Home Assignment 1: Temporal Logic with Binding
Solutions

To hand in before or on November 2, 2011.
The penalty for delays is 2 points per day.

October						1	2
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November		1	2	3	4	5	

Electronic versions (PDF only) can be sent by email to ⟨schmitz@lsv.ens-cachan.fr⟩, paper versions should be handed in on the 2nd or put in my mailbox at LSV, ENS Cachan.

This assignment is concerned with an extension of temporal logics that allows to save visited time points in a temporal structure (through a *binding* operation $\mathsf{B}x$) and revisit them later (through a *jumping* operation J_x).

This logic is actually called *hybrid temporal logic*, and the $\mathsf{B}x$ and J_x constructs are usually noted $\downarrow x$ and $@_x$ respectively.

1 Temporal Logic with Binding

Instead of the usual set of atomic propositions, we consider two disjoint infinite countable sets of atomic symbols: propositions P and variables X, and define accordingly $A \stackrel{\text{def}}{=} P \uplus X$.

Syntax. Formulæ of temporal logic with binding are defined through the syntax

$$\varphi ::= \top \mid a \mid \mathsf{J}_x\varphi \mid \mathsf{B}x.\varphi \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \mathbin{\mathsf{U}} \varphi \mid \varphi \mathbin{\mathsf{S}} \varphi$$

where a is in A and x in X. Thus, compared to the usual temporal logic $\mathrm{TL}(P, \mathsf{U}, \mathsf{S})$, the temporal logic with binding $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ features two new syntactic constructs $\mathsf{B}x$ and J_x, and uses an extended set of atomic symbols A.

We define as usual $\bot \stackrel{\text{def}}{=} \neg\top$, $\varphi \vee \varphi' \stackrel{\text{def}}{=} \neg(\neg\varphi \wedge \neg\varphi')$, $\mathsf{F}\varphi \stackrel{\text{def}}{=} \top\ \mathsf{U}\ \varphi$, $\mathsf{P}\varphi \stackrel{\text{def}}{=} \top\ \mathsf{S}\ \varphi$, $\mathsf{G}\varphi \stackrel{\text{def}}{=} \neg\mathsf{F}\neg\varphi$, $\mathsf{H}\varphi \stackrel{\text{def}}{=} \neg\mathsf{P}\neg\varphi$, $\mathsf{X}\varphi \stackrel{\text{def}}{=} \bot\ \mathsf{U}\ \varphi$, and $\mathsf{Y}\varphi \stackrel{\text{def}}{=} \bot\ \mathsf{S}\ \varphi$. Fragments of the temporal logic with binding are denoted by changing the elements in TL(...): for instance $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U})$ is the logic without the since modality S, $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$ is the logic without U nor S but with F and P, $\mathrm{TL}(P, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ the logic without atomic variables (i.e. where a ranges over P in the syntax definition), etc.

Semantics. Given a temporal flow $(\mathbb{T}, <)$ where $\mathbb{T}$ is a set and $<$ a transitive irreflexive relation over $\mathbb{T}$, we consider temporal structures $w = (\mathbb{T}, <, h)$ where $h : P \to 2^{\mathbb{T}}$ associates to each atomic proposition a set of time points in $\mathbb{T}$. An *assignment* for w is a partial function $\nu : X \to \mathbb{T}$.

The *satisfaction* relation is defined inductively between an extended temporal structure $w = (\mathbb{T}, <, h)$, an assignment ν, a time point i in $\mathbb{T}$, and a $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ formula φ by

$$\begin{array}{ll}
w, i \models_\nu \top & \text{always}\\
w, i \models_\nu p & \text{iff } i \in h(p)\\
w, i \models_\nu \neg\varphi & \text{iff } w, i \not\models_\nu \varphi\\
w, i \models_\nu \varphi \wedge \varphi' & \text{iff } w, i \models_\nu \varphi \text{ and } w, i \models_\nu \varphi'\\
w, i \models_\nu \varphi\ \mathsf{U}\ \varphi' & \text{iff } \exists k. i < k \text{ and } w, k \models_\nu \varphi' \text{ and } \forall j. (i < j < k) \to w, j \models \varphi\\
w, i \models_\nu \varphi\ \mathsf{S}\ \varphi' & \text{iff } \exists k. i > k \text{ and } w, k \models_\nu \varphi' \text{ and } \forall j. (i > j > k) \to w, j \models \varphi\\
w, i \models_\nu x & \text{iff } i = \nu(x)\\
w, i \models_\nu \mathsf{J}_x\varphi & \text{iff } w, \nu(x) \models_\nu \varphi\\
w, i \models_\nu \mathsf{B}x.\varphi & \text{iff } w, i \models_{\nu[x \leftarrow i]} \varphi
\end{array}$$

where p ranges over P, x over X, and j, k over $\mathbb{T}$. The first six clauses of this definition are the same as for the basic temporal logic $\mathrm{TL}(P, \mathsf{U}, \mathsf{S})$. The next clause interprets variables as propositions that should hold at the current time point, J_x jumps to the time point denoted by x, and $\mathsf{B}x$ binds variable x to the current time point.

We denote by $\mathrm{FV}(\varphi)$ the set of free variables of a formula φ, i.e. the set of variables x that appear in φ, either as such or in a jump J_x, without being under the scope of a binder $\mathsf{B}x$. A temporal logic *sentence* is a formula φ with $\mathrm{FV}(\varphi) = \emptyset$, whereas a *query* is a formula with at least one free variable, usually written $\varphi(x_1, \ldots, x_n)$ if $\{x_1, \ldots, x_n\} \subseteq \mathrm{FV}(\varphi)$.

We write that w *satisfies* φ at point i, noted $w, i \models \varphi$, if there exists an assignment ν for every free variable of φ, i.e. with $\mathrm{FV}(\varphi)$ as domain, s.t. $w, i \models_\nu \varphi$.

Let φ and φ' be two formulæ with the same set of free variables. We write that φ is *equivalent* to φ', written $\varphi \equiv \varphi'$, if for all temporal structures w, time points i, and assignments ν for the free variables in $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi')$, $w, i \models_\nu \varphi$ iff $w, i \models_\nu \varphi'$.

[1] **Exercise 1** (Basic Equivalences)**.** Show the following equivalences for all formulæ φ:

$$\neg \mathsf{J}_x\varphi \equiv \mathsf{J}_x\neg\varphi \qquad\qquad \neg \mathsf{B}x.\varphi \equiv \mathsf{B}x.\neg\varphi$$

For any temporal structure w, time point i, and assignment ν for $\mathrm{FV}(\mathsf{J}_x\varphi)$:

$$w, i \models_\nu \neg\mathsf{J}_x\varphi \text{ iff } w, i \not\models_\nu \mathsf{J}_x\varphi \text{ iff } w, \nu(x) \not\models_\nu \varphi \text{ iff } w, \nu(x) \models_\nu \neg\varphi \text{ iff } w, i \models_\nu \mathsf{J}_x\neg\varphi\ .$$

(Note that this does not depend on i!) Now with an assignment ν for $\mathrm{FV}(\mathsf{B}x.\varphi)$:

$$w, i \models_\nu \neg\mathsf{B}x.\varphi \text{ iff } w, i \not\models_\nu \mathsf{B}x.\varphi \text{ iff } w, i \not\models_{\nu[x\leftarrow i]} \varphi \text{ iff } w, i \models_{\nu[x\leftarrow i]} \neg\varphi \text{ iff } w, i \models_\nu \mathsf{B}x.\neg\varphi\ .$$

[3] **Exercise 2** (Natural Language Queries)**.** Temporal logics have historically been invented for the representation of temporal events in formal semantics. Using atomic propositions such as *Mary_run*, *Droopy_laugh*, or *it_rain*, we can model e.g. the sentence "Mary ran" as $\mathsf{P}(\mathit{Mary\_run})$.

