MPRI 2-09-1: Exam
Games That Cannot Go on Forever!

Duration: 2 hours 15 minutes. You may answer in French or in English at your convenience. The numbers in brackets in the margin are indications of length or difficulty.

Fellows and Rosamond (2025) recently proposed a game to be played with children over a wqo. The general setup is a wqo $(X, \leq_X)$ with a “size” function $|\cdot|_X \colon X \to \mathbb{N}$, such that $(X, |\cdot|_X)$ forms a combinatorial class, i.e., for all $n \in \mathbb{N}$, $X_{=n} \stackrel{\text{def}}{=} \{x \in X \mid |x|_X = n\}$ is finite.

Given an initial size $n_0 \in \mathbb{N}$, a *game* is a sequence of elements $x_0, x_1, x_2, \ldots$ from X such that, at each step j,

1. for all $i < j$, $x_i \not\leq_X x_j$, and
2. $|x_j|_X = n_0 + j$.

The game stops before step j if no element x_j satisfying the two conditions exists in X. In the *short* game, the players are attempting to minimise the duration of the game, while in the *long* game, they are trying to maximise it.

Exercise 1. Long games over tuples of natural numbers.

We are interested in this exercise in long games played on the simplest wqo considered by Fellows and Rosamond: the *integer sum games with T channels*, i.e., the games played over $\mathbb{N}^T$ for some finite $T \in \mathbb{N}$, endowed with the product ordering $\leq$, and using the 1-norm defined by $|\vec{u}|_1 \stackrel{\text{def}}{=} \sum_{1 \leq k \leq T} \vec{u}(k)$ for all $\vec{u} \in \mathbb{N}^T$.

[0.5] **(a)** Show that any game played over $\mathbb{N}^T$, $\leq$, and $|\cdot|_1$ that starts with an initial size n_0 is an (amortised) (H, n_0)-controlled bad sequence over the normed wqo $(\mathbb{N}^T, \leq, |\cdot|_\infty)$ (where $|\vec{u}|_\infty \stackrel{\text{def}}{=} \max_{1 \leq k \leq T} \vec{u}(k)$) when using the control function $H(x) \stackrel{\text{def}}{=} x + 1$.

> **Answer:** Given a game $\vec{u}_0, \vec{u}_1, \vec{u}_2, \ldots$ over $\mathbb{N}^T$, for all $i < j$ we have $\vec{u}_i \not\leq u_j$, so this is a bad sequence. Furthermore, $|\vec{u}|_\infty \leq |\vec{u}|_1$ for all $\vec{u} \in \mathbb{N}^T$, hence for all j we have $|\vec{u}_j|_\infty \leq H^j(n_0) = n_0 + j$ as desired.

As seen in class, there is therefore an upper bound on the length of long games with initial size n_0: their length is bounded by $L_{\mathbb{N}^T, H}(n_0)$, which is itself bounded by $h_{\omega^T}(n_0 T)$ in the Cichoń hierarchy for $h(x) \stackrel{\text{def}}{=} x + T$. This function $h_{\omega^T}(n_0 T)$ is a primitive-recursive function in $\mathscr{F}_{T+1}$ for each fixed T (see Theorem 2.8 in the lecture notes). This is a very large upper bound, hence one might wonder whether such long games are indeed possible.

A good candidate in order to obtain a very large lower bound on the length of long games is to play instead over the ordinal ω^T, endowed with the usual ordinal ordering $<$. Any such ordinal $\alpha \in \omega^T$ can be written uniquely as $\alpha = \omega^{T-1} \cdot c_{T-1} + \cdots + \omega^0 \cdot c_0$ in Cantor Normal Form, with $0 \leq c_{T-1}, \ldots, c_0 < \omega$; its size will be $|\alpha|_1 \stackrel{\text{def}}{=} \sum_{0 \leq j < T} c_j$.

[0.5] **(b)** Show that any long game played over ω^T starting with initial size n_0 is also a game played over $\mathbb{N}^T$ with the same initial size, when mapping each $\alpha = \omega^{T-1} \cdot c_{T-1} + \cdots + \omega^0 \cdot c_0 \in \omega^T$ to $\vec{u}(\alpha) \stackrel{\text{def}}{=} (c_{T-1}, \ldots, c_0) \in \mathbb{N}^T$.

