TD 4

1 Synchronous Büchi Transducers

Exercise 1. Give synchronous Büchi transducers for the following formulæ:

1. $\mathsf{F}'q$,
2. $\mathsf{G}q$,
3. $\mathsf{G}'q$,
4. $p\mathsf{S}q$,
5. $p\mathsf{S}'q$,
6. $p\mathsf{U}'q$,
7. $\mathsf{G}(p \rightarrow \mathsf{F}q)$.

2 Recognizable Languages

Recall from the course that a language of infinite words in Σ^ω is *recognizable* iff there exists a Büchi automaton for it.

Exercise 2 (Basic Closure Properties)**.** Show that $\mathsf{Rec}(\Sigma^\omega)$ is closed under

1. finite union, and
2. finite intersection.

Exercise 3 (Ultimately Periodic Words)**.** An *ultimately periodic word* over Σ is a word of form $u \cdot v^\omega$ with u in Σ^* and v in Σ^+.

Prove that any nonempty recognizable language in $\mathsf{Rec}(\Sigma^\omega)$ contains an ultimately periodic word.

Exercise 4 (Rational Languages)**.** A *rational language* L of infinite words over Σ is a finite union

$$L = \bigcup X \cdot Y^\omega$$

where X is in $\mathsf{Rat}(\Sigma^*)$ and Y in $\mathsf{Rat}(\Sigma^+)$. We denote the set of *rational* languages of infinite words by $\mathsf{Rat}(\Sigma^\omega)$.

Show that $\mathsf{Rec}(\Sigma^\omega) = \mathsf{Rat}(\Sigma^\omega)$.

Exercise 5 (Deterministic Büchi Automata)**.** A Büchi automaton is *deterministic* if $|I| \leq 1$, and for each state q in Q and symbol a in Σ, $|\{(q, a, q') \in T \mid q' \in Q\}| \leq 1$.

1. Give a nondeterministic Büchi automaton for the language in $\{a, b\}^\omega$ described by the expression $(a + b)^* a^\omega$.
2. Show that there does not exist any deterministic Büchi automaton for this language.
3. Let $A = (Q, \Sigma, T, q_0, F)$ be a finite deterministic automaton that recognizes the language of finite words $L \subseteq \Sigma^*$. We can also interpret $\mathcal{A}$ as a deterministic Büchi automaton with a language $L' \subseteq \Sigma^\omega$; our goal here is to relate the languages of finite and infinite words defined by $\mathcal{A}$.

   Let the *limit* of a language $L \subseteq \Sigma^*$ be

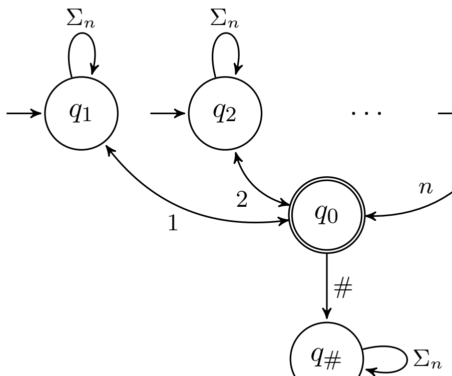
   $$\overrightarrow{L} = \{w \in \Sigma^\omega \mid w \text{ has infinitely many prefixes in } L\} .$$

   Characterize the language L' of infinite words of $\mathcal{A}$ in terms of its language of finite words L and of the limit operation.

3 Büchi Complementation

Exercise 6 (Lower Bound on Büchi Complementation)**.** The best known lower bound on the size of a Büchi automaton for the complement $\overline{L}$ of a language, compared to that of the Büchi automaton for L, is $\Omega\big((0.76\,n)^n\big)$ [Yan, LMCS 4(1:5), 2008], with a matching upper bound modulo a quadratic factor [Schewe, STACS 2009]. We see in this exercise an easier to obtain lower bound of $\Omega(n!)$.

