On the Length of Strongly Monotone Descending Chains over $\mathbb{N}^d$

Sylvain Schmitz & Lia Schütze

MAX PLANCK INSTITUTE FOR SOFTWARE SYSTEMS

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Outline

context

- coverability in vector addition systems (VAS)
- breakthrough on the complexity [Künnemann, Mazowiecki, Schütze, Sinclair-Banks, and Węgrzycki, ICALP 2023]

motivation

- generic algorithm: (dual) backward coverability [Lazić and S., 2021]

results

- structural result
- generic upper bounds
- applications: branching or alternating VAS, strongly increasing or invertible affine nets

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finite set of actions $A \subseteq \mathbb{Z}^d$

Example

$A \overset{\text{def}}{=} \{$  $\}$

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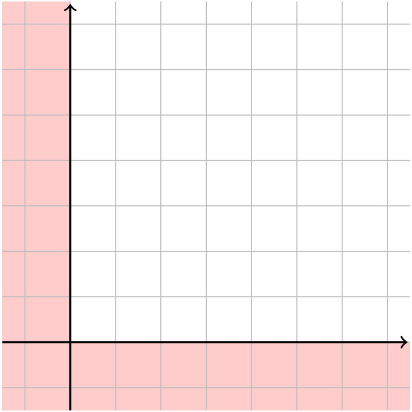
Semantics (over $\mathbb{N}^d$



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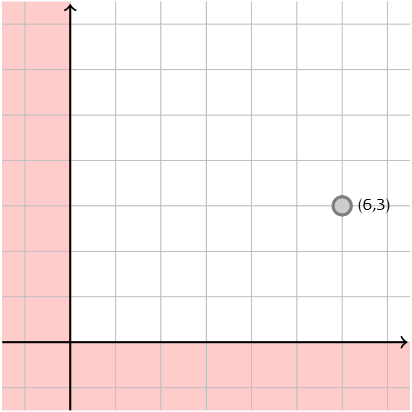

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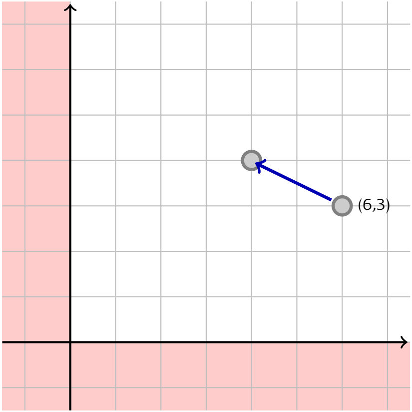

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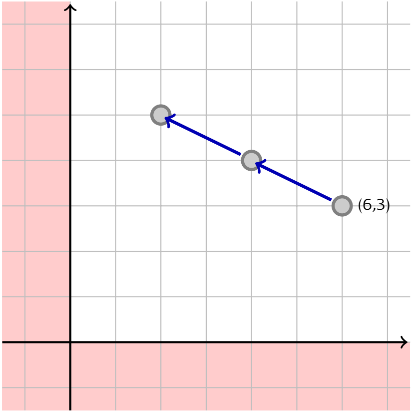

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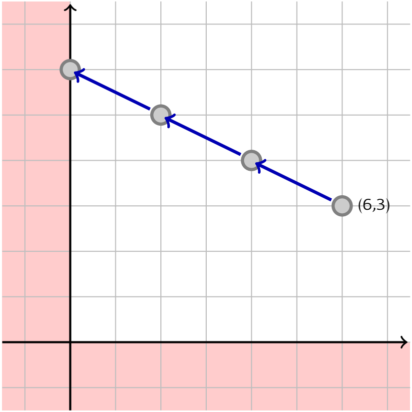

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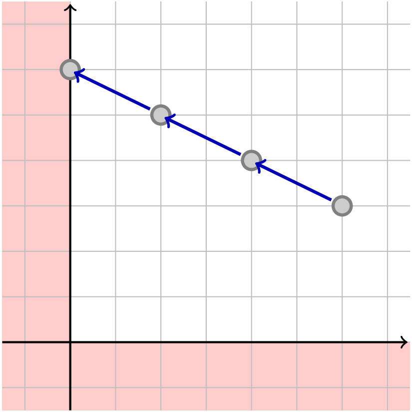

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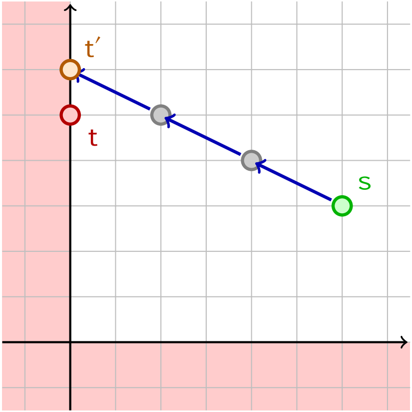

