RAMDOM REALS AND POSSIBLY INFINITE COMPUTATIONS.
PART I: RANDOMNESS IN $\emptyset'$

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Abstract

Using possibly infinite computations on universal monotone Turing machines, we prove Martin-Löf randomness in $\emptyset'$ of the probability that the output be in some set $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ under complexity assumptions about $\mathcal{O}$.

1 Randomness in the spirit of Rice's theorem for computability

Let $\mathbf{2}^*$ be the set of all finite strings in the binary alphabet $\mathbf{2} = \{0, 1\}$.
Let $\mathbf{2}^\omega$ be the set of all infinite binary sequences. For $\mathcal{X} \subseteq \mathbf{2}^\omega$ the Lebesgue set-theoretic measure of $\mathcal{X}$ is denoted by $\mu(\mathcal{X})$. For a particular string $s \in \mathbf{2}^*$, $\mu(s\mathbf{2}^\omega) = 2^{-|s|}$. If $X \subseteq \mathbf{2}^*$ is a prefix-free set then $\mu(X\mathbf{2}^\omega) = \sum_{s\in X} 2^{-|s|} \leq 1$.
As usual (cf.[25] p.451), $\emptyset^{(n)}$ denotes the n-th jump of $\emptyset$, which is a Σ^0_n complete set of integers.

1.1 A problem about randomness and finite computations

Randomness will mean Martin-Löf randomness (relative to possible oracles), which is equivalent to the definition of randomness given by the theory prefix-free program-size complexity. In this theory one considers Turing machines with prefix-free domains and a particular notion of universality: U is universal by "prefix adjunction" if for every Turing machine M with prefix-free domain, there is a word $\mathtt{e}$ such that,

$$\forall p \in \mathbf{2}^* \ (M(p) \text{ halts } \Leftrightarrow \ (U(\mathtt{e}p) \text{ halts and } M(p) = U(\mathtt{e}p)))$$

All along the paper, U denotes a machine universal by prefix adjunction. As pointed by Chaitin ([11], p.109), his randomness results do rely on the fact that U is universal by prefix-adjunction.
Chaitin [9, 11] introduces, for every subset $\mathcal{O}$ of $\mathbf{2}^*$, the real

$$\Omega_U[\mathcal{O}] = \mu(U^{-1}(\mathcal{O})\mathbf{2}^\omega) = \sum_{p \in U^{-1}(\mathcal{O})} 2^{-|p|}$$

which is the probability that, on an *infinite* input, U halts in finite time (reading only finitely many symbols) and produces an output in $\mathcal{O}$.
Chaitin's celebrated result [9], 1975, states that $\Omega = \Omega_U[\mathbf{2}^*]$ is a random real. Chaitin also proves randomness in the case $\mathcal{O}$ is Σ^0_1 ([11], 1987, stated without proof in last assertion of Note p.141). A proof is given in §6.2 below.

Theorem 1.1 (Chaitin, 1987). *If $\mathcal{O}$ is a non empty recursively enumerable subset of $\mathbf{2}^*$ then $\Omega_U[\mathcal{O}]$ is a random real.*

A somewhat surprising corollary of the randomness of Ω is the following.

Corollary 1.2. *There exists a recursive prefix-free set $X \subset \mathbf{2}^*$ such that $\mu(X\mathbf{2}^\omega)$ is a random real.*

Proof. Observe that $\Omega = \mu(domain(U)\mathbf{2}^\omega)$ where $domain(U)$ is recursively enumerable. Conclude using Prop.2.2. □

In the spirit of Rice's theorem for computability, a naive conjecture would state that $\Omega[\mathcal{O}]$ is random for every non empty subset $\mathcal{O}$ of $\mathbf{2}^*$. However, this has been recently disproved by Joe Miller [22] for some Δ^0_2 sets $\mathcal{O}$.

Theorem 1.3 (Miller, 2004). *There exists a non empty Δ^0_2 set $\mathcal{O} \subset \mathbf{2}^*$ such that $\Omega_U[\mathcal{O}]$ is not random. Moreover, such an $\Omega_U[\mathcal{O}]$ can even be a rational number.*

The case where $\mathcal{O}$ is Π^0_1 is still open. This leaves out the following general problem.

Problem 1.4 (Finite computations). *Find conditions on $\mathcal{O} \subseteq \mathbf{2}^*$ in order that $\Omega_U[\mathcal{O}]$ be random (resp. random in $\emptyset^{(n)}$).*

1.2 A problem about randomness and possibly infinite computations

Investigations on prefix program-size complexity with possibly infinite computations with a *monotone Turing machine U universal by prefix adjunction* (cf. Def.4.1) have been initiated by Chaitin [10] and Solovay [32] and continued in [2, 1]. Since the output may be a finite string or a recursive infinite sequence, the output space is $\mathbf{2}^{\leq\omega} = \mathbf{2}^* \cup \mathbf{2}^\omega$. As for the input space, we can consider either self-delimited finite inputs or infinite inputs (cf. Def.4.7). This leads to maps

$$U_{\bowtie} : \mathbf{2}^* \rightarrow \mathbf{2}^{\leq\omega} \quad , \quad U_\infty : \mathbf{2}^\omega \rightarrow \mathbf{2}^{\leq\omega}$$

Of course, the range of $U_{\bowtie}$ is included in $\mathbf{2}^* \cup Rec(\mathbf{2}^\omega)$ where $Rec(\mathbf{2}^\omega)$ is the set of recursive infinite sequences.
Using such machines, one can consider, for every $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$, the sets

$$U_{\bowtie}^{-1}(\mathcal{O}) = U_{\bowtie}^{-1}(\mathcal{O} \cap (\mathbf{2}^* \cup Rec(\mathbf{2}^\omega))) \quad , \quad U_\infty^{-1}(\mathcal{O})$$

The real $\mu(U_\infty^{-1}(\mathcal{O}))$ (resp. $\mu(U_{\bowtie}^{-1}(\mathcal{O})\mathbf{2}^\omega) = \sum_{p \in U_{\bowtie}^{-1}(\mathcal{O})} 2^{-|p|}$) is the probability that, on an *infinite* input, the machine produces an output in $\mathcal{O}$ (resp. and reads only a finite prefix of the input).
Randomness results using $U_{\bowtie}$ have been obtained by Becher, Daicz, Chaitin in [2], 2001, and Becher & Chaitin in [1], 2002.

Theorem 1.5 (Becher, Daicz & Chaitin, 2001). *The probability that the computation reads finitely many symbols of an infinite input and produces a finite output, i.e. $\mu(U_{\bowtie}^{-1}(\mathbf{2}^*)\mathbf{2}^\omega)$, is random in $\emptyset'$.*

Identifying the word 10^n1 with the integer n, we associate to any infinite word α the set $\theta(\alpha)$ of n's such that 10^n1 is a factor of α. Let COF be the set of infinite words such that $\theta(\alpha)$ is cofinite.

Theorem 1.6 (Becher & Chaitin, 2002). *The probability that the computation reads finitely many symbols of an infinite input and produces (via θ) a cofinite set of integers, i.e. $\mu(U_{\bowtie}^{-1}(COF)\mathbf{2}^\omega)$, is random in $\emptyset''$.*

As pointed to us by the referee, a simple application of a classical result, due to Sacks, gives a *non randomness* result for some $\mu(U_\infty^{-1}(\mathcal{O}))$'s.

Proposition 1.7. *Let $\mathcal{O}$ be a countable family of non recursive elements of $\mathbf{2}^\omega$. Then $\mu(U_\infty^{-1}(\mathcal{O})) = 0$ (hence is not random).*

Proof. Observe that $\beta = U_\infty(\alpha)$ is recursive in α. By Sacks's result (cf. [26] p.272 or [27] p.154), $\mu\{\alpha : \beta \text{ is recursive in } \alpha\} = 0$ if β is non recursive. $\square$

Let's state the analog of Problem 1.4 for possibly infinite computations, i.e. halting or non-halting computations.

Problem 1.8. *1.* (Possibly infinite computations on self-delimited finite inputs) *Find conditions on $\mathcal{O} \subseteq \mathbf{2}^* \cup Rec(\mathbf{2}^\omega)$ in order that $\mu(U_{\bowtie}^{-1}(\mathcal{O})\mathbf{2}^\omega)$ be random (resp. random in $\emptyset^{(n)}$).*
2. (Possibly infinite computations on infinite inputs) *Find conditions on $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ in order that $\mu(U_\infty^{-1}(\mathcal{O}))$ be random (resp. random in $\emptyset^{(n)}$).*

1.3 Main theorems

In this paper we present three theorems which give positive answers to Problem 1.8 for large classes of sets. They rely on diverse notions and tools that are recalled and/or developed in §2–5. The proofs are postponed to §6 (cf. 6.3–6.5). They share the same pattern which is that of the proof of an abstract theorem presented in §6.1. Applications of these theorems are stated in §1.4 and proved in §7.
The first theorem (Thm.1.9) deals with the map U_∞ and plain randomness (as opposed to randomness in $\emptyset'$). The two last theorems, the main ones, deal with the respective maps $U_{\bowtie}$ and U_∞ and randomness in $\emptyset'$.

Theorem 1.9. *Let $\mathcal{O} = Y\mathbf{2}^{\leq\omega}$ for some r.e. set $Y \subseteq \mathbf{2}^*$. If $\mathcal{O} \neq \emptyset$ and $\mathcal{O} \neq \mathbf{2}^{\leq\omega}$ then the real $\mu(U_\infty^{-1}(\mathcal{O}))$ is random.*

Theorem 1.10 (1st main theorem). *Suppose $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ contains a finite string or an infinite recursive sequence. If $U_{\bowtie}^{-1}(\mathcal{O})$ is Σ_2^0 definable in $\mathbf{2}^*$ then the real $\mu(U_{\bowtie}^{-1}(\mathcal{O})\mathbf{2}^\omega)$ is random in $\emptyset'$.*

The key condition in the second main theorem is a hardness condition relative to what we call semicomputable Wadge semireduction. This is an appropriate variant of classical Wadge reduction based on the topological properties of the maps associated to Turing machines performing possibly infinite computations (cf. §5 and the forthcoming paper [4]). As studied in [5], these maps – which we call semicomputable maps – are not continuous but merely lower semicontinuous. Based on the effectivization of lower semicontinuous maps, we introduce *semicomputable Wadge semireductions* of sets in $\mathbf{2}^\omega$ to sets in $\mathbf{2}^{\leq\omega}$.