Answer: One can check that $\alpha \not\leq \beta$ implies $\vec{u}(\alpha) \not\leq \vec{u}(\beta)$: in such a case, we actually have $\alpha > \beta$ since the order is total, hence there is an index $0 \leq j < T$ such that $\vec{u}(\alpha)(j) > \vec{u}(\beta)(j)$ and indeed $\vec{u}(\alpha) \not\leq \vec{u}(\beta)$. Furthermore, $|\alpha|_1 = |\vec{u}(\alpha)|_1$.

Thus any lower bound on the length of long games over ω^T is also a lower bound on the length of long games over $\mathbb{N}^T$.

Exercise 2. Long games over ordinals in ω^ω.

Let us write $G_\alpha(n_0)$ for the length of the long games played over an ordinal $\alpha \in \omega^\omega$ with size function $|\cdot|_1$ and initial size n_0. Our ultimate goal of this exercise is to establish a lower bound for $G_{\omega^T}(n_0)$.

Note that we have "index monotonicity:" if $\alpha \leq \beta$ are ordinals in ω^ω, then $G_\alpha(n_0) \leq G_\beta(n_0)$ for all initial sizes n_0, as any game over α is also a game over $\beta \supseteq \alpha$.

[1] **(a)** Provide a long game over ω^T in the case $T = 2$ and $n_0 = 3$, with length $G_{\omega^2}(3) = 4$ (no proof required).

Answer: Here is a long game over ω^T in the case $T = 2$ and $n_0 = 3$:

$$\omega \cdot 3 + 0\ ,\ \omega \cdot 2 + 2\ ,\ \omega \cdot 1 + 4\ ,\ \omega \cdot 0 + 6\ .$$

[3] **(b)** **i.** Prove the following *descent equation* for all ordinals $\alpha \in \omega^\omega$ and $n_0 \in \mathbb{N}$:

$$G_\alpha(n_0) = \max_{\beta < \alpha \text{ s.t. } |\beta|_1 = n_0} 1 + G_\beta(n_0 + 1)\ . \tag{1}$$

ii. Which choice of β maximises eq. (1) when $\alpha = \omega^T$?

Answer: Let us fix α, n_0, and β maximising $G_\beta(n_0 + 1)$ as in eq. (1).

Assume $\alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_{\ell-1}$ is a long game played over α with initial size n_0, thus of length $\ell = G_\alpha(n_0)$. Then $\alpha_0 < \alpha$, $|\alpha_0|_1 = n_0$, and for all j, $|\alpha_{j+1}|_1 = (n_0 + 1) + j$ and $\alpha_{j+1} < \alpha_0$, hence $\alpha_1 > \alpha_2 > \cdots$ is a game played over α_0 with initial size $(n_0 + 1)$. Since β maximises the length of such games, this suffix has length $\ell - 1 \leq G_\beta(n + 1)$ and therefore

$$G_\alpha(n_0) \leq 1 + G_\beta(n_0 + 1)\ .$$

Conversely, let $\beta_0 > \beta_1 > \cdots > \beta_{\ell'-1}$ be a long game played over β with initial size $(n_0 + 1)$, thus of length $\ell' = G_\beta(n_0 + 1)$. Then $\beta > \beta_0 > \beta_1 > \cdots > \beta_{\ell'-1}$ is a game played over α with initial size n_0, thus of length $1 + \ell' \leq G_\alpha(n_0)$ and therefore

$$1 + G_\beta(n_0 + 1) \leq G_\alpha(n_0)\ .$$

In the case $\alpha = \omega^T$,

$$\beta \stackrel{\text{def}}{=} \omega^{T-1} \cdot n_0$$

is such that $\beta < \alpha$, $|\beta|_1 = n_0$, and for all $\gamma < \alpha$ with $|\gamma|_1 = n_0$ we have $\gamma \leq \beta$, therefore by index monotonicity of G, $G_\gamma(n_0+1) \leq G_\beta(n_0+1)$: this value of β maximises eq. (1):

$$G_{\omega^T}(n_0) = 1 + G_{\omega^{T-1}\cdot n_0}(n_0+1) \ .$$

Consider any ordinal $\beta \geq \omega$ in ω^ω. It can be written in Cantor Normal Form as $\beta = \gamma + \omega^{d+1} + m$ for some $\gamma \in \omega^\omega$ and $d, m < \omega$. Define its *maximal next ordinal* $\text{maxnext}(\beta)$ by

$$\text{maxnext}(\gamma + \omega^{d+1} + m) \stackrel{\text{def}}{=} \gamma + \omega^d \cdot (m+2) \ . \tag{2}$$

[2] **(c)** **i.** Show that for all $n \in \mathbb{N}$ and $\beta \geq \omega$ in ω^ω such that $|\beta|_1 = n$,

$$G_\beta(n+1) = 1 + G_{\text{maxnext}(\beta)}(n+2) \ . \tag{3}$$

ii. What is $G_\beta(n+1)$ when $|\beta|_1 = n$ and $\beta < \omega$?

