Demystifying Reachability in Vector Addition Systems

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Outline

vector addition systems (VAS)
and their reachability problem

… solved by the KLMST algorithm
of Sacerdote and Tenney (1977), Mayr (1981), Kosaraju (1982), and Lambert (1992)

decomposition theorem
the KLMST algorithm constructs an ideal decomposition of the set of runs

upper bound theorem
VAS reachability is in cubic Ackermann

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Vector Addition Systems (VAS)

(Karp and Miller, 1969)

Syntax

- dimension $\mathrm{d} \in \mathbb{N}$
- finite set $\mathbf{A} \subseteq_{\mathrm{fin}} \mathbb{Z}^{\mathrm{d}}$ of actions $\mathbf{a} \in \mathbf{A}$

Semantics

- configurations $\mathbf{u}, \mathbf{v}, \ldots \in \mathbb{N}^{\mathrm{d}}$
- transitions $\mathbf{u} \xrightarrow{\mathbf{a}} \mathbf{v} \in \mathbb{N}^{\mathrm{d}} \times \mathbf{A} \times \mathbb{N}^{\mathrm{d}}$ with $\mathbf{v} = \mathbf{u} + \mathbf{a}$

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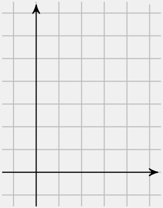
Example VAS

Example

$d = 2$

$\mathbf{A} = \{$  ,  $\}$



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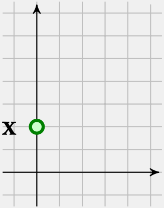

$\mathbf{x}=(0,2)$

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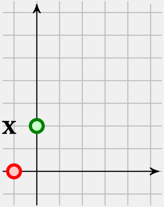

$\mathbf{x} = (0,2) \longrightarrow (-1,0) \notin \mathbb{N}^2$

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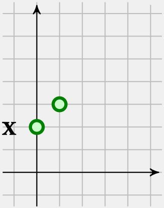

$\mathbf{x}=(0,2)$  $\longrightarrow (1,3)$

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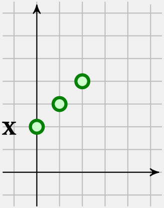

$\mathbf{x} = (0,2)$  $\longrightarrow (1,3)$  $\longrightarrow (2,4)$

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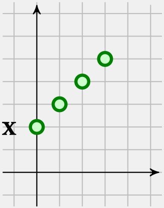

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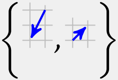
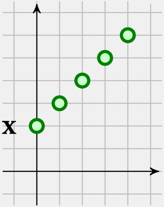
$\mathbf{A} =$ 



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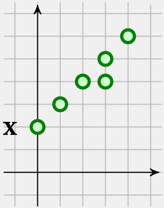

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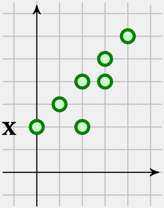

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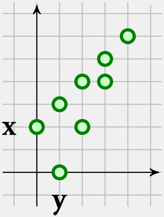
$d = 2$ $\mathbf{A} = \{ \swarrow, \nearrow \}$



$\mathbf{x} = (0,2) \xrightarrow{\nearrow} (1,3) \xrightarrow{\nearrow} (2,4) \xrightarrow{\nearrow} (3,5) \xrightarrow{\nearrow} (4,6) \xrightarrow{\swarrow} (3,4) \xrightarrow{\swarrow} (2,2) \xrightarrow{\swarrow} (0,1) = \mathbf{y}$

Runs and Preruns

Definition (Prerun)

A prerun is an element

$$(\mathbf{u},\ (\mathbf{u}_1,\mathbf{a}_1,\mathbf{v}_1)\cdots(\mathbf{u}_k,\mathbf{a}_k,\mathbf{v}_k),\ \mathbf{v})$$

from $\text{PreRuns}_{\mathbf{A}} \stackrel{\text{def}}{=} \mathbb{N}^d \times (\mathbb{N}^d \times \mathbf{A} \times \mathbb{N}^d)^* \times \mathbb{N}^d$