Theorem 1.11 (2nd main theorem). *Let $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ satisfy the following conditions:*

- $\mathcal{O}$ is semicomputably Wadge hard for Σ^0_2 subsets of $\mathbf{2}^\omega$ (cf. Def.5.5),

- $U_\infty^{-1}(\mathcal{O}) \subseteq \mathbf{2}^\omega$ is Σ^0_2 definable in $\mathbf{2}^\omega$.

Then $\mu(U_\infty^{-1}(\mathcal{O}))$ and $\mu(U_\infty^{-1}(\mathbf{2}^{\leq\omega} \setminus \mathcal{O}))$ are random in $\emptyset'$.

The following simple remarks and proposition stress the role of some of the hypothesis and delimitate the scope of the above theorems. Randomness in Thm.1.9 cannot be improved to randomness in $\emptyset'$, cf. the following easy result (proved in §7.3).

Proposition 1.12. *Suppose $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ is of the form $\mathcal{O} = X \cup Y\mathbf{2}^{\leq\omega}$ where $X, Y \subseteq \mathbf{2}^*$ are Σ^0_1 and X is the union of finitely many prefix-free sets. Then $\mu(U_\infty^{-1}(\mathcal{O}))$ is not random in $\emptyset'$.*

Remark 1.13 *(*About Thm.1.10). 1. If $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ contains no finite string nor any infinite recursive sequence then $U_\bowtie^{-1}(\mathcal{O}) = \emptyset$.
2. We suppose that $U_\bowtie^{-1}(\mathcal{O})$ is Σ^0_2 in $\mathbf{2}^*$ to insure that $\mu(U_\bowtie^{-1}(\mathcal{O})\mathbf{2}^\omega)$ is left c.e. in $\emptyset'$, which is a key point in the proof of randomness.
3. Nothing can be stated about $\mu(U_\bowtie^{-1}(\mathbf{2}^{\leq\omega} \setminus \mathcal{O})\mathbf{2}^\omega)$, contrary to what is done in Thm.1.11. The reason is that $\mu(U_\infty^{-1}(\mathbf{2}^{\leq\omega})) = 1$ but $\mu(U_\bowtie^{-1}(\mathbf{2}^{\leq\omega})\mathbf{2}^\omega) \neq 1$ since it is, in fact, random in $\emptyset'$ as we shall see in Corollary 1.17. The stumbling block is that, up to now, there is no known general technique to deal with differences of random reals.

Remark 1.14 *(*About Thm.1.11). 1. We suppose that $U_\infty^{-1}(\mathcal{O})$ is Σ^0_2 definable in $\mathbf{2}^\omega$ to insure that $\mu(U_\infty^{-1}(\mathcal{O}))$ is left c.e. in $\emptyset'$, which is again a key point in the proof of randomness.
2. The hypothesis $\mathcal{O} \cap (\mathbf{2}^* \cup Rec(\mathbf{2}^\omega)) \neq \emptyset$ in Thm.1.10 is much weaker than that of Σ^0_2-hardness in Thm.1.11 and is not sufficient for Thm.1.11, cf. Prop.1.12 above.

Thm.1.16 below (proved in §5.4.) gives quite simple topological conditions (cf. Def.1.15) on a given set $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ which are sufficient to prove the key semicomputable Wadge Σ^0_2-hardness condition in Thm.1.11.

Definition 1.15. *1. $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ satisfies condition $(*)$ if there exists a recursive increasing chain of words (with respect to the prefix ordering) in $\mathcal{O}$, the*

limit of which is not in $\mathcal{O}$.
2. $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ satisfies condition $(**)$ if there exists $u \in \mathbf{2}^*$ such that $\mathcal{O}$ is effectively dense for $u\mathbf{2}^*$ in $\mathbf{2}^{\leq\omega}$ and effectively codense for $u\mathbf{2}^*$ in $\mathbf{2}^\omega$, i.e. if there exist total computable maps $F : \mathbf{2}^* \to \mathcal{O}$ and $G : \mathbf{2}^* \to \mathbf{2}^\omega \setminus \mathcal{O}$ such that, for all $v \in \mathbf{2}^*$, uv is a prefix of $F(v)$ and $G(v)$.

Theorem 1.16. *If $\mathcal{O}$ satisfies $(*)$ or $\mathcal{O}$ is Σ^0_2 in $\mathbf{2}^{\leq\omega}$ and satisfies $(**)$ then $\mathcal{O}$ is semicomputably Wadge hard for Σ^0_2 subsets of $\mathbf{2}^\omega$.*

1.4 Applications of the main theorems on $\emptyset'$-randomness

The main theorems dealing with $\emptyset'$-randomness have diverse applications, the proofs of which are given in §7. First, an application of Thm.1.10.

Corollary 1.17. *Let $\mathcal{O} = X \cup Y\mathbf{2}^{\leq\omega}$ for some Σ^0_2 sets $X, Y \subseteq \mathbf{2}^*$ such that $X \cup Y \neq \emptyset$. Then $\mu(U_{\bowtie}^{-1}(\mathcal{O})\mathbf{2}^\omega)$ is random in $\emptyset'$.*
In particular, letting $\mathcal{O} = \mathbf{2}^{\leq\omega}$, we see that the probability $\mu(U_{\bowtie}^{-1}(\mathbf{2}^{\leq\omega})\mathbf{2}^\omega) = \mu(domain(U_{\bowtie})\mathbf{2}^\omega)$ that an infinite word contains some self-delimited prefix is random in $\emptyset'$.

Remark 1.18. The set of self-delimited inputs in the prefix-free domain of a universal machine relative to halting computations is a recursively enumerable set (cf. the proof of Cor.1.2). However, the set $U_{\bowtie}^{-1}(\mathbf{2}^{\leq\omega})$ of self-delimited inputs relative to infinite computations of a universal prefix-free machine is merely $\Sigma^0_1 \wedge \Pi^0_1$, cf. Prop.4.9. In fact, this set cannot be not r.e. since then its associated measure would be left c.e., hence recursive in in $\emptyset'$, which is not the case since it is random in $\emptyset'$.

Thm.1.11 and Thm.1.16 have the following corollary which answers a question raised by An.A. Muchnik [24]).

Corollary 1.19. *1. Let $\mathcal{O} = X \cup Y\mathbf{2}^{\leq\omega}$ for some Σ^0_2 sets $X, Y \subseteq \mathbf{2}^*$. Suppose $\mathcal{O}$ satisfies one of the two conditions $(*)$ or $(**)$ described in Def.1.15. Then $\mu(U_\infty^{-1}(\mathcal{O}))$ and $\mu(U_\infty^{-1}(\mathbf{2}^{\leq\omega} \setminus \mathcal{O}))$ are random in $\emptyset'$.*
In particular, letting $\mathcal{O} = \mathbf{2}^*$, the probability $\mu(U_\infty^{-1}(\mathbf{2}^\omega))$ (resp. $\mu(U_\infty^{-1}(\mathbf{2}^*))$) that the output is infinite (resp. finite) is random in $\emptyset'$.

A direct corollary of Thm.1.11 dealing with Π^0_2 sets, but not contained in Cor.1.19, is as follows.

Corollary 1.20. *Let $\mathcal{O} = X \cup Y\mathbf{2}^{\leq\omega} \cup \mathcal{Z}$ where $X, Y \subseteq \mathbf{2}^*$ are Σ^0_1 subsets of $\mathbf{2}^*$ and X is the union of finitely many prefix-free sets and $\mathcal{Z}$ is a Π^0_2 subset of $\mathbf{2}^\omega$. Suppose $Y\mathbf{2}^\omega \cup \mathcal{Z}$ is semicomputably Wadge hard for Π^0_2 subsets of*

$\mathbf{2}^\omega$. *Then* $\mu(U_\infty^{-1}(\mathcal{O}))$ *and* $\mu(U_\infty^{-1}(\mathbf{2}^{\leq\omega} \setminus \mathcal{O}))$ *are random in* $\emptyset'$.
In particular, letting $\mathcal{Z} = \mathbf{2}^\omega \setminus \mathbf{2}^*0^\omega$ *and* $X = Y = \emptyset$, *the probability* $\mu(U_\infty^{-1}(\mathbf{2}^\omega \setminus \mathbf{2}^*0^\omega))$ *that the output contains infinitely many* 1*'s is random in* $\emptyset'$.

1.5 Higher order randomness and possibly infinite computations

The main theorems on randomness in $\emptyset'$ proved in this paper suggest that adding a hardness condition on $\mathcal{O}$ relative to semicomputable Wadge semireduction leads to randomness in the successive jumps. Such extensions of Thm.1.10 and Thm.1.11 obtained by replacing all Σ_2^0 assumptions by their Σ_n^0 analogs are considered in a forthcoming paper [3].

2 From $\mathbf{2}^\omega$ to $\mathbf{2}^{\leq\omega}$ topological spaces

We consider on $\mathbf{2}^\omega$ the usual compact Cantor topology generated by the countable family of basic open sets $s\mathbf{2}^\omega$ where s varies over $\mathbf{2}^*$. If $X \subseteq \mathbf{2}^*$ then $X\mathbf{2}^\omega$ denotes the open subset of $\mathbf{2}^\omega$ whose elements have an initial segment in X. If $\alpha \in \mathbf{2}^\omega$ we denote by $\alpha\restriction n$ the prefix of α of length n. For $a \in \mathbf{2}^*$, $|a|$ denotes the length of a. The empty string is denoted by λ. If $a \in \mathbf{2}^*$, $a\restriction n$ is the prefix of a with length $\min(n, |a|)$. We assume the prefix ordering $\preceq$ in $\mathbf{2}^*$, and we write $a \preceq b$ if a is a prefix of b, and $a \prec b$ if a is a proper prefix of b.

2.1 Prefix-free sets

$X \subseteq \mathbf{2}^*$ is *prefix-free* if and only if no proper extension of an element of X belongs to X. We denote by $\min(X)$ the prefix-free set consisting of all minimal elements of X with respect to the prefix ordering $\preceq$. A prefix-free set $X \subset \mathbf{2}^*$ is *maximal* iff for any $a \notin X$, $X \cup \{a\}$ is not prefix-free. For example, the sets $\{\lambda\}$ and $\{0^n1 : n \geq 0\}$ are both maximal prefix-free.
If $X \subset \mathbf{2}^*$ is prefix-free and every sequence $\alpha \in \mathbf{2}^\omega$ has an initial segment in X then X is maximal and $\sum_{a\in X} 2^{-|a|} = 1$. The converse is not true: 1^*0 is maximal prefix-free and $\sum_{a\in\{1\}^*0} 2^{-|a|} = 1$ but 1^*0 contains no prefix of the sequence 1^ω. In fact, a simple application of König's Lemma proves that finiteness is required.