One might however want to explicit three (or more) points in a typical natural language sentence, which we might want to query:

- the point of *speech* s, i.e. the speech utterance time, typically taken to be the point of satisfaction,
- the point of *event* e, i.e. the point at which the event occurred, typically taken to be the point where the main proposition holds,
- the point of *reference* r, i.e. the point we are talking about.

For instance, the past perfect sentence "Mary was going to run" can be queried over $X = \{s, e, r\}$ by

$$\varphi(s, r, e) = s \wedge \mathsf{P}(r \wedge \mathsf{F}(e \wedge \mathit{Mary\_run}))\ ,$$

and similarly "Mary had run" by

$$\varphi(s, r, e) = s \wedge \mathsf{P}(r \wedge \mathsf{P}(e \wedge \mathit{Mary\_run}))\ .$$

1. Propose $\mathrm{TL}(P, X, \mathsf{F}, \mathsf{P})$ queries for the sentences "Mary runs", "Mary will run", "Mary will be going to run", and "Mary will have run".

   As usual with specifications, and even more so with formal semantics of natural language, there is no "perfect answer".

$$\begin{aligned}\text{“Mary runs”: } & s \wedge r \wedge e \wedge \mathit{Mary\_run}\\ \text{“Mary will run”: } & s \wedge r \wedge \mathsf{F}(e \wedge \mathit{Mary\_run})\\ \text{“Mary will be going to run”: } & s \wedge \mathsf{F}(r \wedge \mathsf{F}(e \wedge \mathit{Mary\_run}))\\ \text{“Mary will have run”: } & s \wedge \mathsf{F}(r \wedge \mathsf{P}(e \wedge \mathit{Mary\_run}))\ .\end{aligned}$$

   The last query could be refined as $s \wedge \mathsf{F}(r \wedge \mathsf{P}(e \wedge \mathsf{P}r \wedge \mathit{Mary\_run}))$ if we want to insist that e takes place after s.

2. What about "Mary will have been going to run"?

   The sentence would require two reference points:

$$s \wedge \mathsf{F}(r_1 \wedge \mathsf{P}(r_2 \wedge \mathsf{F}(e \wedge \mathit{Mary\_run}))) \ .$$

3. Let us assume that our temporal flow is endowed with a binary relation $\mathsf{nextday}$ corresponding to the notion of "next day". For instance, we could assume to be working over $(\mathbb{Q}, <)$ and that i $\mathsf{nextday}$ j iff $\lfloor i \rfloor + 1 = \lfloor j \rfloor$. Add a modality Xd with semantics $w, i \models_\nu \mathsf{Xd}\varphi$ iff $\exists j.i$ $\mathsf{nextday}$ j and $w, j \models_\nu \varphi$. How would you model "Mary had run yesterday", "Mary had run the previous day", and "Mary will run yesterday"?

   Taking an internal perspective (more natural with temporal logic), we translate "yesterday" as $\mathsf{Xd}s$ and "the previous day" as $\mathsf{Xd}r$ in the context of the event or of the reference point (which one is modified is indeed unsure).

$$\begin{aligned}
\text{"Mary had run yesterday": } & s \wedge \mathsf{P}(r \wedge \mathsf{P}(e \wedge \mathsf{Xd}s \wedge \mathit{Mary_run}))\\
\text{or (there are two possible readings): } & s \wedge \mathsf{P}(r \wedge \mathsf{Xd}s \wedge \mathsf{P}(e \wedge \mathit{Mary_run}))\\
\text{"Mary had run the previous day": } & s \wedge \mathsf{P}(r \wedge \mathsf{P}(e \wedge \mathsf{Xd}r \wedge \mathit{Mary_run}))\\
\text{"Mary will run yesterday": } & s \wedge r \wedge \mathsf{F}(e \wedge \mathsf{Xd}s \wedge \mathit{Mary_run}) \ .
\end{aligned}$$

   As expected, the last query is unsatisfiable.

2 Temporal Logic Fragments

Exercise 3 (Expressiveness of TL$(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$).

[1] 1. Express $\varphi \mathrel{\mathsf{U}} \varphi'$ by an equivalent formula using φ, φ', atomic variables, and the B, J, and F constructs. (Note that the case of S is symmetric, using P instead of F.)

   For fresh variables $x, y \notin \mathrm{FV}(\varphi \mathrel{\mathsf{U}} \varphi')$:

$$\varphi \mathrel{\mathsf{U}} \varphi' \equiv \mathsf{B}x.\mathsf{F}(\varphi' \wedge \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi)) \ .$$

[2] 2. Prove the equivalence of your formula with $\varphi \mathrel{\mathsf{U}} \varphi'$.

   Let ν be an assignment for $\mathrm{FV}(\varphi \mathrel{\mathsf{U}} \varphi')$. Then, for all w, i, ν, using the semantics:

$$\begin{aligned}
& w, i \models_\nu \mathsf{B}x.\mathsf{F}(\varphi' \wedge \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi))\\
\text{iff } & w, i \models_{\nu[x\leftarrow i]} \mathsf{F}(\varphi' \wedge \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi))\\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu[x\leftarrow i]} \varphi' \wedge \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi)\\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu[x\leftarrow i]} \varphi' \text{ and } w, k \models_{\nu[x\leftarrow i]} \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi)\\
\text{iff } & \exists k.i < k \text{ and } w, k \models_\nu \varphi' \text{ and } w, k \models_{\nu[x\leftarrow i]} \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi)\\
& \hfill \text{(since } x \notin \mathrm{FV}(\varphi'))
\end{aligned}$$