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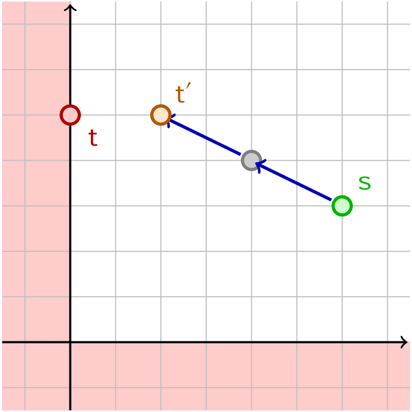

Complexity of VAS Coverability

Lipton 1976: EXPSPACE-hard

Rackoff 1978: in EXPSPACE

Complexity of (Unary) VAS Coverability

Minimal covering path

$$\sup_{s \in \mathbb{N}^d} (\text{length of shortest path } s \to^* t' \geqslant t)$$

Lipton 1976: $n^{2^{\Omega(d)}}$ length

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[Künnemann et al., 2023]

- by induction: $L_{d-1} > L_{d-2} > \ldots$ are bounds on minimal covering paths in dimension $d-1, d-2, \ldots$
- let $N_i \stackrel{\text{def}}{=} n \cdot L_{i-1}$ for all $i \leqslant d$
- a vector $u \in \mathbb{N}^d$ is thin if there is a permutation $\sigma \in S_d$ s.t. $\forall i$:

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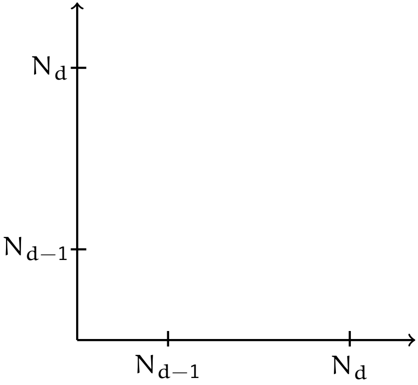

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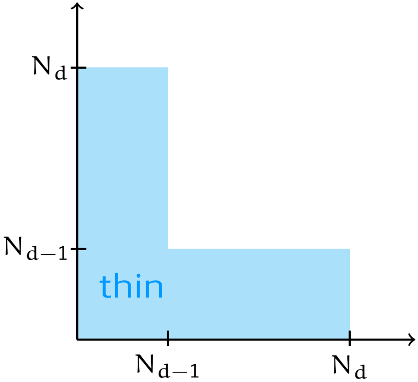

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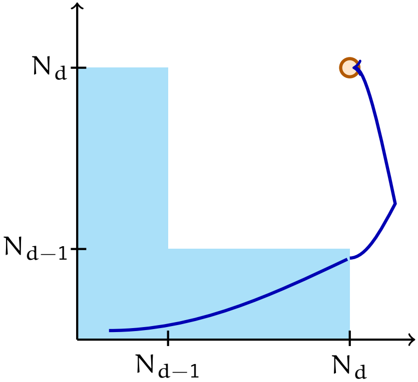

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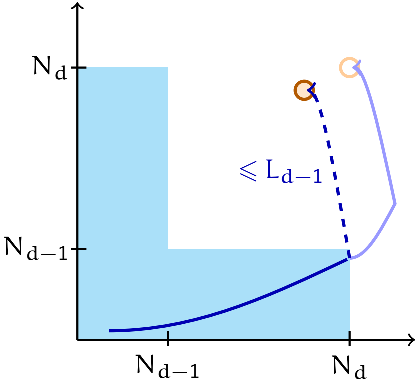

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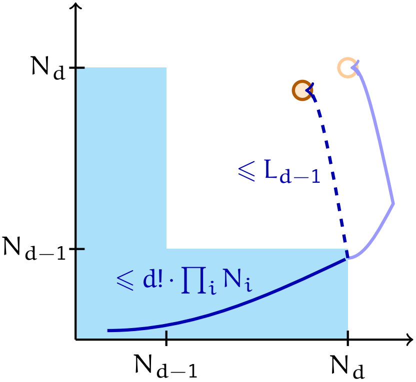