Proposition 2.1. *Let* $X \subseteq \mathbf{2}^*$. *Then* $X\mathbf{2}^\omega = \mathbf{2}^\omega$, *if and only if* X *contains a finite maximal prefix-free set. In particular, if* X *is prefix-free then* $X\mathbf{2}^\omega = \mathbf{2}^\omega$ *if and only if* X *is finite and maximal prefix-free.*

Proof. The $\Leftarrow$ direction is easy. For the $\Rightarrow$ direction, suppose $X \subseteq \mathbf{2}^*$ contains no finite maximal prefix-free and define inductively $\alpha \in \mathbf{2}^\omega$ such that for all $n \in \mathbb{N}$ the set $X^{(n)} = \{p \in \mathbf{2}^* : (\alpha \restriction n)p \in X\}$ contains no finite maximal prefix-free set. Equality $X\mathbf{2}^\omega = \mathbf{2}^\omega$ insures $\alpha \in X\mathbf{2}^\omega$, hence there is an n such that $\alpha \restriction n \in X$. Whence, $\lambda \in X^{(n)}$ and the singleton set $\{\lambda\}$ is a finite maximal prefix-free subset of $X^{(n)}$. A contradiction. □

Proposition 2.2. *If $X \subseteq \mathbf{2}^*$ is r.e. then there exists a recursive prefix-free set $Y \subset \mathbf{2}^*$ such that $X\mathbf{2}^\omega = Y\mathbf{2}^\omega$. Moreover, one can recursively go from an r.e. code for X to r.e. codes for Y and $\mathbf{2}^* \setminus Y$.*

Note. *In general,* $\min(X)$ *is not r.e., hence cannot be the wanted Y.*

Proof. Let f be a partial recursive function with domain X. Let X_t be the set of strings with length $\leq t$ on which f is defined and converges in at most t computation steps. Set $Y = \bigcup_{t \in \mathbb{N}} Y_t$ where

$$Y_t = \{u \in \mathbf{2}^* : |u| = t + \max_{v \in X_t} |v| \wedge \exists v \in X_t \; v \preceq u \wedge \forall i < t \; \forall w \in X_i \; \neg(w \preceq u)\}$$

An easy induction shows that $X_t\mathbf{2}^\omega = (\bigcup_{i \leq t} Y_i)\mathbf{2}^\omega$ for all t, whence $X\mathbf{2}^\omega = Y\mathbf{2}^\omega$. Also, the Y_t's are finite and prefix-free and their elements are pairwise incomparable, so that Y is also prefix-free.
Moreover, Y is recursive since a string of length k is in Y if and only if it is in Y_t for some $t \leq k$.
Finally, the passage from X to Y and $\mathbf{2}^* \setminus Y$ is clearly effective. □

2.2 Arithmetical and Borel hierarchies on $\mathbf{2}^\omega$

We shall use the classical representation of *effective open* subsets of the Cantor space.

Proposition 2.3. *The three following conditions are equivalent.*
- *$\mathcal{X}$ is a Σ^0_1 subset of the Cantor space $\mathbf{2}^\omega$,*
- *$\mathcal{X} = X\mathbf{2}^\omega$ for some recursively enumerable $X \subseteq \mathbf{2}^*$,*
- *$\mathcal{X} = Y\mathbf{2}^\omega$ for some prefix-free recursive $Y \subseteq \mathbf{2}^*$.*

Moreover, one can recursively go from X to Y in the above equivalences.

2.3 Topology and Arithmetical Hierarchy for the $\mathbf{2}^{\leq\omega}$ space

We extend to $\mathbf{2}^{\leq\omega}$ the prefix partial order on $\mathbf{2}^*$. For $\xi, \eta \in \mathbf{2}^{\leq\omega}$, $\xi \preceq \eta$ if and only if $\xi, \eta \in \mathbf{2}^*$ and $\xi \preceq \eta$ or $\xi \in \mathbf{2}^*$, $\eta \in \mathbf{2}^\omega$ and $\eta \restriction |\xi| = \xi$.

We consider on $\mathbf{2}^{\leq\omega}$ the compact zero dimensional metrizable topology generated by the basic open sets $\{s\}$ and $s\mathbf{2}^{\leq\omega} = \{\xi \in \mathbf{2}^{\leq\omega} : s \preceq \xi\}$, where s varies over $\mathbf{2}^*$ (Boasson & Nivat, [6], Tom Head [16, 17], Becher& Grigorieff [5]). The induced topology on the subspace $\mathbf{2}^*$ is the discrete topology and that on the subspace $\mathbf{2}^\omega$ is the compact Cantor topology. As a subset of $\mathbf{2}^{\leq\omega}$, $\mathbf{2}^*$ is open and dense, hence not closed. So that $\mathbf{2}^\omega$ is closed and not open.

As for the Cantor space, the Arithmetical Hierarchy can be extended to subsets of the topological space $\mathbf{2}^{\leq\omega}$ by effectivization of the finite levels of the Borel hierarchy. Let's mention the representation of open and F_σ (resp. Σ^0_1 and Σ^0_2) subsets of $\mathbf{2}^{\leq\omega}$ which will be used in §5.3, 7.2.

Proposition 2.4. *Let $\mathcal{X} \subset \mathbf{2}^{\leq\omega}$.*
1. The three following conditions are equivalent.
i. $\mathcal{X}$ is open (resp. Σ^0_1) in $\mathbf{2}^{\leq\omega}$,
ii. $\mathcal{X} = X \cup Y\mathbf{2}^{\leq\omega}$ for some $X, Y \subseteq \mathbf{2}^*$ (resp. r.e. X, Y),
iii. $\mathcal{X} = Z \cup T\mathbf{2}^{\leq\omega}$ for some $Z, T \subseteq \mathbf{2}^*$ (resp. r.e. Z and recursive T), such that T is prefix-free.
Moreover, one can recursively go from X, Y to Z, T in the above equivalences.
2. $\mathcal{X}$ is clopen (i.e. closed and open) in $\mathbf{2}^{\leq\omega}$ if and only if it is of the form $\mathcal{X} = X \cup Y\mathbf{2}^{\leq\omega}$ where $X, Y \subseteq \mathbf{2}^*$ are finite.

Proposition 2.5. *If $X \subseteq \mathbb{N} \times \mathbf{2}^*$ and $i \in \mathbb{N}$ then $X_i = \{u \in \mathbf{2}^* : (i, u) \in X\}$. For $\mathcal{X} \subseteq \mathbf{2}^{\leq\omega}$, the three following conditions are equivalent.*
i. $\mathcal{X}$ is Σ^0_2 in $\mathbf{2}^{\leq\omega}$,
ii. $\mathcal{X} = \bigcup_{i\in\mathbb{N}} \mathbf{2}^{\leq\omega} \setminus (X_i \cup\ Y_i\mathbf{2}^{\leq\omega})$ where $X, Y \subseteq \mathbb{N} \times \mathbf{2}^*$ are r.e.
iii. $\mathcal{X} = \bigcup_{i\in\mathbb{N}} \mathbf{2}^{\leq\omega} \setminus (Z_i \cup\ T_i\mathbf{2}^{\leq\omega})$ where $Z, T \subseteq \mathbb{N} \times \mathbf{2}^*$, Z is r.e. and T is recursive prefix-free.
Moreover, one can recursively go from X, Y to Z, T in the above equivalences.

The relation between the arithmetical hierarchies relative to $\mathbf{2}^*$, $\mathbf{2}^\omega$ and $\mathbf{2}^{\leq\omega}$ is as follows.

Proposition 2.6 ([5]). *1. Let $n \geq 2$ and $\mathcal{X} \subseteq \mathbf{2}^{\leq\omega}$. Then*

$$\begin{array}{lcl}\mathcal{X} \text{ is } \Sigma^0_n(\mathbf{2}^{\leq\omega}) & \Leftrightarrow & \mathcal{X} \cap \mathbf{2}^* \text{ is } \Sigma^0_n(\mathbf{2}^*) \;\wedge\; \mathcal{X} \cap \mathbf{2}^\omega \text{ is } \Sigma^0_n(\mathbf{2}^\omega) \\ \mathcal{X} \text{ is } \Pi^0_n(\mathbf{2}^{\leq\omega}) & \Leftrightarrow & \mathcal{X} \cap \mathbf{2}^* \text{ is } \Pi^0_n(\mathbf{2}^*) \;\wedge\; \mathcal{X} \cap \mathbf{2}^\omega \text{ is } \Pi^0_n(\mathbf{2}^\omega)\end{array}$$

2. For $n = 1$ we only have

$$\begin{array}{lcl}\mathcal{X} \subseteq \mathbf{2}^* & \Rightarrow & (\mathcal{X} \text{ is } \Sigma^0_1(\mathbf{2}^*) \;\Leftrightarrow\; \mathcal{X} \text{ is } \Sigma^0_1(\mathbf{2}^{\leq\omega})) \\ \mathcal{X} \subseteq \mathbf{2}^\omega & \Rightarrow & (\mathcal{X} \text{ is } \Pi^0_1(\mathbf{2}^\omega) \;\Leftrightarrow\; \mathcal{X} \text{ is } \Pi^0_1(\mathbf{2}^{\leq\omega}))\end{array}$$

Remark 2.7. For counterexamples to Point 1 with $n = 1$, consider $\mathcal{X} = \mathbf{2}^*$ and $\mathcal{X} = \mathbf{2}^\omega$.

The following straightforward corollary of Prop.2.6 is used in application of the randomness theorems of this paper.

Proposition 2.8. *Let $n \geq 2$ and $X, Y \subseteq \mathbf{2}^*$ be Σ^0_n (resp. Π^0_n). Then $X \cup Y\mathbf{2}^{\leq\omega}$ is an open Σ^0_n (resp. Π^0_n) subset of $\mathbf{2}^{\leq\omega}$.*

Remark 2.9. As already noticed, the above proposition fails for Π^0_1 : $\mathbf{2}^*$ is Π^0_1 in $\mathbf{2}^*$ but it is not closed in $\mathbf{2}^{\leq\omega}$, hence not Π^0_1 in $\mathbf{2}^{\leq\omega}$.

3 Computably enumerable random reals

3.1 Computably enumerable reals

Infinite binary sequences can be identified with real numbers in $[0, 1]$, when the sequence is taken as the binary expansion of a real number. Hence, every real in $[0, 1]$ has a corresponding sequence in $\mathbf{2}^\omega$. This sequence is unique except for dyadic rational numbers of the form $k2^{-i}$, for natural numbers i, k, for which there are two of them. Since they form a set of measure 0, this fact does not affect the considerations over probabilities that we make in this work.
A real $\mathtt{x}$ is computable if its fractional part $\mathtt{x} - \lfloor \mathtt{x} \rfloor$ has recursive binary expansion.

