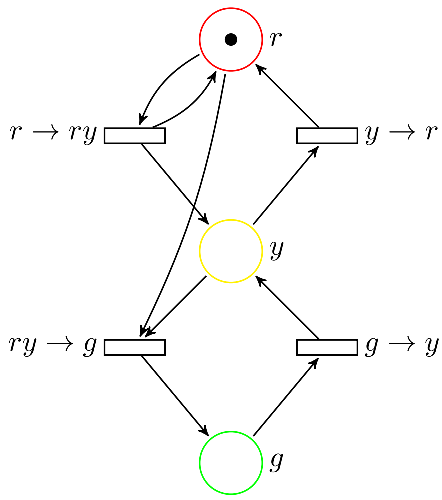
TD 6: Petri Nets

1 Modeling Using Petri Nets

Exercise 1 (Traffic Lights)**.** Consider again the traffic lights example from the lecture notes:



1. How can you correct this Petri net to avert unwanted behaviours (like $r \to ry \to rr$) in a 1-safe manner?
2. Extend your Petri net to model two traffic lights handling a street intersection.

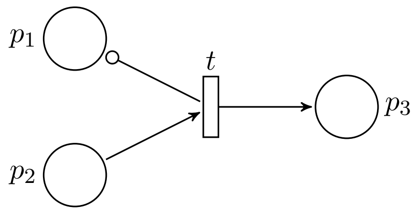
Exercise 2 (Producer/Consumer)**.** A producer/consumer system gathers two types of processes:

producers who can make the actions *produce* (p) or *deliver* (d), and

consumers with the actions *receive* (r) and *consume* (c).

All the producers and consumers communicate through a single unordered channel.

1. Model a producer/consumer system with two producers and three consumers. How can you modify this system to enforce a maximal capacity of ten simultaneous items in the channel?
2. An *inhibitor arc* between a place p and a transition t makes t firable only if the current marking at p is zero. In the following example, there is such an inhibitor arc between p_1 and t. A marking $(0, 2, 1)$ allows to fire t to reach $(0, 1, 2)$, but $(1, 1, 1)$ does not allow to fire t.



Using inhibitor arcs, enforce a priority for the first producer and the first consumer on the channel: the other processes can use the channel only if it is not currently used by the first producer and the first consumer.

2 Model Checking Petri Nets

Exercise 3 (Upper Bounds). Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider as usual propositional LTL, with a set of atomic propositions AP equal to P the set of places of the Petri net. We define proposition p to hold in a marking m in $\mathbb{N}^P$ if $m(p) > 0$.

The models of our LTL formulæ are *computations* $m_0 m_1 \cdots$ in $(\mathbb{N}^P)^\omega$ such that, for all $i \in \mathbb{N}$, $m_i \to_{\mathcal{N}} m_{i+1}$ is a transition step of the Petri net $\mathcal{N}$.

1. We want to prove that state-based LTL model checking can be performed in polynomial space for 1-safe Petri nets. For this, prove that one can construct an exponential-sized Büchi automaton $\mathcal{B}_{\mathcal{N}}$ from a 1-safe Petri net that recognizes all the infinite computations of $\mathcal{N}$ starting in m_0.

2. In the general case, state-based LTL model checking is undecidable. Prove it for Petri nets with at least two unbounded places, by a reduction from the halting problem for 2-counter Minsky machines.

3. We consider now a different set of atomic propositions, such that $\Sigma = 2^{\mathrm{AP}}$, and a labeled Petri net, with a labeling homomorphism $\lambda : T \to \Sigma$. The models of our LTL formulæ are infinite words $a_0 a_1 \cdots$ in Σ^ω such that $m_0 \xrightarrow{t_0}_{\mathcal{N}} m_1 \xrightarrow{t_1}_{\mathcal{N}} m_2 \cdots$ is an execution of $\mathcal{N}$ and $\lambda(t_i) = a_i$ for all i.