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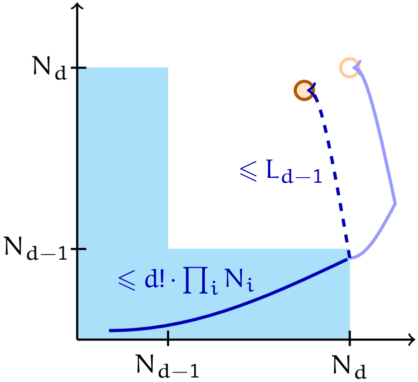

$$L_d \stackrel{\text{def}}{=} d! \cdot \prod N_i + L_{d-1}$$

Well Structured Transition Systems

[Abdulla, Čerāns, Jonsson & Tsay '00; Finkel & Schnoebelen '01]

- general algorithmic framework
- algorithms for several verification problems
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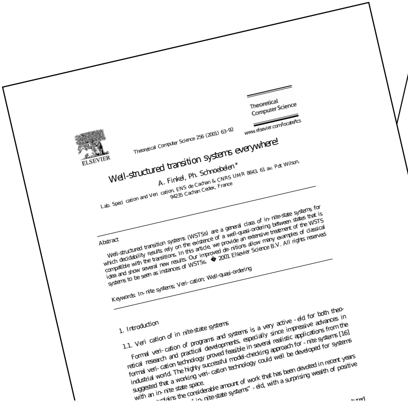

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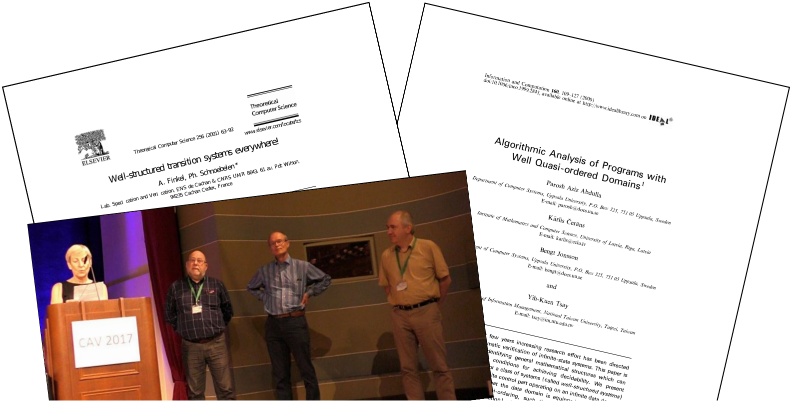

DUAL BACKWARD COVERABILITY ALGORITHM

[Lazić and S., 2021]

Configurations that do not cover t in $\leqslant k$ steps:

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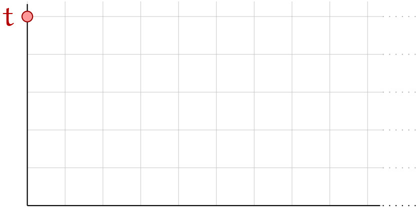

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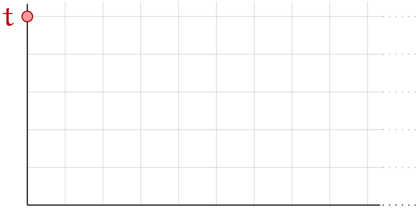

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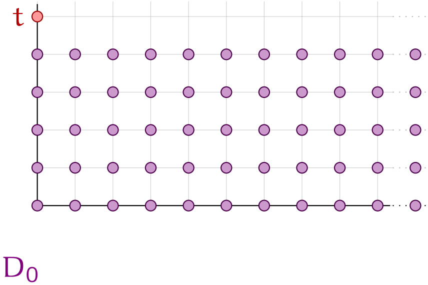

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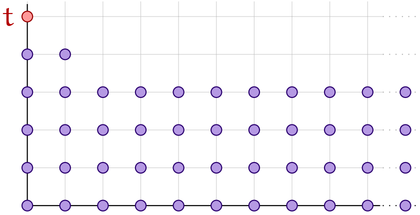

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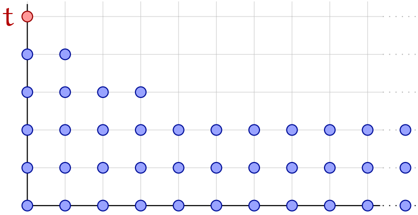

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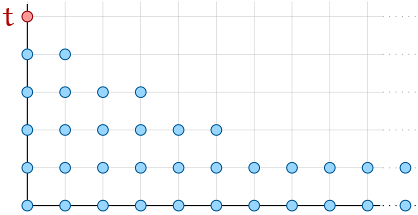

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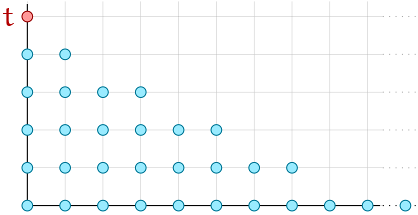

