Implicational Relevance Logic is 2-ExpTime-Complete

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RTA-TLCA, July 14th 2014

Outline

- Implicational relevance logic $\mathbf{R}_{\to}$
  the "oldest" relevance logic (Moh, 1950; Church, 1951)

Theorem
Provability in $\mathbf{R}_{\to}$ is 2-ExpTime*-complete.*

- Branching VASS
  as a means to prove algorithmic results

Fact (Demri et al., 2013)
Coverability in BVASS is 2-ExpTime*-complete.*

- Inter-reductions
  between counter systems and substructural logics

Branching Vector Addition Systems

$\mathcal{B} = \langle Q, d, T_u, T_s \rangle$ Q finite set of states, d dimension in $\mathbb{N}$

configurations $q, \mathbf{v} \in Q \times \mathbb{N}^d$

unary rules $q \xrightarrow{\mathbf{u}} q' \in T_u \subseteq_{\text{fin}} Q \times \mathbb{Z}^d \times Q$

$$\frac{q, \mathbf{v}}{q', \mathbf{v} + \mathbf{u}} \text{ (unary)}$$

split rules $q \to q_1 + q_2 \in T_s \subseteq Q^3$

$$\frac{q, \mathbf{v}_1 + \mathbf{v}_2}{q_1, \mathbf{v}_1 \quad q_2, \mathbf{v}_2} \text{ (split)}$$

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$$\frac{q, \mathbf{v}}{q', \mathbf{v} + \mathbf{u} \geqslant \mathbf{0}} \text{(unary)}$$

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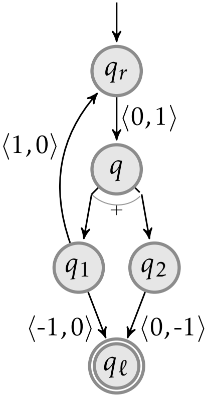
Example



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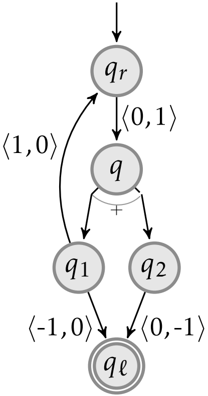
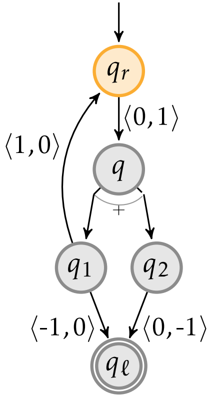


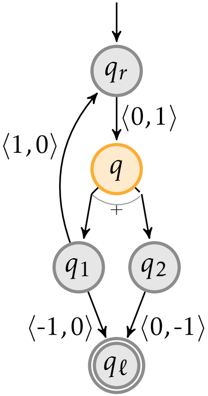
$q_r, 0, 0$

Example



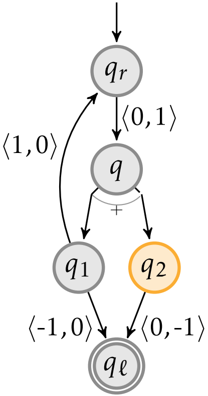
$$\frac{q_r, 0, 0}{q, 0, 1}$$

Example



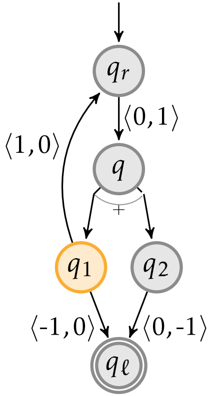
$$\cfrac{\cfrac{q_r, 0, 0}{q, 0, 1}}{q_1, 0, 0 \qquad\qquad q_2, 0, 1}$$

Example



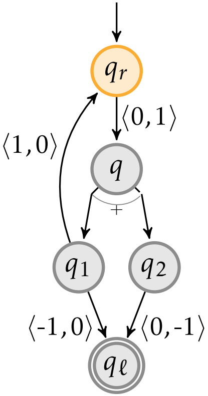
$$\cfrac{\cfrac{q_r,0,0}{q,0,1}}{q_1,0,0 \qquad \cfrac{q_2,0,1}{q_\ell,0,0}}$$

Example



$$\begin{array}{ccc} & \dfrac{q_r,0,0}{q,0,1} & \\ \hline \dfrac{\color{orange}{q_1,0,0}}{q_r,1,0} & & \dfrac{q_2,0,1}{q_\ell,0,0} \end{array}$$

