mathbf{V}_2 \# \cdots \# \mathbf{V}_m \#$ (aka codes):

$$\pi(x) = \omega^{\alpha(\mathbf{V}_1 \cdots \mathbf{V}_m)} + \omega^{\alpha(\mathbf{V}_1 \cdots \mathbf{V}_{m-1})} + \cdots + \omega^{\alpha(\mathbf{V}_1)}$$

Encoded Hardy Computations

Work on codes instead of ordinals:

Example

$$\pi + 1, n \to \pi, n + 1$$

is encoded as

$$\#x, n \to x, n + 1 \tag{R1}$$

Encoded Hardy Computations

Example

Let $k > 1$; the initial step of the encoded Hardy computation for $H^{\Omega}(k)$ is

$$x_0, k \to x_1, k$$

with codes

$$x_0 = (\mathbf{1}_{k-1})^k \# \,; \qquad \pi(x_0) = (\Omega_k)_k$$

$$x_1 = (\mathbf{1}_{k-1})^{k-1}(\mathbf{1}_{k-2})^k \# \,; \quad \pi(x_1) = \omega^{\omega^{\omega^{k-1}\cdot(k-1)+\omega^{k-2}\cdot k}}$$

Properties

Proposition (Correctness)

$x, n \to y, m$ *and* x *pure imply* $H^{\pi(x)}(n) = H^{\pi(y)}(m)$.

Proposition (Robustness)

Let x, x' *be pure codes and* $n' > 0$. *If* x' *is* n'*-trim and* $x, n \sqsubseteq x', n'$, *then* $H^{\pi(x)}(n) \leqslant H^{\pi(x')}(n')$.

Concluding Remarks

- answers the open complexity questions on enriched nets
- first "natural" problem complete for $F_{\omega^{\omega^{\omega}}}$
- also expressiveness corollaries (see paper)

Abdulla, P.A., Delzanno, G., and Van Begin, L., 2007. Comparing the expressive power of well-structured transition systems. In Duparc, J. and Henzinger, T.A., editors, *CSL 2007*, volume 4646 of *LNCS*, pages 99–114. doi:10.1007/978-3-540-74915-8_11.

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Lazić, R., Newcomb, T., Ouaknine, J., Roscoe, A., and Worrell, J., 2008. Nets with tokens which carry data. *Fund. Inform.*, 88(3):251–274.

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Why Cumulative Encodings

Non-Robustness of Cantor Normal Form

Encoding as "$\dot{+}$"-separated sequences $p = \mathbf{V}_1 \dot{+} \cdots \dot{+} \mathbf{V}_m$:

$$\chi(p) = \omega^{\alpha(\mathbf{V}_1)} + \cdots + \omega^{\alpha(\mathbf{V}_m)}$$

Example

For $k = 1$:

$$\chi(1 \dot{+} 0) = \omega^{\omega} + \omega \qquad \chi(10) = \omega^{\omega+1}$$

$$\begin{aligned} H^{\chi(1\dot{+}0)}(n) &= H^{\omega^{\omega}+\omega}(n) \\ &< H^{\omega^{\omega\cdot(n-1)+\omega^{n}}}(n) = H^{\chi(10)}(n) \end{aligned}$$