Answer: Assume $\beta \geq \omega$ is in ω^ω with $|\beta|_1 = n$. Write β as $\gamma + \omega^{d+1} + m$ for some $\gamma \in \omega^\omega$ and $d, m < \omega$; then $n = |\beta|_1 = |\gamma|_1 + 1 + m$ and $|\text{maxnext}(\beta)|_1 = |\gamma + \omega^d \cdot (m+2)|_1 = |\gamma|_1 + m + 2 = n+1$ and $\text{maxnext}(\beta)$ is the maximal ordinal with this property. By "index monotonicity" of the G_α functions, we have therefore that $\text{maxnext}(\beta)$ maximises $G_\delta(n+2)$ over all choices of $\delta < \beta$ with $|\delta|_1 = n+1$, and therefore by the descent equation (1), $G_\beta(n+1) = 1 + G_{\text{maxnext}(\beta)}(n+2)$.

In the case $\beta < \omega$, $|\beta|_1 = n$ entails $\beta = n$. Then there does not exist any ordinal $\delta < n$ with $|\delta|_1 = n+1$, thus by the descent equation (1), $G_\beta(n+1) = 0$.

A "Cichoń-like" Family of Functions. Equations (2) and (3) are not so convenient, so we define for all $\alpha \in \omega^\omega$ and $x \in \mathbb{N}$

$$E_0(x) \stackrel{\text{def}}{=} 0 \ , \tag{4}$$

$$E_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + E_\alpha(x+2) \ , \tag{5}$$

$$E_\lambda(x) \stackrel{\text{def}}{=} 1 + E_{\lambda(x+1)}(0) \ , \tag{6}$$

where, as in the lecture notes, the assignment of fundamental sequences $\lambda(x)$ is defined as $\gamma + \omega^d \cdot (x+1)$ when $\lambda = \gamma + \omega^{d+1}$ (which is the only case occurring for $\lambda < \omega^\omega$). Recall that ordinals have a "left quotient with remainder" operation, so that $\alpha = \omega \cdot q(\alpha) + r(\alpha)$ for some uniquely defined remainder $r(\alpha) < \omega$ and quotient $q(\alpha)$.

[3] **(d)** Show that $G_\alpha(|\alpha|_1 + 1) = E_{q(\alpha)}(r(\alpha))$ for all $\alpha \in \omega^\omega$.

Answer: We proceed by transfinite induction over $q(\alpha) \in \omega^\omega$. Let $n \stackrel{\text{def}}{=} |\alpha|_1$ in each case.

- If $q(\alpha) = 0$, then $\alpha < \omega$ and $r(\alpha) = \alpha = n$. Then as seen in **(2.c)**, $G_\alpha(n+1) = 0 \stackrel{(4)}{=} E_0(r(\alpha))$.

- If $q(\alpha) = \beta + 1$ is a successor ordinal, then $\alpha = \omega \cdot \beta + \omega + r(\alpha)$ and $\text{maxnext}(\alpha) \overset{(2)}{=} \omega \cdot \beta + \omega^0 \cdot (r(\alpha)+2)$ thus $G_\alpha(n+1) \overset{(3)}{=} 1 + G_{\text{maxnext}(\alpha)}(n+2) \overset{\text{i.h.}}{=} 1 + E_\beta(r(\alpha)+2) \overset{(5)}{=} E_{\beta+1}(r(\alpha))$.
- If $q(\alpha) = \lambda = \gamma + \omega^{d+1}$ is a limit ordinal, then $\alpha = \omega \cdot \lambda + r(\alpha) = \omega \cdot \gamma + \omega^{d+2} + r(\alpha)$. Thus $\text{maxnext}(\alpha) \overset{(2)}{=} \omega \cdot \gamma + \omega^{d+1} \cdot (r(\alpha)+2) = \omega \cdot (\gamma + \omega^d \cdot (r(\alpha)+2))$ and $G_\alpha(n+1) \overset{(3)}{=} 1 + G_{\text{maxnext}(\alpha)}(n+2) \overset{\text{i.h.}}{=} 1 + E_{(\gamma+\omega^d \cdot (r(\alpha)+2))}(0) \overset{(6)}{=} E_\lambda(0)$.