Example



$$\begin{array}{c} q_r,0,0 \\ \hline q,0,1 \\ \hline \begin{array}{ccc} \begin{array}{c} q_1,0,0 \\ \hline q_r,1,0 \\ \hline q,1,1 \end{array} & \qquad & \begin{array}{c} q_2,0,1 \\ \hline q_\ell,0,0 \\ \end{array} \end{array} \end{array}$$

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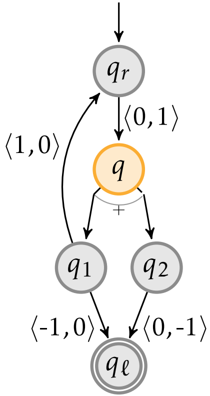

$$
\dfrac{\dfrac{\dfrac{\dfrac{q_1,1,0 \qquad q_2,0,1}{q,1,1}}{q_r,1,0}}{q_1,0,0} \qquad \dfrac{q_\ell,0,0}{q_2,0,1}}{\dfrac{q,0,1}{q_r,0,0}}
$$

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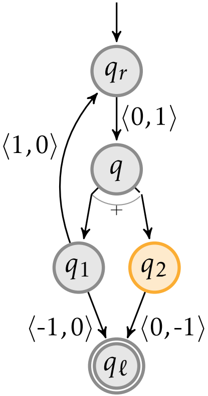
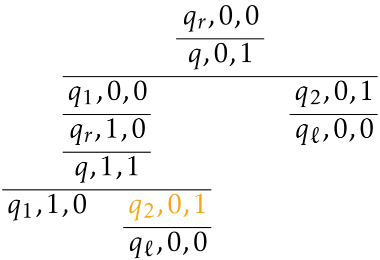


q_r
⟨0,1⟩
⟨1,0⟩
q
+
q_1
q_2
⟨-1,0⟩
⟨0,-1⟩
q_ℓ



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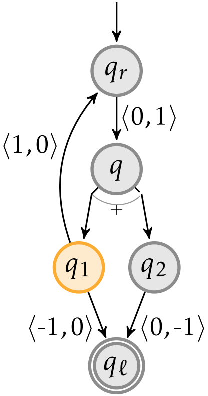

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Some Application Domains

computational linguistics (survey in S., 2010)

- dominance links (Rambow, 1994)
- abstract categorial grammars (de Groote, 2001)
- minimal grammars (Salvati, 2011)

linear logic inter-reductions with MELL (de Groote et al., 2004; Lazić and S., 2014)

protocol verification Horn deduction modulo AC (Verma and Goubault-Larrecq, 2005)

data logics for XML $\mathrm{FO}^2(<,+1,\sim)$ (Bojańczyk et al., 2009; Dimino et al., 2013)

parallel programming (Bouajjani and Emmi, 2013)

Decision Problems

Given $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$ and $q_r, q_\ell \in Q$.

Reachability

Does there exist a deduction tree rooted by $q_r, \mathbf{0}$ and with $q_\ell, \mathbf{0}$ as leaves?

- Tower-hard (Lazić and S., 2014, Friday 12:15 at CSL-LICS)
- decidability open, recursively equivalent to MELL provability (de Groote et al., 2004)

(Root) Coverability

Does there exist a deduction tree rooted by $q_r, \mathbf{v}$ for some $\mathbf{v} \in \mathbb{N}^d$ and with $q_\ell, \mathbf{0}$ as leaves?

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- parametric complexity: doubly exponential in dimension d, but polynomial in $|Q|$ and $\|T_u\|_\infty$

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Implicational Relevance Logic $\mathbf{R}_{\to}$

see talk by A. Urquhart, Wednesday 10:45 at LATD

Example: $A \to (B \to A)$

"if it's raining (A), then if your favorite color is green (B) then it's raining (A)"

A theorem in classical logic, not in relevance logic.

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\,(\mathrm{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,(\mathrm{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\,(\to_{\mathrm{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to_{\mathrm{R}})$$

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Some History

Independently defined Hilbert-style axiomatic systems by Moh (1950) and Church (1951)

Weak Deduction Theorem (Church, 1951)

If $A_1,\ldots,A_{n-1},A_n \vdash B$ and A_n is relevant, then $A_1,\ldots,A_{n-1} \vdash A_n \to B$.

Proved decidable by Kripke (1959) (wqo argument)

Theorem (Kripke, 1959)

If $\vdash A$ is a theorem of $\mathbf{R}_{\to}$, then there exists an irredundant proof for it.

