TD 9: Vector Addition Systems; Unfoldings

1 Vector Addition Systems

Exercise 1 (VASS)**.** An n-dimensional *vector addition system with states* (VASS) is a tuple $\mathcal{V} = \langle Q, \delta, q_0 \rangle$ where Q is a finite set of states, $q_0 \in Q$ the initial state, and $\delta \subseteq Q \times \mathbb{Z}^n \times Q$ the transition relation. A configuration of $\mathcal{V}$ is a pair (q, v) in $Q \times \mathbb{N}^n$. An execution of $\mathcal{V}$ is a sequence of configurations $(q_0, v_0)(q_1, v_1) \cdots (q_m, v_m)$ such that $v_0 = \bar{0}$, and for $0 < i \leq m$, $(q_{i-1}, v_i - v_{i-1}, q_i)$ is in δ.

1. Show that any VASS can be simulated by a Petri net.
2. Show that, conversely, any Petri net can be simulated by a VASS.

Exercise 2 (VAS)**.** An n-dimensional *vector addition system* (VAS) is a pair (v_0, W) where $v_0 \in \mathbb{N}^n$ is the initial vector and $W \subseteq \mathbb{Z}^n$ is the set of transition vectors. An execution of (v_0, W) is a sequence $v_0 v_1 \cdots v_m$ where $v_i \in \mathbb{N}$ for all $0 \leq i \leq m$ and $v_i - v_{i-1} \in W$ for all $0 < i \leq m$.

We want to show that any n-dimensional VASS $\mathcal{V}$ can be simulated by an $(n+3)$-dimensional VAS (v_0, W).

Hint: Let $k = |Q|$, and define the two functions $a(i) = i + 1$ and $b(i) = (k+1)(k-i)$. Encode a configuration (q_i, v) of $\mathcal{V}$ as the vector $(v(1), \ldots, v(n), a(i), b(i), 0)$. For every state q_i, $0 \leq i < k$, we add two transition vectors to W:

$$t_i = (0, \ldots, 0, -a(i), a(k-i) - b(i), b(k-i))$$
$$t'_i = (0, \ldots, 0, b(i), -a(k-i), a(i) - b(k-i))$$

For every transition $d = (q_i, w, q_j)$ of $\mathcal{V}$, we add one transition vector to W:

$$t_d = (w(1), \ldots, w(n), a(j) - b(i), b(j), -a(i))$$

1. Show that any execution of $\mathcal{V}$ can be simulated by (v_0, W) for a suitable v_0.
2. Conversely, show that this VAS (v_0, W) simulates $\mathcal{V}$ faithfully.

2 Unfoldings

Exercise 3 (Adequate Partial Orders)**.** A partial order $\prec$ between events is *adequate* if the three following conditions are verified:

(a) $\prec$ is well-founded,

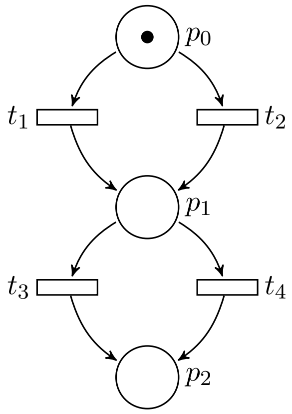
(b) $C_t \subsetneq C_{t'}$ implies $t \prec t'$, and

(c) $\prec$ is preserved by finite extensions: as in the lecture notes, if $t \prec t'$ and $M_t = M_{t'}$, and E and E' are two isomorphic extensions of C_t and $C_{t'}$ with $C_u = C_t \oplus E$ and $C_{u'} = C_{t'} \oplus E'$, then $u \prec u'$.

As you can guess, adequate partial orders result in complete unfoldings.

1. Show that $\prec_s$ defined by $t \prec_s t'$ iff $|C_t| < |C_{t'}|$ is adequate.

2. Construct the finite unfolding of the following Petri net using $\prec_s$; how does the size of this unfolding relate to the number of reachable markings?



3. Suppose we define an arbitrary total order $\ll$ on the transitions T of the Petri net, i.e. they are $t_1 \ll \cdots \ll t_n$. Given a set S of events and conditions of $\mathcal{Q}$, $\varphi(S)$ is the sequence $t_1^{i_1} \cdots t_n^{i_n}$ in T^* where i_j is the number of events labeled by t_j in S. We also note $\ll$ for the lexicographic order on T^*.

   Show that $\prec_e$ defined by $t \prec_e t'$ iff $|C_t| < |C_{t'}|$ or $|C_t| = |C_{t'}|$ and $\varphi(C_t) \ll \varphi(C_{t'})$ is adequate. Construct the finite unfolding for the previous Petri net using $\prec_e$.

4. There might still be examples where $\prec_e$ performs poorly. One solution would be to use a *total* adequate order. Give a 1-safe Petri net that shows that $\prec_e$ is not total.

Exercise 4 (LTL(U) Model Checking)**.** We consider again the problem of model checking state-based LTL formulæ against the reachable markings of a 1-safe Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. The LTL(U) formulæ we consider use a subset of the places as atomic propositions: AP $\subseteq P$. An atomic proposition p in AP holds in a marking m in $\mathbb{N}^P$ (written $m \models p$) if $m(p) > 0$.

