TD 9: Vector Addition Systems; Unfoldings

1 Vector Addition Systems

Exercise 1 (VASS). An $n$-dimensional vector addition system with states (VASS) is a tuple $V = (Q, \delta, q_0)$ where $Q$ is a finite set of states, $q_0 \in Q$ the initial state, and $\delta \subseteq Q \times \mathbb{Z}^n \times Q$ the transition relation. A configuration of $V$ is a pair $(q, v)$ in $Q \times \mathbb{N}^n$. An execution of $V$ is a sequence of configurations $(q_0, v_0)(q_1, v_1) \cdots (q_m, v_m)$ such that $v_0 = \bar{0}$, and for $0 < i \leq m$, $(q_{i-1}, v_i - v_{i-1}, q_i)$ is in $\delta$.

1. Show that any VASS can be simulated by a Petri net.

2. Show that, conversely, any Petri net can be simulated by a VASS.

Exercise 2 (VAS). An $n$-dimensional vector addition system (VAS) is a pair $(v_0, W)$ where $v_0 \in \mathbb{N}^n$ is the initial vector and $W \subseteq \mathbb{Z}^n$ is the set of transition vectors. An execution of $(v_0, W)$ is a sequence $v_0v_1 \cdots v_m$ where $v_i \in \mathbb{N}$ for all $0 \leq i \leq m$ and $v_i - v_{i-1} \in W$ for all $0 < i \leq m$.

We want to show that any $n$-dimensional VASS $V$ can be simulated by an $(n + 3)$-dimensional VAS $(v_0, W)$.

Hint: Let $k = |Q|$, and define the two functions $a(i) = i + 1$ and $b(i) = (k + 1)(k - i)$.

Encode a configuration $(q_i, v)$ of $V$ as the vector $(v(1), \ldots, v(n), a(i), b(i), 0)$. For every state $q_i$, $0 \leq i < k$, we add two transition vectors to $W$:

$t_i = (0, \ldots, 0, -a(i), a(k - i) - b(i), b(k - i))$

$t'_i = (0, \ldots, 0, b(i), -a(k - i), a(i) - b(k - i))$

For every transition $d = (q_i, w, q_j)$ of $V$, we add one transition vector to $W$:

$t_d = (w(1), \ldots, w(n), a(j) - b(i), b(j), -a(i))$

1. Show that any execution of $V$ can be simulated by $(v_0, W)$ for a suitable $v_0$.

2. Conversely, show that this VAS $(v_0, W)$ simulates $V$ faithfully.

2 Unfoldings

Exercise 3 (Adequate Partial Orders). A partial order $\prec$ between events is adequate if the three following conditions are verified:

(a) $\prec$ is well-founded,

...
(b) $C_t \subset C_{t'}$ implies $t \prec t'$, and

(c) $\prec$ is preserved by finite extensions: as in the lecture notes, if $t \prec t'$ and $M_t = M_{t'}$, and $E$ and $E'$ are two isomorphic extensions of $C_t$ and $C_{t'}$ with $C_u = C_t \oplus E$ and $C_{u'} = C_{t'} \oplus E'$, then $u \prec u'$.

As you can guess, adequate partial orders result in complete unfoldings.

1. Show that $\prec_s$ defined by $t \prec_s t'$ iff $|C_t| < |C_{t'}|$ is adequate.

2. Construct the finite unfolding of the following Petri net using $\prec_s$; how does the size of this unfolding relate to the number of reachable markings?

3. Suppose we define an arbitrary total order $\preceq$ on the transitions $T$ of the Petri net, i.e. they are $t_1 \preceq \cdots \preceq t_n$. Given a set $S$ of events and conditions of $Q$, $\varphi(S)$ is the sequence $t_1^{i_1} \cdots t_n^{i_n}$ in $T^*$ where $i_j$ is the number of events labeled by $t_j$ in $S$. We also note $\preceq$ for the lexicographic order on $T^*$.

Show that $\prec_e$ defined by $t \prec_e t'$ iff $|C_t| < |C_{t'}|$ or $|C_t| = |C_{t'}|$ and $\varphi(C_t) \preceq \varphi(C_{t'})$ is adequate. Construct the finite unfolding for the previous Petri net using $\prec_e$.

4. There might still be examples where $\prec_e$ performs poorly. One solution would be to use a total adequate order. Give a 1-safe Petri net that shows that $\prec_e$ is not total.