Definition 3.1 (Soare, 1965 [30]). *A real is left (resp. right) computably enumerable (in short c.e.) if and only if its left (resp. right) Dedekind cut is r.e. The definition extends in an obvious way to sequences of reals.*

Much information about c.e. reals can be found in Downey's lectures [12] or Downey & Hirschfeldt's book [13]. We shall use the following result, due to Calude & Hertlind & Khoussainov & Wang, 1998 [8], and Downey & Laforte [15], 2002.

Proposition 3.2 ([8],[15]). *The following conditions on a real $\mathtt{a} \in [0, 1]$ are equivalent.*

i. $\mathtt{a}$ is left c.e.

ii. There exists an r.e. prefix-free set X such that $\mathtt{a} = \mu(X\mathbf{2}^\omega)$.

iii. There exists a recursive prefix-free set X such that $\mathtt{a} = \mu(X\mathbf{2}^\omega)$.

Moreover, the passage between these conditions is effective.

The following result is one of the tools we shall use to prove all theorems about randomness.

Proposition 3.3. *1. If $(a_i)_{i\in\mathbb{N}}$ is recursive in $\emptyset^{(n)}$ then $\sup_{i\in\mathbb{N}} a_i$ and $\inf_{i\in\mathbb{N}} a_i$ are respectively left and right c.e. in $\emptyset^{(n)}$, hence recursive in $\emptyset^{(n+1)}$.*
2i. If $\mathcal{X} \subseteq \mathbf{2}^\omega$ is Σ^0_n (resp. Π^0_n, resp. Δ^0_n) then $\mu(\mathcal{X})$ is left $\emptyset^{(n-1)}$-c.e. (resp. right $\emptyset^{(n-1)}$-c.e., resp. $\emptyset^{(n-1)}$-computable).
2ii. If $i \mapsto \mathcal{X}_i$ is a Σ^0_n (resp. Π^0_n) sequence of subsets of $\mathbf{2}^\omega$ then $\sup_{i\in\mathbb{N}} \mu(\mathcal{X}_i)$ (resp. $\inf_{i\in\mathbb{N}} \mu(\mathcal{X}_i)$) is left (resp. right) c.e. in $\emptyset^{(n-1)}$.

Proof. 1. Straightforward.
2i-ii. Initial case $n = 1$: Direct application of Prop.2.3 and Prop.3.2.
Induction step. Suppose that the property is true for n and let $\mathcal{X}$ be Σ^0_{n+1}. Then $\mathcal{X} = \bigcup_{i\in\mathbb{N}} \mathcal{X}_i$ for some Π^0_n increasing sequence $(\mathcal{X}_i)_i$. The induction hypothesis insures that the sequence $(\mu(\mathcal{X}_i))_i$ is right c.e. in $\emptyset^{(n-1)}$ hence recursive in $\emptyset^{(n)}$. Thus, $\mu(\mathcal{X}) = \sup_i \mu(\mathcal{X}_i)$ is left c.e. in $\emptyset^{(n)}$. Idem with sequences of Π^0_{n+1} sets. □

3.2 Random reals

We assume the notion of randomness (and randomness in an oracle) for elements of $\mathbf{2}^\omega$ as introduced by Martin-Löf, [21] 1966, and Schnorr's characterization using the prefix-free program-size complexity function H introduced by Chaitin, [9] 1975. Cf. textbooks [20, 13, 11, 7].
Randomness for real numbers $\mathtt{x}$ is defined via the corresponding binary sequences of their fractional parts (i.e. $\mathtt{x} - \lfloor \mathtt{x} \rfloor$). The definition is given for the alphabet $\{0, 1\}$, but it can be shown to be invariant under any alphabet. That is, the property of being random is inherent to the number and it is independent of the system in which it is represented.
The existence of random reals can be established by a measure-theoretic argument. As stated in §1.1, explicit random reals have been found by Chaitin, cf. Thm.1.1.

3.3 Combining random reals

Recall Solovay's reducibility and its classical relation to prefix-free program-size complexity function H and randomness (cf. [13, 12, 14]).

Definition 3.4 (Solovay, [31] 1975). *Let $\mathtt{a}, \mathtt{b} \in [0, 1]$ be c.e. reals. Let's denote by $lc(\mathtt{x}) = \{q \in \mathbb{Q} : q < \mathtt{x}\}$ the Dedekind left cut of $\mathtt{x}$. We say that $\mathtt{a}$ is Solovay reducible to $\mathtt{b}$ if there exists some constant c and some partial computable function $f : lc(\mathtt{b}) \to lc(\mathtt{a})$ with domain $lc(\mathtt{b})$ such that, for all $q \in lc(\mathtt{b})$,*

$$c(\mathtt{b} - q) > \mathtt{a} - f(q)$$

Theorem 3.5 (Solovay, [31] 1975). *Let* $\mathtt{a}, \mathtt{b} \in [0,1]$ *be c.e. reals associated to* $\alpha, \beta \in \mathbf{2}^\omega$. *If* $\mathtt{a}$ *is Solovay reducible to* $\mathtt{b}$ *then there exists some constant* d *such that, for all* n, $H(\alpha \restriction n) \leq H(\beta \restriction n) + d$. *In particular, if* $\mathtt{a}$ *is random then so is* $\mathtt{b}$.

As an easy corollary, we get the following result on which we shall rely for the proof of the main theorems (cf. §6).

Proposition 3.6. *If* $\mathtt{a}, \mathtt{b}$ *are both left (resp. right) c.e. and* $\mathtt{a}$ *is random then* $\mathtt{a} + \mathtt{b}$ *is random.*

Proof. We prove that $\mathtt{a}$ is Solovay reducible to $\mathtt{a} + \mathtt{b}$. Dividing $\mathtt{a}, \mathtt{b}$ by some power of 2, we reduce to the case $\mathtt{a} + \mathtt{b} < 1$. Clearly, $\mathtt{a} + \mathtt{b}$ is left c.e. Let $q < \mathtt{a} + \mathtt{b}$. Since $\mathtt{a}, \mathtt{b}$ are c.e., we can recursively enumerate the left Dedekind cuts of $\mathtt{a}, \mathtt{b}$ and find q_0, q_1 in these cuts such that $q_0 + q_1 \geq q$. Then $\mathtt{a} + \mathtt{b} > q_0 + q_1 \geq q$, hence $\mathtt{a} + \mathtt{b} - q > \mathtt{a} + \mathtt{b} - (q_0 + q_1) > \mathtt{a} - q_0$. Letting $c = 1$ and $f(q) = q_0$, we see that $\mathtt{a}$ is Solovay reducible to $\mathtt{a} + \mathtt{b}$. Considering $1 - \mathtt{a}, 1 - \mathtt{b}$, the right c.e. case reduces to the left c.e. one. □

Corollary 3.7. *Let* $n \geq 1$. *If* $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \subseteq \mathbf{2}^\omega$ *where* $\mathcal{X}_1, \mathcal{X}_2$ *are disjoint and* $\Sigma^0_n(\mathbf{2}^\omega)$ *(resp.* $\Pi^0_n(\mathbf{2}^\omega)$*) and* $\mu(\mathcal{X}_1)$ *is random in* $\emptyset^{(n-1)}$ *then* $\mu(\mathcal{X})$ *is random in* $\emptyset^{(n-1)}$.

Proof. Apply Prop.3.3 and Prop.3.6 relativized to oracle $\emptyset^{(n-1)}$. □

4 Different maps associated to the same Turing machine

4.1 Monotone Turing machines

In the case of halting computations different architectures of Turing machines are irrelevant in terms of computability. Turing machines, under any architecture whatsoever, compute exactly all partial recursive functions. However, architectural decisions on the moving abilities of the output head and the possibility of overwriting the output do affect the class of functions that become computable via possibly infinite computations.
In this paper we consider solely monotone Turing machines. This was indeed Turing's original assumption [33], insuring that in the limit of time the output of a non halting computation always converges, either to a finite or an infinite sequence. This concept was also considered by Levin [19], Schnorr [28, 29], see [20] p.276.

Definition 4.1. *A Turing machine is monotone if its output tape is one-way and write-only (hence no erasing nor overwriting is possible).*
Thus, the sequence of symbols written on the output tape increases monotonically with respect to the prefix ordering as the number of computation steps grows.

Remark 4.2. A sequence $\beta \in \mathbf{2}^\omega$ is the output of some monotone Turing machine with input $\alpha \in \mathbf{2}^\omega$ if and only if β is recursive in α.

All the material in this paper goes through mutatis mutandis when oracles are added to monotone Turing machines.

4.2 Maps representing machine behavior

A possibly infinite computation on a Turing machine is either a halting or a non halting computation. The output may be finite or infinite, and the input actually read by the machine may also be finite or infinite. This leads to consider $\mathbf{2}^*$ or $\mathbf{2}^\omega$ as the set of inputs, and $\mathbf{2}^{\leq\omega}$ as the set of outputs. Hence to represent the machine behavior as maps $\mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ or $\mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$. Whereas there is a unique notion of computability for maps with values in $\mathbf{2}^\omega$, when values in $\mathbf{2}^{\leq\omega}$ are allowed there are two notions: computability and semicomputability [5].

Definition 4.3. *Let $\mathcal{S}$ be among the sets $\mathbf{2}^*$, $\mathbf{2}^\omega$ and $\mathbf{2}^{\leq\omega}$ and let $F : \mathcal{S} \to \mathbf{2}^{\leq\omega}$ be a total map.*
1. F is semicomputable *if it is the input/output behaviour of some monotone Turing machine with inputs in $\mathcal{S}$ and possibly infinite computations.*
2. F is computable *if it is the output/output behaviour of some Turing machine with inputs in $\mathcal{S}$ and possibly infinite computations which halts in case the output is finite.*

Remark 4.4. 1. It is clear that total computable maps $\mathbf{2}^* \to \mathbf{2}^*$ are exactly the recursive ones. However, as concerns semicomputability, infinite computations really add. For instance, consider $F : \mathbf{2}^* \to \{\lambda, 0\}$ such that $F(0^n) = \lambda$ (the empty word) and $F(0^n 1 s)$ is 0 if $\varphi_n(n) \downarrow$, else undefined, where $\varphi : \mathbb{N}^2 \to \mathbb{N}$ is a universal partial recursive function
2. The "semi" character comes from the fact that for $\alpha \in \mathbf{2}^\omega$, if $F(\alpha)$ is a finite string with length $< n$ then the computation can nevertheless go on forever: though the output is completely written at some finite time, we never know that there is no more output to expect. Thus, to decide whether $F(\alpha)$ has length greater than n, we have to compute $F(\alpha)$ up to the moment (if there is any) the output has length $> n$. This is not a decision algorithm

but merely a semi-decision one. Moreover, the decision of whether $F(\alpha)$ is finite is a Σ^0_2 problem.