and focusing on the latter satisfaction relation,

$$
\begin{aligned}
& w,k \models_{\nu[x\leftarrow i]} \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi) \\
\text{iff } & w,k \models_{\nu[x\leftarrow i, y\leftarrow k]} \mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi) \\
\text{iff } & w,i \models_{\nu[x\leftarrow i, y\leftarrow k]} \mathsf{G}(\mathsf{F}y \to \varphi) \\
\text{iff } & w,i \models_{\nu[y\leftarrow k]} \mathsf{G}(\mathsf{F}y \to \varphi) && (\text{since } x \notin \mathrm{FV}(\mathsf{G}(\mathsf{F}y \to \varphi))) \\
\text{iff } & \forall j.(i<j) \to w,j \models_{\nu[y\leftarrow k]} \mathsf{F}y \to \varphi \\
\text{iff } & \forall j.(i<j) \to ((w,j \models_{\nu[y\leftarrow k]} \mathsf{F}y) \to (w,j \models_{\nu[y\leftarrow k]} \varphi)) \\
\text{iff } & \forall j.(i<j) \to ((\exists \ell.j<\ell \text{ and } w,\ell \models_{\nu[y\leftarrow k]} y) \to (w,j \models_{\nu[y\leftarrow k]} \varphi)) \\
\text{iff } & \forall j.(i<j) \to ((\exists \ell.j<\ell \text{ and } \ell = k) \to (w,j \models_{\nu[y\leftarrow k]} \varphi)) \\
\text{iff } & \forall j.(i<j<k) \to w,j \models_{\nu[y\leftarrow k]} \varphi \\
\text{iff } & \forall j.(i<j<k) \to w,j \models_{\nu} \varphi && (\text{since } y \notin \mathrm{FV}(\varphi))
\end{aligned}
$$

which yields the proof:

$$
\begin{aligned}
& w,i \models_\nu \mathsf{B}x.\mathsf{F}(\varphi' \wedge \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to \varphi)) \\
\text{iff } & \exists k.i<k \text{ and } w,k \models_\nu \varphi' \text{ and } \forall j.(i<j<k) \to w,j \models_\nu \varphi \\
\text{iff } & w,i \models_\nu \varphi \mathbin{\mathsf{U}} \varphi' \,.
\end{aligned}
$$

[1] 3. Prove that for any formula φ of $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$, there exists an equivalent translated formula $(\varphi)_{\mathsf{U}}$ of $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$.

We proceed by induction on φ. The cases where φ is neither an U nor an S formula are trivial by induction hypothesis. If it is an U formula (the case of S is symmetric), i.e. if $\varphi = \psi \mathbin{\mathsf{U}} \psi'$, then by induction hypothesis we can find $(\psi)_{\mathsf{U}} \equiv \psi$ and $(\psi')_{\mathsf{U}} \equiv \psi'$ in $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$, thus $\varphi \equiv (\psi)_{\mathsf{U}} \mathbin{\mathsf{U}} (\psi')_{\mathsf{U}}$. Exercise 3, Question 2 shows that $(\varphi)_{\mathsf{U}} \stackrel{\text{def}}{=} \mathsf{B}x.\mathsf{F}((\psi')_{\mathsf{U}} \wedge \mathsf{B}y.\mathsf{J}_x\mathsf{G}(\mathsf{F}y \to (\psi)_{\mathsf{U}}))$ is a formula $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$ equivalent to φ.

[1] **Exercise 4** (Expressiveness of $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{F}, \mathsf{P})$)**.** Show that for any formula φ of $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$, there exists a formula $(\varphi)_{\mathsf{J}}$ of $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{F}, \mathsf{P})$ which is equivalent over *linear* temporal flows $(\mathbb{T}, <)$, i.e. flows s.t. that $<$ is a strict linear ordering over $\mathbb{T}$.

First, let us show

$$\mathsf{J}_x\varphi \equiv \mathsf{P}(x \wedge \varphi) \vee (x \wedge \varphi) \vee \mathsf{F}(x \wedge \varphi)$$

over linear temporal flows: for any linear temporal structure w, time point i, and assignment ν for the variables in $\mathrm{FV}(\mathsf{J}_x\varphi)$,

$$\begin{aligned}
&w,i \models_\nu \mathsf{J}_x\varphi\\
\text{iff } & w,\nu(x) \models_\nu \varphi\\
\text{iff } & \exists j.\nu(x)=j \text{ and } w,j \models_\nu \varphi\\
\text{iff } & \exists j.w,j \models_\nu x \text{ and } w,j \models_\nu \varphi\\
\text{iff } & \exists j.w,j \models_\nu x\wedge\varphi\\
\text{iff } & \exists j.(j<i \text{ or } j=i \text{ or } j>i) \text{ and } w,j \models_\nu x\wedge\varphi \qquad \text{(since $<$ is linear)}\\
\text{iff } & (\exists j.(j<i) \text{ and } w,j \models_\nu x\wedge\varphi) \text{ or } (\exists j.(j=i) \text{ and } w,j \models_\nu x\wedge\varphi)\\
& \text{ or } (\exists j.(j>i) \text{ and } w,j \models_\nu x\wedge\varphi)\\
\text{iff } & w,i \models_\nu \mathsf{P}(x\wedge\varphi) \text{ or } w,i \models_\nu x\wedge\varphi \text{ or } w,i \models_\nu \mathsf{F}(x\wedge\varphi)\\
\text{iff } & w,i \models_\nu \mathsf{P}(x\wedge\varphi)\vee(x\wedge\varphi)\vee\mathsf{F}(x\wedge\varphi)\ .
\end{aligned}$$

Second, by induction on φ in $\mathrm{TL}(P,X,\mathsf{B},\mathsf{J},\mathsf{F},\mathsf{P})$, we construct the desired formula $(\varphi)_{\mathsf{J}}$ in $\mathrm{TL}(P,X,\mathsf{B},\mathsf{F},\mathsf{P})$ in the obvious way.

Linearity is essential: Consider the temporal structure

$$w = (\{i,j,k\},\{(j,i),(j,k)\},p\mapsto\{i\})$$

for the set of atomic propositions $P=\{p\}$, and the formula

$$\varphi = \mathsf{B}x.\mathsf{PF}(\neg p\wedge \mathsf{J}_x p)\ .$$

Clearly, $w,i\models\varphi$, but neither $w,i\models \mathsf{B}x.\mathsf{PF}(\neg p\wedge\mathsf{P}(x\wedge p))$ (because $w,k\not\models_{x\mapsto i}\mathsf{P}(x\wedge p)$ since $w,j\not\models_{x\mapsto i} x\wedge p$), nor $w,i\models\mathsf{B}x.\mathsf{PF}(\neg p\wedge x\wedge p)$ (it is unsatisfiable), nor $w,i\models \mathsf{B}x.\mathsf{PF}(\neg p\wedge\mathsf{F}(x\wedge p))$ (there is no point $>k$).

Exercise 5 (The N Modality)**.** We restrict ourselves in this exercise to the $(\mathbb{N},<)$ temporal flow. A temporal structure w can then be seen as an infinite word in Σ^ω, where $\Sigma=2^P$. Let us note $w[i)$ for the suffix of w starting at position i and $w[i]$ of the symbol of w at position i.

We consider the "now" modality N, with semantics $w,i\models_\nu \mathsf{N}\varphi$ iff $w[i),0\models_\nu\varphi$. Thus, this modality "forgets" about the past.

1. Let $P_{n+1}=\{p_0,\dots,p_n\}=P_n\cup\{p_n\}$ be a set of atomic propositions, defining the alphabet $\Sigma_{n+1}=2^{P_{n+1}}$. We want to show the existence of an $O(n)$-sized $\mathrm{TL}(P_{n+1},\mathsf{N},\mathsf{U},\mathsf{S})$ formula such that any equivalent $\mathrm{TL}(P_{n+1},\mathsf{U},\mathsf{S})$ formula is of size $\Omega(2^n)$.