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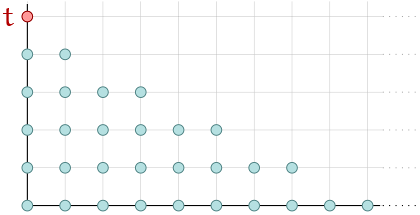

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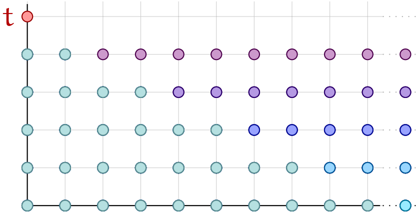

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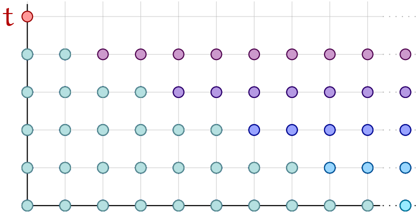

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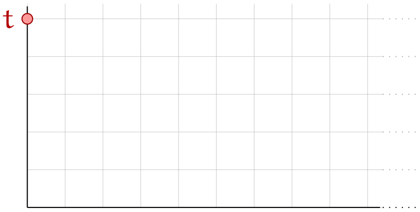
Ideal Representations

- downwards-closed sets over a wqo have a unique decomposition as finite unions of ideals
- over $\mathbb{N}^d$: ideals as vectors in $(\mathbb{N} \cup \{\omega\})^d$



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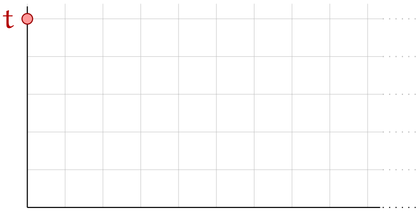

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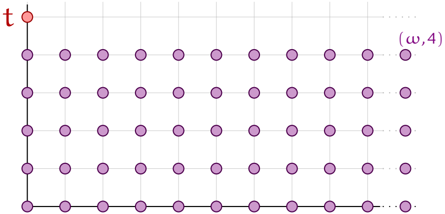

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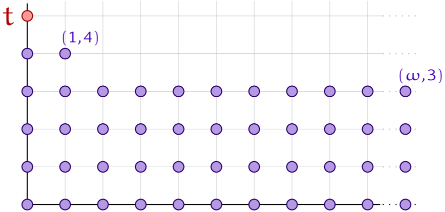

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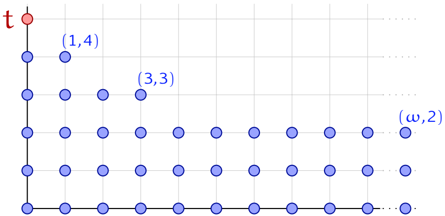

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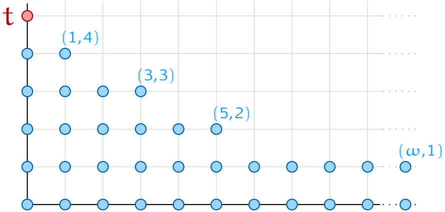

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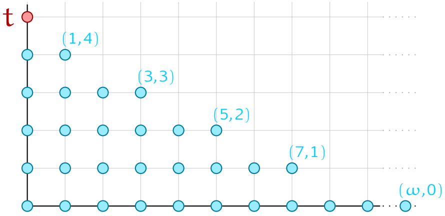

$D_4 = \{(1,4),(3,3),(5,2),(7,1),(\omega,0)\}$

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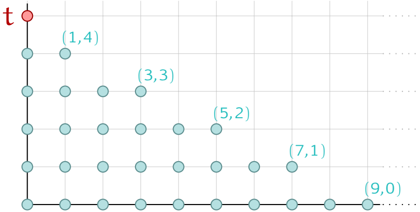

$$D_5 = \{(1,4),(3,3),(5,2),(7,1),(9,0)\}$$

Coverability in VAS Extensions

VAS
AVAS (top-down)
BVAS (bottom-up)
affine nets
strictly incr. affine nets
invertible affine nets

- the backward coverability algorithm applies
- generic complexity upper bounds for the (dual) backward coverability algorithm [Lazić and S., 2021]
- here: generic complexity upper bounds in $n^{2^{O(d)}}$

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Dimension of Ideals over $\mathbb{N}^d$

For an ideal I seen as a vector in $(\mathbb{N} \cup \{\omega\})^d$

$$\omega(I) \stackrel{\text{def}}{=} \{1 \leqslant i \leqslant d \mid I(i) = \omega\}$$

$$\dim I \stackrel{\text{def}}{=} |\omega(I)|$$

Example

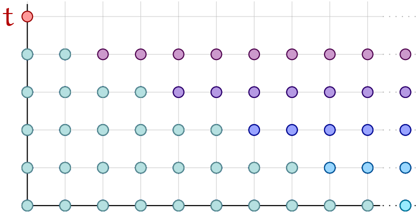
For $d = 3$, $\omega((2,10,\omega)) = \{3\}$ and $\dim(2,10,\omega) = 1$.