Semicomputable maps have a very simple characterization as limits of monotone maps $\mathbf{2}^* \to \mathbf{2}^*$ (cf. [5] for more developments).

Proposition 4.5. *A map $F : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ is semicomputable if and only if there exists a total recursive monotone increasing map $f : \mathbf{2}^* \to \mathbf{2}^*$ such that $F(\alpha) = \lim_{t\to\infty} f(\alpha \upharpoonright t)$.*

Proof. $\Leftarrow$ is straightforward. As for $\Rightarrow$, let M semicompute F. Observe that on input $u\alpha$, the current output of M at step $|u|$ does not depend on α because the input head has read $\leq |u|$ symbols. This allows us to define a total recursive f as follows: $f(u)$ is the current output of M on input u at step $|u|$. Clearly, f is monotone increasing and $F = \lim f$. □

4.3 Maps with prefix-free domain

For purposes in the theory of program-size complexity Chaitin [9] introduced the notion of *self-delimiting* inputs for halting computations on Turing machines. Instead of the usual assumption on Turing machines that the input tape contains a finite string followed by a blank symbol marking the end of the input, one now assumes no blanks, nor any other external way of input delimitation. An input must contain in itself the information to know where it ends, so the machine can realize when to finish reading the input tape; this is what self-delimiting means. Formally, an input p is self-delimiting for M if during its computation M reads p entirely and makes no attempt to move beyond the last symbol of p.
In order to properly deal with the case of an empty input, we suppose that the input tape contains a first dummy cell which receives no symbol and which is scanned by the head when the computation starts.
The following result characterizes these computations.

Theorem 4.6 (Chaitin, [9] Thm 2.1). *A partial recursive function has prefix-free domain if and only if it is the input/output behavior of some Turing machine on halting computations on its self-delimiting inputs.*

Chaitin [10] also developed the notion of self-delimiting inputs for possibly infinite computations. As the sole condition for these computations, he requires the input p to be finite and self-delimiting: p has to be entirely read and the head of the input tape should make no attempt to read beyond the last symbol of p. These computations determine maps $\mathbf{2}^* \to \mathbf{2}^{\leq\omega}$

with prefix-free domains which we shall call self-delimiting semicomputable maps.
We will refer to the following different maps associated to the same Turing machine M.

Definition 4.7. *Let M be a monotone Turing machine.*
1. (Chaitin [9]) *$M : \mathbf{2}^* \to \mathbf{2}^*$ is the partial recursive map associated to halting computations of M on the set of its self-delimited inputs. The domain of M is a prefix-free set.*
2. $M_{\bowtie} : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ is the self-delimiting semicomputable map associated to possibly infinite computations of M on the set of its self-delimited inputs. The domain of $M_{\bowtie}$ is a prefix-free set.
When defined, $M_{\bowtie}(p) \in \mathbf{2}^{\leq\omega}$ is the limit in $\mathbf{2}^{\leq\omega}$ of the monotone increasing sequence of current outputs at successive steps.
3. $M_\infty : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ is the total semicomputable map (cf. Def.4.3) for possibly infinite computations of M provided with inputs in $\mathbf{2}^\omega$. If α has no prefix in domain($M_{\bowtie}$) then the computation reads α entirely, else it reads only this prefix $\alpha\upharpoonright i$ and M_∞ is constant on $(\alpha\upharpoonright i)\mathbf{2}^\omega$.

The domains of M and $M_{\bowtie}$ can be described in terms of computations on infinite words. This is the contents of the following straightforward proposition.

Proposition 4.8. *Let M be a monotone Turing machine. Then*

$$\begin{aligned}
domain(M) &= \{p : \textit{for some infinite input } \alpha \succ p,\ M_\infty \textit{ halts} \\
&\qquad\qquad \textit{and reads exactly the finite prefix } p \textit{ of its input}\} \\
&= \{p : \textit{for all infinite input } \alpha \succ p,\ M_\infty \textit{ halts} \\
&\qquad\qquad \textit{and reads exactly the finite prefix } p \textit{ of its input}\} \\
domain(M_{\bowtie}) &= \{p : \textit{for some infinite input } \alpha \succ p, \\
&\qquad\qquad M_\infty \textit{ reads exactly the finite prefix } p \textit{ of its input}\} \\
&= \{p : \textit{for all infinite input } \alpha \succ p, \\
&\qquad\qquad M_\infty \textit{ reads exactly the finite prefix } p \textit{ of its input}\}
\end{aligned}$$

The next proposition gives the syntactical complexity of the domains of M and $M_{\bowtie}$.

Proposition 4.9. *Let M be a monotone Turing machine. Then*
- *domain(M) is $\Sigma^0_1(\mathbf{2}^*)$.*
- *domain($M_{\bowtie}$) is $(\Sigma^0_1 \wedge \Pi^0_1)(\mathbf{2}^*)$ and this bound can not be improved.*

Proof. Observe that the definition of $domain(M_{\bowtie})$ involves the conjunction of an existential condition with a universal one, namely:

- *at some computation step the input has been entirely read,*
- *the head of the input tape never moves beyond the end of the input.*

To see that this complexity bound is sharp consider the Busy Beaver function $bb : \mathbb{N} \to \mathbb{N}$ where $bb(n)$ is the maximum number of 0's that can be produced by some Turing machine with no input having n states and which halts. It is easy to devise a monotone Turing machine M such that $domain(M_{\bowtie}) = \{0^n 1p : |p| = bb(n)\}$. To conclude, recall that bb is not recursive but recursive in $\emptyset'$. □

4.4 Universal machines and simulation by prefix adjunction

Assume an effective enumeration of all tables of instructions of monotone machines. This determines an effective enumeration $k \mapsto M_k$.

Definition 4.10. *1. The universal monotone Turing machine U is defined as follows:*
- U reads the input looking for a prefix of the form $0^k 1$ for some $k \in \mathbb{N}$,
- if it finds some, U simulates M_k on the remaining part of the input.
2. We denote by U^A the machine with oracle A which is similarly obtained.

The above universal machine has very fine simulation abilities.

Proposition 4.11 (Simulation by prefix adjunction). *1. By prefix adjunction to the input, U simulates any Turing machine for finite computations as well as for infinite ones: for all $k \in \mathbb{N}$, $p, q \in \mathbf{2}^*$, $\alpha \in \mathbf{2}^\omega$, $\xi \in \mathbf{2}^{\leq\omega}$,*
i. $p \in domain(M_k)$ (i.e. M_k halts on p and p is self-delimited for M_k) and $M_k(p) = q$ if and only if $0^k 1p \in domain(U)$ and $U(0^k 1p) = q$.
ii. $p \in domain((M_k)_{\bowtie})$ (i.e. p is self-delimited for M_k) and $(M_k)_{\bowtie}(p) = \xi$ if and only if $0^k 1p \in domain(U_{\bowtie})$ and $U_{\bowtie}(0^k 1p) = \xi$.
iii. $(M_k)_\infty(\alpha) = \xi$ if and only if $U_{\bowtie}(0^k 1\alpha) = \xi$.
2. Let f be a total recursive function. By prefix adjunction to the input, U simulates $M_{f(k)}$: there exists $\eta \in \mathbf{2}^*$ such that for all k, p, q, α, ξ, the above equivalences and equalities hold with $\eta 0^k 1$ in place of $0^k 1$.

Proof. 1. Trivial from the definition of U.
2. Set $\eta = 0^\ell 1$ where M_ℓ is the Turing machine which behaves as follows:
- it reads the input looking for a prefix of the form $0^k 1$ for some $k \in \mathbb{N}$,
- if such a prefix exists then it computes $f(k)$,
- it then simulates machine $M_{f(k)}$ on the part of the input not yet read. □

5 Semicomputable Wadge semireductions

5.1 Semicomputability and lower semicontinuity

As is well known, computable maps $\mathbf{2}^\omega \to \mathbf{2}^\omega$ are continuous for the usual Cantor topology. Indeed, for maps $\mathbf{2}^\omega \to \mathbf{2}^\omega$, computability is the effectivization of continuity. However, as we developed in another paper [5], for maps into $\mathbf{2}^{\leq\omega}$ the topological counterparts of computability and semicomputability are respectively continuity and lower semicontinuity. This last notion is the analog of the classical notion of lower semicontinuity for real valued functions, but for functions with values in $\mathbf{2}^{\leq\omega}$ with respect to the prefix ordering on this space.

Definition 5.1. *Let $\mathcal{S}$ be $\mathbf{2}^\omega$ or $\mathbf{2}^{\leq\omega}$. A map $F : \mathcal{S} \to \mathbf{2}^{\leq\omega}$ is lower semicontinuous at $\xi \in \mathcal{S}$ if, for all $n \in \mathbb{N}$, there exists a neighborhood $\mathcal{V}$ of ξ such that $F(\eta) \restriction n \succeq F(\xi) \restriction n$ for all $\eta \in \mathcal{V}$.*

The following easy proposition (cf. [5]) shows that lower semicontinuity differs from continuity only at points with finite image.

Proposition 5.2. *Let $F : \mathcal{S} \to \mathbf{2}^{\leq\omega}$ and $\xi \in \mathcal{S}$.*
1. If $F(\xi) \in \mathbf{2}^\omega$ then F is lower semicontinuous at ξ if and only if F is continuous at ξ.
2. If $F(\xi) \in \mathbf{2}^*$ then F is continuous (resp. lower semicontinuous) at ξ if and only if there exists a neighborhood $\mathcal{V}$ of ξ such that $F(\eta) = F(\xi)$ (resp. $F(\eta) \succeq F(\xi)$) for all $\eta \in \mathcal{V}$.

Proof. 1. If $F(\xi) \in \mathbf{2}^\omega$ (or merely $|F(\xi)| \geq n$) then the condition $F(\eta) \restriction n \succeq F(\xi) \restriction n$ exactly means $F(\eta) \restriction n = F(\xi) \restriction n$, which is the condition for continuity.
2. Recall that finite strings are isolated points in the $\mathbf{2}^{\leq\omega}$ space. $\Leftarrow$ is trivial. As for $\Rightarrow$, let $n = |F(\xi)|$. □

The following proposition is easy.