Definition (Run)

A prerun is connected (is a run) if

(source) $\mathbf{u} = \mathbf{u}_1$

(transitions) $\forall 1 \leqslant j \leqslant k, \mathbf{u}_j + \mathbf{a}_j = \mathbf{v}_j$

(contiguity) $\forall 1 < j \leqslant k, \mathbf{v}_{j-1} = \mathbf{u}_j$

(target) $\mathbf{v}_k = \mathbf{v}$

The Reachability Problem

$\text{Runs}_{\mathbf{A}}(\mathbf{x},\mathbf{y}) \stackrel{\text{def}}{=} \{\rho \in \text{PreRuns}_{\mathbf{A}} \mid \rho \text{ is a run with source } \mathbf{x} \text{ and target } \mathbf{y}\}$

VAS Reachability

input $\mathbf{A} \subseteq_{\text{fin}} \mathbb{Z}^{d}, \mathbf{x}, \mathbf{y} \in \mathbb{N}^{d}$

question Is $\mathbf{y}$ reachable from $\mathbf{x}$ in $\mathbf{A}$?
I.e., is $\text{Runs}_{\mathbf{A}}(\mathbf{x},\mathbf{y}) \neq \emptyset$?

Theorem (Mayr, 1981; Kosaraju, 1982; Lambert, 1992; Leroux, 2011)
VAS Reachability is decidable.

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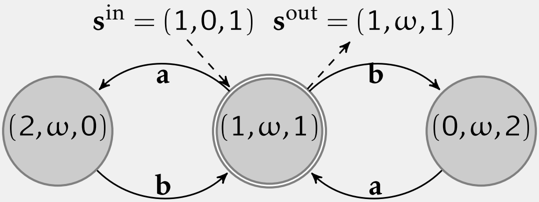
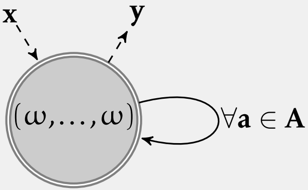
Marked Graph

Example

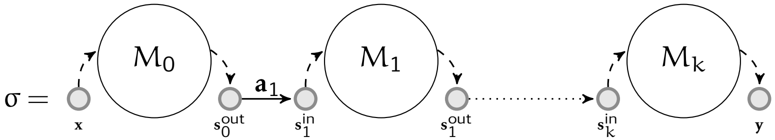
$$\mathbf{A} = \{\mathbf{a} = (1,1,-1), \mathbf{b} = (-1,0,1))$$



Example (Initial graph)



Marked Graph Sequence



- associated set of runs $\Omega_\sigma \subseteq \mathrm{Runs}_{\mathbf{A}}(\mathbf{x},\mathbf{y})$
- perfectness condition (aka θ condition): decidable semantic condition ensuring $\Omega_\sigma \neq \emptyset$
- effective decomposition of imperfect sequences: $\Omega_\sigma = \bigcup_{\sigma' \in \mathrm{decompose}(\sigma)} \Omega_{\sigma'}$

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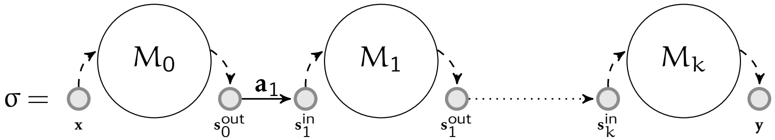

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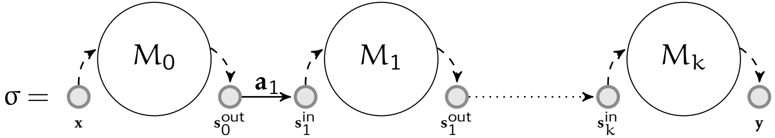