[2] (a) First consider the following $\mathrm{TL}(P_{n+1},\mathsf{U})$ formula of exponential size:

$$\bigwedge_{S\subseteq P_n}\left(\begin{array}{l}(\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j \wedge p_n) \to \mathsf{G}((\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j) \to p_n) \\ \wedge(\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j \wedge \neg p_n) \to \mathsf{G}((\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j) \to \neg p_n)\end{array}\right) \qquad (\varphi_n)$$

Provide a TL($P_{n+1}, \mathsf{U}, \mathsf{S}$) formula ψ_n of size $O(n)$ *initially* equivalent to φ_n.
Define

$$\mathsf{G}\left(\left(\bigwedge_{i=0}^{n-1} p_i \leftrightarrow \mathsf{P}'(\mathsf{H}\bot \wedge p_i)\right) \to (p_n \leftrightarrow \mathsf{P}'(\mathsf{H}\bot \wedge p_n))\right) \qquad (\psi_n)$$

Claim 5.1. For any φ in TL($P, X, \mathsf{B}, \mathsf{J}, \mathsf{S}, \mathsf{U}$), any w in Σ^ω, any assignment ν, and any k in $\mathbb{N}$,

$$w, k \models_\nu \mathsf{P}'(\varphi \wedge \mathsf{H}\bot) \text{ iff } w, 0 \models_\nu \varphi\ .$$

Proof of Claim 5.1.

$$\begin{aligned} & w, k \models_\nu \mathsf{P}'(\varphi \wedge \mathsf{H}\bot) \\ \text{iff } & \exists j. j \leq k \text{ and } w, j \models_\nu \varphi \text{ and } w, j \models_\nu \mathsf{H}\bot \\ \text{iff } & \exists j. j \leq k \text{ and } w, j \models_\nu \varphi \text{ and } j = 0 \\ \text{iff } & w, 0 \models_\nu \varphi\ . \end{aligned}$$

□

Given a in Σ_{n+1}, i.e. $a \subseteq P_{n+1}$, we write $a|_n$ for the projection of a over P_n. For the initial equivalence, for all w in Σ_{n+1}^ω,

$$\begin{aligned} & w, 0 \models \varphi_n \\ \text{iff } & \bigwedge_{S\subseteq P_n} (w[0]|_n = S \wedge w, 0 \models p_n \to (\forall k. k > 0 \to (w[k]|_n = S \to w, k \models p_n))) \\ & \quad \wedge (w[0] = S \wedge w, 0 \models \neg p_n \to (\forall k. k > 0 \to (w[k]|_n = S \to w, k \models \neg p_n))) \\ \text{iff } & \forall k. k > 0 \to \bigwedge_{S\subseteq P_n} (w[0]|_n = S \wedge w[k]|_n = S) \to (w, 0 \models p_n \leftrightarrow w, k \models p_n) \\ \text{iff } & \forall k. k > 0 \to \left(\left(\bigwedge_{i=0}^{n-1} w, 0 \models p_i \leftrightarrow w, k \models p_i\right) \to (w, 0 \models p_n \leftrightarrow w, k \models p_n)\right) \\ \text{iff } & \forall k. k > 0 \to \left(\left(w, k \models \bigwedge_{i=0}^{n-1} \mathsf{P}'(\mathsf{H}\bot \wedge p_i) \leftrightarrow p_i\right) \to (w, k \models \mathsf{P}'(\mathsf{H}\bot \wedge p_n) \leftrightarrow p_n)\right) \\ & \hfill \text{(by Claim 5.1)} \\ \text{iff } & \forall k. k > 0 \to w, k \models \left(\bigwedge_{i=0}^{n-1} p_i \leftrightarrow \mathsf{P}'(\mathsf{H}\bot \wedge p_i)\right) \to (p_n \leftrightarrow \mathsf{P}'(\mathsf{H}\bot \wedge p_n)) \\ \text{iff } & w, 0 \models \psi_n\ . \end{aligned}$$

[1] (b) Provide a $\mathrm{TL}(P_{n+1}, \mathsf{N}, \mathsf{U}, \mathsf{S})$ formula of size $O(n)$ equivalent to $\mathsf{G}\varphi_n$.

The formula $\mathsf{GN}\psi_n$ is equivalent to $\mathsf{G}\varphi_n$: for any w in Σ_{n+1}^ω and any i:

$$
\begin{aligned}
& w, i \models \mathsf{GN}\psi_n \\
\text{iff } & \forall k.k > i \to w, k \models \mathsf{N}\psi_n \\
\text{iff } & \forall k.k > i \to w[k], 0 \models \psi_n \\
\text{iff } & \forall k.k > i \to w[k], 0 \models \varphi_n && \text{(by Exercise 5, Question 1a)} \\
\text{iff } & \forall k.k > i \to w, k \models \varphi_n && \text{(because } \varphi_n \text{ is pure future)} \\
\text{iff } & w, i \models \mathsf{G}\varphi_n \ .
\end{aligned}
$$

Exercise 6 of TD 2 shows that any formula of $\mathrm{TL}(P_{n+1}, \mathsf{U}, \mathsf{S})$ equivalent to $\mathsf{G}\varphi_n$ has size $\Omega(2^n)$, which allows to conclude.

[4] 2. Give a translation from $\mathrm{TL}(P, \mathsf{N}, \mathsf{U}, \mathsf{S})$ formulæ into equivalent $\mathrm{TL}(P, \{x, y\}, \mathsf{B}, \mathsf{F}, \mathsf{P})$ sentences with two (non-free) variables x and y. What is your translation for the formula $p \wedge \mathsf{F}(\neg \mathsf{NP}p)$?

The idea is to always keep the point of origin bound by the variable x. Let $\mathsf{P}'\varphi \stackrel{\text{def}}{=} \varphi \vee \mathsf{P}\varphi$; we define the inductive translation $(.)_\mathsf{N}$ by

$$
\begin{aligned}
(\top)_\mathsf{N} &\stackrel{\text{def}}{=} \top & (p)_\mathsf{N} &\stackrel{\text{def}}{=} p \\
(\neg\varphi)_\mathsf{N} &\stackrel{\text{def}}{=} \neg(\varphi)_\mathsf{N} & (\varphi \wedge \varphi')_\mathsf{N} &\stackrel{\text{def}}{=} (\varphi)_\mathsf{N} \wedge (\varphi')_\mathsf{N} \\
(\mathsf{N}\varphi)_\mathsf{N} &\stackrel{\text{def}}{=} \mathsf{B}x.(\varphi)_\mathsf{N} \\
(\varphi \mathbin{\mathsf{U}} \varphi')_\mathsf{N} &\stackrel{\text{def}}{=} (\varphi)_\mathsf{N} \mathbin{\mathsf{U}} (\varphi')_\mathsf{N} & (\varphi \mathbin{\mathsf{S}} \varphi')_\mathsf{N} &\stackrel{\text{def}}{=} (\varphi)_\mathsf{N} \mathbin{\mathsf{S}} ((\varphi')_\mathsf{N} \wedge \mathsf{P}'x) \ .
\end{aligned}
$$

