TD 3: Ehrenfeucht-Fraïssé Games

Exercise 1 (Non-Strict Until)**.**

1. Show that U is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}', \mathsf{U}')$ over $(\mathbb{R}, <)$.
2. Show that U is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}', \mathsf{U}')$ over $(\mathbb{N}, <)$.

Exercise 2 (Periodic Properties)**.**

1. Show that the fact that a finite temporal time flow is of "even length" cannot be expressed in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$.
2. Recall Exercise 3 of TD 2: Show that the set $(\{p\}\Sigma)^\omega$ cannot be expressed in $\mathrm{TL}(\{p\}, \mathsf{S}, \mathsf{U})$ over $(\mathbb{N}, <)$.

Exercise 3 (Linear Orders with Gaps)**.** In this exercise we assume $(\mathbb{T}, <)$ to be a linear time flow.

1. Let us define a new unary "gap" modality gap:

$$\begin{aligned} w, i \models \mathsf{gap}\varphi \text{ iff } \forall k.k > i \to (\exists \ell.k < \ell \wedge \forall j.i < j < \ell \to w, j \models \varphi) \\ \vee (\exists j.i < j < k \wedge w, j \models \neg\varphi) \\ \wedge \exists k_1.k_1 > i \wedge \forall j.i < j \leq k_1 \to w, j \models \varphi \\ \wedge \exists k_2.k_2 > i \wedge w, k_2 \models \neg\varphi\ . \end{aligned}$$

   The intuition behind gap is that φ should hold for some time until a gap occurs in the time flow, after which $\neg\varphi$ holds at points arbitrarily close to the gap.

   (a) Show that, if $(\mathbb{T}, <)$ is Dedekind-complete, then $\mathsf{gap}p$ for $p \in \mathrm{AP}$ cannot be satisfied.

   (b) Express $\mathsf{gap}\varphi$ using the standard U modality.

2. Consider the temporal flow $(\{0\} \times \mathbb{Z}_{<0} \times \mathbb{Z} \cup \{1\} \times \mathbb{Z} \times \mathbb{Z}, <)$ where $<$ is the lexicographic ordering and $\mathrm{AP} = \{p\}$. Let n be an even integer in $\mathbb{Z}$, and define

$$\begin{aligned} h_0(p) &= \{(0, i, j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1, i, j) \in \mathbb{T} \mid i \text{ is odd}\} \\ h_1(p) &= \{(0, i, j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1, i, j) \in \mathbb{T} \mid i > n \text{ is odd}\}\ . \end{aligned}$$

   (a) Show that $w_0, (x, i, j) \models \mathsf{gap}p$ for any $x \in \{0, 1\}$, odd i, and j.

   (b) Show that no $\mathrm{TL}(\{p\}, \mathsf{S}, \mathsf{U})$ formula can distinguish between $(w_0, (0, -1, 0))$ and $(w_1, (0, -1, 0))$.

(c) Here is the definition of the Stavi "until" modality:

$$
\begin{aligned}
w, i \models \varphi \mathbin{\bar{\mathsf{U}}} \psi \text{ iff } \exists \ell . i < \ell & \\
& \wedge \forall k . i < k < \ell \rightarrow [\exists j_1 . k < j_1 \wedge \forall j . i < j < j_1 \rightarrow w, j \models \varphi] \\
& \qquad \vee [(\forall j_2 . k < j_2 < \ell \rightarrow w, j_2 \models \psi) \\
& \qquad\quad \wedge (\exists j_3 . i < j_3 < k \wedge w, j_3 \models \neg\varphi)] \\
& \wedge \exists k_1 . i < k_1 < \ell \wedge w, k_1 \models \neg\varphi \\
& \wedge \exists k_2 . i < k_2 < \ell \wedge \forall j . i < j < k_2 \rightarrow w, j \models \varphi
\end{aligned}
$$

This modality is quite similar to $\mathsf{gap}\varphi$, but further requires ψ to hold for some time after the gap (the "j_2" condition above).

Show that $w_1, (0, -1, 0) \models p \mathbin{\bar{\mathsf{U}}} \neg \mathsf{gap}\, p$ but $w_0, (0, -1, 0) \not\models p \mathbin{\bar{\mathsf{U}}} \neg \mathsf{gap}\, p$.

Exercise 4 (Stuttering and LTL(U'))**.** In the time flow $(\mathbb{N}, <)$, i.e. when working with words σ in Σ^ω, *stuttering* denotes the existence of consecutive symbols, like *aaaa* and *bb* in *baaaabb*. Concrete systems tend to stutter, and thus some argue that verification properties should be stutter invariant.

A *stuttering function* $f : \mathbb{N} \rightarrow \mathbb{N}_+$ from the positive integers to the strictly positive integers. Let $\sigma = a_0 a_1 \cdots$ be an infinite word of Σ^ω and f a stuttering function, we denote by $\sigma[f]$ the infinite word $a_0^{f(0)} a_1^{f(1)} \cdots$, i.e. where the i-th symbol of σ is repeated $f(i)$ times. A language $L \subseteq \Sigma^\omega$ is *stutter invariant* if, for all words σ in Σ^ω and all stuttering functions f,

$$\sigma \in L \text{ iff } \sigma[f] \in L\ .$$

1. Prove that if φ is a TL(AP, U') formula, then $L(\varphi)$ is stutter-invariant.

2. A word $\sigma = a_0 a_1 \cdots$ in Σ^ω is *stutter-free* if, for all i in $\mathbb{N}$, either $a_i \neq a_{i+1}$, or $a_i = a_j$ for all $j \geq i$. We note $\mathrm{sf}(L)$ for the set of stutter-free words in a language L.

   Show that, if L and L' are two stutter invariant languages, then $\mathrm{sf}(L) = \mathrm{sf}(L')$ iff $L = L'$.

3. Let φ be a TL(AP, X, U') formula such that $L(\varphi)$ is stutter invariant. Construct inductively a formula $\tau(\varphi)$ of TL(AP, U') such that $\mathrm{sf}(L(\varphi)) = \mathrm{sf}(L(\tau(\varphi)))$, and thus such that $L(\varphi) = L(\tau(\varphi))$ according to the previous question. What is the size of $\tau(\varphi)$ (there exists a solution of size $O(|\varphi| \cdot 2^{|\varphi|})$)?