Instead of constructing an exponential-sized Büchi automaton $\mathcal{B}_\mathcal{N}$ (based on the reachability graph of $\mathcal{N}$) and its intersection with $\mathcal{B}_{\neg\varphi}$, we want to construct a Petri net $\mathcal{N}_{\neg\varphi}$ for the product of $\mathcal{N}$ and $\mathcal{B}_{\neg\varphi}$, and check its emptiness using unfolding techniques.

1. Describe how to construct this product Petri net $\mathcal{N}_{\neg\varphi}$ if $\mathrm{AP} = P$. Are unfolding techniques going to be efficient on this product?

2. Let us suppose $\mathrm{AP} \subsetneq P$. A transition t of T is *visible* if there exists p in AP such that $W(t,p) - W(p,t) \neq 0$.

   We only have to synchronize visible actions with the transitions of $\mathcal{B}_{\neg\varphi}$, but we need to distinguish two forms of acceptance: let I and L be two sets of *illegal* and *livelock* transitions of $\mathcal{N}_{\neg\varphi}$, such that all the transitions of I are visible.

   (a) An infinite execution $m_0 \xrightarrow{t_1} m_1 \xrightarrow{t_2} \cdots$ of $\mathcal{N}_{\neg\varphi}$ is *illegal* if t_i is infinitely often in I.

   (b) An infinite execution $m_0 \xrightarrow{t_1} m_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} m_i \xrightarrow{t_{i+1}} \cdots$ of $\mathcal{N}_{\neg\varphi}$ is a *livelock* if t_i is in L and no subsequent transition t_{i+j} for $j > 0$ is visible.

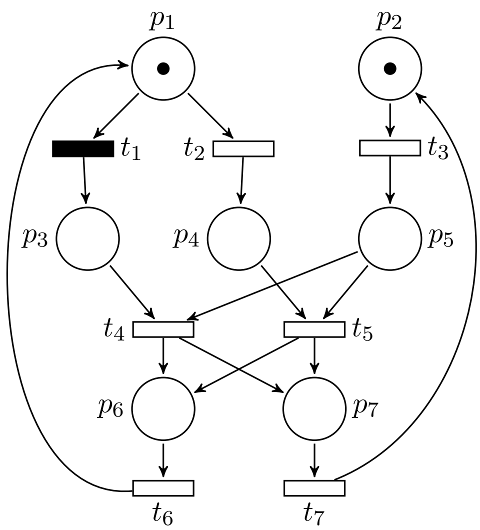
   Propose a construction for $\mathcal{N}_{\neg\varphi}$ such that $\mathcal{N} \models \varphi$ a LTL(U) formula iff $\mathcal{N}_{\neg\varphi}$ has no illegal nor livelock infinite executions.

3. Let us treat illegal executions in $\mathcal{N}_{\neg\varphi}$. Given a set of events and conditions S, we denote by $|S|_I$ the number of events of S labeled by some transition in I. An event e is a *repeat* with respect to some adequate partial ordering $\prec$, if there exists another event e' (its *companion*) with $M_e = M_{e'}$ and either

   (a) $e' < e$, or

   (b) $\neg(e' < e)$, $e' \prec e$, and $|C_{e'}|_I \geq |C_e|_I$.

   A repeat e is *terminal* if there does not exist another repeat e' with $e' < e$. A repeat e with companion e' is *successful* if $e' < e$ and $||\lfloor e \rfloor \backslash \lfloor e' \rfloor||_I > 0$. A *tableau* is an unfolding where we cut off at terminal events. Construct the tableau for the following Petri net where $I = \{t_1\}$ and the order $\prec_e$ of the previous exercise:



4. Show that the existence of a successful repeat in the unfolding of $\mathcal{N}_{\neg\varphi}$ implies the existence of an illegal execution.

5. Let us prove that we do not need to unfold $\mathcal{N}_{\neg\varphi}$ indefinitely.

   (a) Let B be the maximal number of tokens that can appear simultaneously in a marking. Show that, for any $k \geq 0$, if a subset of events E of a configuration C contains strictly more than $k \cdot B$ events, then there exists a chain $e_1 < \cdots < e_{k+1}$ in E.

   (b) Let K be the number of distinct reachable markings of $\mathcal{N}_{\neg\varphi}$. Show that a tableau cannot contain more than $K^2 \cdot B$ non terminal events.

6. Let us prove that we can always witness an illegal execution thanks to a successful repeat in the tableau for $\mathcal{N}_{\neg\varphi}$.

   (a) Denote a configuration as *bad* if it contains more than $(K \cdot B) + 1$ I-events. Show that $\mathcal{N}_{\neg\varphi}$ exhibits an illegal execution iff its infinite unfolding contains a bad configuration.

   (b) Show that a bad configuration contains at least one terminal.

   (c) Prove that, given a bad configuration C_t of the unfolding of $\mathcal{N}_{\neg\varphi}$, either C_t contains a successful terminal of the tableau, or there exists another bad configuration $C_{t'}$ with $t' \prec t$.

   (d) Conclude.