Let $\Sigma_n = \{\#, 1, 2, \ldots, n\}$ be our alphabet, and L_n the language of the following Büchi automaton (note the two-ways transitions):



1. Let $a_1 \cdots a_k$ be a fixed, finite word in $\{1, \ldots, n\}^*$. Prove that any infinite word in

   $$(\Sigma_n^* a_1 a_2 \Sigma_n^* a_2 a_3 \Sigma_n^* \cdots \Sigma_n^* a_{k-1} a_k \Sigma_n^* a_k a_1)^\omega$$

   is also a word of L_n.

2. Let $(i_1, \ldots, i_n)$ be a permutation of $\{1, \ldots, n\}$. Show that the infinite word
$$(i_1 \cdots i_n \#)^\omega$$
is not in L_n.

3. Consider two different permutations $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ of $\{1, \ldots, n\}$. As shown in the previous question, the two infinite words $\rho = (i_1 \cdots i_n \#)^\omega$ and $\sigma = (j_1 \cdots j_n \#)^\omega$ are in $\overline{L_n}$.

   Suppose that $\mathcal{B}$ is a Büchi automaton that recognizes $\overline{L_n}$; show that if ρ eventually loops forever in a subset R of the states of $\mathcal{B}$, and σ does the same in a subset S, then R and S are disjoint sets.

4. Conclude.

Exercise 7 (Closure by Complementation)**.** The purpose of this exercise is to prove that $\mathrm{Rec}(\Sigma^\omega)$ is closed under complement. We consider for this a Büchi automaton $A = (Q, \Sigma, T, I, F)$, and want to prove that its complement language $\overline{L(A)}$ is in $\mathrm{Rec}(\Sigma^\omega)$.

We note $q \xrightarrow{u} q'$ for q, q' in Q and $u = a_1 \cdots a_n$ in Σ^* if there exists a sequence of states $q_0, \ldots, q_n$ such that $q_0 = q$, $q_n = q'$ and for all $0 \leq i < n$, (q_i, a_{i+1}, q_{i+1}) is in T. We note in the same way $q \xrightarrow{u}_F q'$ if furthermore at least one of the states $q_0, \ldots, q_n$ belongs to F.

We define the *congruence* $\sim_A$ over Σ^* by
$$u \sim_A v \text{ iff } \forall q, q' \in Q,\ (q \xrightarrow{u} q' \Leftrightarrow q \xrightarrow{v} q') \text{ and } (q \xrightarrow{u}_F q' \Leftrightarrow q \xrightarrow{v}_F q')\ .$$

1. Show that $\sim_A$ has finitely many congruence classes $[u]$, for u in Σ^*.

2. Show that each $[u]$ for u in Σ^* is in $\mathrm{Rec}(\Sigma^*)$, i.e. is a regular language of finite words.

3. Consider the language $K(L)$ for $L \subseteq \Sigma^\omega$
$$K(L) = \{[u][v]^\omega \mid u, v \in \Sigma^*, [u][v]^\omega \cap L \neq \emptyset\}\ .$$
   Show that $K(L)$ is in $\mathrm{Rec}(\Sigma^\omega)$ for any $L \subseteq \Sigma^\omega$.

4. Show that $K(L(A)) \subseteq L(A)$ and $K(\overline{L(A)}) \subseteq \overline{L(A)}$.

5. Prove that for any infinite word σ in Σ^ω there exist u and v in Σ^* such that σ belongs to $[u][v]^\omega$. The following theorem might come in handy when applied to couples of positions (i, j) inside σ:

   Theorem 1 (Ramsey, infinite version)**.** *Let X be some countably infinite set, n an integer, and $c : X^{(n)} \to \{1, \ldots, k\}$ a k-*coloring *of the n-tuples of X. Then there exists some infinite* monochromatic *subset M of X such that all the n-tuples of M have the same image by c.*

6. Conclude.