Tree Transducers

Home assignment to hand in before or on November 2, 2018.

October	1	2	3	4	5	6	7
	8	9	10	11	12	13	14
	15	16	17	18	19	20	21
	22	23	24	25	26	27	28
	29	30	31				
November				1	2	3	4

Electronic versions (PDF only) can be sent by email to ⟨sylvain.schmitz@lsv.fr⟩; paper versions should be handed in on the 2nd or put in my mailbox at LSV, ENS Paris-Saclay. **No delays**. The numbers in the margins next to exercises are indications of time and difficulty, not necessarily of the points you might earn answering them.

We learn in this homework about some of the (many) notions of *transducers* over finite trees. Let $\mathcal{F}$ and $\mathcal{F}'$ be two finite ranked alphabets. A transducer τ realises a relation $[\![\tau]\!] \subseteq T(\mathcal{F}) \times T(\mathcal{F}')$.

Tree transducers find many applications in computational linguistics, compilers, XML processing, and logics, whenever we need to model such transformations over finite trees. Chapter 6 of *Tree Automata Techniques and Applications* gives a quick overview of two of the models we shall encounter in this homework, along with motivating examples.

As seen in class, one of the simplest means of defining tree transformations is through *tree homomorphisms*. Recall the following facts: tree homomorphisms do not preserve recognisability (c.f. *TATA* Exa. 1.4.2), but linear homomorphisms do (c.f. *TATA* Thm. 1.4.3) and inverse homomorphisms also do (c.f. *TATA* Thm. 1.4.4).

1 Top-Down Tree Transducers

TATA, Sec. 6.4.2 Let $\mathcal{X}$ be a countable set of variables. For Q a ranked alphabet with arities greater than zero, we abuse notations and write $Q(\mathcal{X})$ for the alphabet of pairs $(q, x) \in Q \times \mathcal{X}$ with $\mathit{arity}(q, x) \stackrel{\text{def}}{=} \mathit{arity}(q) - 1$. This is just for convenience, and $(q, x)(t_1, \ldots, t_n)$ is really the term $q(x, t_1, \ldots, t_n)$.

A *top-down tree transducer* (NDTT) is a tuple $\mathcal{D} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ where Q is a finite set of states, all of arity 1, $\mathcal{F}$ and $\mathcal{F}'$ are finite ranked alphabets, $I \subseteq Q$ is a set of root states, and Δ is a finite set of term rewriting rules of the form $q(f(x_1, \ldots, x_n)) \to e$ where $q \in Q$, $f \in \mathcal{F}_n$ for some $n \in \mathbb{N}$, and $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n))$. Note that any such e is a term from $T(Q \cup \mathcal{F}', \mathcal{X})$.

The transduction $[\![\mathcal{D}]\!]$ realised by $\mathcal{D}$ is defined through rewriting semantics:

$$[\![\mathcal{D}]\!] \stackrel{\text{def}}{=} \{(t,t') \in T(\mathcal{F}) \times T(\mathcal{F}') \mid \exists q \in I \,.\, q(t) \to^* t'\} \;.$$

TATA, Exa. 6.4.2.2

Exercise 1 (Example). Let $\mathcal{F} \stackrel{\text{def}}{=} \{a^{(1)}, \$^{(0)}\}$ and $\mathcal{F}' \stackrel{\text{def}}{=} \{f^{(2)}, a^{(1)}, b^{(1)}, \$^{(0)}\}$. Consider the NDTT $\mathcal{D} = (\{q, q'\}, \mathcal{F}, \mathcal{F}', \Delta, \{q\})$ with Δ the set of rules

$$q(a(x_1) \to f(q'(x_1), q'(x_1)) \qquad\qquad q'(\$) \to \$$$
$$q'(a(x_1)) \to a(q'(x_1)) \qquad\qquad q'(a(x_1)) \to b(q'(x_1))$$

Then we have for instance the following derivation:

$$\begin{aligned} q(a(a(a(\$)))) &\to f(\ q'(a(a(\$))),\ q'(a(a(\$)))\ ) \\ &\to f(\ a(q'(a(\$))),\ q'(a(a(\$)))\ ) \\ &\to f(\ a(b(q'(\$))),\ q'(a(a(\$)))\ ) \\ &\to f(\ a(b(\$)),\ q'(a(a(\$)))\ ) \\ &\to f(\ a(b(\$)),\ b(q'(a(\$)))\ ) \\ &\to f(\ a(b(\$)),\ b(b(q'(\$)))\ ) \\ &\to f(\ a(b(\$)),\ b(b(\$))\ ) \end{aligned}$$

showing that $(\ a(a(a(\$))),\ f(a(b(\$)), b(b(\$)))\ ) \in [\![\mathcal{D}]\!]$.

[2] Show that $[\![\mathcal{D}]\!] = \{(a(t), f(t_1,t_2)) \mid t \in T(\mathcal{F}), t_1, t_2 \in T(\{a,b,\$\}),$ and $\mathit{height}(t) = \mathit{height}(t_1) = \mathit{height}(t_2)\}$.

Using ideas similar to those of Thm. 1.4.3 of *TATA*, one can show:

Fact 1 (Linear NDTTs). *An NDTT $\mathcal{D} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ is* linear*, if every rule $q(f(x_1), \ldots, x_n)) \to e$ from Δ is such that e is a linear term in $T(\mathcal{F}' \cup Q, \mathcal{X}_n)$. If $\mathcal{D}$ is linear and L is a recognisable tree language over $\mathcal{F}$, then $[\![\mathcal{D}]\!](L) \stackrel{\text{def}}{=} \{t' \in T(\mathcal{F}) \mid \exists t \in L, (t,t') \in [\![\mathcal{D}]\!]\}$ is recognisable over $\mathcal{F}'$.*

2 Bottom-Up Tree Transducers

TATA, Sec. 6.4.1, and exercises from 2017 in TD3 and the exam

Let $\mathcal{X}$ be a countable set of variables.

A *bottom-up tree transducer* (NUTT) is a tuple $\mathcal{U} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ where Q is a finite set of states, all of arity 1, $\mathcal{F}$ and $\mathcal{F}'$ are finite ranked alphabets, $I \subseteq Q$ is a set of root states, and Δ is a finite set of term rewriting rules of the form $f(q_1(x_1), \ldots, q_n(x_n)) \to q(e)$ where $q, q_1, \ldots, q_n \in Q$, $f \in \mathcal{F}_n$ for some $n \in \mathbb{N}$, and $e \in T(\mathcal{F}', \mathcal{X}_n)$.