Proposition 5.3. *Every semicomputable map $F : \mathcal{S} \to \mathbf{2}^{\leq\omega}$ (cf. Def.4.3) is lower semicontinuous.*

Remark 5.4. Semicomputable maps $\mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ are not continuous in general. For instance, let $erase(\alpha)$ be obtained by erasing all zeros in α. Then $erase : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ is semicomputable and discontinuous at all points $\alpha \in \mathbf{2}^*0^\omega$.

5.2 Wadge semireducibility

The classical Wadge hierarchy (cf. textbooks: Moschovakis [23], Kechris [18]) provides a refinement of the Borel hierarchy based on the simple topological notion of inverse image by a continuous function. The notion of Wadge reduction has best properties with zero-dimensional Polish spaces, in particular with the compact spaces $\mathbf{2}^\omega$ and $\mathbf{2}^{\leq\omega}$. Effectivizing continuous maps by the computable ones (cf. Def.4.3), one can also consider computable Wadge reductions.
Associated to lower semicontinuous maps into $\mathbf{2}^{\leq\omega}$ we introduce the notion of *Wadge semireduction* and its effectivization by the semicomputable maps, which is the kind of effectivization yielded by possibly infinite computations on monotone Turing machines, cf. the forthcoming paper [4].

Definition 5.5. *Let $\mathcal{S},\mathcal{T}$ be $\mathbf{2}^\omega$ or $\mathbf{2}^{\leq\omega}$, $\mathcal{X} \subseteq \mathcal{S}$, $\mathcal{Y} \subseteq \mathcal{T}$.*
1. (Wadge, 1972 [34, 35]) $\mathcal{X}$ is Wadge reducible to $\mathcal{Y}$ (denoted $\mathcal{X} \preceq_W \mathcal{Y}$) if there exists a continuous map $F : \mathcal{S} \to \mathcal{T}$ such that $\mathcal{X} = F^{-1}(\mathcal{Y})$.
2. In case $\mathcal{T} = \mathbf{2}^{\leq\omega}$, Wadge semireducibility $\preceq_{sW}$ is defined similarly with lower semicontinuous maps.
3. Computable Wadge reducibility $\preceq_W^{eff}$ and semicomputable Wadge semireducibility $\preceq_{sW}^{s-eff}$ are similarly defined with computable and semicomputable maps.
4. Let C be a class of subsets of $\mathcal{S}$. Relative to any one of the above reducibilities, $\mathcal{Y}$ is C-hard if every set $\mathcal{X} \in C$ is reducible to $\mathcal{Y}$.

The following proposition is straightforward.

Proposition 5.6. *Relative to any one of the above reducibilities, if $\mathcal{X}$ is reducible to $\mathcal{Y}$ then the complement of $\mathcal{X}$ is reducible to that of $\mathcal{Y}$.*
If $\mathcal{X}$ is hard for a class $\mathcal{C}$ then the complement of $\mathcal{X}$ is hard for the class of complements of sets in $\mathcal{C}$.

5.3 Wadge hardness

Let's denote by $\mathbf{\Sigma}^0_n(\mathbf{2}^\omega)$ and $\mathbf{\Pi}^0_n(\mathbf{2}^\omega)$ the finite levels of the Borel hierarchy on $\mathbf{2}^\omega$. As a well known consequence of the hierarchy theorem, if a subset of $\mathbf{2}^\omega$ is Wadge hard for the class $\mathbf{\Sigma}^0_n(\mathbf{2}^\omega)$ (resp. $\mathbf{\Pi}^0_n(\mathbf{2}^\omega)$) then it cannot be in $\mathbf{\Pi}^0_n(\mathbf{2}^\omega)$ (resp. $\mathbf{\Sigma}^0_n(\mathbf{2}^\omega)$). One of the key results in Wadge's theory is that the converse is also true.

Theorem 5.7 (Wadge [34, 35], cf.[23] or [18]). *Let $n \geq 1$ and $\mathcal{X} \subseteq \mathbf{2}^\omega$.*

$$\mathcal{X} \text{ is } \preceq_W \mathbf{\Sigma}^0_n(\mathbf{2}^\omega)\text{-hard} \Leftrightarrow \mathcal{X} \text{ is } \preceq_W \Sigma^0_n(\mathbf{2}^\omega)\text{-hard} \Leftrightarrow \mathcal{X} \notin \mathbf{\Pi}^0_n(\mathbf{2}^\omega)$$
$$\mathcal{X} \text{ is } \preceq_W \mathbf{\Pi}^0_n(\mathbf{2}^\omega)\text{-hard} \Leftrightarrow \mathcal{X} \text{ is } \preceq_W \Pi^0_n(\mathbf{2}^\omega)\text{-hard} \Leftrightarrow \mathcal{X} \notin \mathbf{\Sigma}^0_n(\mathbf{2}^\omega)$$

A naive expectation is that the same result is true for hardness with respect to *computable* Wadge reducibility and the effective Σ^0_n or Π^0_n classes of $\mathbf{2}^\omega$ subsets. But this is false. Only the $\Rightarrow$ implication of the last equivalence remains true (which is the straightforward direction).
The quite classical result of Point 1 of the next proposition leads to a somewhat surprising result (Point 3) concerning hardness relative to semicomputable semireductions from $\mathbf{2}^\omega$ to $\mathbf{2}^{\leq\omega}$ (cf.[4] for more developments).

Proposition 5.8. *1.* $\mathbf{2}^*0^\omega$ *is a* Σ^0_2 *subset of* $\mathbf{2}^\omega$ *which is* $\preceq^{eff}_W$*-hard for* $\Sigma^0_2(\mathbf{2}^\omega)$*, hence* $\preceq_W$*-hard for* $\mathbf{\Sigma}^0_2(\mathbf{2}^\omega)$*.*
2. If $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ *and* $\mathbf{2}^*0^\omega \preceq^{s-eff}_{sW} \mathcal{O}$ *then* $\mathcal{O}$ *is* $\preceq^{s-eff}_{sW}$*-hard for* Σ^0_2 *subsets of* $\mathbf{2}^\omega$*, hence* $\preceq_{sW}$*-hard for* $\mathbf{\Sigma}^0_2(\mathbf{2}^\omega)$*.*
3. $\mathbf{2}^*$ *is a* Σ^0_1 *subset of* $\mathbf{2}^{\leq\omega}$ *which is* $\preceq^{s-eff}_{sW}$*-hard for* Σ^0_2 *subsets of* $\mathbf{2}^\omega$*, hence* $\preceq_{sW}$*-hard for* $\mathbf{\Sigma}^0_2(\mathbf{2}^\omega)$*.*
4. $\mathbf{2}^\omega$ *is a* Π^0_1 *subset of* $\mathbf{2}^{\leq\omega}$ *which is* $\preceq^{s-eff}_{sW}$*-hard for* Π^0_2 *subsets of* $\mathbf{2}^\omega$*, hence* $\preceq_{sW}$*-hard for* $\mathbf{\Pi}^0_2(\mathbf{2}^\omega)$*.*

Proof. 1. Let $\mathcal{X} \subseteq \mathbf{2}^\omega$ be Σ^0_2. Using the classical representation of Π^0_2 subsets of $\mathbf{2}^\omega$ via the quantifier $\exists^\infty$ (cf. Rogers [26] Thm. XVIII p.328), there exists some recursive relation $R \subseteq \mathbf{2}^*$ such that

$$\alpha \in \mathcal{X} \Leftrightarrow \{i : R(\alpha \upharpoonright i)\} \text{ is finite}$$

Set $G(\alpha)(n) = 1$ if $R(\alpha \upharpoonright n)$ holds. Then $G : \mathbf{2}^\omega \to \mathbf{2}^\omega$ is a computable map such that $G^{-1}(\mathbf{2}^*0^\omega) = \mathcal{X}$.
2. Suppose $\mathbf{2}^*0^\omega = G^{-1}(\mathcal{O})$ where $G : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ is semicomputable. If $F : \mathbf{2}^\omega \to \mathbf{2}^\omega$ and $\mathcal{X} = F^{-1}(\mathbf{2}^*0^\omega)$ then $\mathcal{X} = (G \circ F)^{-1}(\mathcal{O})$. Finally, observe that if F is computable then $G \circ F : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ is semicomputable.
3. Observe that $\mathbf{2}^*0^\omega = erase^{-1}(\mathbf{2}^*)$ where $erase : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ is the semicomputable function which erases all 0's.
4. Straightforward from Point 3. □

5.4 Getting semicomputable Wadge hardness: proof of Thm.1.16

We now prove that conditions $(*)$ and $(**)$ on a subset of $\mathbf{2}^{\leq\omega}$ introduced in Def.1.15 imply $\preceq^{s-eff}_{sW}$-hardness for the class of Σ^0_2 subsets of $\mathbf{2}^\omega$.
Due to Prop.5.8, it is sufficient to prove that $\mathbf{2}^*0^\omega \preceq^{s-eff}_{sW} \mathcal{O}$.
Case of condition $(*)$. Assume $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ satisfies $(*)$. Let $(s_i)_{i\in\mathbb{N}}$ be a recursive increasing chain of words in $\mathcal{O}$ with respect to the prefix ordering with limit not in $\mathcal{O}$. Let $g : \mathbf{2}^* \to \mathbf{2}^*$ be such that $g(u) = s_i$ where i is the number of 1's in u. Clearly, g is total recursive and monotone increasing with respect to the prefix ordering. Set $G(\alpha) = \lim_{i\to\infty} g(\alpha \upharpoonright i)$. Then

$G : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ is semicomputable and

$$\begin{array}{lcl} \alpha \in \mathbf{2}^*0^\omega & \Rightarrow & G(\alpha) = s_i \in \mathcal{O} \text{ where } i \text{ is the number of 1's in } \alpha \\ \alpha \notin \mathbf{2}^*0^\omega & \Rightarrow & G(\alpha) = \lim\limits_{i\to\infty} s_i \notin \mathcal{O} \end{array}$$

Thus, $\mathbf{2}^*0^\omega = G^{-1}(\mathcal{O})$ and $\mathbf{2}^*0^\omega \preceq_{sW}^{s-eff} \mathcal{O}$.
Case of condition $(**)$. Assume now $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ is $\Sigma^0_2(\mathbf{2}^{\leq\omega})$ and satisfies condition $(**)$ and let $u \in \mathbf{2}^*$ and $F : \mathbf{2}^* \to \mathcal{O}$ and $G : \mathbf{2}^* \to \mathbf{2}^\omega \setminus \mathcal{O}$ be total computable maps such that, for all $v \in \mathbf{2}^*$, uv is a prefix of $F(v)$ and $G(v)$. Since $\mathcal{O}$ is $\Sigma^0_2(\mathbf{2}^{\leq\omega})$, Prop.2.6 insures that $\mathbf{2}^\omega \setminus \mathcal{O}$ is $\Pi^0_2(\mathbf{2}^\omega)$. Use the classical representation of Π^0_2 subsets of $\mathbf{2}^\omega$ via the quantifier $\exists^\infty$ (cf. Rogers [26] Thm. XVIII p.328) to get a recursive relation $R \subseteq \mathbf{2}^*$ such that

$$\alpha \in \mathbf{2}^\omega \setminus \mathcal{O} \iff \exists^\infty n \; R(\alpha \upharpoonright n)$$