   Prove that action-based LTL model checking can be performed in polynomial space for labeled 1-safe Petri nets.

3 Unfoldings

Exercise 4 (Adequate Partial Orders). A partial order $\prec$ between events is *adequate* if the three following conditions are verified:

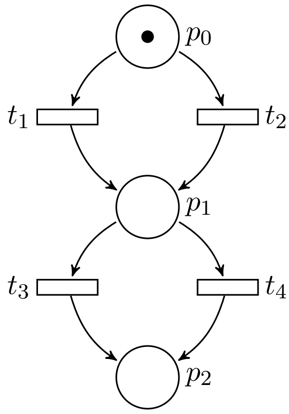
(a) $\prec$ is well-founded,

(b) $C_t \subsetneq C_{t'}$ implies $t \prec t'$, and

(c) $\prec$ is preserved by finite extensions: as in the lecture notes, if $t \prec t'$ and $B(t) = B(t')$, and E and E' are two isomorphic extensions of C_t and $C_{t'}$ with $C_u = C_t \oplus E$ and $C_{u'} = C_{t'} \oplus E'$, then $u \prec u'$.

As you can guess, adequate partial orders result in complete unfoldings.

1. Show that $\prec_s$ defined by $t \prec_s t'$ iff $|C_t| < |C_{t'}|$ is adequate.
2. Construct the finite unfolding of the following Petri net using $\prec_s$; how does the size of this unfolding relate to the number of reachable markings?



3. Suppose we define an arbitrary total order $\ll$ on the transitions T of the Petri net, i.e. they are $t_1 \ll \cdots \ll t_n$. Given a set S of events and conditions of $\mathcal{Q}$, $\varphi(S)$ is the sequence $t_1^{i_1} \cdots t_n^{i_n}$ in T^* where i_j is the number of events labeled by t_j in S. We also note $\ll$ for the lexicographic order on T^*.

   Show that $\prec_e$ defined by $t \prec_e t'$ iff $|C_t| < |C_{t'}|$ or $|C_t| = |C_{t'}|$ and $\varphi(C_t) \ll \varphi(C_{t'})$ is adequate. Construct the finite unfolding for the previous Petri net using $\prec_e$.
4. There might still be examples where $\prec_e$ performs poorly. One solution would be to use a *total* adequate order; why? Give a 1-safe Petri net that shows that $\prec_e$ is not total.

4 Coverability Graphs

Exercise 5 (Dickson's Lemma)**.** A *quasi-order* $(A, \leq)$ is a set A endowed with a reflexive and transitive ordering relation $\leq$. A *well quasi order* (wqo) is a quasi order $(A, \leq)$ s.t., for any infinite sequence $a_0 a_1 \cdots$ in A^ω, there exist indices $i < j$ with $a_i \leq a_j$.

1. Let $(A, \leq)$ be a wqo and $B \subseteq A$. Show that $(B, \leq)$ is a wqo.
2. Show that $(\mathbb{N} \uplus \{\omega\}, \leq)$ is a wqo.
3. Let $(A, \leq)$ be a wqo. Show that any infinite sequence $a_0 a_1 \cdots$ in A^ω embeds an infinite increasing subsequence $a_{i_0} \leq a_{i_1} \leq a_{i_2} \leq \cdots$ with $i_0 < i_1 < i_2 < \cdots$.

4. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two wqo's. Show that the cartesian product $(A \times B, \leq_\times)$, where the product ordering is defined by $(a, b) \leq_\times (a', b')$ iff $a \leq_A a'$ and $b \leq_B b'$, is a wqo.

Exercise 6 (Coverability Graph)**.** The *coverability problem* for Petri nets is the following decision problem:

Instance: A Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$ and a marking m_1 in $\mathbb{N}^P$.

Question: Does there exist m_2 in $\mathsf{reach}_\mathcal{N}(m_0)$ such that $m_1 \leq m_2$?