By applying the previous exercises we can later express our formulæ into $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{F}, \mathsf{P})$. In order to use only two variables x, y however, an alternative translation of $\varphi \mathbin{\mathsf{U}} \varphi'$ and $\varphi \mathbin{\mathsf{S}} \varphi'$ is required, which exploits the fact that y is only used at the "root" of the formula and can later be overloaded:

$$
\begin{aligned}
(\varphi \mathbin{\mathsf{U}} \varphi')_\mathsf{N} &\stackrel{\text{def}}{=} \mathsf{B}y.\mathsf{F}((\varphi')_\mathsf{N} \wedge \mathsf{H}(\mathsf{P}y \to (\varphi)_\mathsf{N})) \\
(\varphi \mathbin{\mathsf{S}} \varphi')_\mathsf{N} &\stackrel{\text{def}}{=} \mathsf{B}y.\mathsf{P}(\mathsf{P}'x \wedge (\varphi')_\mathsf{N} \wedge \mathsf{G}(\mathsf{F}y \to (\varphi)_\mathsf{N})) \ .
\end{aligned}
$$

One can check that y never appears free in a translated formula. The equivalence between the two translations of U and S formulæ can be proved as in Exercise 3: for φ, φ' two formulæ of $\mathrm{TL}(P, \{x, y\}, \mathsf{B}, \mathsf{F}, \mathsf{P})$ where $y \notin \mathrm{FV}(\varphi) \cup \mathrm{FV}(\varphi')$, w in Σ^ω, and $i \geq 0$:

$$
\begin{aligned}
& w, i \models_\nu \mathsf{B}y.\mathsf{F}(\varphi' \wedge \mathsf{H}(\mathsf{P}y \to \varphi)) \\
\text{iff } & w, i \models_{\nu[y\leftarrow i]} \mathsf{F}(\varphi' \wedge \mathsf{H}(\mathsf{P}y \to \varphi)) \\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu[y\leftarrow i]} \varphi' \text{ and } w, k \models_{\nu[y\leftarrow i]} \mathsf{H}(\mathsf{P}y \to \varphi) \\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu[y\leftarrow i]} \varphi' \text{ and } w, k \models_{\nu[y\leftarrow i]} \mathsf{H}(\mathsf{P}y \to \varphi) \\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu} \varphi' \text{ and } w, k \models_{\nu[y\leftarrow i]} \mathsf{H}(\mathsf{P}y \to \varphi) && (\text{since } y \notin \mathrm{FV}(\varphi')) \\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu} \varphi' \text{ and } \forall j.j < k \to w, j \models_{\nu[y\leftarrow i]} \mathsf{P}y \to \varphi \\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu} \varphi' \text{ and } \forall j.i < j < k \to w, j \models_{\nu[y\leftarrow i]} \varphi \\
\text{iff } & \exists k.i < k \text{ and } w, k \models_{\nu} \varphi' \text{ and } \forall j.i < j < k \to w, j \models \varphi && (\text{since } y \notin \mathrm{FV}(\varphi)) \\
\text{iff } & w, i \models_\nu \varphi \mathbin{\mathsf{U}} \varphi' \ ;
\end{aligned}
$$

and similarly for the two translations of S.

Claim 5.2. For all formulæ φ in $\mathrm{TL}(P, \mathsf{N}, \mathsf{U}, \mathsf{S})$, for all w in Σ^ω, and all $i \geq k$ in $\mathbb{N}$,

$$w[k], i - k \models \varphi \text{ iff } w, i \models_{x\mapsto k} (\varphi)_{\mathsf{N}} \ .$$

Note that Claim 5.2 allows to conclude. Define $\mathsf{F}'\varphi \stackrel{\text{def}}{=} \varphi \vee \mathsf{F}\varphi$; setting $k = 0$ in Claim 5.2 yields

$$
\begin{aligned}
& w, i \models \varphi \\
\text{iff } & w[0], i \models \varphi && (\text{since } w[0] = w) \\
\text{iff } & w, i \models_{x\mapsto 0} (\varphi)_{\mathsf{N}} && (\text{by Claim 5.2}) \\
\text{iff } & w, 0 \models_{x\mapsto 0, y\mapsto i} \mathsf{F}'(y \wedge (\varphi)_{\mathsf{N}}) \\
\text{iff } & w, 0 \models_{y\mapsto i} \mathsf{B}x.\mathsf{F}'(y \wedge (\varphi)_{\mathsf{N}}) \\
\text{iff } & w, i \models_{y\mapsto i} \mathsf{P}'(\mathsf{H}\bot \wedge \mathsf{B}x.\mathsf{F}'(y \wedge (\varphi)_{\mathsf{N}})) && (\text{by Claim 5.1}) \\
\text{iff } & w, i \models \mathsf{B}y.\mathsf{P}'(\mathsf{H}\bot \wedge \mathsf{B}x.\mathsf{F}'(y \wedge (\varphi)_{\mathsf{N}}))
\end{aligned}
$$

for all w in Σ^ω, $i \in \mathbb{N}$, and φ in $\mathrm{TL}(P, \mathsf{N}, \mathsf{U}, \mathsf{S})$, hence proving the desired equivalent sentence of $\mathrm{TL}(P, \{x, y\}, \mathsf{B}, \mathsf{U}, \mathsf{S})$.

A handful fact is the following:

Claim 5.3. For all w in Σ^ω, and all i, k in $\mathbb{N}$,

$$w, i \models_{x\mapsto k} \mathsf{P}'x \text{ iff } i \geq k \ .$$

Proof of Claim 5.3.

$$
\begin{aligned}
& w, i \models_{x\mapsto k} x \vee \mathsf{P}x \\
\text{iff } & w, i \models_{x\mapsto k} x \text{ or } w, i \models_{x\mapsto k} \mathsf{P}x \\
\text{iff } & k = i \text{ or } \exists j.j < i \text{ and } w, j \models_{x\mapsto k} x \\
\text{iff } & k = i \text{ or } \exists j.j < i \text{ and } j = k \\
\text{iff } & k = i \text{ or } k < i \ .
\end{aligned}
$$

□

Proof of Claim 5.2. We proceed by induction on formulæ of TL($P, \mathsf{N}, \mathsf{U}, \mathsf{S}$). Most cases are very easy, and we only treat the cases of $\mathsf{N}\varphi$ and $\varphi \mathbin{\mathsf{S}} \varphi'$. Consider any w in Σ^ω and any $i \geq k$ in $\mathbb{N}$:

For $\mathsf{N}\varphi$:

$$\begin{aligned}
& w, i \models_{x\mapsto k} \mathsf{B}x.(\varphi)_\mathsf{N} \\
\text{iff } & w, i \models_{x\mapsto i} (\varphi)_\mathsf{N} \\
\text{iff } & w[i], 0 \models \varphi && \text{(by ind. hyp. since } i \geq i) \\
\text{iff } & w[k], i-k \models \mathsf{N}\varphi \;.
\end{aligned}$$