The transduction $[\![\mathcal{U}]\!]$ realised by $\mathcal{U}$ is defined through rewriting semantics:

$$[\![\mathcal{U}]\!] \stackrel{\text{def}}{=} \{(t,t') \in T(\mathcal{F}) \times T(\mathcal{F}') \mid \exists q \in I \,.\, t \to^* q(t')\} \;.$$

TATA, Exa. 6.4.2.2 **Exercise 2** (Example). Let $\mathcal{F} \stackrel{\text{def}}{=} \{a^{(1)}, \$^{(0)}\}$ and $\mathcal{F}' \stackrel{\text{def}}{=} \{f^{(2)}, a^{(1)}, b^{(1)}, \$^{(0)}\}$. Consider the NDTT $\mathcal{U} = (\{q, q'\}, \mathcal{F}, \mathcal{F}', \Delta, \{q\})$ with Δ the set of rules

$$a(q'(x_1)) \to q(f(x_1, x_1)) \qquad \$ \to q'(\$)$$
$$a(q'(x_1)) \to q'(a(x_1)) \qquad a(q'(x_1)) \to q'(b(x_1))$$

Then we have for instance the following derivation:

$$\begin{aligned} a(a(a(\$))) &\to a(a(a(q'(\$)))) \\ &\to a(a(q'(b(\$)))) \\ &\to a(q'(a(b(\$)))) \\ &\to q(f(\ a(b(\$)), a(b(\$))\ )) \end{aligned}$$

showing that $(\ a(a(a(\$))),\ f(a(b(\$)), a(b(\$)))\ ) \in [\![\mathcal{U}]\!]$.

[2] Show that $[\![\mathcal{U}]\!] = \{(a(t), f(t', t')) \mid t \in T(\mathcal{F}), t' \in T(\{a, b, \$\}), \text{ and } \mathit{height}(t) = \mathit{height}(t')\}$.

Here is an analogue of Fact 1 for linear NUTTs:

Fact 2 (Linear NUTTs). *A NUTT $\mathcal{U} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ is* linear*, if every rule $f(q_1(x_1), \ldots, q_n(x_n)) \to q(e)$ is such that e is a linear term in $T(\mathcal{F}', \mathcal{X}_n)$. If $\mathcal{U}$ is linear and L is a recognisable tree language over $\mathcal{F}$, then $[\![\mathcal{U}]\!](L)$ is a recognisable tree language over $\mathcal{F}'$.*

3 Macro Tree Transducers

3.1 Definitions

Let $\mathcal{X}$ be a countable set of variables and $\mathcal{Y}$ a countable set of parameters; we assume $\mathcal{X}$ and $\mathcal{Y}$ to be disjoint.

Syntax. A *macro tree transducer* (NMTT) is a tuple $\mathcal{M} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ where Q is a finite set of states, all of arity ≥ 1, $\mathcal{F}$ and $\mathcal{F}'$ are finite ranked alphabets, $I \subseteq Q_1$ is a set of root states of arity one, and Δ is a finite set of term rewriting rules of the form $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e$ where $q \in Q_{1+p}$ for some $p \geq 0$, $f \in \mathcal{F}_n$ for some $n \in \mathbb{N}$, and $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n), \mathcal{Y}_p)$.

Inside-Out Semantics. Given a NMTT, the *inside-out* rewriting relation over trees in $T(\mathcal{F} \cup \mathcal{F}' \cup Q)$ is defined by: $t \xrightarrow{\text{IO}} t'$ if there exist a rule $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e$ in Δ, a context $C \in C(\mathcal{F} \cup \mathcal{F}' \cup Q)$, and two substitutions $\sigma\colon \mathcal{X} \to T(\mathcal{F})$ and $\rho\colon \mathcal{Y} \to T(\mathcal{F}')$ such that $t = C[q(f(x_1, \ldots, x_n), y_1, \ldots, y_p)\sigma\rho]$ and $t' = C[e\sigma\rho]$. In other words, in inside-out rewriting, when applying a rewriting rule $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e$, the parameters $y_1, \ldots, y_p$ must be mapped to trees in $T(\mathcal{F}')$, with no remaining states from Q.

The *inside-out* transduction $[\![\mathcal{M}]\!]_{\mathrm{IO}}$ realised by $\mathcal{M}$ is defined through inside-out rewriting semantics:

$$[\![\mathcal{M}]\!]_{\mathrm{IO}} \stackrel{\text{def}}{=} \{(t,t') \in T(\mathcal{F}) \times T(\mathcal{F}') \mid \exists q \in I \,.\, q(t) \xrightarrow{\mathrm{IO}}{}^{*} t'\} \,.$$

Exercise 3 (Example)**.** Let $\mathcal{F} \stackrel{\text{def}}{=} \{a^{(1)}, \$^{(0)}\}$ and $\mathcal{F}' \stackrel{\text{def}}{=} \{f^{(3)}, a^{(1)}, b^{(1)}, \$^{(0)}\}$. Consider the NMTT $\mathcal{M} = (\{q^{(1)}, q'^{(3)}\}, \mathcal{F}, \mathcal{F}', \Delta, \{q\})$ with Δ the set of rules

$$\begin{aligned} q(a(x_1)) &\to q'(x_1, \$, \$) & q'(\$, y_1, y_2) &\to f(y_1, y_1, y_2)\\ q'(a(x_1), y_1, y_2) &\to q'(x_1, a(y_1), a(y_2)) & q'(a(x_1), y_1, y_2) &\to q'(x_1, a(y_1), b(y_2))\\ q'(a(x_1), y_1, y_2) &\to q'(x_1, b(y_1), a(y_2)) & q'(a(x_1), y_1, y_2) &\to q'(x_1, b(y_1), b(y_2)) \end{aligned}$$

Then we have for instance the following derivation:

$$\begin{aligned} q(a(a(a(\$)))) &\xrightarrow{\mathrm{IO}} q'(a(a(\$)), \$, \$)\\ &\xrightarrow{\mathrm{IO}} q'(a(\$), b(\$), b(\$))\\ &\xrightarrow{\mathrm{IO}} q'(\$, a(b(\$)), b(b(\$)))\\ &\xrightarrow{\mathrm{IO}} f(a(b(\$)), a(b(\$)), b(b(\$))) \end{aligned}$$

showing that $(\ a(a(a(\$))),\ f(a(b(\$)), a(b(\$)), b(b(\$)))\ ) \in [\![\mathcal{M}]\!]$.