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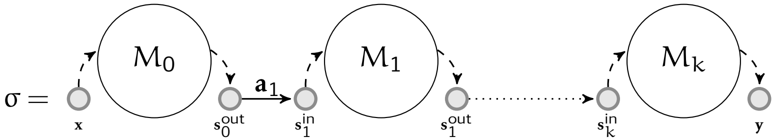

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KLMST Algorithm (Schematically)

Construct a sequence $S_0, S_1, \ldots$ of finite sets of marked witness graph sequences with $\forall n$

$$\Omega_n \stackrel{\text{def}}{=} \bigcup_{\sigma \in S_n} \Omega_\sigma = \text{Runs}_{\mathbf{A}}(\mathbf{x}, \mathbf{y})$$

init S_0 is s.t. $\text{Runs}_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) = \Omega_0$

$\forall n$

- if $S_n = \{\sigma\} \uplus S$ and $\neg\text{perfect}(\sigma)$

  $$S_{n+1} \stackrel{\text{def}}{=} S \cup (\text{decompose}(\sigma))$$

- otherwise stop: $\text{Runs}_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) \neq \emptyset$

terminates via a ranking function r

$$\forall \sigma' \in \text{decompose}(\sigma) \,.\, r(\sigma) > r(\sigma')$$

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1. conceptual complexity

   the complexity of the proofs (especially of Mayr, 1981) wrap the result in mystery use of their original ideas has been made

2. computational complexity
   - ExpSpace-hard problem (Lipton, 1976),
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Deciphering the statement (upcoming slides)

- definition of a well quasi order (wqo) over preruns (Jančar, 1990)
- wqo ideals (Finkel and Goubault-Larrecq, 2009, 2012)

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Significance

- entails decidability of VAS Reachability:

$$\mathsf{Runs}_{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \emptyset \text{ iff } \downarrow \mathsf{Runs}_{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \emptyset$$

- generalises Habermehl et al. (2010)'s result on the computability of downward-closures of VAS languages
- template for decidability proofs in extensions (unordered data nets, branching VAS, pushdown VAS, ...)?

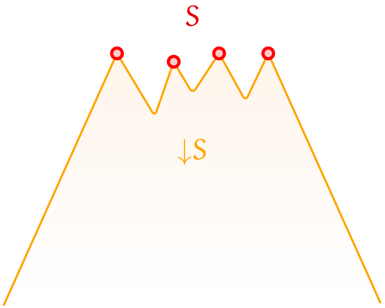
Well quasi orders

- Downward closure
  $\downarrow S \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S \,.\, x \leqslant s\}$

over a quasi-order $(X, \leqslant)$

- Descending Chain Property
  A quasi-order $(X, \leqslant)$ is a well quasi order if every descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ of downwards-closed subsets of X is finite.

- Examples
  - finite sets with equality
  - $\mathbb{N}$ with the naturals ordering
  - $A \times B$ with the product ordering (Dickson's Lemma)
  - A^* with scattered subword ordering (Higman's Lemma)



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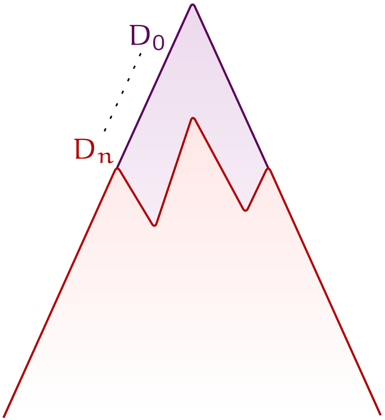

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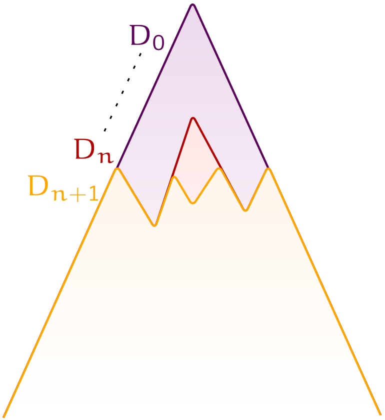