For $\varphi \mathbin{\mathsf{S}} \varphi'$:

$$\begin{aligned}
& w, i \models_{x\mapsto k} (\varphi)_\mathsf{N} \mathbin{\mathsf{S}} (\mathsf{P}'x \wedge (\varphi')_\mathsf{N}) \\
\text{iff } & \exists j. j < i \text{ and } w, j \models_{x\mapsto k} (\varphi')_\mathsf{N} \text{ and } w, j \models_{x\mapsto k} \mathsf{P}'x \\
& \qquad \text{and } \forall \ell. j < \ell < i \to w, \ell \models_{x\mapsto k} (\varphi)_\mathsf{N} \\
\text{iff } & \exists j. k \leq j < i \text{ and } w, j \models_{x\mapsto k} (\varphi')_\mathsf{N} \\
& \qquad \text{and } \forall \ell. j < \ell < i \to w, \ell \models_{x\mapsto k} (\varphi)_\mathsf{N} && \text{(by Claim 5.3)} \\
\text{iff } & \exists j. k \leq j < i \text{ and } w[k], j-k \models \varphi' \\
& \qquad \text{and } \forall \ell. j < \ell < i \to w[k], \ell - k \models \varphi \\
& && \text{(by ind. hyp. since } \ell > j \geq k) \\
\text{iff } & w[k], i-k \models \varphi \mathbin{\mathsf{S}} \varphi' \;. && \square
\end{aligned}$$

The translation of $p \wedge \mathsf{F}(\neg\mathsf{NFP}p)$ is

$$\begin{aligned}
& \mathsf{B}y.\mathsf{P}'(\mathsf{H}\bot \wedge \mathsf{B}x.\mathsf{F}'(y \wedge p \wedge \mathsf{F}(\neg\mathsf{B}x.\mathsf{FP}(p \wedge \mathsf{P}'x)))) \\
\equiv\ & p \wedge \mathsf{F}(\neg\mathsf{B}x.\mathsf{FP}(p \wedge \mathsf{P}'x)) \\
\equiv\ & p \wedge \mathsf{F}(\mathsf{B}x.\mathsf{GH}(\mathsf{P}'x \to \neg p)) \\
\equiv\ & p \wedge \mathsf{FG}\neg p \;.
\end{aligned}$$

3 Guarded First-Order Logic

For a temporal structure $(\mathbb{T}, <, h)$ over $P = \{p_1, \ldots, p_m\}$, we consider as usual the signature $(\mathbb{T}, <, P_{p_1}, \ldots, P_{p_m})$ where each P_p for p in P is a unary relation such that $P_p(i)$ holds in time point i iff $i \in h(p)$. The *guarded fragment* of first-order logic over this signature is usually defined by restricting existential quantifications to be of form $\exists x. g \wedge \psi$ where g is an atomic formula (i.e. of form $x = y$, $x < y$, or $P_p(x)$) with $\mathrm{FV}(\psi) \subseteq \mathrm{FV}(g)$. We consider here a slightly different syntactic restriction: let

$$\begin{aligned}
g &::= x = z \mid x < z && \text{(guards)} \\
\psi &::= x = y \mid x < y \mid P_p(x) \mid \neg\psi \mid \psi \wedge \psi \mid \exists x.(g \wedge \psi) && \text{(formulæ)}
\end{aligned}$$

where p ranges over P, x, y, z over X with $x \neq z$, and existential quantification assumes $x \in \mathrm{FV}(g)$ ($\mathrm{FV}(\psi) \subseteq \mathrm{FV}(g)$ is not required). We note $\mathrm{gFO}_P(<)$ for the set of fomulæ in this fragment.

This is known in the literature as the *bounded* fragment of FO.

Exercise 6 (Guarded Completeness)**.** We want to prove the expressive completeness of temporal logic with binding with the strongly guarded fragment we just defined. The equivalence between a $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ formula φ and a $\mathrm{gFO}_P(<)$ formula $\psi(x)$ is defined as usual by: for any temporal structure w, any time point i, and any assignment ν for $\mathrm{FV}(\psi) = \mathrm{FV}(\varphi) \uplus \{x\}$,

$$w, i \models_\nu \varphi \text{ iff } w, \nu[x \leftarrow i] \models \psi(x)\ .$$

[2] 1. Show that any $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ formula φ can be translated into an equivalent $\mathrm{gFO}_P(<)$ formula $(\varphi)_{\mathrm{FO}}(x)$ with $\mathrm{FV}((\varphi)_{\mathrm{FO}}(x)) = \{x\} \cup \mathrm{FV}(\varphi)$.

We merely need to extend the usual first-order translation for temporal logics. Thanks to Exercise 3 we can define this translation for φ in $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$:

$$\begin{aligned}
(\top)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} (x = x) & & \\
(y)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} x = y & (p)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} P_p(x) \\
(\mathsf{J}_y\varphi)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} (\varphi)_{\mathrm{FO}}(y) & (\mathsf{B}y.\varphi)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} \exists y. x = y \wedge (\varphi)_{\mathrm{FO}}(x) \\
(\neg\varphi)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} \neg(\varphi)_{\mathrm{FO}}(x) & (\varphi \wedge \varphi')_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} (\varphi)_{\mathrm{FO}}(x) \wedge (\varphi')_{\mathrm{FO}}(x) \\
(\mathsf{F}\varphi)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} \exists x'. x < x' \wedge (\varphi)_{\mathrm{FO}}(x') & (\mathsf{P}\varphi)_{\mathrm{FO}}(x) &\overset{\mathrm{def}}{=} \exists x'. x' < x \wedge (\varphi)_{\mathrm{FO}}(x')\ ,
\end{aligned}$$

where $x \neq x'$ are fresh variables and $y \in \mathrm{FV}(\varphi)$ (thus $x \neq y$ in the translation of $\mathsf{B}y.\varphi$).

The equivalence is proved by induction over $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{F}, \mathsf{P})$: for all w, i, ν,

For $\top$:

$$\begin{aligned}
&w, \nu[x \leftarrow i] \models (x = x)(x) \\
\text{iff } &w, i \models_\nu \top\ .
\end{aligned}$$

For y in X:

$$\begin{aligned}
&w, \nu[x \leftarrow i] \models x = y \\
\text{iff } &\nu(y) = i \\
\text{iff } &w, i \models_\nu y\ .
\end{aligned}$$

For p in P:

$$\begin{aligned}
&w, \nu[x \leftarrow i] \models P_p(x) \\
\text{iff } &i \in h(p) \\
\text{iff } &w, i \models_\nu p\ .
\end{aligned}$$

For $\mathsf{J}_y\varphi$:

$$\begin{aligned}
&w,\nu[x\leftarrow i]\models(\varphi)_{\mathrm{FO}}(y) &\\
\text{iff } &w,\nu(y)\models_{\nu[x\leftarrow i]}\varphi & \text{(by ind. hyp.)}\\
\text{iff } &w,\nu(y)\models_{\nu}\varphi & \text{(since } x\notin\mathrm{FV}(\varphi))\\
\text{iff } &w,i\models_{\nu}\mathsf{J}_y\varphi\;. &
\end{aligned}$$