[2] Show that $[\![\mathcal{M}]\!] = \{(a(t), f(t_1, t_1, t_2)) \mid t \in T(\mathcal{F}), t_1, t_2 \in T(\{a, b, \$\}),$ and $height(t) = height(t_1) = height(t_2)\}$.

3.2 Comparison with Top-Down and Bottom-Up Transductions

Exercise 4 (Monadic trees)**.** Let $\mathcal{F}' \stackrel{\text{def}}{=} \{a^{(1)}, b^{(1)}, \$^{(0)}\}$. Observe that trees in $T(\mathcal{F}')$ are in bijection with contexts in $C(\mathcal{F}')$ and words over $\{a, b\}^*$.

[1] 1. Show that, if $\mathcal{D}$ is a NDTT from some $\mathcal{F}$ to $\mathcal{F}'$ and L is a recognisable tree language over $\mathcal{F}$, then $[\![\mathcal{D}]\!](L)$ is also recognisable over $\mathcal{F}'$. *Hint: The same kind of argument shows that $[\![\mathcal{U}]\!](L)$ is recognisable for a NUTT from $\mathcal{F}$ to $\mathcal{F}'$ and L a recognisable tree language.*

   Let $\mathcal{D}$ be a NDTT from $\mathcal{F}$ to $\mathcal{F}'$. In any rule $q(f(x_1, \ldots, x_n)) \to e$ of $\mathcal{D}$, e belongs to $T(\mathcal{F}' \cup Q(\mathcal{X}_n))$. Since the maximal arity in $\mathcal{F}'$ is one, e has at most one occurrence of any variable in $\mathcal{X}_n$, thus $\mathcal{D}$ is linear. By Fact 1, $[\![\mathcal{D}]\!](L)$ is recognisable.

   The same reasoning applies to a NUTT $\mathcal{U}$ into $\mathcal{F}'$, using Fact 2. □

2. For a context C from $C(\mathcal{F}')$, we write C^R for its *mirror context*, read from the leaf to the root. For instance, if $C = a(b(a(a(\Box))))$, then $C^R = a(a(b(a(\Box))))$. Formally, let $n \in \mathbb{N}$ be such that $\operatorname{dom} C = \{0^m \mid m \leq n\}$; then $C(0^n) = \Box$ and $C(0^m) \in \{a, b\}$ for $m < n$. Then C^R is defined by $\operatorname{dom} C^R \stackrel{\text{def}}{=} \operatorname{dom} C$, $C^R(0^n) \stackrel{\text{def}}{=} \Box$, and $C^R(0^m) \stackrel{\text{def}}{=} C^R(0^{n-m})$ for all $m < n$.

[3] Give an NMTT $\mathcal{M}$ from $\mathcal{F}'$ to $\mathcal{F}'$ such that $[\![\mathcal{M}]\!]_{\text{IO}} = \{(C[\$], C[C^R[\$]]) \mid C \in C(\mathcal{F}')\}$. In terms of words over $\{a, b\}^*$, this transducer maps w to the palindrome ww^R. Is $[\![\mathcal{M}]\!]_{\text{IO}}(T(\mathcal{F}))$ a recognisable tree language?

$\mathcal{M} \stackrel{\text{def}}{=} (Q, \mathcal{F}', \mathcal{F}', \Delta, I)$ where $Q \stackrel{\text{def}}{=} \{q_i^{(1)}, q^{(2)}\}$, $I \stackrel{\text{def}}{=} \{q_i\}$, and Δ is the set of rules

$$q_i(\$) \to \$ \qquad q_i(a(x_1)) \to a(q(x_1, a(\$))) \qquad q_i(b(x_1)) \to b(q(x_1, b(\$)))$$
$$q(\$, y_1) \to y_1 \quad q(a(x_1), y_1) \to a(q(x_1, a(y_1))) \quad q(b(x_1), y_1) \to b(q(x_1, b(y_1)))\ .$$

We leave the proof of correctness to the reader.

This macro tree transducer is actually input- and parameter-linear, deterministic, and complete. Because a tree language over $\mathcal{F}'$ is recognisable if and only if the corresponding word language over $\{a, b\}$ is recognisable, $[\![\mathcal{M}]\!]_{\text{IO}}(T(\mathcal{F}))$ is not a recognisable tree language. In turn, this shows that recognisable tree languages are not closed under (input-)linear macro transductions, not even the complete deterministic ones. □

⚠ Several copies used a rule of the shape $q(\$, y_1, y_2) \to q'(y_1, y_2)$. This is not allowed by teh definition of NMTTs: any occurrence of a state symbol on the right-hand side of a rule **must** use one of the variables x_i; here there are no variables since $ is a constant symbol, thus there cannot be a state symbol on the right-hand side of a rule from $q(\$, y_1, y_2)$.

[1] **Exercise 5** (From NDTTs to NMTTs)**.** Show that, if $\mathcal{D}$ is an NDTT, then we can construct an NMTT $\mathcal{M}$ such that $[\![\mathcal{D}]\!] = [\![\mathcal{M}]\!]_{\text{IO}}$.
Let $\mathcal{D} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$. Observe that this is also an NMTT, where all the states have arity one. It remains to note that IO rewriting captures general rewriting in this case: this is because there are no parameters, thus the restriction on ρ in the definition of $\xrightarrow{\text{IO}}$ is trivial. □

[4] **Exercise 6** (From NUTTs to NMTTs)**.** Show that, if $\mathcal{U}$ is an NUTT, then we can construct an NMTT $\mathcal{M}$ such that $[\![\mathcal{U}]\!] = [\![\mathcal{M}]\!]_{\text{IO}}$. *Hint: For a NUTT $\mathcal{U} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$, define*

$$Q_\Delta \stackrel{\text{def}}{=} \{\bar{e}^{(n+1)} \mid n > 0, \exists f \in \mathcal{F}_n, \exists q_1, \ldots, q_n, q \in Q\,.\,(f(q_1(x_1), \ldots, q_n(x_n)) \to q(e)) \in \Delta\}\ .$$