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  $\downarrow S \overset{\text{def}}{=} \{x \in X \mid \exists s \in S \,.\, x \leqslant s\}$
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  A quasi-order $(X, \leqslant)$ is a well quasi order if every descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ of downwards-closed subsets of X is finite.
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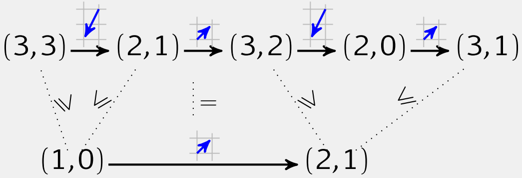
Prerun Embedding

Construct the ordering $\trianglelefteq$ over preruns inductively; recall

$$\mathsf{PreRuns}_{\mathbf{A}} \stackrel{\text{def}}{=} \mathbb{N}^d \times (\mathbb{N}^d \times \mathbf{A} \times \mathbb{N}^d)^* \times \mathbb{N}^d$$

Example (Run Embedding $\trianglelefteq$)



Lemma (Jančar, 1990)

$(\mathsf{PreRuns}_{\mathbf{A}}, \trianglelefteq)$ *is a wqo.*

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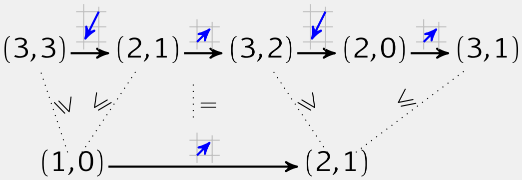

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Ideals as Canonical Bases

Lemma (Canonical Ideal Decomposition; Bonnet, 1975)

Every downward-closed subset $D \subseteq X$ of a wqo $(X, \leqslant)$ is the union of a unique finite family of incomparable (for the inclusion) ideals.

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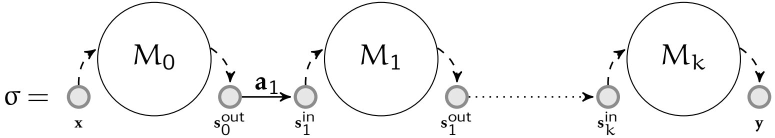
Marked Graph Sequence (Ideal View)



A representation for (particular) prerun ideals with

$$I_\sigma \supseteq \downarrow\Omega_\sigma$$

Theorem (Perfectness as Ideal Adherence)
If σ is perfect then $I_\sigma = \downarrow\Omega_\sigma$.

KLMST Algorithm (Ideal View)

Construct a sequence $D_0, D_1, \ldots$ of downwards-closed sets, represented as finite sets of ideals, with $\forall n$

$$D_n \stackrel{\text{def}}{=} \bigcup_{\sigma \in S_n} I_\sigma \supseteq \downarrow \mathsf{Runs}_{\mathbf{A}}(\mathbf{x}, \mathbf{y})$$

init $D_0 \stackrel{\text{def}}{=} \mathsf{PreRuns}_{\mathbf{A}}$

$\forall n$
- if $D_n = I \sqcup D$ and $\neg\mathsf{perfect}(I)$,

  $D_{n+1} \stackrel{\text{def}}{=} D \sqcup \mathsf{decompose}(I)$

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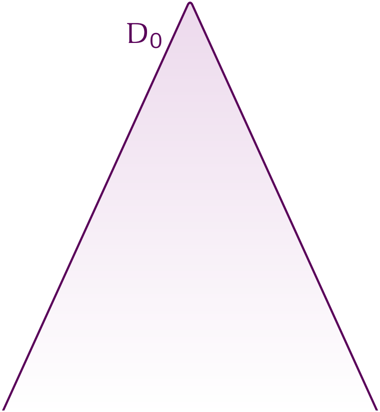