Inhabitation of Simple Types

$\tau ::= a \mid \tau \to \tau$ $\quad$ a ranges over atomic types

λI-Calculus (Church, 1930's)

Given τ, does there exist a λI term with type τ?

$$\frac{}{x : \tau \vdash x : \tau}\ (\text{var})$$

$$\frac{\Gamma, x : \tau \vdash t : \tau' \quad x \text{ occurs free in } t}{\Gamma \vdash (\lambda x.t) : \tau \to \tau'}\ (\text{abs})$$

$$\frac{\Gamma \vdash t : \tau \to \tau' \quad \Delta \vdash t' : \tau}{\Gamma, \Delta \vdash (tt') : \tau'}\ (\text{app})$$

Inhabitation of Simple Types

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Combinatory Logic (see Curry and Craig, 1953)

Given τ, does there exist a term built from combinators B, C, I, W with type τ?

$$\begin{aligned}
Bfgx &= f(gx) : (B \to C) \to ((A \to B) \to (A \to C)) \\
Cfxy &= fyx : (A \to (B \to C)) \to (B \to (A \to C)) \\
Ix &= x : A \to A \\
Wxy &= xyy : (A \to (A \to B)) \to (A \to B)
\end{aligned}$$

Complexity of the Decision Problem

Theorem (Urquhart, 1990)

Provability in $\mathbf{R}_{\rightarrow}$ is ExpSpace*-hard and* Ackermann*-easy.*

Theorem (Urquhart, 1999)

Provability in $\mathbf{R}_{\rightarrow,\wedge}$ is Ackermann*-complete.*

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From $\mathbf{R}_{\rightarrow}$ to BVASS (1/2)

- subformula property: given a formula F, set

$$Q = \text{Subformulæ}(F) \cup \ldots \qquad d = |\text{Subformulæ}(F)|\,.$$

- a sequent $\Gamma \vdash A$ becomes a configuration $A, \mathbf{v}_\Gamma \in Q \times \mathbb{N}^d$
- rules implement proof search:

$$\frac{}{A \vdash A}\ (\mathsf{Id})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\ (\to_{\mathsf{L}})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to_{\mathsf{R}})$$

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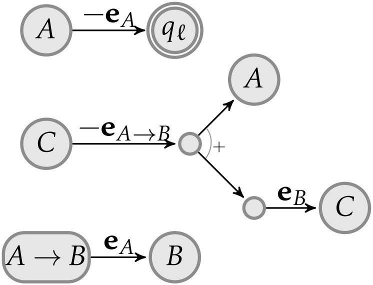

FROM $\mathbf{R}_{\rightarrow}$ TO BVASS (2/2)

What about contraction? (Urquhart, 1999)

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (\mathsf{C}) \qquad \frac{q, \mathbf{v} + \mathbf{e}_i}{q, \mathbf{v} + 2\mathbf{e}_i}\ (\text{expansion})$$

Proposition

$\mathbf{R}_{\rightarrow} <_{\text{LOGSPACE}}$ Expansive BVASS Reachability

Proposition

Expansive BVASS Reachability $<_{\text{PSPACE}}$ BVASS Coverability

Thankfully, the exponential blow-up only impacts the state space of the constructed BVASS

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From BVASS to $\mathbf{R}_{\to}$ (1/2)

Given a BVASS $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$, wlog. $T_u \subseteq Q \times \{\mathbf{e}_i, -\mathbf{e}_i \mid 1 \leqslant i \leqslant d\}$

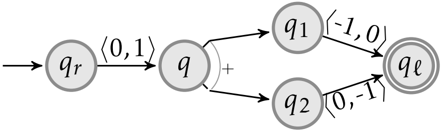
- atomic formulæ $Q \uplus \{e_i \mid 1 \leqslant i \leqslant d\}$
- encoding of a vector $\mathbf{v} = c_1\mathbf{e}_1 + \cdots + c_d\mathbf{e}_d$: $\Gamma_{\mathbf{v}} = e_1^{c_1}, \ldots, e_d^{c_d}$
- encoding of a set of rules: $\Delta_{\{t_1,\ldots,t_k\}} = \ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner$

$$\ulcorner q \xrightarrow{\mathbf{e}_i} q_1 \urcorner = (e_i \to q_1) \to q$$

$$\ulcorner q \xrightarrow{-\mathbf{e}_i} q_1 \urcorner = q_1 \to (e_i \to q)$$

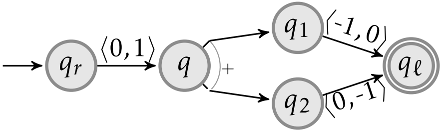
$$\ulcorner q \to q_1 + q_2 \urcorner = q_1 \to (q_2 \to q)$$