For 1-safe Petri nets, coverability coincides with reachability, and is thus PSPACE-complete.

One way to decide the general coverability problem is to use Karp and Miller's coverability graph (see the lecture notes). Indeed, we have the equivalence between the two statements:

i. there exists m_2 in $\mathsf{reach}_\mathcal{N}(m_0)$ such that $m_1 \leq m_2$, and

ii. there exists m_3 in $\mathsf{CoverabilityGraph}_\mathcal{N}(m_0)$ such that $m_1 \leq m_3$.

1. In order to prove that (i) implies (ii), we will prove a stronger statement: for a marking m in $(\mathbb{N} \uplus \{\omega\})^P$, write $\Omega(m) = \{p \in P \mid m(p) = \omega\}$ be the set of ω-places of m.

   Show that, if $m_0 \xrightarrow{u}_\mathcal{N} m_2$ in the Petri net $\mathcal{N}$ for some u in T^*, then there exists m_3 in $(\mathbb{N} \uplus \{\omega\})^P$ such that $m_2(p) = m_3(p)$ for all p in $P \setminus \Omega(m_3)$ and $m_0 \xrightarrow{u}_G m_3$ in the coverability graph.

2. Let us prove that (ii) implies (i). The idea is that we can find reachable markings that agree with m_3 on its finite places, and that can be made arbitrarily high on its ω-places. For this, we need to identify the graph nodes where new ω values were introduced, which we call *ω-nodes.*

   (a) The *threshold* $\Theta(u)$ of a transition sequence u in T^* is the minimal marking m in $\mathbb{N}^P$ s.t. u is enabled from m. Show how to compute $\Theta(u)$. Show that $\Theta(u \cdot v) \leq \Theta(u) + \Theta(v)$ for all u, v in T^*.

   (b) Recall that an ω value is introduced in the coverability graph thanks to Algorithm 1.

   Let $\{v_1, \ldots, v_\ell\}$ be the set of "v" sequences found on line 3 of the algorithm that resulted in adding at least one ω value to m' on line 5 during a single call to AddOmegas(m, m', V) on line 8 of the CoverabilityGraph algorithm from the course notes. Let $w = v_1 \cdots v_\ell$. Show that, for any k in $\mathbb{N}$, the marking ν_k defined by

$$\nu_k(p) = \begin{cases} m'(p) & \text{if } p \in P \setminus \Omega(m) \\ \Theta(w^k)(p) & \text{if } p \in \Omega(m) \end{cases}$$

repeat
  saved $\leftarrow m'$
  foreach $m'' \in V$ *s.t.* $\exists v \in T^+, m'' \xrightarrow{v}_G m$ **do**
    if $m'' < m'$ **then**
      $m' \leftarrow m' + ((m' - m'') \cdot \omega)$
    end
  end
until *saved* $= m'$
return m'

Algorithm 1: $\textsc{AddOmegas}(m, m', V)$

allows to fire w^k. How does the marking ν'_k with $\nu_k \xrightarrow{w^k}_{\mathcal{N}} \nu'_k$ compare to ν_k?

(c) Prove that, if $m_0 \xrightarrow{u}_G m_3$ for some u in T^* in the coverability graph and m' in $\mathbb{N}^{\Omega(m_3)}$ is a partial marking on the places of $\Omega(m_3)$, then there are
- n in $\mathbb{N}$,
- a decomposition $u = u_1 u_2 \cdots u_{n+1}$ with each u_i in T^* (where the markings μ_i reached by $m \xrightarrow{u_1 \cdots u_i}_G \mu_i$ for $i \leq n$ have new ω values),
- sequences $w_1, \ldots, w_n$ in T^+,
- numbers $k_1, \ldots, k_n$ in $\mathbb{N}$,

such that $m_0 \xrightarrow{u_1 w_1^{k_1} u_2 \cdots u_n w_n^{k_n} u_{n+1}}_{\mathcal{N}} m_2$ with $m_2(p) = m_3(p)$ for all p in $P \setminus \Omega(m_3)$ and $m_2(p) \geq m'(p)$ for all p in $\Omega(m_3)$.