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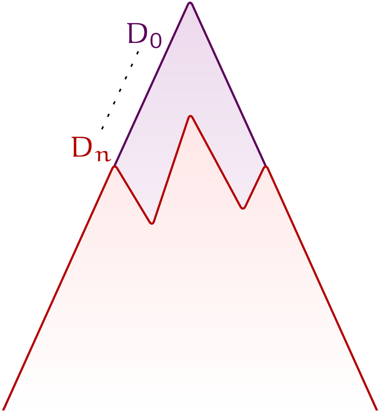

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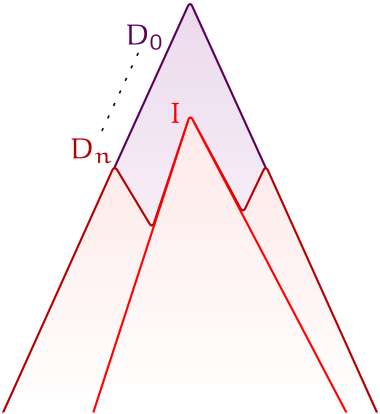

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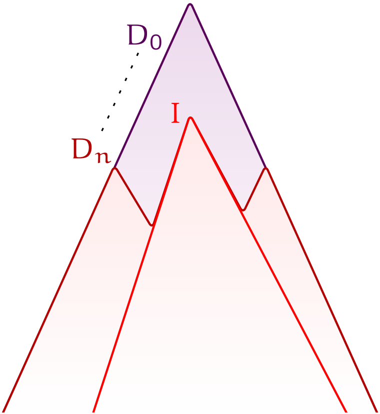

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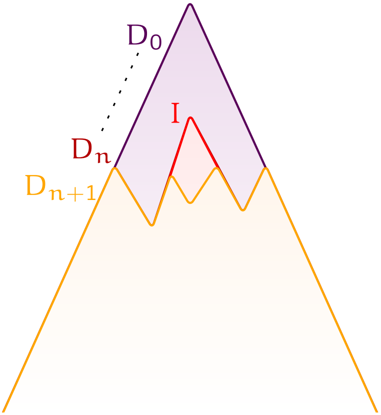

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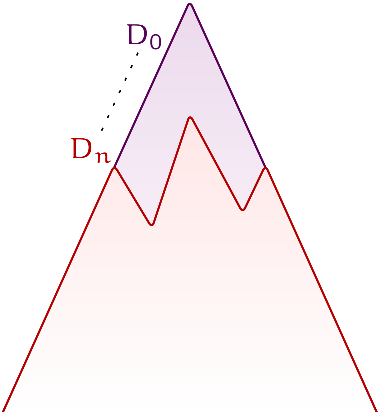

Mystery 2: Computational Complexity

Theorem (Upper Bound Theorem)

VAS Reachability is in $\mathbf{F}_{\omega^3}$.

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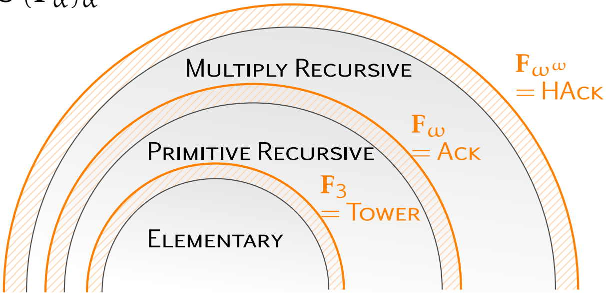