A "Hardy-like" Family of Functions. Equation (6) is still not quite "Cichoń-like" and we would also prefer working with a "Hardy-like" definition. Consider the functions defined for $\alpha \in \omega^\omega$ and $x \in \mathbb{N}$ by

$$E^0(x) \overset{\text{def}}{=} x \; , \tag{7}$$

$$E^{\alpha+1}(x) \overset{\text{def}}{=} E^\alpha(x+2) \; , \tag{8}$$

$$E^\lambda(x) \overset{\text{def}}{=} E^{\lambda(x+1)}(0) \tag{9}$$

again with the usual assignment of fundamental sequences (so that $(\gamma + \omega^{d+1})(x+1) = \gamma + \omega^d \cdot (x+2)$).

[1.5] **(e)** Show that, for all $\alpha \in \omega^\omega$ and $x \in \mathbb{N}$, $E^\alpha(x) = E_\alpha(x) + x + |\alpha|_1$.

Answer: By transfinite induction over α.

- For the zero case, $E^0(x) \overset{(7)}{=} x = 0 + x + 0 \overset{(4)}{=} E_0(x) + x + |0|_1$.
- For the successor case, $E^{\alpha+1}(x) \overset{(8)}{=} E^\alpha(x+2) \overset{\text{i.h.}}{=} E_\alpha(x+2) + x + 2 + |\alpha|_1 = 1 + E_\alpha(x+2) + x + |\alpha|_1 + 1 \overset{(5)}{=} E_{\alpha+1}(x) + x + |\alpha+1|_1$.
- For the limit case $\lambda = \gamma + \omega^{d+1}$, $E^\lambda(x) \overset{(9)}{=} E^{\gamma+\omega^d \cdot (x+2)}(0) \overset{\text{i.h.}}{=} E_{\gamma+\omega^d \cdot (x+2)}(0) + 0 + |\gamma|_1 + x + 2 = 1 + E_{\gamma+\omega^d \cdot (x+2)}(0) + x + |\gamma|_1 + 1 \overset{(6)}{=} E_\lambda(x) + x + |\gamma + \omega^{d+1}|_1$.

We will use the following facts about these functions: if $x \leq y$, then for all $\alpha \in \omega^\omega$

$$E^\alpha(x) \leq E^\alpha(y) \tag{10}$$

and if $\alpha + \beta$ is a term in Cantor Normal Form, then for all x

$$E^{\alpha+\beta}(x) = E^\alpha\big(E^\beta(x)\big) \; . \tag{11}$$

[2] **(f)** Show that $E^{\omega^d \cdot x}(0) \geq 2x$ for all $d, x \in \mathbb{N}$.

Answer: By recurrence over $d \in \mathbb{N}$.

- Base case $d = 0$: $E^x(0) = 2x$ by eq. (7) when $x = 0$ and by a straightforward recurrence over x using eq. (8) otherwise.
- Recurrence step $d + 1$: by recurrence over x.

- If $x = 0$ then $E^{\omega^{d+1}\cdot 0}(0) \overset{(7)}{=} 0 = 2x$.
- Otherwise, if $x > 0$, $E^{\omega^{d+1}\cdot x}(0) \overset{(11)}{=} E^{\omega^{d+1}}\big(E^{\omega^{d+1}\cdot(x-1)}(0)\big)$. By recurrence hypothesis on $x-1 < x$, $E^{\omega^{d+1}\cdot(x-1)}(0) \geq 2x-2$, hence by eq. (10) $E^{\omega^{d+1}}\big(E^{\omega^{d+1}\cdot(x-1)}(0)\big) \geq E^{\omega^{d+1}}(2x-2) \overset{(9)}{=} E^{\omega^d\cdot(2x)}(0)$. By recurrence hypothesis on $d < d+1$, the latter is $\geq 4x \geq 2x$.