Construct an NMTT $\mathcal{M} \stackrel{\text{def}}{=} (Q \uplus Q_\Delta \uplus \{q_0^{(0)}\}, \mathcal{F}, \mathcal{F}', \Delta', \{q_0\})$. The set of rules Δ' should contain in particular the rules

$$\bar{e}(g(x_1, \ldots, x_m), y_1, \ldots, y_n) \to e[y_i/x_i]_{1 \leq i \leq n}$$

for all $\bar{e} \in Q_\Delta$ and $g \in \mathcal{F}_m$ for $m \geq 0$, which drop the argument $g(x_1, \ldots, x_m)$ and substitute y_i for x_i in e; note that $e[y_i/x_i]_{1\leq i\leq n}$ is a term in $T(\mathcal{F}', \mathcal{Y}) \subseteq T(\mathcal{F}' \cup Q(\mathcal{X}), \mathcal{Y})$. Your NMTT should be defined such that, for all $t \in T(\mathcal{F})$, $t' \in T(\mathcal{F}')$, and $q \in Q$,

$$t \to^* q(t') \text{ in } \mathcal{U} \qquad \iff \qquad q(t) \xrightarrow{\text{IO}}{}^* t' \text{ in } \mathcal{M} . \tag{$*$}$$

Let $\Delta' \stackrel{\text{def}}{=} \Delta_0 \uplus \Delta_1$ where Δ_0 is the set of rules for q_0 and Δ_1 the remainder of the rules:

$$\begin{aligned}
\Delta_0 \stackrel{\text{def}}{=} & \{q_0(a) \to t \mid \exists q \in I \,.\, a \to q(t) \in \Delta, a \in \mathcal{F}_0\} \\
& \cup \{q_0(f(x_1, \ldots, x_n)) \to \bar{e}(x_1, q_1(x_1), \ldots, q_n(x_n)) \\
& \qquad \mid \exists q \in I, f(q_1(x_1), \ldots, q_n(x_n)) \to q(e) \in \Delta, n > 0\} \, ; \\
\Delta_1 \stackrel{\text{def}}{=} & \{q(a) \to t \mid a \to q(t) \in \Delta, a \in \mathcal{F}_0\} && (1) \\
& \cup \{q(f(x_1, \ldots, x_n)) \to \bar{e}(x_1, q_1(x_1), \ldots, q_n(x_n)) \\
& \qquad \mid f(q_1(x_1), \ldots, q_n(x_n)) \to q(e) \in \Delta, n > 0\} && (2) \\
& \cup \{\bar{e}(g(x_1, \ldots, x_m), y_1, \ldots, y_n) \to e[y_i/x_i]_{1\leq i\leq n} \\
& \qquad \mid m \in \mathbb{N}, g \in \mathcal{F}_m, \bar{e} \in Q_\Delta\} \, . && (3)
\end{aligned}$$

We are going to prove $(*)$ by induction over t, and using the rules of Δ_1 exclusively. By definition of Δ_0, the outcome will be that, for all $t \in T(\mathcal{F})$ and $t' \in T(\mathcal{F}')$, $q_0(t) \xrightarrow{\text{IO}}{}^* t'$ if and only if there exists $q \in I$ such that $t \to^* q(t')$.

base case $a \in \mathcal{F}_0$: by (1), $a \to q(t') \in \Delta$ if and only if $q(a) \to t' \in \Delta_1$ if and only if $q(a) \xrightarrow{\text{IO}} t'$.

induction step $f(t_1, \ldots, t_n)$, $n > 0$: we consider each direction in $(*)$ separately:

$\Longrightarrow$ if $t = f(t_1, \ldots, t_n) \to^* q(t')$, then $t \to^* f(q_1(t'_1), \ldots, q_n(t'_n)) \to t'$ for $t' = q(e[t'_i/x_i]_{1\leq i\leq n})$, a rule $(f(q_1(x_1), \ldots, q_n(x_n)) \to q(e)) \in \Delta$, and derivations $t_i \to^* q_i(t'_i)$ for all $1 \leq i \leq n$. Thus, in $\mathcal{M}$,

$$\begin{aligned}
q(t) &= q(f(t_1, t_2, \ldots, t_n)) \\
&\xrightarrow{\text{IO}} \bar{e}(t_1, q_1(t_1), q_2(t_2), \ldots, q_n(t_n)) && \text{(using (2))} \\
&\xrightarrow{\text{IO}}{}^* \bar{e}(t_1, t'_1, q_2(t_2), \ldots, t'_n) && \text{(by ind. hyp., } q_1(t_1) \xrightarrow{\text{IO}}{}^* t'_1) \\
&\ \vdots \\
&\xrightarrow{\text{IO}}{}^* \bar{e}(t_1, t'_1, t'_2, \ldots, t'_n) && \text{(by ind. hyp., } q_n(t_n) \xrightarrow{\text{IO}}{}^* t'_n) \\
&\xrightarrow{\text{IO}} e[y_i/x_i]_{1\leq i\leq n}[t'_i/y_i]_{1\leq i\leq n} && \text{(using (3))} \\
&= e[t'_i/x_i]_{1\leq i\leq n} && \text{(by composition of the substitutions)} \\
&= t' \, .
\end{aligned}$$

$\Longleftarrow$ if $q(t) = q(f(t_1,\dots,t_n)) \xrightarrow{\text{IO}}{}^* t'$, then this derivation must start with a step $q(f(t_1,\dots,t_n)) \xrightarrow{\text{IO}} \bar{e}(t_1, q_1(t_1),\dots,q_n(t_n))$ using a rule of the form (2) for some $(f(q_1(x_1),\dots,q_n(x_n)) \to q(e)) \in \Delta$.

Because we are in an inside-out derivation, we must handle the subtrees $q_i(t_i)$ first, hence we must continue with $\bar{e}(t_1, q_1(t_1),\dots,q_n(t_n)) \xrightarrow{\text{IO}}{}^* \bar{e}(t_1, t'_1,\dots,t'_n)$ with $q_i(t_i) \xrightarrow{\text{IO}}{}^* t'_i \in T(\mathcal{F}')$ for all $1 \le i \le n$.