From BVASS to $\mathbf{R}_{\to}$ (2/3)



$$q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r$$

From BVASS to $\mathbf{R}_\rightarrow$ (2/3)



$$\frac{q_\ell, e_1, \rightarrow (e_1 \rightarrow q_1), q_\ell, q_\ell \rightarrow (e_2 \rightarrow q_2), q_1 \rightarrow (q_2 \rightarrow q) \vdash e_2 \rightarrow q \qquad q_r \vdash q_r}{q_\ell, e_1, q_\ell \rightarrow (e_1 \rightarrow q_1), q_\ell, q_\ell \rightarrow (e_2 \rightarrow q_2), q_1 \rightarrow (q_2 \rightarrow q), (e_2 \rightarrow q) \rightarrow q_r \vdash q_r} (\rightarrow_\mathsf{L})$$

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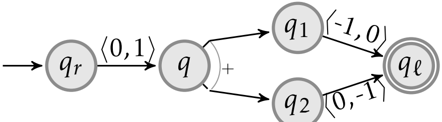

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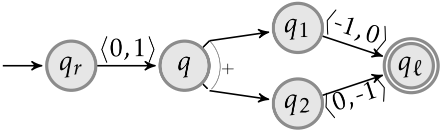

$$\dfrac{\dfrac{\dfrac{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1 \qquad q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}\,(\to_L)}{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q}\,(\to_R) \qquad q_r \vdash q_r}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r}\,(\to_L)$$

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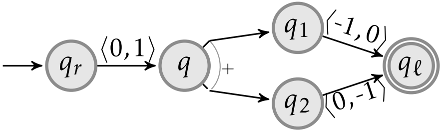

$$
\dfrac{
\dfrac{
q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1
\qquad
\dfrac{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2 \qquad q \vdash q}{q_\ell, q_\ell \to (e_2 \to q_2), {\color{orange} q_2 \to q}, e_2 \vdash q}\,(\to_L)
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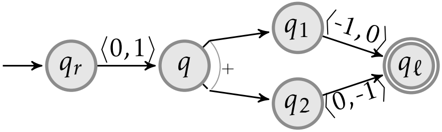
\qquad q_r \vdash q_r
}{
q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r
}\,(\to_L)
$$

From BVASS to $\mathbf{R}_{\rightarrow}$ (2/3)

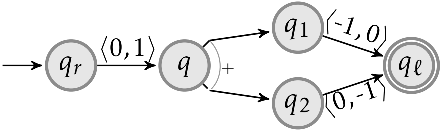


$$\dfrac{\dfrac{q_\ell \vdash q_\ell \qquad e_1, e_1 \to q_1 \vdash q_1}{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1}(\to_L) \qquad \dfrac{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2 \qquad q \vdash q}{q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}(\to_L)}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}(\to_L)$$

$$\dfrac{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q}(\to_R)$$

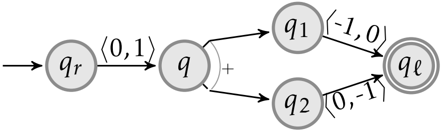
$$\dfrac{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q \qquad q_r \vdash q_r}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r}(\to_L)$$

From BVASS to $\mathbf{R}_{\to}$ (2/3)



$$
\dfrac{
\dfrac{
q_\ell \vdash q_\ell \qquad
\dfrac{e_1 \vdash e_1 \qquad q_1 \vdash q_1}{e_1, e_1 \to q_1 \vdash q_1}\,(\to_L)
}{
q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1
}\,(\to_L)
\qquad
\dfrac{
q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2 \qquad q \vdash q
}{
q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q
}\,(\to_L)
}{
\dfrac{
\dfrac{
q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q
}{
q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q
}\,(\to_R)
\qquad q_r \vdash q_r
}{
q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r
}\,(\to_L)
}\,(\to_L)
$$

From BVASS to $\mathbf{R}_{\to}$ (2/3)



$$\frac{e_1 \vdash e_1 \qquad q_1 \vdash q_1}{e_1, e_1 \to q_1 \vdash q_1}\,(\to_L) \qquad \frac{q_\ell \vdash q_\ell \qquad e_2 \to q_2, e_2 \vdash q_2}{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2}\,(\to_L)$$

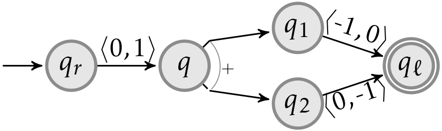
$$\frac{q_\ell \vdash q_\ell \qquad e_1, e_1 \to q_1 \vdash q_1}{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1}\,(\to_L) \qquad \frac{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2 \qquad q \vdash q}{q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}\,(\to_L)$$

$$\frac{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1 \qquad q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}\,(\to_L)$$

$$\frac{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q}\,(\to_R)$$

$$\frac{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q \qquad q_r \vdash q_r}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r}\,(\to_L)$$