Exercise 7 (Decidability of Model-checking Action-based LTL)**.**

1. Let $\mathcal{N}$ be Petri net, G its coverability graph, and m some marking in $\mathbb{N}^P$. An infinite *computation* is a sequence $m_0 m_1 \cdots$ in $(\mathbb{N}^P)^\omega$ where for all $i \in \mathbb{N}$, $m_i \to_{\mathcal{N}} m_{i+1}$ is a transition step. The *effect* $\Delta(u)$ of a transition sequence u in T^* is defined by $\Delta(\varepsilon) = 0^P$ and $\Delta(ut) = \Delta(u) - W(P, t) + W(t, P)$.

   Show that there exists an infinite computation s.t. $m \leq m_i$ for infinitely many indices i iff there exists an accessible loop $m' \xrightarrow{v}_G m'$ in G s.t. $m \leq m'$ and $\Delta(v) \geq 0^P$.

2. Show that action-based LTL model-checking is decidable for labeled Petri nets.

Exercise 8 (Rackoff's Algorithm)**.** A rather severe issue with the coverability graph construction is that it can generate a graph of Ackermannian size compared to that of the original Petri net. We show here a much more decent $\textsc{ExpSpace}$ upper bound, which is matched by an $\textsc{ExpSpace}$ hardness proof by Lipton.

Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider *generalized markings* in $\mathbb{Z}^P$. A *generalized computation* is a sequence $\mu_1 \cdots \mu_n$ in $(\mathbb{Z}^P)^*$ such that, for all $1 \leq i < n$, there is a transition t in T with $\mu_{i+1}(p) = \mu_i(p) - W(p,t) + W(t,p)$ for all $p \in P$ (i.e. we do not enforce enabling conditions). For a subset I of P, a generalized sequence is *I-admissible* if furthermore $\mu_i(p) \geq W(p,t)$ for all p in I at each step $1 \leq i < n$. For a value B in $\mathbb{N}$, it is *I–B-bounded* if furthermore $\mu_i(p) < B$ for all p in I at each step $1 \leq i \leq n$. A generalized sequence is an *I-covering* for m_1 if $\mu_1 = m_0$ and $\mu_n(p) \geq m_1(p)$ for all p in I.

Thus a computation is a P-admissible generalized computation, and a P-admissible P-covering for m_1 answers the coverability problem.

For a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$ and a marking m_1 in $\mathbb{N}^P$, let $\ell(\mathcal{N}, m_1)$ be the length of the shortest P-admissible P-covering for m_1 in $\mathcal{N}$ if one exists, and otherwise $\ell(\mathcal{N}, m_1) = 0$. For L, k in $\mathbb{N}$, define

$$M_L(k) = \sup\{\ell(\mathcal{N}, m_1) \mid |P| = k, \max_{p \in P, t \in T} W(p,t) + \max_{p \in P} m_1(p) \leq L\}$$

the maximal $\ell(\mathcal{N}, m_1)$ over *all* Petri nets $\mathcal{N}$ of dimension k and all markings m_1 to cover, under some restrictions on incoming weights $W(p,t)$ in $\mathcal{N}$ and values in m_1.

1. Show that $M_L(0) \leq 1$.

2. We want to show that

   $$M_L(k) \leq (L \cdot M_L(k-1))^k + M_L(k-1)$$

   for all $k \geq 1$. To this end, we prove that, for every marking m_1 in $\mathbb{N}^P$ for a Petri net $\mathcal{N}$ with $|P| = k$,

   $$\ell(\mathcal{N}, m_1) \leq (L \cdot M_L(k-1))^k + M_L(k-1)\ . \tag{$*$}$$

   Let

   $$B = M_L(k-1) \cdot \max_{p \in P, t \in T} W(p,t) + \max_{p \in P} m_1(p)\ .$$

   and suppose that there exists a P-admissible P-covering $w = \mu_1 \cdots \mu_n$ for m_1 in $\mathcal{N}$.