Finally, we must finish with (3), and we obtain $t' = e[y_i/x_i]_{1\le i\le n}[t'_i/y_i]_{1\le i\le n} = e[t'_i/x_i]_{1\le i\le n}$ by composition of the substitutions. Thus, in $\mathcal{U}$,

$$\begin{aligned} t = f(t_1,t_2,\dots,t_n) &\to^* f(q_1(t'_1),t_2,\dots,t_n) && \text{(by ind. hyp., } t_1 \to^* q_1(t_1))\\ &\vdots\\ &\to^* f(q_1(t'_1),q_2(t'_2),\dots,q_n(t'_n)) && \text{(by ind. hyp., } t_n \to^* q_n(t_n))\\ &\to q(e[t'_i/x_i]_{1\le i\le n}) && \text{(using } f(q_1(x_1),\dots,q_n(x_n)) \to q(e))\\ &= t'\ . \end{aligned}$$

□

Exercise 7 (Image intersection)**.** Let $\mathcal{M} = (Q,\mathcal{F},\mathcal{F}',\Delta,I)$ be an NMTT and $\mathcal{A}' = (Q',\mathcal{F}',\Delta',I')$ be a complete DFTA. The purpose of this exercise is to construct an NMTT $\mathcal{M}'$ such that, for all $t \in T(\mathcal{F})$ and $t' \in T(\mathcal{F}')$, $(t,t') \in [\![\mathcal{M}']\!]_{\text{IO}}$ if and only if $(t,t') \in [\![\mathcal{M}]\!]_{\text{IO}}$ and $t' \in L(\mathcal{A}')$.

Let $Q'' \stackrel{\text{def}}{=} \bigcup_{p\in\mathbb{N}} Q_{p+1} \times Q'^{p+1}$. Each symbol $\langle q, q_0q_1\cdots q_p\rangle$ of Q'' pairs a $(p+1)$-ary state from $\mathcal{M}$ with a vector of $p+1$ states from $\mathcal{A}'$, and has arity $p+1$. We want to construct $\mathcal{M}'$ with Q'' as set of states such that, for all $p \in \mathbb{N}$, $\langle q, q_0\cdots q_p\rangle \in Q''$, $t \in T(\mathcal{F})$, and $t'_0,t'_1,\dots,t'_p \in T(\mathcal{F}')$ such that $\forall 1 \le j \le p$, $t'_j \to^* q_j$ in $\mathcal{A}'$,

$$\begin{aligned} &q(t,t'_1,\dots,t'_p) \xrightarrow{\text{IO}}{}^* t'_0 \text{ in } \mathcal{M} \text{ and } t'_0 \to^* q_0 \text{ in } \mathcal{A}'\\ &\qquad\iff \langle q, q_0\cdots q_p\rangle(t,t'_1,\dots,t'_p) \xrightarrow{\text{IO}}{}^* t'_0 \text{ in } \mathcal{M}'. \qquad (\dagger) \end{aligned}$$

[7] Show how to construct $\mathcal{M}'$.

Hint: Consider some arbitrary $n \in \mathbb{N}$ and vector $q_0q_1\cdots q_p \in Q'^{p+1}$ for some $p \in \mathbb{N}$. For such fixed n and p, $\mathcal{X}_n$ and $\mathcal{Y}_p$ can be treated as finite alphabets of symbols of rank 0. Let $\mathcal{Z}$ be a countable set of fresh variable names. Define the NUTT $\mathcal{U}_{n,q_0q_1\cdots q_p} \stackrel{\text{def}}{=} (Q' \uplus \{q_\mathcal{X}\}, \mathcal{F}' \cup \mathcal{X}_n \cup \mathcal{Y}_p \cup Q, \mathcal{F}' \cup \mathcal{X}_n \cup \mathcal{Y}_p \cup Q'', \Delta_{n,q_0q_1\cdots q_p}, \{q_0\})$ with

$$\begin{aligned} \Delta_{n,q_0q_1\cdots q_p} \stackrel{\text{def}}{=} &\{y_j \to q_j(y_j) \mid 1 \le j \le p\} && (u_\mathcal{Y})\\ &\cup \{x_i \to q_\mathcal{X}(x_i) \mid 1 \le i \le n\} && (u_\mathcal{X})\\ &\cup \{f(q'_1(z_1),\dots,q'_m(z_m)) \to q'(f(z_1,\dots,z_m))\\ &\qquad \mid f(q'_1,\dots,q'_m) \to q' \in \Delta'\} && (u_{\mathcal{F}'})\\ &\cup \{q(q_\mathcal{X}(z_0),q'_1(z_1),\dots,q'_m(z_m)) \to q'_0(\langle q, q'_0q'_1\cdots q'_m\rangle(z_0,\dots,z_m))\\ &\qquad \mid m \ge 0, q \in Q_{m+1}, q'_0,q'_1,\dots,q'_m \in Q'\}\ . && (u_Q) \end{aligned}$$

This NUTT defines a transduction from terms $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n), \mathcal{Y}_p)$ *to terms* $e' \in T(\mathcal{F}' \cup Q''(\mathcal{X}_n), \mathcal{Y}_p)$.

Let $\mathcal{M}' \stackrel{\text{def}}{=} (Q'', \mathcal{F}, \mathcal{F}', \Delta'', I \times I')$, where

$$\Delta'' \stackrel{\text{def}}{=} \{\langle q, q_0 q_1 \cdots q_p\rangle(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e' \\ \mid q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e \in \Delta \text{ and } (e, e') \in [\![\mathcal{U}_{n, q_0 q_1 \cdots q_p}]\!]\}$$

Let us first observe that (†) implies the result. Indeed, for all $t \in T(\mathcal{F})$ and $t' \in T(\mathcal{F}')$, $(t, t') \in [\![\mathcal{M}']\!]_{\text{IO}}$ if and only if there exists $\langle q, q'\rangle \in I \times I'$ such that $\langle q, q'\rangle(t) \xrightarrow{\text{IO}}{}^{*} t'$ in $\mathcal{M}'$, which by (†) is if and only if there exist $q \in I$ such that $q(t) \xrightarrow{\text{IO}}{}^{*} t'$ in $\mathcal{M}$ and $q' \in I'$ such that $t \to^* q'$ in $\mathcal{A}'$, thus if and only if $(t, t') \in [\![\mathcal{M}]\!]_{\text{IO}}$ and $t' \in L(\mathcal{A}')$.