From BVASS to $\mathbf{R}_{\to}$ (2/3)



$$\dfrac{e_2 \vdash e_2 \quad q_2 \vdash q_2}{e_2 \to q_2, e_2 \vdash q_2}\,(\to_L)$$

$$\dfrac{e_1 \vdash e_1 \quad q_1 \vdash q_1}{e_1, e_1 \to q_1 \vdash q_1}\,(\to_L) \qquad \dfrac{q_\ell \vdash q_\ell \quad e_2 \to q_2, e_2 \vdash q_2}{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2}\,(\to_L)$$

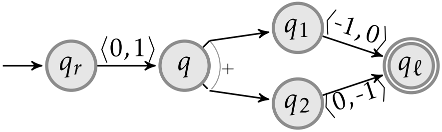
$$\dfrac{q_\ell \vdash q_\ell \quad e_1, e_1 \to q_1 \vdash q_1}{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1}\,(\to_L) \qquad \dfrac{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2 \quad q \vdash q}{q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}\,(\to_L)$$

$$\dfrac{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1 \quad q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}\,(\to_L)$$

$$\dfrac{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q}\,(\to_R)$$

$$\dfrac{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q \quad q_r \vdash q_r}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r}\,(\to_L)$$

From BVASS to $\mathbf{R}_{\to}$ (2/3)



$$\frac{e_2 \vdash e_2 \qquad q_2 \vdash q_2}{e_2 \to q_2, e_2 \vdash q_2}\,(\to_L)$$

$$\frac{q_\ell \vdash q_\ell \qquad e_2 \to q_2, e_2 \vdash q_2}{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2}\,(\to_L)$$

$$\frac{e_1 \vdash e_1 \qquad q_1 \vdash q_1}{e_1, e_1 \to q_1 \vdash q_1}\,(\to_L)$$

$$\frac{q_\ell \vdash q_\ell \qquad e_1, e_1 \to q_1 \vdash q_1}{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1}\,(\to_L)$$

$$\frac{q_\ell, q_\ell \to (e_2 \to q_2), e_2 \vdash q_2 \qquad q \vdash q}{q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}\,(\to_L)$$

$$\frac{q_\ell, e_1, q_\ell \to (e_1 \to q_1) \vdash q_1 \qquad q_\ell, q_\ell \to (e_2 \to q_2), q_2 \to q, e_2 \vdash q}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}\,(\to_L)$$

$$\frac{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), e_2 \vdash q}{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q}\,(\to_R)$$

$$\frac{q_\ell, e_1, \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q) \vdash e_2 \to q \qquad q_r \vdash q_r}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r}\,(\to_L)$$

$$\frac{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell, q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r}{q_\ell, e_1, q_\ell \to (e_1 \to q_1), q_\ell \to (e_2 \to q_2), q_1 \to (q_2 \to q), (e_2 \to q) \to q_r \vdash q_r}\,(C)$$

From BVASS to $\mathbf{R}_{\to}$ (3/3)

Lemma

A sequent $q_\ell, \Gamma_{\mathbf{v}}, \Delta_T \vdash q$ is provable in $\mathbf{R}_{\to}$ iff there exists an expansive deduction tree of $\mathcal{B}$ with

1. *leaves labeled by $q_\ell, \mathbf{0}$,*
2. *root labeled by $q, \mathbf{v}$,*
3. *each rule in T employed at least once.*

Comprehensive Reachability

Every rule in $T_u \cup T_s$ is used at least once in the witness deduction tree.

Proposition

BVASS Coverability $<_{\text{LogSpace}}$ Comprehensive Expansive BVASS Reachability

Extensions

Larger fragments of $\mathbf{R}$ and contractive intuitionistic linear logic:

Theorem
Provability in $\mathbf{R}^{\mathbf{t}}_{\rightarrow}$, $\mathbf{IMLLC}$, *and* $\mathbf{IMELZC}$ *is 2-*EXPTIME*-complete.*

Full Paper Goodies

http://arxiv.org/abs/1402.0705

Appendix A A focusing proof sequent calculus for $\mathbf{R}_{\to}$

Appendix B A parameterized analysis of 2-ExpTime-easiness for BVASS (instead of BVAS as in Demri et al., 2013)

Perspectives

- employing a BVASS coverability tool for $\mathbf{R}_{\rightarrow}$ (Majumdar and Wang, 2013)?
- what about the complexity of $\mathbf{T}_{\rightarrow}$ (Bimbó and Dunn, 2013; Padovani, 2013)?

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