[2] **(g)** Recall that the Hardy functions relativised to the successor function $H(x) \overset{\text{def}}{=} x+1$ are defined through $H^0(x) \overset{\text{def}}{=} x$, $H^{\alpha+1}(x) \overset{\text{def}}{=} H^\alpha(x+1)$, and $H^\lambda(x) \overset{\text{def}}{=} H^{\lambda(x)}(x)$. Show that $E^\alpha(2x) \geq H^\alpha(x)$ for all $\alpha \in \omega^\omega$ and $x \in \mathbb{N}$.

Answer: By transfinite induction over α.

- For the zero case, $E^0(2x) \overset{(7)}{=} 2x \geq x = H^0(x)$.
- For the successor case, $E^{\alpha+1}(2x) \overset{(8)}{=} E^\alpha(2x+2) \overset{\text{i.h.}}{\geq} H^\alpha(x+1) = H^{\alpha+1}(x)$.
- For the limit case $\lambda = \gamma + \omega^{d+1}$, we have

$$\begin{aligned}
E^{\gamma+\omega^{d+1}}(2x) &\overset{(9)}{=} E^{\gamma+\omega^d\cdot(x+1)+\omega^d\cdot(x+1)}(0)\\
&\overset{(11)}{=} E^{\gamma+\omega^d\cdot(x+1)}\big(E^{\gamma+\omega^d\cdot(x+1)}(0)\big)\\
&\overset{(10)}{\geq} E^{\gamma+\omega^d\cdot(x+1)}(2x) && \text{as } E^{\gamma+\omega^d\cdot(x+1)}(0) \overset{\textbf{(2.f)}}{\geq} 2x+2 \geq 2x\\
&\overset{\text{i.h.}}{\geq} H^{\gamma+\omega^d\cdot(x+1)}(x)\\
&= H^\lambda(x)\;.
\end{aligned}$$

[1.5] **(h)** Recall that one may define the Ackermann function as $\text{Ack}(n) \overset{\text{def}}{=} H^{\omega^{n+1}}(n)$. In order to conclude, show that

$$\text{Ack}(n) - 2n - 1 \leq G_{\omega^T}(n_0)$$

when setting $T \overset{\text{def}}{=} n+2$ and $n_0 \overset{\text{def}}{=} 2n+2$; this is also a lower bound for the length of long integer sum games with T channels and initial size n_0 by Exercise **(1.b)**.

Answer: We have

$$\begin{aligned}
\text{Ack}(n) - 2n - 1 &= H^{\omega^{n+1}}(n) - 2n - 1\\
&\overset{\textbf{(2.g)}}{\leq} E^{\omega^{n+1}}(2n) - 2n - 1\\
&\overset{\textbf{(2.e)}}{=} E_{\omega^{n+1}}(2n)\\
&\overset{(6)}{=} 1 + E_{\omega^n\cdot(2n+2)}(0)\\
&\overset{\textbf{(2.d)}}{=} 1 + G_{\omega^{n+1}\cdot(2n+2)}(2n+3)\\
&\overset{\textbf{(2.b)}}{=} G_{\omega^{n+2}}(2n+2)\;.
\end{aligned}$$

Exercise 3. Short games over tuples of natural numbers.

Let us return to $\mathbb{N}^T$ with the product ordering and 1-norm. Let us write $S_X(n_0)$ for the length of the short games played over a subset $X \subseteq \mathbb{N}^T$ with starting size n_0. By a reasoning analogous to Question **(2.b)**, we have a descent equation

$$\begin{aligned} S_X(n_0) &= 0 && \text{if } X_{=n_0} = \emptyset\ , \\ S_X(n_0) &= \min_{\vec{u}\in X_{=n_0}} 1 + S_{X\setminus\uparrow\vec{u}}(n_0+1) && \text{otherwise}\ . \end{aligned} \tag{12}$$

[2] **(a)** Consider the family of vectors $(\vec{u}_k)_{1\leq k\leq T}$ defined by $\vec{u}_k(k) \stackrel{\text{def}}{=} n_0+k-1$ and $\vec{u}_k(i) \stackrel{\text{def}}{=} 0$ for $i \neq k$. The set $(\cdots((\mathbb{N}^T \setminus \uparrow\vec{u}_1) \setminus \uparrow\vec{u}_2)\cdots) \setminus \uparrow\vec{u}_T$ is a downwards-closed subset of $\mathbb{N}^T$. What is its ideal decomposition?