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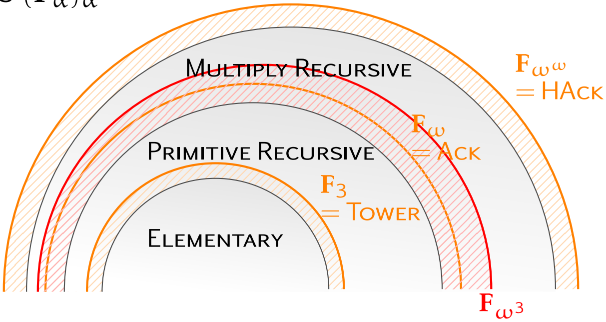

Concluding Remarks

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- new understanding of the KLMST decomposition
  extension to other models (BVASS, PDVAS,...)?
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Blondin, M., Finkel, A., Göller, S., Haase, C., and McKenzie, P., 2015. Reachability in two-dimensional vector addition systems with states is PSPACE-complete. In *Proc. LICS 2015*. IEEE Press. http://arxiv.org/abs/1412.4259. To appear.

Bonnet, R., 1975. On the cardinality of the set of initial intervals of a partially ordered set. In *Infinite and finite sets: to Paul Erdős on his 60th birthday, Vol. 1*, Coll. Math. Soc. János Bolyai, pages 189–198. North-Holland.

Figueira, D., Figueira, S., Schmitz, S., and Schnoebelen, Ph., 2011. Ackermannian and primitive-recursive bounds with Dickson's Lemma. In *LICS 2011*, pages 269–278. IEEE Press. doi:10.1109/LICS.2011.39.

Finkel, A. and Goubault-Larrecq, J., 2009. Forward analysis for WSTS, part I: Completions. In *Proc. STACS 2009*, volume 3 of *LIPIcs*, pages 433–444. LZI. doi:10.4230/LIPIcs.STACS.2009.1844.

Finkel, A. and Goubault-Larrecq, J., 2012. Forward analysis for WSTS, part II: Complete WSTS. *Logic. Meth. in Comput. Sci.*, 8(3:28):1–35. doi:10.2168/LMCS-8(3:28)2012.

Habermehl, P., Meyer, R., and Wimmel, H., 2010. The downward-closure of Petri net languages. In *Proc. ICALP 2010*, volume 6199 of *LNCS*, pages 466–477. Springer. doi:10.1007/978-3-642-14162-1_39.

Jančar, P., 1990. Decidability of a temporal logic problem for Petri nets. *Theor. Comput. Sci.*, 74(1):71–93. doi:10.1016/0304-3975(90)90006-4.

Karp, R.M. and Miller, R.E., 1969. Parallel program schemata. *Journal of Computer and System Sciences*, 3(2): 147–195. doi:10.1016/S0022-0000(69)80011-5.

Kosaraju, S.R., 1982. Decidability of reachability in vector addition systems. In *Proc. STOC'82*, pages 267–281. ACM. doi:10.1145/800070.802201.

Lambert, J.L., 1992. A structure to decide reachability in Petri nets. *Theor. Comput. Sci.*, 99(1):79–104. doi:10.1016/0304-3975(92)90173-D.

Leroux, J., 2011. Vector addition system reachability problem: a short self-contained proof. In *Proc. POPL 2011*, pages 307–316. ACM. doi:10.1145/1926385.1926421.

Lipton, R., 1976. The reachability problem requires exponential space. Technical Report 62, Yale University. http://cpsc.yale.edu/sites/default/files/files/tr63.pdf.

Mayr, E.W., 1981. An algorithm for the general Petri net reachability problem. In *Proc. STOC'81*, pages 238–246. ACM. doi:10.1145/800076.802477.

Müller, H., 1985. The reachability problem for VAS. In *Advances in Petri Nets 1984*, volume 188 of *LNCS*, pages 376–391. Springer. doi:10.1007/3-540-15204-0_21.

Sacerdote, G.S. and Tenney, R.L., 1977. The decidability of the reachability problem for vector addition systems. In *Proc. STOC'77*, pages 61–76. ACM. doi:10.1145/800105.803396.