**Exercise 4** (LTL(U) Model Checking). We consider again the problem of model checking state-based LTL formulæ against the reachable markings of a 1-safe Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. The LTL(U) formulæ we consider use a subset of the places as atomic propositions: $AP \subseteq P$. An atomic proposition $p$ in $AP$ holds in a marking $m$ in $\mathbb{N}^P$ (written $m \models p$) if $m(p) > 0$.

Instead of constructing an exponential-sized Büchi automaton $\mathcal{B}_\mathcal{N}$ (based on the reachability graph of $\mathcal{N}$) and its intersection with $\mathcal{B}_{\neg \varphi}$, we want to construct a Petri net $\mathcal{N}_\neg \varphi$ for the product of $\mathcal{N}$ and $\mathcal{B}_{\neg \varphi}$, and check its emptiness using unfolding techniques.
1. Describe how to construct this product Petri net $N_{\sim \varphi}$ if $AP = P$. Are unfolding techniques going to be efficient on this product?

2. Let us suppose $AP \subset P$. A transition $t$ of $T$ is visible if there exists $p$ in $AP$ such that $W(t, p) - W(p, t) \neq 0$.

We only have to synchronize visible actions with the transitions of $B_{\sim \varphi}$, but we need to distinguish two forms of acceptance: let $I$ and $L$ be two sets of illegal and livelock transitions of $N_{\sim \varphi}$, such that all the transitions of $I$ are visible.

(a) An infinite execution $m_0 \xrightarrow{t_1} m_1 \xrightarrow{t_2} \cdots$ of $N_{\sim \varphi}$ is illegal if $t_i$ is infinitely often in $I$.

(b) An infinite execution $m_0 \xrightarrow{t_1} m_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} m_i \xrightarrow{t_{i+1}} \cdots$ of $N_{\sim \varphi}$ is a livelock if $t_i$ is in $L$ and no subsequent transition $t_{i+j}$ for $j > 0$ is visible.

Propose a construction for $N_{\sim \varphi}$ such that $N \models \varphi$ a LTL($U$) formula iff $N_{\sim \varphi}$ has no illegal nor livelock infinite executions.

3. Let us treat illegal executions in $N_{\sim \varphi}$. Given a set of events and conditions $S$, we denote by $|S|_I$ the number of events of $S$ labeled by some transition in $I$. An event $e$ is a repeat with respect to some adequate partial ordering $\prec$, if there exists another event $e'$ (its companion) with $M_e = M_{e'}$ and either

(a) $e' < e$, or

(b) $\neg(e' < e)$, $e' \prec e$, and $|C_{e'}|_I \geq |C_e|_I$.

A repeat $e$ is terminal if there does not exist another repeat $e'$ with $e' < e$. A repeat $e$ with companion $e'$ is successful if $e' < e$ and $|e| \setminus |e'|_I > 0$. A tableau is an unfolding where we cut off at terminal events. Construct the tableau for the following Petri net where $I = \{t_1\}$ and the order $\prec_e$ of the previous exercise:
4. Show that the existence of a successful repeat in the unfolding of $\mathcal{N}_\varphi$ implies the existence of an illegal execution.

5. Let us prove that we do not need to unfold $\mathcal{N}_\varphi$ indefinitely.
   
   (a) Let $B$ be the maximal number of tokens that can appear simultaneously in a marking. Show that, for any $k \geq 0$, if a subset of events $E$ of a configuration $C$ contains strictly more than $k \cdot B$ events, then there exists a chain $e_1 < \cdots < e_{k+1}$ in $E$.

   (b) Let $K$ be the number of distinct reachable markings of $\mathcal{N}_\varphi$. Show that a tableau cannot contain more than $K^2 \cdot B$ non terminal events.

6. Let us prove that we can always witness an illegal execution thanks to a successful repeat in the tableau for $\mathcal{N}_\varphi$.
   
   (a) Denote a configuration as bad if it contains more than $(K \cdot B) + 1$ $I$-events. Show that $\mathcal{N}_\varphi$ exhibits an illegal execution iff its infinite unfolding contains a bad configuration.

   (b) Show that a bad configuration contains at least one terminal.

   (c) Prove that, given a bad configuration $C_t$ of the unfolding of $\mathcal{N}_\varphi$, either $C_t$ contains a successful terminal of the tableau, or there exists another bad configuration $C_{t'}$ with $t' \prec t$.

   (d) Conclude.