Let us now prove (†). We consider the two directions of the implication independently.

$\Longrightarrow$ By induction over $t \in T(\mathcal{F})$; assume that $q(t, t'_1, \ldots, t'_p) \xrightarrow{\text{IO}}{}^{*} t'_0$ in $\mathcal{M}$ and $\forall 0 \le j \le p \,.\, t'_j \to^* q_j$ in $\mathcal{A}'$; let $\rho \stackrel{\text{def}}{=} [t'_j / y_j]_{1 \le j \le p}$.

Let us first show that, for all $n \in \mathbb{N}$, terms $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n), \mathcal{Y}_p)$, substitutions σ from $\mathcal{X}_n$ to strict subtrees of t, trees $t' \in T(\mathcal{F}')$, and states $q' \in Q'$,

$$e\sigma\rho \xrightarrow{\text{IO}}{}^{*} t' \text{ in } \mathcal{M} \text{ and } t' \to^* q' \text{ in } \mathcal{A}' \\ \Longrightarrow \exists e' \,.\, e \to^* q'(e') \text{ in } \mathcal{U}_{n, q_0 \cdots q_p} \text{ and } e'\sigma\rho \xrightarrow{\text{IO}}{}^{*} t' \text{ in } \mathcal{M}' \,. \quad (\stackrel{e}{\Rightarrow})$$

We prove $(\stackrel{e}{\Rightarrow})$ by induction over the term $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n), \mathcal{Y}_p)$.

base case $e = y_j$**,** $1 \le j \le p$**:** The derivation in $\mathcal{M}$ is empty: $e\sigma\rho = t'_j = t'$. Since $\mathcal{A}'$ is deterministic and we have both $t' \to^* q'$ and $t' \to^* q_j$, $q = q_j$.

In $\mathcal{U}_{n, q_0 q_1 \cdots q_p}$, we have $e = y_j \to q_j(y_j) = q'(y_j)$ using $(u_{\mathcal{Y}})$. Finally, $e' \stackrel{\text{def}}{=} y_j = e$ is such that $e'\sigma\rho = t'_j = t'$ and has an empty derivation in $\mathcal{M}'$.

inductive step $e = f(e_1, \ldots, e_m)$**,** $f \in \mathcal{F}'_m$**,** $m \ge 0$**:** The inside-out derivation in $\mathcal{M}$ must be of the form $e\sigma\rho = f(e_1\sigma\rho, \ldots, e_m\sigma\rho) \xrightarrow{\text{IO}}{}^{*} f(t''_1, \ldots, t''_m) = t'$ where $e_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ for all $1 \le k \le m$. The derivation in $\mathcal{A}'$ can similarily be decomposed as $t' = f(t''_1, \ldots, t''_m) \to^* f(q'_1, \ldots, q'_m) \to q'$ with $t''_k \to^* q'_k$ in $\mathcal{A}'$ for every $1 \le k \le m$.

By the ind. hyp. on $(\stackrel{e}{\Rightarrow})$ applied to the e_k, there exists for each $1 \le k \le m$ a term e'_k such that $e_k \to^* q'_k(e'_k)$ in $\mathcal{U}_{n, q_0 q_1 \cdots q_p}$ and $e'_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ in $\mathcal{M}'$. These derivations in $\mathcal{U}_{n, q_0 q_1 \cdots q_p}$ entail that $e = f(e_1, \ldots, e_m) \to^* f(q'_1(e'_1), \ldots, q'_m(e'_m)) \to q'(f(e'_1, \ldots, e'_m))$ using $(u_{\mathcal{F}'})$.

Finally, $e' \stackrel{\text{def}}{=} f(e'_1, \ldots, e'_m)$ is such that $e'\sigma\rho = f(e'_1\sigma\rho, \ldots, e'_m\sigma\rho) \xrightarrow{\text{IO}}{}^{*} f(t''_1, \ldots, t''_m) = t'$ in $\mathcal{M}'$.

inductive step $e = q''(x_i, e_1, \dots, e_m)$**,** $1 \leq i \leq n$**:** the inside-out derivation in $\mathcal{M}$ must be of the form $q''(x_i\sigma, e_1\sigma\rho, \dots, e_m\sigma\rho) \xrightarrow{\text{IO}}{}^{*} q''(x_i\sigma, t''_1, \dots, t''_m) \xrightarrow{\text{IO}}{}^{*} t'$, where $e_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ in $\mathcal{M}$ for all $1 \leq k \leq m$. As $\mathcal{A}'$ is complete, there exist states $(q'_k)_{1\leq k\leq m}$ such that $t''_k \to^* q'_k$ in $\mathcal{A}$ for all $1 \leq k \leq m$.

By the ind. hyp. on ($\overset{e}{\Rightarrow}$) applied to the e_k, there exists for each $1 \leq k \leq m$ a term e'_k such that $e_k \to^* q'_k(e'_k)$ in $\mathcal{U}_{n,q_0q_1\cdots q_p}$ and $e'_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ in $\mathcal{M}'$. These derivations in $\mathcal{U}_{n,q_0q_1\cdots q_p}$ entail that $e = q''(x_i, e_1, \dots, e_m) \to^* q''(q_{\mathcal{X}}(x_i), q'_1(e'_1), \dots, q'_m(e'_m)) \to q'(\langle q'', q'q'_1\cdots q'_m\rangle(x_i, e'_1, \dots, e'_m))$ using $(u_{\mathcal{X}})$ and (u_Q).

There remains to show that $e' \overset{\text{def}}{=} \langle q'', q'q'_1\cdots q'_m\rangle(x_i, e'_1, \dots, e'_m)$ is such that $e'\sigma\rho \xrightarrow{\text{IO}}{}^{*} t'$ in $\mathcal{M}'$: we already have $e'\sigma\rho \xrightarrow{\text{IO}}{}^{*} \langle q'', q'q'_1\cdots q'_m\rangle(x_i\sigma, t''_1, \dots, t''_m)$. As we have seen that $q''(x_i\sigma, t''_1, \dots, t''_m) \xrightarrow{\text{IO}}{}^{*} t'$ and $t' \to^* q'$ in $\mathcal{A}'$, we can apply the ind. hyp. on (†) over $\sigma(x_i)$ to deduce the remaining steps $\langle q'', q'q'_1\cdots q'_m\rangle(x_i\sigma, t''_1, \dots, t''_m) \xrightarrow{\text{IO}}{}^{*} t'$ in $\mathcal{M}'$.