Monotonicity

[Lazić and S., 2021]

- at every step k, there must exist an ideal in D_k but not in D_{k+1}: we say it is proper at step k
- the chain is strongly monotone if, $\forall I_{k+1}$ proper at step $k+1$, $\exists I_k$ proper at step k s.t.

$$\dim I_{k+1} \leqslant \dim I_k$$



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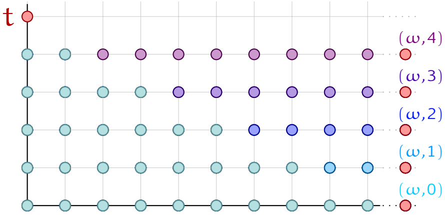

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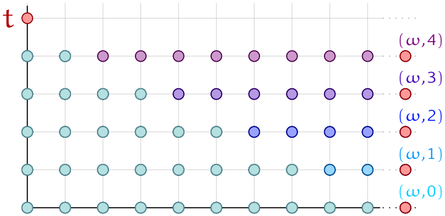
$$\omega(I_{k+1}) \subseteq \omega(I_k)$$



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[Novikov and Yakovenko, 1999; Benedikt et al., 2017]

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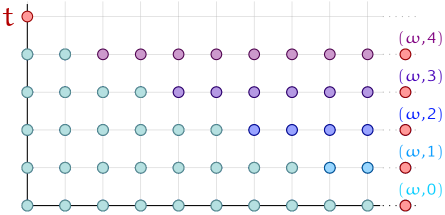

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Monotonicity in VAS Extensions

For the descending chains of the dual backward coverability algorithm:

	ω-monotone	strongly monotone
VAS	✓	✓
AVAS (top-down)	✓	✓
BVAS (bottom-up)	✓	✓
affine nets	✗	✗
strictly incr. affine nets	✓	✓
invertible affine nets	✗	✓

The Length of Descending Chains

Issue

The length can be arbitrary (also for strongly monotone chains): for all n,

$$\{(0,\omega)\} \supsetneq \{(0,n)\} \supsetneq \{(0,n-1)\} \supsetneq \cdots \supsetneq \{(0,1)\} \supsetneq \{(0,0)\}$$

Control

$$|D| \stackrel{\text{def}}{=} \max_{I \in D} |I|$$

$$|I| \stackrel{\text{def}}{=} \max_{i \notin \omega(I)} I(i)$$

For $g\colon \mathbb{N} \to \mathbb{N}$ and $n_0 \in \mathbb{N}$: a chain $D_0 \supsetneq D_1 \supsetneq \cdots$ is (g, n_0)-controlled if, $\forall k$,

$$|D_k| \leqslant g^k(n_0)$$

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In our VAS Extensions

The descending chains of the dual backward coverability algorithm are (g, n_0)-controlled by

$$g(x) \stackrel{\text{def}}{=} x + n \qquad n_0 \stackrel{\text{def}}{=} n$$

(n the size of the coverability instance)

Main Results

- set up suitable $L_d > L_{d-1} > \cdots$ and $N_d > N_{d-1} > \cdots$, that depend on the control (g, n_0)
- extend the definition of thinness to ideals in $(\mathbb{N} \cup \{\omega\})^d$
- consider a (g, n_0)-controlled strongly monotone descending chain $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell$

Lemma

Every ideal in the decompositions of the D_k is thin.

Theorem

The length of the chain satisfies $\ell \leqslant L_d + 1$.

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In our VAS Extensions

For $g(x) \stackrel{\text{def}}{=} x + n$ and $n_0 \stackrel{\text{def}}{=} n$, this yields

$$\ell \leqslant L_d + 1 \in n^{2^{O(d)}}$$

and the same bound applies to the running time of the (dual) backward coverability algorithm.

Concluding Remarks

- generic approach to the complexity of coverability problems
- explanatory: 🪄 thinness is an inherent property of the backward coverability algorithm
- thanks to the conditional lower bounds of [Künnemann et al., 2023]: optimality of the backward coverability algorithm
- applications beyond coverability problems?

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