Answer: In general, in order to compute an ideal decomposition for $D \setminus \uparrow\vec{u}_k$ where D is a downwards-closed subset of $\mathbb{N}^T$, it suffices to compute one for $D_k \stackrel{\text{def}}{=} \mathbb{N}^T \setminus \uparrow\vec{u}_k$ and then to intersect with D.

Observe that $D_k = (\downarrow\vec{v}_k) \cap \mathbb{N}^T$ where $\vec{v}_k(k) \stackrel{\text{def}}{=} n_0+k-2$ and $\vec{v}_k(i) = \omega$ for $i \neq k$. Hence the decomposition we are looking for is $\mathbb{N}^T \cap D_1 \cap \cdots \cap D_T = (\downarrow\vec{v}) \cap \mathbb{N}^T$ where $\vec{v}$ is a (finite) vector defined by $\vec{v}(i) \stackrel{\text{def}}{=} n_0+i-2$ for all $1 \leq i \leq T$.

[1] **(b)** Show that $S_{\mathbb{N}^T}(n_0) \leq T + (n_0+T)^T$.

Answer: Here is a strategy ensuring the game uses at most that many steps.

1. For the first $1 \leq k \leq T$ steps, choose $\vec{u}_k$ in the descent equation (12). Then $S_{\mathbb{N}^T}(n_0) \leq T + S_X(n_0+T)$ where X is the downwards-closed set of the previous question.
2. By the previous question, X is a *finite* set, of cardinal $|X| \leq \prod_{k=0}^{T-1}(n_0-2+k) \leq (n_0+T)^T$, hence $S_X(n_0+T) \leq (n_0+T)^T$.

References

Michael R. Fellows and Frances A. Rosamond. Games that cannot go on forever! Active participation in research is the main issue for kids. In *Procedings of the 7th International Conference on Creative Mathematical Sciences Communication (CMSC 2024)*, volume 15229 of *Lecture Notes in Computer Science*, pages 15–35. Springer, 2025. doi:10.1007/978-3-031-73257-7_2.

Extras

…*Finished already? If you are bored, you might as well prove equations* (11) *and* (10).

Answer: Let us prove eq. (11) by transfinite induction over β: for all x,

- for the zero case, $E^{\alpha+0}(x) = E^{\alpha}(x) \stackrel{(7)}{=} E^{\alpha}\big(E^0(x)\big)$;

- for the successor case, $E^{\alpha+\beta+1}(x) \stackrel{(8)}{=} E^{\alpha+\beta}(x+2) \stackrel{\text{i.h.}}{=} E^{\alpha}\big(E^{\beta}(x+2)\big) \stackrel{(8)}{=} E^{\alpha}\big(E^{\beta+1}(x)\big)$;
- for the limit case, $E^{\alpha+\lambda}(x) \stackrel{(9)}{=} E^{\alpha+\lambda(x+1)}(0) \stackrel{\text{i.h.}}{=} E^{\alpha}\big(E^{\lambda(x+1)}(0)\big) \stackrel{(9)}{=} E^{\alpha}\big(E^{\lambda}(x)\big)$.

Let us prove eq. (10) by transfinite induction over α: for all $y \geq x$,

- for the zero case, $E^0(y) \stackrel{(7)}{=} y \geq x \stackrel{(7)}{=} E^0(x)$;
- for the successor case, $E^{\alpha+1}(y) \stackrel{(8)}{=} E^{\alpha}(y+2) \stackrel{\text{i.h.}}{\geq} E^{\alpha}(x+2) \stackrel{(8)}{=} E^{\alpha+1}(x)$;
- for the limit case, let $z \stackrel{\text{def}}{=} y - x$, then

$$\begin{aligned}
E^{\gamma+\omega^{d+1}}(y) &\stackrel{(9)}{=} E^{\gamma+\omega^{d}\cdot(y+2)}(0)\\
&\stackrel{(11)}{=} E^{\gamma+\omega^{d}\cdot(x+2)}\big(E^{\omega^d\cdot z}(0)\big)\\
&\stackrel{\text{i.h.}}{\geq} E^{\gamma+\omega^{d}\cdot(x+2)}(0) && \text{because } E^{\omega^d\cdot z}(0)\geq 0\\
&\stackrel{(9)}{=} E^{\gamma+\omega^{d+1}}(x)\;.
\end{aligned}$$