   (a) Show that, if w is P–B-bounded, then $(*)$ holds.

   (b) Assume the contrary: we can split w as $w_1 w_2$ such that w_1 is P–B-bounded and w_2 starts with a marking μ_j with a place p such that $\mu_j(p) \geq B$. Show that $(*)$ also holds.

3. Show that $M_L(|P|) \leq L^{(3 \cdot |P|)!}$ for $L \geq 2$.

4. Given a Petri net $\mathcal{N} = \langle P, T, W, m_0 \rangle$ and a marking m_1, set $L = 2 + \max_{p \in P, t \in T} W(p,t) + \max_{p \in P} m_1(p)$. Assuming that the size n of the instance $(\mathcal{N}, m_1)$ of the coverability problem is more than

   $$\max(\log L, |P|, \max_{p \in P, t \in T} \log W(t,p))\ ,$$

deduce that we can guess a P-admissible P-covering for m_1 of length at most $2^{2^{c \cdot n \log n}}$ for some constant c. Conclude that coverability can be solved in ExpSpace.

5 Vector Addition Systems

Exercise 9 (VASS)**.** An n-dimensional *vector addition system with states* (VASS) is a tuple $\mathcal{V} = \langle Q, \delta, q_0 \rangle$ where Q is a finite set of states, $q_0 \in Q$ the initial state, and $\delta \subseteq Q \times \mathbb{Z}^n \times Q$ the transition relation. A configuration of $\mathcal{V}$ is a pair (q, v) in $Q \times \mathbb{N}^n$. An execution of $\mathcal{V}$ is a sequence of configurations $(q_0, v_0)(q_1, v_1) \cdots (q_m, v_m)$ such that $v_0 = \bar{0}$, and for $0 < i \leq m$, $(q_{i-1}, v_i - v_{i-1}, q_i)$ is in δ.

1. Show that any VASS can be simulated by a Petri net.
2. Show that, conversely, any Petri net can be simulated by a VASS.

Exercise 10 (VAS)**.** An n-dimensional *vector addition system* (VAS) is a pair (v_0, W) where $v_0 \in \mathbb{N}^n$ is the initial vector and $W \subseteq \mathbb{Z}^n$ is the set of transition vectors. An execution of (v_0, W) is a sequence $v_0 v_1 \cdots v_m$ where $v_i \in \mathbb{N}$ for all $0 \leq i \leq m$ and $v_i - v_{i-1} \in W$ for all $0 < i \leq m$.

We want to show that any n-dimensional VASS $\mathcal{V}$ can be simulated by an $(n+3)$-dimensional VAS (v_0, W).

Hint: Let $k = |Q|$, and define the two functions $a(i) = i + 1$ and $b(i) = (k+1)(k-i)$. Encode a configuration (q_i, v) of $\mathcal{V}$ as the vector $(v(1), \ldots, v(n), a(i), b(i), 0)$. For every state q_i, $0 \leq i < k$, we add two transition vectors to W:

$$t_i = (0, \ldots, 0, -a(i), a(k-i) - b(i), b(k-i))$$
$$t'_i = (0, \ldots, 0, b(i), -a(k-i), a(i) - b(k-i))$$

For every transition $d = (q_i, w, q_j)$ of $\mathcal{V}$, we add one transition vector to W:

$$t_d = (w(1), \ldots, w(n), a(j) - b(i), b(j), -a(i))$$

1. Show that any execution of $\mathcal{V}$ can be simulated by (v_0, W) for a suitable v_0.
2. Conversely, show that this VAS (v_0, W) simulates $\mathcal{V}$ faithfully.