Returning to (†), let now $t = f(t_1, \dots, t_n)$ for some $n \in \mathbb{N}$. Then the inside-out derivation in $\mathcal{M}$ is necessarily of the form

$$q(f(t_1, \dots, t_n), t'_1, \dots, t'_p) \xrightarrow{\text{IO}} e[t_i/x_i]_{1\leq i\leq n}\rho \xrightarrow{\text{IO}}{}^{*} t'_0$$

for some rule $q(f(x_1, \dots, x_n), y'_1, \dots, y'_p) \to e \in \Delta$. Thus, by ($\overset{e}{\Rightarrow}$), there exists e' such that $e \to^* q_0(e')$ in $\mathcal{U}_{n,q_0\cdots q_p}$, i.e. $(e, e') \in [\![\mathcal{U}_{n,q_0\cdots q_p}]\!]$, and $e'[t_i/x_i]_{1\leq i\leq n}\rho \xrightarrow{\text{IO}}{}^{*} t'_0$ in $\mathcal{M}'$. Since Δ' contains $\langle q, q_0q_1\cdots q_p\rangle(f(x_1, \dots, x_n), y_1, \dots, y_p) \to e'$, we obtain the desired derivation in $\mathcal{M}'$.

$\Longleftarrow$ By induction over $t \in T(\mathcal{F})$; assume that $\langle q, q_0\cdots q_p\rangle(t, t'_1, \dots, t'_p) \xrightarrow{\text{IO}}{}^{*} t'_0$ in $\mathcal{M}'$ and $t'_j \to^* q_j$ in $\mathcal{A}'$ for $1 \leq j \leq p$; let $\rho \overset{\text{def}}{=} [t'_j/y_j]_{1\leq j\leq p}$.

Let us first show that, for all $n \in \mathbb{N}$, terms $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n), \mathcal{Y}_p)$ and $e' \in T(\mathcal{F}' \cup Q''(\mathcal{X}_n), \mathcal{Y}_p)$, substitutions σ from $\mathcal{X}_n$ to strict subtrees of t, trees $t' \in T(\mathcal{F}')$, and states $q' \in Q'$,

$$e \to^* q'(e') \text{ in } \mathcal{U}_{n,q_0\cdots q_p} \text{ and } e'\sigma\rho \xrightarrow{\text{IO}}{}^{*} t' \text{ in } \mathcal{M}' \Longrightarrow e\sigma\rho \xrightarrow{\text{IO}}{}^{*} t' \text{ in } \mathcal{M} \text{ and } t' \to^* q' \text{ in } \mathcal{A}' . \quad (\overset{e}{\Leftarrow})$$

We prove ($\overset{e}{\Leftarrow}$) by induction over the term $e \in T(\mathcal{F}' \cup Q(\mathcal{X}_n), \mathcal{Y}_p)$.

base case $e = y_j$**,** $1 \leq j \leq p$**:** The derivation in $\mathcal{U}_{n,q_0\cdots q_p}$ is $e = y_j \to q_j(y_j)$ by $(u_{\mathcal{Y}})$, thus $e' = y_j = e$ and $y_j\sigma\rho = t'_j = t'$. Thus the derivations in $\mathcal{M}$ and $\mathcal{M}'$ coincide and $t'_j \to^* q_j$ in $\mathcal{A}$ by assumption.

inductive step $e = f(e_1, \dots, e_m)$**,** $f \in \mathcal{F}'_m$**,** $m \geq 0$**:** Let us decompose the derivation in $\mathcal{U}_{n,q_0\cdots q_p}$ as $e = f(e_1, \dots, e_m) \to^* f(q'_1(e'_1), \dots, q'_m(e'_m)) \to q'(f(e'_1, \dots, e'_m))$

for some $f(q'_1, \ldots, q'_m) \to q' \in \Delta'$ by $(u_{\mathcal{F}'})$, where $e_k \to^* q'_k(e'_k)$ in $\mathcal{U}_{n,q_0\cdots q_p}$ for each $1 \leq k \leq m$ and $e' = f(e'_1, \ldots, e'_m)$.

The inside-out derivation in $\mathcal{M}'$ is thus necessarily of the form $e'\sigma\rho = f(e'_1\sigma\rho, \ldots, e'_m\sigma\rho) \xrightarrow{\text{IO}}{}^{*} f(t''_1, \ldots, t''_m) = t'$ where $e'_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ for all $1 \leq k \leq m$.

By ind. hyp. on $(\overset{e}{\Leftarrow})$ applied to each e_k for $1 \leq k \leq m$, $e_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ in $\mathcal{M}$ and $t''_k \to^* q'_k$ in $\mathcal{A}'$.

Thus, in $\mathcal{M}$, $e\sigma\rho = f(e_1\sigma\rho, \ldots, e_m\sigma\rho) \xrightarrow{\text{IO}}{}^{*} f(t''_1, \ldots, t''_m) = t'$. Furthermore, in $\mathcal{A}'$, $t' = f(t''_1, \ldots, t''_m) \to^* f(q'_1, \ldots, q'_m) \to q'$.

inductive step $e = q''(x_i, e_1, \ldots, e_m)$, $1 \leq i \leq n$: Let us decompose the derivation in $\mathcal{U}_{n,q_0\cdots q_p}$ as $e = q''(x_i, e_1, \ldots, e_m) \to^* q''(q_{\mathcal{X}}(x_i), q'_1(e'_1), \ldots, q'_m(e'_m)) \to q'(\langle q'', q'q'_1 \cdots q'_m\rangle(x_i, e'_1, \ldots, e'_m))$ using $(u_{\mathcal{X}})$ and (u_Q), where $e_k \to^* q'_k(e'_k)$ in $\mathcal{U}_{n,q_0\cdots q_p}$ for each $1 \leq k \leq m$ and $e' = \langle q'', q'q'_1 \cdots q'_m\rangle(x_i, e'_1, \ldots, e'_m)$.