Schmitz, S., 2014. Complexity bounds for ordinal-based termination. In *Proc. RP 2014*, volume 8762 of *LNCS*, pages 1–19. Springer. doi:10.1007/978-3-319-11439-2_1.

A Bit of Magic...

Theorem (Length Function Theorem, S., 2014)

(g,n)-controlled decreasing sequences $\alpha_0, \alpha_1, \ldots$ of ordinals $< \alpha$ with $n \geqslant |\alpha|$ are of length bounded by $g^\alpha(n)$ in the Hardy hierarchy.

Claim (KLMST control, using Figueira et al.*, 2011)*

In the KLMST algorithm, the sequence of ranks along any branch is controlled by $(H^{\omega^{d+1}}, |\mathbf{A}|)$.

As a result, the KLMST algorithm runs in space $\left(H^{\omega^{d+1}}\right)^{\omega^{\omega^3}}(|\mathbf{A}|)$, which is in $\mathbf{F}_{\omega^3}$.

A Bit of Magic: Controlled Sequences

Definition (Ordinal Norm)

For an ordinal $\alpha < \varepsilon_0$, define the *norm* $|\alpha|$ as the maximal coefficient appearing in its Cantor normal form $\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_n} \cdot c_n$:

$$|\alpha| \stackrel{\text{def}}{=} \max_{1 \leqslant j \leqslant n} (c_j, |\alpha_j|) .$$

Definition (Controlled Sequence)

Let $g\colon \mathbb{N} \to \mathbb{N}$ strictly monotone and $n \in \mathbb{N}$. A sequence of ordinals $\alpha_0, \alpha_1, \ldots$ is (g,n)-*controlled* if

$$\forall i \,.\, |\alpha_i| \leqslant g^i(n) .$$

In particular, $|\alpha_0| \leqslant n$.

A Bit of Magic: Hardy Functions

Ordinal-indexed hierarchy of functions $h^\alpha: \mathbb{N} \to \mathbb{N}$

Definition

Fix $h: \mathbb{N} \to \mathbb{N}$ strictly monotone:

$$h^0(x) \stackrel{\text{def}}{=} x \qquad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^\alpha(h(x)) \qquad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda(x)}(x)$$

where $\lambda(0) < \lambda(1) < \cdots < \lambda$ is the standard fundamental sequence for the limit ordinal λ, e.g. $\omega(x) = x + 1$, $\omega^2(x) = \omega \cdot (x+1)$, $\omega^\omega(x) = \omega^{x+1}$.

Example

For instance for $H(x) \stackrel{\text{def}}{=} x + 1$:

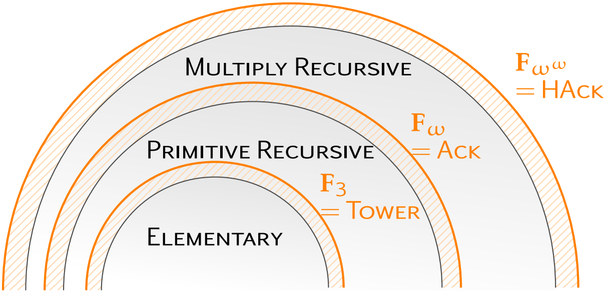
$$H^\omega(x) = 2x + 1 \qquad H^{\omega^2}(x) = 2^{x+1}(x+1) - 1$$

$$H^{\omega^3}(x) \approx 2^{\cdot^{\cdot^{\cdot^2}}}\Big\}x \text{ times} \qquad H^{\omega^\omega}(x) \approx \text{ackermann}(x)$$

A Bit of Magic: Fast-Growing Complexity

For $\alpha \geqslant 3$:

$$\mathscr{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta<\omega^\alpha} \text{FDTime}(H^\beta(n)) \,, \quad \mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{p\in\mathscr{F}_{<\alpha}} \text{DTime}(H^{\omega^\alpha}(p(n)))$$



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