Towards Complexity Bounds for VAS Reachability

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Meet the Dragon:

- **VAS**: vector addition systems
  $A \subseteq \mathbb{Z}^d$ finite set of actions

- **reachability**: given $x, y \in \mathbb{N}^d$ and A,
  does there exist a run
  $$x = c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} \cdots \xrightarrow{a_m} c_m = y$$
  where $c_{i+1} = c_i + a_i$ and $a_i \in A \ \forall i$.

- **decidable**:
  - Mayr 1981
  - Kosaraju 1982
  - Lambert 1992

  ] KLMST decomposition

  Leroux 2009 & 2011] Presburger inductive invariants

The Weapon:

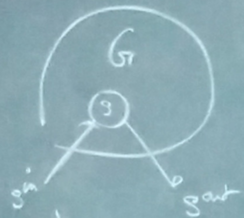
- length function theorems

  bounds on the length of controlled bad sequences

→ complexity for wqo-based termination

- classify F_{ω^3}

Marked Graphs



$G = (V, E, s)$

$V \subseteq \mathbb{N}_\omega^d$

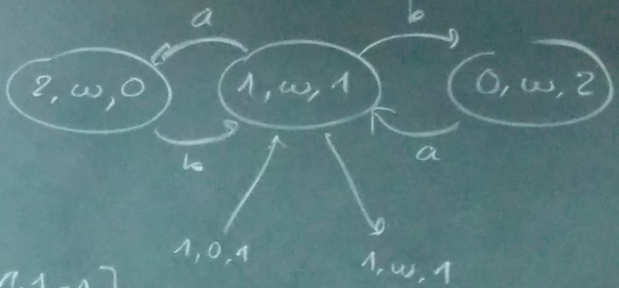
$E \subseteq V \times A \times V$

$M = (s^{in}, G, s^{out})$

$s^{in} \leq_\omega s$

$s^{out} \leq_\omega s$

$$\begin{tikzcd}[column sep=small] & F & \\ F_{in} \arrow[ur, phantom, sloped, "\subseteq"] & & F_{out} \arrow[ul, phantom, sloped, "\supseteq"] \\ & \{1, \ldots, d\} \arrow[ul, phantom, sloped, "\supseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & \end{tikzcd}$$



$$\begin{bmatrix} a : 1,1,-1 \\ b = -1,0,1 \end{bmatrix} = A$$

Set of runs of M:

$$\Omega_M = \left\{ x \xrightarrow{\sigma} y \;\middle|\; \begin{array}{l} x \le_\omega s_{in} \\ y \le_\omega s_{out} \end{array} \right\}$$

$$1,0,1 \xrightarrow{w_1 \dots w_n} 1,n,1$$

$$w_i \in \{ab, ba\}$$

Pumpability

$$s^{in} \xrightarrow{\sigma_+} s_1 \xrightarrow{\sigma_+} s_2 \cdots s_m \xrightarrow{\sigma} s'_m \xrightarrow{\sigma_-} \cdots \xrightarrow{\sigma_-} s'_1 \xrightarrow{\sigma_-} s^{out}$$

MGTS

$$\xi = M_0\, a_1\, M_1\, a_2 \cdots M_k$$

$$s_0^{in} = x \qquad s_k^{out} = y$$

$$\Omega_\xi \quad \underbrace{x_0 \xrightarrow{\sigma_0} y_0}_{\in\, \Omega_{M_0}} \xrightarrow{a_1} \underbrace{x_1 \xrightarrow{\sigma_1}}_{\in\, \Omega_{M_1}} \cdots\; y_k$$

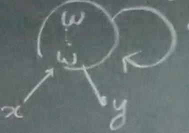
System of linear equations:

unboundedness of solutions

- on the edges
- on the input/output ω-components

<u>TH</u>: if ξ is perfect then $\Omega_\xi \neq \emptyset$

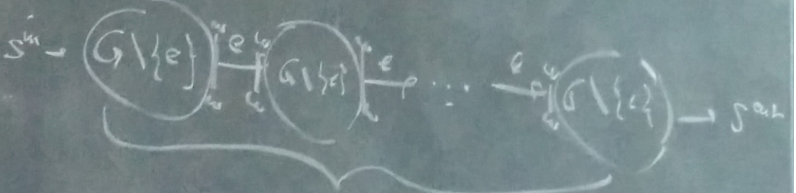
KLMST algorithm



$\forall a \in A$

$$\xi \longrightarrow \mathrm{dec}(\xi)$$

$$\Omega_\xi = \bigcup_{\xi' \in \mathrm{dec}(\xi)} \Omega_{\xi'}$$



$\leq C$

Ranking function

$$r(\xi) \in \omega^{\omega^3}$$

$$\forall \xi \; \forall \xi' \in \mathrm{dec}(\xi),\; r(\xi) > r(\xi')$$

$$M \mapsto \beta_M$$

$$\stackrel{\mathrm{def}}{=} \omega^2 \cdot (d - |F|) + \omega \cdot |t| + \left(2d - |F^{\mathrm{in}}| - |F^{\mathrm{out}}|\right)$$

$$r(M_0 \, a_1 \, M_1 \ldots M_k) \stackrel{\mathrm{def}}{=} \bigoplus_{1 \leq j \leq k} \omega^{\beta_j}$$