Observe that, for every $v \in u\mathbf{2}^*$, since $G(v) \in \mathbf{2}^\omega \setminus \mathcal{O}$, there are infinitely many prefixes of $G(v)$ in R. Let $\varphi : \mathbf{2}^* \times \mathbb{N} \to \mathbf{2}^*$ be such that $\varphi(v)$ is the least prefix of $G(v)$ which is in R and has length $\geq |v|$. Since G is computable, φ is total recursive.
Recall that λ denotes the empty word and, for $\xi \in \mathbf{2}^{\leq\omega}$, $\xi \upharpoonright i$ is defined as the prefix of ξ with length $\min(i, length(\xi))$. We define $\ell : \mathbf{2}^* \to \mathbf{2}^*$ as follows: for $i \geq 1$, $v \in \mathbf{2}^*$,

$$\begin{array}{lcllcl} \ell(\lambda) & = & \lambda & \ell(v1) & = & \varphi(\ell(v)) \\ \ell(0^i) & = & F(\lambda)\upharpoonright i & \ell(v10^i) & = & F(\ell(v1))\upharpoonright i \end{array}$$

Let $L : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ be the map induced by ℓ : $L(\alpha) = \lim_{i\to\infty} \ell(\alpha \upharpoonright i)$. Clearly, L is semicomputable. If $\alpha = v0^\omega$ where $v = \lambda$ or $v \in \mathbf{2}^*1$ then $L(\alpha) = F(\ell(v)) \in \mathcal{O}$. If α contains infinitely many 1's then $L(\alpha)$ has infinitely many prefixes in R, hence is not in $\mathcal{O}$. Thus, $L^{-1}(\mathcal{O}) = \mathbf{2}^*0^\omega$ so that $\mathbf{2}^*0^\omega \preceq_{sW}^{s-eff} \mathcal{O}$.
This finishes the proof of Thm.1.16.

Remark 5.9. Conditions $(*)$ and $(**)$ are independent. In fact, any boolean combination of these conditions is true for some set $\mathcal{O}$ as shown by the following examples.
$(*) \wedge (**)$. Let $\mathcal{O} = \mathbf{2}^*$.
$(*) \wedge \neg(**)$. $\mathcal{O} = ((00)^*(11)^*)^*$ fails the density condition. Another example is $\mathcal{O} = \mathbf{2}^{\leq\omega} \setminus \{\alpha\}$, for a recursive $\alpha \in \mathbf{2}^\omega$, which fails the codensity condition.
$\neg(*) \wedge (**)$. Let $\mathcal{O} = \mathbf{2}^*0^\omega$, which fails $(*)$ because it contains no finite strings.
$\neg(*) \wedge \neg(**)$. Let $\mathcal{O} \subset \mathbf{2}^*$ be any finite set or $u\mathbf{2}^{\leq\omega}$, for some $u \in \mathbf{2}^*$.

6 Proofs of randomness theorems and their corollaries

As said in §1, we shall use a monotone machine universal by prefix adjunction such as that of Def.4.10 and the associated partial recursive map $U : \mathbf{2}^* \to \mathbf{2}^*$, self-delimiting semicomputable map $U_{\bowtie} : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ and total semicomputable map $U_\infty : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ (Def. 4.7). We shall also admit $\emptyset^{(n)}$ as oracle and consider the map $U^{\emptyset^{(n)}} : \mathbf{2}^* \to \mathbf{2}^*$ which is partial recursive in $\emptyset^{(n)}$ and universal by prefix adjunction.

6.1 Proof pattern of Thm.1.1, 1.9, 1.10, 1.11

Proofs of these theorems all have the same pattern which we now describe as the proof of a general abstract result.

Theorem 6.1. *Let $V : \mathcal{S} \to \mathcal{T}$ be either $U : \mathbf{2}^* \to \mathbf{2}^*$ or $U_{\bowtie} : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ or $U_\infty : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ and let $\mathcal{O} \subseteq \mathcal{T}$ and $n \in \mathbb{N}$. For $\mathcal{X} \subseteq \mathcal{S}$, we let $\mathcal{C}(\mathcal{X}) = \mathcal{X}$ in case $\mathcal{S} = \mathbf{2}^\omega$ and $\mathcal{C}(\mathcal{X}) = \mathcal{X}\mathbf{2}^\omega$ in case $\mathcal{S} = \mathbf{2}^*$.*
If $V^{-1}(\mathcal{O})$ is $\Sigma^0_{n+1}(\mathcal{S})$ and there exists a partial self-delimiting (in case $\mathcal{S} = \mathbf{2}^*$) or total (in case $\mathcal{S} = \mathbf{2}^\omega$) semicomputable map $F : \mathcal{S} \to \mathcal{T}$ such that $\mathcal{C}(F^{-1}(\mathcal{O})) = \mathcal{C}(domain(U^{\emptyset(n)}))$ then $\mu(\mathcal{C}(V^{-1}(\mathcal{O})))$ is random in $\emptyset^{(n)}$.

Proof. Using the assumption of universality by prefix adjunction of V, there exists $\sigma \in \mathbf{2}^*$ such that $F(\xi) = V(\sigma\xi)$ for all $\xi \in \mathcal{S}$ (and $domain(F) = \{\xi \in \mathcal{S} : \sigma\xi \in domain(V)\}$ in case $\mathcal{S} = \mathbf{2}^*$). In particular, $V^{-1}(\mathcal{O}) \cap \sigma\mathcal{S} = \sigma F^{-1}(\mathcal{O})$. Hence, we get the partition of sets

$$
\begin{aligned}
V^{-1}(\mathcal{O}) &= (V^{-1}(\mathcal{O}) \cap \sigma\mathcal{S}) \cup (V^{-1}(\mathcal{O})) \setminus \sigma\mathcal{S}) \\
&= (\sigma F^{-1}(\mathcal{O})) \cup (V^{-1}(\mathcal{O})) \setminus \sigma\mathcal{S}) \\
\mathcal{C}(V^{-1}(\mathcal{O})) &= \sigma\mathcal{C}(F^{-1}(\mathcal{O})) \cup \mathcal{C}((V^{-1}(\mathcal{O})) \setminus \sigma\mathcal{S}) \\
\mathcal{C}(V^{-1}(\mathcal{O})) &= \sigma\mathcal{C}(domain(U^{\emptyset(n)})) \cup \mathcal{C}((V^{-1}(\mathcal{O})) \setminus \sigma\mathcal{S})
\end{aligned}
$$

and that of the associated measures

$$
\begin{aligned}
\mu(\mathcal{C}(V^{-1}(\mathcal{O}))) &= 2^{-|\sigma|}\mu(\mathcal{C}(domain(U^{\emptyset(n)}))) + \mu(\mathcal{C}(V^{-1}(\mathcal{O}) \setminus \sigma\mathcal{S})) \\
&= 2^{-|\sigma|}\Omega^{\emptyset^{(n)}} + \mu(\mathcal{C}(V^{-1}(\mathcal{O}) \setminus \sigma\mathcal{S}))
\end{aligned}
$$

The $\Sigma^0_{n+1}(\mathcal{S})$ character of $V^{-1}(\mathcal{O})$ insures that of $V^{-1}(\mathcal{O}) \setminus \sigma\mathcal{S}$. Which, in turn, insures the $\Sigma^0_{n+1}(\mathbf{2}^\omega)$ character of $\mathcal{C}(V^{-1}(\mathcal{O}) \setminus \sigma\mathcal{S})$, hence that its measure is left c.e. in $\emptyset^{(n)}$. Since $domain(U^{\emptyset^{(n)}})$ is Σ^0_{n+1}, the real $2^{-|\sigma|}\Omega^{\emptyset^{(n)}}$

is also left c.e. in $\emptyset^{(n)}$. Chaitin's Thm.1.1 relativized to oracle $\emptyset^{(n)}$ insures that $\mu(\mathcal{C}(domain(U^{\emptyset^{(n)}}))) = \Omega^{\emptyset^{(n)}}$ is random in $\emptyset^{(n)}$, hence also its product by the dyadic rational $2^{-|\sigma|}$. Finally, Cor.3.7 insures that $\mu(\mathcal{C}(V^{-1}(\mathcal{O})))$ is random in $\emptyset^{(n)}$. □

6.2 Proof of Chaitin's Thm.1.1

We apply Thm.6.1 with $n = 0$ and $U : \mathbf{2}^* \to \mathbf{2}^*$ as $V : \mathcal{S} \to \mathcal{T}$. Since $\mathcal{O} \neq \emptyset$, we can consider some fixed $a \in \mathcal{O}$. We let $F : \mathbf{2}^* \to \mathbf{2}^*$ be the partial self-delimiting semicomputable map defined on $domain(U)$ which, on this domain, is constant with value a. Clearly, $F^{-1}(\mathcal{O}) = domain(U)$. Also, since $\mathcal{O}$ is $\Sigma^0_1(\mathbf{2}^*)$ so is $V^{-1}(\mathcal{O})$. Thus, the conditions of Thm.6.1 hold, so that $\mu(\mathcal{C}(V^{-1}(\mathcal{O}))) = \Omega[\mathcal{O}]$ is random.

6.3 Proof of Thm.1.9 (plain randomness)

We shall use Thm.7.4 which does not rely on results of this §. Let $\mathcal{O} = Y\mathbf{2}^{\leq\omega}$ where $Y \subseteq \mathbf{2}^*$ is Σ^0_1 and $\mathcal{O} \neq \mathbf{2}^{\leq\omega}, \emptyset$, i.e. $\lambda \notin Y$ and $Y \neq \emptyset$. We apply Thm.6.1 with $n = 0$ and $U_\infty : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ as $V : \mathcal{S} \to \mathcal{T}$. Let u be any word in Y. Necessarily, $u \neq \lambda$. We let $F : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ be the total semicomputable map such that $F(\alpha) = \lambda$ if U does not halt on any finite prefix of α and $F(\alpha) = u$ otherwise. Clearly, $F^{-1}(\mathcal{O}) = domain(U)\mathbf{2}^\omega$. Also, since Y is $\Sigma^0_1(\mathbf{2}^*)$, Thm.7.4 (line 1a of Table 1) insures that $V^{-1}(\mathcal{O}) = U_\infty^{-1}(Y\mathbf{2}^{\leq\omega})$ is $\Sigma^0_1(\mathbf{2}^\omega)$. Thus, the conditions of Thm.6.1 hold, so that $\mu(V^{-1}(\mathcal{O})) = \mu(U_\infty^{-1}(\mathcal{O}))$ is random.