For $\mathsf{B}x.\varphi$:

$$\begin{aligned}
&w,\nu[x\leftarrow i]\models\exists y.x=y\wedge(\varphi)_{\mathrm{FO}}(x) &\\
\text{iff } &w,\nu[x\leftarrow i,y\leftarrow i]\models(\varphi)_{\mathrm{FO}}(x) &\\
\text{iff } &w,i\models_{\nu[y\leftarrow i]}\varphi & \text{(by ind. hyp.)}\\
\text{iff } &w,i\models_{\nu}\mathsf{B}y.\varphi\;. &
\end{aligned}$$

The cases of $\neg\varphi$ and $\varphi\wedge\varphi'$ are immediate by induction hypothesis, and it remains to treat $\mathsf{F}\varphi$ ($\mathsf{P}\varphi$ is symmetric):

$$\begin{aligned}
&w,\nu[x\leftarrow i]\models\exists x'.x'<x\wedge(\varphi)_{\mathrm{FO}}(x') &\\
\text{iff } &\exists j.i<j \text{ and } w,\nu[x\leftarrow i,x'\leftarrow j]\models(\varphi)_{\mathrm{FO}}(x') &\\
\text{iff } &\exists j.i<j \text{ and } w,j\models_{\nu[x\leftarrow i]}\varphi & \text{(by ind. hyp. since } x'\notin\mathrm{FV}(\varphi))\\
\text{iff } &w,i\models_{\nu[x\leftarrow i]}\mathsf{F}\varphi &\\
\text{iff } &w,i\models_{\nu}\mathsf{F}\varphi\;. & \text{(since } x\notin\mathrm{FV}(\varphi))
\end{aligned}$$

[4] 2. Prove the converse: for any $\mathrm{gFO}_P(<)$ formula ψ with a free variable x, there exists an equivalent formula $\mathsf{B}x.(\psi)_{\mathrm{TL}}$ in $\mathrm{TL}(P,X,\mathsf{B},\mathsf{J},\mathsf{U},\mathsf{S})$ with $\mathrm{FV}(\psi)=\mathrm{FV}((\psi)_{\mathrm{TL}})$.

Let us first consider the different possibilities for $\exists x.(g\wedge\psi)$. Since $x\in\mathrm{FV}(g)$, there exists a variable $y\neq x$ s.t. $g=(x<y)$ or $g=(y<x)$ or $g=(x=y)$. The translation $(.)_{\mathrm{TL}}$ from $\mathrm{gFO}_P(<)$ formulæ is then defined inductively:

$$\begin{aligned}
(x=y)_{\mathrm{TL}} &\stackrel{\text{def}}{=}\mathsf{J}_x y & (x<y)_{\mathrm{TL}} &\stackrel{\text{def}}{=}\mathsf{J}_x\mathsf{F}y\\
(P_p(x))_{\mathrm{TL}} &\stackrel{\text{def}}{=}\mathsf{J}_x p & (\exists x.(x=y)\wedge\psi)_{\mathrm{TL}} &\stackrel{\text{def}}{=}\mathsf{J}_y\mathsf{B}x.(\psi)_{\mathrm{TL}}\\
(\exists x.(x<y)\wedge\psi)_{\mathrm{TL}} &\stackrel{\text{def}}{=}\mathsf{J}_y\mathsf{P}\mathsf{B}x.(\psi)_{\mathrm{TL}} & (\exists x.(y<x)\wedge\psi)_{\mathrm{TL}} &\stackrel{\text{def}}{=}\mathsf{J}_y\mathsf{F}\mathsf{B}x.(\psi)_{\mathrm{TL}}\\
(\neg\psi)_{\mathrm{TL}} &\stackrel{\text{def}}{=}\neg(\psi)_{\mathrm{TL}} & (\psi\wedge\psi')_{\mathrm{TL}} &\stackrel{\text{def}}{=}(\psi)_{\mathrm{TL}}\wedge(\psi)_{\mathrm{TL}}\;.
\end{aligned}$$

Because every $(\varphi)_{\mathrm{TL}}$ formula is guarded by a J modality, it will be easier to prove

$$w\models_{\nu}(\varphi)_{\mathrm{TL}}\text{ iff } w,\nu\models\varphi\;, \tag{$*$}$$

i.e. to ignore time points i. This immediately yields the desired equivalence between ψ and $\mathsf{B}x.(\psi)_{\mathrm{TL}}$.

The equivalence $(*)$ is proven by induction on $\mathrm{gFO}_P(<)$ formulæ:

For $x = y$:

$$
\begin{aligned}
& w \models_\nu \mathsf{J}_x y \\
\text{iff } & w, \nu(x) \models_\nu y \\
\text{iff } & \nu(x) = \nu(y) \\
\text{iff } & w, \nu \models x = y \;.
\end{aligned}
$$

For $x < y$:

$$
\begin{aligned}
& w \models_\nu \mathsf{J}_x \mathsf{F} y \\
\text{iff } & w, \nu(x) \models_\nu \mathsf{F} x \\
\text{iff } & \exists j . \nu(x) < j \text{ and } w, j \models_\nu y \\
\text{iff } & \exists j . \nu(x) < j \text{ and } j = \nu(y) \\
\text{iff } & \nu(x) < \nu(y) \\
\text{iff } & w, \nu \models x < y \;.
\end{aligned}
$$

For $P_p(x)$:

$$
\begin{aligned}
& w \models \mathsf{J}_x p \\
\text{iff } & w, \nu(x) p \\
\text{iff } & \nu(x) \in h(p) \\
\text{iff } & w, \nu \models P_p(x) \;.
\end{aligned}
$$

For $\exists x.(x < y) \wedge \psi$ (the other existential quantifications are similar):

$$
\begin{aligned}
& w \models_\nu \mathsf{J}_y \mathsf{P} \mathsf{B} x.(\psi)_{\mathrm{TL}} \\
\text{iff } & w, \nu(y) \models_\nu \mathsf{P} \mathsf{B} x.(\psi)_{\mathrm{TL}} \\
\text{iff } & \exists j . j < \nu(y) \text{ and } w, j \models_\nu \mathsf{B} x.(\psi)_{\mathrm{TL}} \\
\text{iff } & \exists j . j < \nu(y) \text{ and } w, j \models_{\nu[x \leftarrow j]} (\psi)_{\mathrm{TL}} \\
\text{iff } & \exists j . j < \nu(y) \text{ and } w, \nu[x \leftarrow j] \models \psi && \text{(by ind. hyp.)} \\
\text{iff } & \exists j . w, \nu[x \leftarrow j] \models x < y \text{ and } w, [x \leftarrow j] \models \psi \\
\text{iff } & \exists j . w, \nu[x \leftarrow j] \models x < y \wedge \psi \\
\text{iff } & w, \nu \models \exists x . x < y \wedge \psi \;.
\end{aligned}
$$

The equivalence holds trivially for boolean connectives.

[3] 3. Deduce that $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ is first-order complete over $(\mathbb{N}, <)$.

Of course, since $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ encompasses $\mathrm{TL}(P, \mathsf{U}, \mathsf{S})$, we already know it is first-order complete over $(\mathbb{N}, <)$. The idea here is to provide a simple proof, which carries over to more complex time flows (it works for instance for branching time flows) and to more than one free variable.