Length Function Theorem

over $(X, \leq)$, a sequence $x_0, x_1, \ldots$ is *bad*

if $\forall i < j$, $x_i \not\leq x_j$

$(X, \leq)$ is *wqo* iff all bad sequences are finite

$\forall n$, $\quad n, n-1, n-2, \ldots, 0$

$\forall n \quad (0,1)\ (n,0)\ (n-1,0) \ldots (0,0)$

$\|\cdot\|_A$, $g : \mathbb{N} \to \mathbb{N}$, $n \in \mathbb{N}$

$x_0, x_1, \ldots$ is *(g,n)-controlled* if

$\forall i \quad \|x_i\|_A \leq g^i(n)$

Fast-Growing Functions

$g : \mathbb{N} \to \mathbb{N}$ monotone expansive

$$F_{g,0}(x) \stackrel{def}{=} g(x)$$

$$F_{g,\alpha+1}(x) \stackrel{def}{=} F_{g,\alpha}^{x+1}(x)$$

$$F_{g,\lambda}(x) \stackrel{def}{=} F_{g,\lambda(x)}(x)$$

Ex:

$$\omega(x) = x+1$$

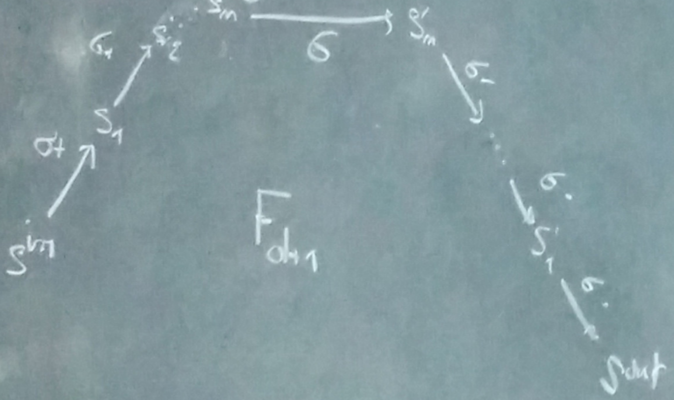
$$\omega^{\omega}(x) = \omega^{x+1}$$

$$F_{F_2,2}(x) = F_4(x)$$

$$F_{F_\alpha,\beta}(x) \neq F_{\alpha+\beta}(x)$$

$$F_{F_n,\omega}(x) \leq F_\omega(F_n(x))$$

Pumpability



TH $\alpha < \varepsilon_0$, $\|\alpha\| \leq n$,

$$L_{\omega^{\alpha},g}(n) \leq F_{g,\alpha}(n)$$

Descent Equation

$$L_{\alpha,g}(n) = \max_{\beta_0<\alpha,\ \|\beta_0\|\leq n} 1 + L_{\beta_0,g}\big(g(n)\big)$$

$\beta_0 > \beta_1 \cdots \in \alpha$

Ranking function

$$r(\xi) \in \omega^{\omega^3}$$

$$\forall \xi\ \forall \xi' \in dec(\xi),\ r(\xi) > r(\xi')$$

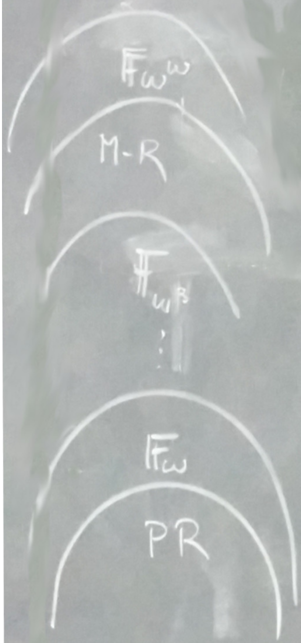
$$M \mapsto \beta_M$$

$$\stackrel{def}{=} \omega^2 \cdot (d - |F|) + \omega \cdot |E| + (2d - |F^{in}| - |F^{out}|)$$

$$r(M_0\, a_1\, M_1 \cdots M_k) \stackrel{def}{=} \bigoplus_{1 \leq j < k} \omega^{\beta_j}$$

$$\leq F_{F_{d+1},\, \omega^3}(\|\xi_0\|)$$

$$\leq F_{\omega^3}(F_{d+1}(\|\xi_0\|))$$



$$\mathbb{F}_{\omega^3} = \bigcup_{\rho \in \mathcal{F}_{<\omega^3}} \mathrm{SPACE}\left(F_{\omega^3}\left(\rho(n)\right)\right)$$