The inside-out derivation in $\mathcal{M}'$ is thus necessarily of the form $e'\sigma\rho = \langle q'', q'q'_1 \cdots q'_m\rangle(x_i\sigma, e'_1\sigma\rho, \ldots, e'_m\sigma\rho) \xrightarrow{\text{IO}}{}^{*} \langle q'', q'q'_1 \cdots q'_m\rangle(x_i\sigma, t''_1, \ldots, t''_m) \xrightarrow{\text{IO}}{}^{*} t'$ where $e'_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ for all $1 \leq k \leq m$.

By ind. hyp. on $(\overset{e}{\Leftarrow})$ applied to each e_k for $1 \leq k \leq m$, $e_k\sigma\rho \xrightarrow{\text{IO}}{}^{*} t''_k$ in $\mathcal{M}$ and $t''_k \to^* q'_k$ in $\mathcal{A}'$.

Thus, in $\mathcal{M}$, $e\sigma\rho = q''(x_i\sigma, e_1\sigma\rho, \ldots, e_m\sigma\rho) \xrightarrow{\text{IO}}{}^{*} q''(x_i\sigma, t''_1, \ldots, t''_m)$. Furthermore, by ind. hyp. on (†) applied to the strict subtree $\sigma(x_i)$ in the derivation $\langle q'', q'q'_1 \cdots q'_m\rangle(x_i\sigma, t''_1, \ldots, t''_m) \xrightarrow{\text{IO}}{}^{*} t'$ in $\mathcal{M}'$, $q''(x_i\sigma, t''_1, \ldots, t''_m) \xrightarrow{\text{IO}}{}^{*} t'$ in $\mathcal{M}$, and $t' \to^* q'$ in $\mathcal{A}'$.

Returning to (†), let now $t = f(t_1, \ldots, t_n)$ for some $n \in \mathbb{N}$. Then the inside-out derivation in $\mathcal{M}'$ is necessarily of the form

$$\langle q, q_0 \cdots q_p\rangle(f(t_1, \ldots, t_n), t'_1, \ldots, t'_p) \xrightarrow{\text{IO}} e'[t_i/x_i]_{1\leq i\leq n}\rho \xrightarrow{\text{IO}}{}^{*} t'_0$$

for some rule $q(f(x_1, \ldots, x_n), y_1, \ldots, y_p) \to e \in \Delta$ and $(e, e') \in [\![\mathcal{U}_{n,q_0q_1\cdots q_p}]\!]$. Thus by $(\overset{e}{\Leftarrow})$, $q(f(t_1, \ldots, t_n), t'_1, \ldots, t'_p) \xrightarrow{\text{IO}} e[t_i/x_i]_{1\leq i\leq n}\rho \xrightarrow{\text{IO}}{}^{*} t'_0$ in $\mathcal{M}$ and $t'_0 \to^* q_0$ in $\mathcal{A}'$. □

A Optional Exercise

As the title of the section indicates, this homework assignment is too long, but I like the following exercise and you might feel like attempting to solve it.

Exercise 8 (Pre-image computation)**.** Let $\mathcal{M} = (Q, \mathcal{F}, \mathcal{F}', \Delta, I)$ be an NMTT. The purpose of this exercise is to construct an *alternating finite tree automaton* (AFTA) $\mathcal{A}$ such that $L(\mathcal{A}) = [\![\mathcal{M}]\!]^{-1}_{\text{IO}}(T(\mathcal{F}') \stackrel{\text{def}}{=} \{t \in T(\mathcal{F}) \mid \exists t' \in T(\mathcal{F}') \,.\, (t, t') \in [\![\mathcal{M}]\!]_{\text{IO}}\}$.

[2] 1. Have a look at Sec. 7.2.2 of *TATA* on your own. Then show how to construct the AFTA $\mathcal{A}$ (without proof).

   Let $\mathcal{A} \stackrel{\text{def}}{=} (Q, \mathcal{F}, \Delta', I)$ where, for all $n \in \mathbb{N}$, $q \in Q$, and $f \in \mathcal{F}_n$,

$$\Delta'(q, f) \stackrel{\text{def}}{=} \bigvee_{(q(f(x_1,\ldots,x_n),y_1,\ldots,y_p)\to e)\in\Delta} \quad \bigwedge_{\substack{q'(x_i,e_1,\ldots,e_m)\in\text{Subterms}(e)\\ m\in\mathbb{N},q'\in Q_{m+1},1\leq i\leq n}} (q', i)$$

   This automaton checks that for at least one right-hand side e, all the 'calls' to some $q'(x_i, e_1, \ldots, e_m)$ succeed.

2. Theorem 7.4.1 from *TATA* shows how to construct a DFTA equivalent to the
[1] AFTA $\mathcal{A}$. Conclude by giving a proof of the following theorem:

Theorem 1 (Inverse macro transductions)**.** *If $\mathcal{M}$ is an NMTT from $\mathcal{F}$ to $\mathcal{F}'$ and L' a recognisable tree language over $\mathcal{F}'$, then $[\![\mathcal{M}]\!]_{\text{IO}}^{-1}(L') \stackrel{\text{def}}{=} \{t \in T(\mathcal{F}) \mid \exists t' \in L' \,.\, (t, t') \in [\![M]\!]_{\text{IO}}\}$ is a recognisable tree language over $\mathcal{F}$.*

By Exercise 7, we can construct from $\mathcal{M}$ and a complete DFTA $\mathcal{A}'$ for L' an NMTT $\mathcal{M}'$ such that, for all $t \in T(\mathcal{F})$, $\exists t' \in T(\mathcal{F}') \,.\, (t, t') \in [\![\mathcal{M}']\!]_{\text{IO}}$ if and only if $\exists t' \in L' \,.\, (t, t') \in [\![\mathcal{M}]\!]_{\text{IO}}$. Applying the construction from Exercise 8 to $\mathcal{M}'$, we obtain an AFTA $\mathcal{A}$ recognising $[\![\mathcal{M}]\!]_{\text{IO}}^{-1}(L')$. Finally, Thm. 7.4.1 from *TATA* shows that $L(\mathcal{A})$ is a recognisable tree language over $\mathcal{F}$. □