TD 3

1 LTL and Büchi Automata

Exercise 1 (Büchi Automaton Construction)**.** Consider the LTL formula $\varphi = \mathsf{G}(\neg p \vee \neg \mathsf{X} p)$.

1. Give a Büchi automaton for the models of φ.
2. Use the LTL to Büchi algorithm seen during the last lecture to construct another Büchi automaton.

Exercise 2 (Simplifying Formulæ)**.** The complexity of LTL to Büchi automata constructions is very sensitive to the size of the input LTL formulæ. It is therefore interesting to simplify formulæ prior to the construction.

1. Propose some simple equivalences on LTL formulæ. Apply them to simplify the following formula:

$$\mathsf{F}(((\mathsf{G}r) \mathbin{\mathsf{U}} p) \wedge (\neg q \mathbin{\mathsf{U}} p)) \vee \mathsf{F}(\neg p \vee \mathsf{F}q) \ .$$

2. Let us consider LTL(X, U) formulæ in negative normal form (NNF). We are looking for a sufficient syntactic condition on formulæ for the following equivalence to hold:

$$\psi \mathbin{\mathsf{U}} \varphi \equiv \varphi \tag{1}$$

   Define a *pure eventuality formula* as an NNF LTL(X, U) formula of form:

   - $\mathsf{F}\psi$ where ψ is an LTL formula in NNF, or
   - $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \mathbin{\mathsf{U}} \psi$, $\mathsf{G}\varphi_1$, $\varphi_1 \mathbin{\mathsf{R}} \varphi_2$ or $\mathsf{X}\varphi_1$, where φ_1 and φ_2 are pure eventuality formulæ and ψ an LTL formula in NNF.

   Prove that the language $L(\varphi)$ of a pure eventuality formula φ verifies $\Sigma^* L(\varphi) = L(\varphi)$.

3. Prove that equivalence (1) holds if φ is a pure eventuality formula.

2 Stuttering

Exercise 3 (Stuttering and LTL(U))**.** In the context of a word σ in Σ^ω, *stuttering* denotes the existence of consecutive symbols, like *aaaa* and *bb* in *baaaabb*. Concrete systems tend to stutter, and thus some argue that verification properties should be stutter invariant.

A *stuttering function* $f : \mathbb{N} \to \mathbb{N}_+$ from the positive integers to the strictly positive integers. Let $\sigma = a_0 a_1 \cdots$ be an infinite word of Σ^ω and f a stuttering function, we denote by $\sigma[f]$ the infinite word $a_0^{f(0)} a_1^{f(1)} \cdots$, i.e. where the i-th symbol of σ is repeated $f(i)$ times. A language $L \subseteq \Sigma^\omega$ is *stutter invariant* if, for all words σ in Σ^ω and all stuttering functions f,

$$\sigma \in L \text{ iff } \sigma[f] \in L\ .$$

1. Prove that if φ is a LTL(U) formula, then $L(\varphi)$ is stutter-invariant.
2. A word $\sigma = a_0 a_1 \cdots$ in Σ^ω is *stutter-free* if, for all i in $\mathbb{N}$, either $a_i \neq a_{i+1}$, or $a_i = a_j$ for all $j \geq i$. We note $\mathsf{sf}(L)$ for the set of stutter-free words in a language L.

   Show that, if L and L' are two stutter invariant languages, then $\mathsf{sf}(L) = \mathsf{sf}(L')$ iff $L = L'$.
3. Let φ be a LTL(X, U) formula such that $L(\varphi)$ is stutter invariant. Construct inductively a formula $\tau(\varphi)$ of LTL(U) such that $\mathsf{sf}(L(\varphi)) = \mathsf{sf}(L(\tau(\varphi)))$, and thus such that $L(\varphi) = L(\tau(\varphi))$ according to the previous question. What is the size of $\tau(\varphi)$ (there exists a solution of size $O(|\varphi| \cdot 2^{|\varphi|})$)?

Exercise 4 (Complexity of LTL(U))**.** We want to prove that the model checking and satisfiability problems for LTL(U) formulæ are both PSPACE-complete.

1. Prove that $\mathsf{MC}^\exists(\mathsf{X}, \mathsf{U})$ can be reduced to $\mathsf{MC}^\exists(\mathsf{U})$: given an instance (M, φ) of $\mathsf{MC}^\exists(\mathsf{X}, \mathsf{U})$, construct a stutter-free Kripke structure M' and an LTL(U) formula $\tau'(\varphi)$. Beware: the τ construction of the previous exercise does not yield a polynomial reduction!
2. Show that $\mathsf{MC}^\exists(\mathsf{X}, \mathsf{U})$ can be reduced to $\mathsf{SAT}(\mathsf{U})$.