Consider a time flow $(\mathbb{T}, <)$ where there exists a minimal element i_0 in $\mathbb{T}$, for instance 0 in $\mathbb{N}$ or the root ε of a branching flow. Then any $\mathrm{FO}_P(<)$ formula ψ can be translated into an equivalent formula

$$\psi \equiv \exists x_0 . (\psi)_g \wedge \neg \exists x . x < x_0 \tag{$\dagger$}$$

where $(\psi)_g$ is in $\mathrm{gFO}_P(<)$ and x_0 does not appear in ψ. By the previous question, $(\psi)_g$ has an equivalent $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ formula $\mathsf{B}x.((\psi)_g)_{\mathrm{TL}}$, hence

$$\psi \equiv \mathsf{B}x.\mathsf{P}'(\mathsf{H}\bot \wedge \mathsf{B}x_0.\mathsf{F}'(x \wedge ((\psi)_g)_{\mathrm{TL}})) \tag{‡}$$

where x is a free variable of ψ.

Let us define the translation $(.)_g$: we set $(\psi)_g \stackrel{\text{def}}{=} \psi$ except for the case of existential quantification, which is

$$(\exists x.\psi)_g \stackrel{\text{def}}{=} (\exists x.x_0 < x \wedge (\psi)_g) \vee (\exists x.x_0 = x \wedge (\psi)_g) \ .$$

Claim 6.1. For every w with minimal point i_0, every $\mathrm{FO}_P(<)$ formula ψ where x_0 does not appear, and every assignment ν for $\mathrm{FV}(\psi)$

$$w, \nu \models \psi \text{ iff } w, \nu[x_0 \leftarrow i_0] \models (\psi)_g \ .$$

Although (†) is not needed for the answer, it is still worth proving as a sanity check, using Claim 6.1: for all w with minimal point i_0 and every assignment ν for $\mathrm{FV}(\psi)$,

$$\begin{aligned}
&w, \nu \models \exists x_0.(\psi)_g \wedge \neg\exists x.x < x_0\\
\text{iff } &\exists i.w, \nu[x_0 \leftarrow i] \models (\psi)_g \text{ and } w, \nu[x_0 \leftarrow i] \models \neg\exists x.x < x_0\\
\text{iff } &\exists i.w, \nu[x_0 \leftarrow i] \models (\psi)_g \text{ and } \forall j.w, \nu[x_o \leftarrow i, x \leftarrow j] \models \neg x < x_0\\
\text{iff } &\exists i.w, \nu[x_0 \leftarrow i] \models (\psi)_g \text{ and } \forall j.j \not< i\\
\text{iff } &\exists i.w, \nu[x_0 \leftarrow i] \models (\psi)_g \text{ and } i = i_0 \qquad \text{(since } i_0 \text{ is the minimal element of } \mathbb{T})\\
\text{iff } &w, \nu[x_0 \leftarrow i_0] \models (\psi)_g\\
\text{iff } &w, \nu \models \psi \ . \qquad \text{(by Claim 6.1)}
\end{aligned}$$

Proof of Claim 6.1. We proceed by induction over $\mathrm{FO}_P(<)$ formulæ. The only case of interest is that of a formula $\exists x.\psi$:

$$\begin{aligned}
&w, \nu[x_0 \leftarrow i_0] \models (\exists x.x_0 < x \wedge (\psi)_g) \vee (\exists x.x_0 = x \wedge (\psi)_g)\\
\text{iff } &\exists i.w, \nu[x_0 \leftarrow i_0, x \leftarrow i] \models (x_0 < x) \vee (x_0 = x) \text{ and } w, \nu[x_0 \leftarrow i_0, x \leftarrow i] \models (\psi)_g\\
\text{iff } &\exists i.(i_0 < i \text{ or } i_0 = i) \text{ and } w, \nu[x \leftarrow i] \models \psi \qquad \text{(by ind. hyp.)}\\
\text{iff } &\exists i.w, \nu[x \leftarrow i] \models \psi \qquad \text{(since } i_0 \text{ is the minimal element)}\\
\text{iff } &w, \nu \models \exists x.\psi \ .
\end{aligned}$$

□

It remains to prove (‡): for all w with minimal element i_0, for all assignments ν

for $\mathrm{FV}(\psi)$, and all time points i,

$$\begin{aligned}
& w, i \models_\nu \mathsf{B}x.\mathsf{P}'(\mathsf{H}\bot \wedge \mathsf{B}x_0.\mathsf{F}'(x \wedge ((\psi)_g)_{\mathrm{TL}})) \\
\text{iff } & \exists j.(j = i \vee j < i) \text{ and } w, j \models_{\nu[x\leftarrow i]} \mathsf{H}\bot \text{ and } w, j \models_{\nu[x\leftarrow i]} \mathsf{B}x_0 \\
& \qquad\qquad \text{and } w, j \models_{\nu[x\leftarrow i]} \mathsf{F}'(x \wedge ((\psi)_g)_{\mathrm{TL}}) \\
\text{iff } & \exists j.(j = i \vee j < i) \text{ and } j = i_0 \text{ and } w, j \models_{\nu[x\leftarrow i]} \mathsf{B}x_0.\mathsf{F}'(x \wedge ((\psi)_g)_{\mathrm{TL}}) \\
& \qquad\qquad \text{(using the same arguments as in Claim 5.1)} \\
\text{iff } & w, i_0 \models_{\nu[x\leftarrow i]} \mathsf{B}x_0.\mathsf{F}'(x \wedge ((\psi)_g)_{\mathrm{TL}}) \qquad \text{(since } i_0 \text{ is the minimal element)} \\
\text{iff } & \exists k.(k = i_0 \vee k > i_0) \text{ and } w, k \models_{\nu[x\leftarrow i, x_0\leftarrow i_0]} x \\
& \qquad\qquad \text{and } w, k \models_{\nu[x\leftarrow i, x_0\leftarrow i_0]} ((\psi)_g)_{\mathrm{TL}} \\
\text{iff } & w, i \models_{\nu[x\leftarrow i, x_0\leftarrow i_0]} ((\psi)_g)_{\mathrm{TL}} \\
\text{iff } & w, \nu[x \leftarrow i, x_0 \leftarrow i_0] \models (\psi)_g \qquad \text{(by } (*)) \\
\text{iff } & w, \nu[x \leftarrow i] \models \psi\ . \qquad \text{(by Claim 6.1)}
\end{aligned}$$

As a conclusion, observe that $\mathrm{TL}(P, X, \mathsf{B}, \mathsf{J}, \mathsf{U}, \mathsf{S})$ with $|P| > 0$ has non elementary satisfiability problem over infinite words, by straightforward reduction from the corresponding problem for $\mathrm{FO}_P(<)$.

Note: I just found a paper on hybrid logics for linear frames:

> M. Franceschet, M. de Rijke, and B.-H. Schlingloff. Hybrid logics on linear structures: Expressivity and complexity. In 10th TIME / 4th ICTL, pages 192–202. IEEE Computer Society, 2003. doi: 10.1109/TIME.2003.1214893.