6.4 Proof of 1st main theorem: Thm.1.10 (randomness in $\emptyset'$)

6.4.1 Harmless overshoot reducibility

We introduce a convenient tool related to "harmless overshoot" (cf. §6.4.2).

Definition 6.2. *Let $X, Y \subseteq \mathbf{2}^*$ be prefix-free. We say that X is "harmless overshoot" reducible to Y, written $X \preceq_{HOS} Y$, if the following conditions hold:*

i. $Y \subseteq X\mathbf{2}^*$, i.e. any word $y \in Y$ extends some word in $x \in X$,
ii. $X\mathbf{2}^\omega = Y\mathbf{2}^\omega$.

Harmless overshoot reducibility can also be expressed as follows.

Proposition 6.3. *$X \preceq_{HOS} Y$ if and only if*

$$Y = \bigcup_{x \in X} xS_x$$

where, for each $x \in X$, $S_x \subset \mathbf{2}^*$ is finite maximal prefix-free.

Proof. $\Rightarrow$. For each $x \in X$, let S_x be the set of u's such that $xu \in Y$. Since Y is prefix-free so are the S_x's. From condition i of Def.6.2 we know that every $y \in Y$ has a prefix in $x \in X$, hence $y \in xS_x$. Thus, $Y = \bigcup_{x \in X} xS_x$ and $Y\mathbf{2}^\omega = \bigcup_{x \in X} xS_x\mathbf{2}^\omega$. Using condition ii, we get $X\mathbf{2}^\omega = \bigcup_{x \in X} x(S_x\mathbf{2}^\omega)$, whence, for each $x \in X$, $x\mathbf{2}^\omega = xS_x\mathbf{2}^\omega$, i.e. $\mathbf{2}^\omega = S_x\mathbf{2}^\omega$. Finally, Prop.2.1 insures that S_x is finite maximal prefix-free.
$\Leftarrow$. Clearly, $Y \subseteq X\mathbf{2}^*$ and $Y\mathbf{2}^\omega = \bigcup_{x \in X} x(S_x\mathbf{2}^\omega)$. Now, Prop.2.1 yields $S_x\mathbf{2}^\omega = \mathbf{2}^\omega$, whence $Y\mathbf{2}^\omega = X\mathbf{2}^\omega$. □

6.4.2 Simulation in the limit and harmless overshoot

Chaitin, [10]) 1976, introduces the *simulation in the limit* technique that tells how to perform a simulation of a computation relative to an oracle, via an infinite computation in a machine that lacks the oracle. The technique requires that the oracle be recursively enumerable. The simulated computation is run in increasing number of steps, using a fake oracle: at step t a question to the oracle is answered "no" unless the question is found to be true in at most t steps. As the number of steps t goes to infinity any finite number of questions will eventually be answered correctly by the fake oracle.
We apply this technique to simulate a computation on $U^{\emptyset'}$ as concerns the input and work tapes, but not that of the output tape. Now, in spite of the fact that, in the limit, the fake oracle realizes its mistakes and provides the correct answers, the simulation may already have read beyond the input. This happens because the domain of the function being simulated, that is $domain(U^{\emptyset'})$, is not recursively enumerable, so the simulation may not know where the input actually ends until it gets the correct oracle answers. In the meantime, extra symbols from the input tape may have been read. However, since we are just interested in discovering, in the limit, whether a computation on $U^{\emptyset'}$ actually halts, the actual value of those extra bits turns out to be irrelevant. Chaitin [10] called this feature *harmless overshoot.*

6.4.3 Proof of Thm.1.10

We now apply simulation in the limit and harmless overshoot to construct the partial self-delimiting semicomputable map $F : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ needed to use Thm.6.1.

Lemma 6.4. *Suppose $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ contains some finite string or some infinite recursive sequence. Then there exists a partial self-delimiting semicomputable map $F : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ such that*

i. $domain(U^{\emptyset'}) \preceq_{HOS} domain(F)$

ii. F is constant on its domain and has value in $\mathcal{O}$.

Proof. Following Prop.4.8, we shall relate a partial self-delimiting semicomputable map $M_{\bowtie} : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ (resp. $M : \mathbf{2}^* \to \mathbf{2}^*$) to the restriction of M_∞ to the set of α such that the computation of M_∞ on input α reads only finitely many symbols of α (resp. and halts).

1. Let ξ be some fixed finite string or infinite sequence in $\mathcal{O}$. We define $F : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ as the $M_{\bowtie}$ map associated to a Turing machine M with infinite inputs and possibly infinite computations which performs the simulation in the limit of $U^{\emptyset'}$ as described below:

> *quota* := 1 (number of steps of $U^{\emptyset'}$ to be simulated)
> If ξ is a finite string then outputs ξ.
> `do forever`
>
> 1. If ξ is an infinite sequence then output the *quota*-th symbol of ξ.
> 2. Simulate $U^{\emptyset'}$ (on the given infinite input) for at most *quota* computation steps. For each question to the oracle of whether $U(q)$ halts, simulate $U(q)$ and take as an answer whether it halts in at most *quota* steps.
> 3. If $U^{\emptyset'}$ did not halt, or if an oracle answer was found to be mistaken (i.e., it changed from its previous value, from "no" to "yes"), or more questions were asked, then move the input head.
> 4. Else do not move it.
> 5. *quota* := *quota* + 1
>
> `end do`

Thus, for each value of *quota*, M does two kinds of simulation.
First, it simulates steps 0,1,2,... of the computation of $U^{\emptyset'}$ (on the given infinite input), up to *quota* or some halting step of the simulation of $U^{\emptyset'}$.
Second, M simulates steps 0,1,2,... of the computation of U for every input for which a question to the oracle was raised, up to *quota* or some halting step of U on this input.

2. Clearly, the output is always ξ, hence always in $\mathcal{O}$. It remains to prove the stated $\preceq_{HOS}$ reducibility.

3. Suppose M reads only a finite part q of its input $\alpha \in \mathbf{2}^\omega$. Then

there exists a finite prefix p of q such that the simulation of $U^{\emptyset'}$ on input p halts and the fake oracle used for that simulation is never found mistaken in the remaining infinite part of the computation (where the input head does not move). Thus, p is indeed in $domain(U^{\emptyset'})$. Which proves $domain(M_{\bowtie}) \subseteq domain(U^{\emptyset'})\mathbf{2}^*$.
4. Suppose $p \in domain(U^{\emptyset'})$. Then $U^{\emptyset'}$ on input p halts in finitely many steps, say N steps, at which it can perform only finitely many oracle questions. Let us call Q the set of programs that are consulted to the oracle. Every $q \in Q$ such that $U(q) \downarrow$, halts in some finite number of steps. Let T be the maximum number of steps required by the halting programs of Q. For values of *quota* less than T, the simulation of some oracle questions may be wrong, but for every value of $quota \geq T$, they will necessarily be correct. Let α be any sequence in $\mathbf{2}^\omega$ and consider the computation of M on input $p\alpha$. Whatever be α, the amount of bits read by the computation of M on input $p\alpha$ will never exceed $\max(T, N)$ (harmless overshoot) and will, of course, be at least $|p|$. Let S_p be the set of $d \in \mathbf{2}^*$ such that on some input $p\alpha$, M reads exactly the prefix pd of α. Since every $\alpha \in \mathbf{2}^\omega$ extends some $d \in s_p$, Prop.2.1 insures that S_p is finite maximal prefix-free. Together with the inclusion proved in 3, this proves

$$domain(M_{\bowtie}) = \bigcup_{p \in domain(U^{\emptyset'})} pS_p$$

Using Prop.6.3, we get $domain(U^{\emptyset'}) \preceq_{HOS} domain(M_{\bowtie})$. □

6.4.4 Proof of Thm.1.10 randomness in $\emptyset'$

Lemma 6.4 gives a partial self-delimited semicomputable map $F : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ such that $domain(U^{\emptyset'})\mathbf{2}^\omega = domain(F)\mathbf{2}^\omega$ and $domain(F) = F^{-1}(\mathcal{O})$. Prop.4.9 insures that $domain(F)$, hence also $F^{-1}(\mathcal{O})$ is Σ^0_2. Thus, letting $V : \mathcal{S} \to \mathcal{T}$ be $U_{\bowtie} : \mathbf{2}^* \to \mathbf{2}^{\leq\omega}$ and $n = 1$, the conditions of Thm.6.1 hold, yielding the conclusion of Thm.1.10.

6.5 Proof of 2d main theorem: Thm.1.11 (randomness in $\emptyset'$)

Let $V : \mathcal{S} \to \mathcal{T}$ be $U_\infty : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ and $n = 1$. Since $domain(U^{\emptyset'})$ is $\Sigma^0_2(\mathbf{2}^*)$, we see that $domain(U^{\emptyset'})\mathbf{2}^\omega$ is $\Sigma^0_2(\mathbf{2}^\omega)$. Using the hypothesis that $\mathcal{O}$ is semicomputable Wadge hard for Σ^0_2 subsets of $\mathbf{2}^\omega$, there exists a total semicomputable map $F : \mathbf{2}^\omega \to \mathbf{2}^{\leq\omega}$ such that $F^{-1}(\mathcal{O}) = domain(U^{\emptyset'})\mathbf{2}^\omega$. Thus, the conditions of Thm.6.1 hold, yielding the conclusion of Thm.1.11.

7 Proof of corollaries

In order to apply the main theorems, we have to bound the syntactical complexity of $U^{-1}_{\bowtie}(\mathcal{O})$ and $U^{-1}_{\infty}(\mathcal{O})$.

7.1 Finite unions of prefix-free sets: the bounded chain condition

The following notion leads to low syntactical complexity for some interesting classes of subsets of $\mathbf{2}^{\leq\omega}$.

Definition 7.1. *A set $X \subseteq \mathbf{2}^*$ satisfies the k-bounded chain condition, in short X is* k-bdd-chain*, if every monotone strictly increasing chain in X (with respect to the prefix ordering) has at most k elements.*
X satisfies the bounded chain condition, in short X is bdd-chain*, if it satisfies the k-bounded chain condition for some k.*

Proposition 7.2.
1. A set $X \subseteq \mathbf{2}^*$ satisfies the k-bounded chain condition if and only if it is the union of at most k many prefix-free sets.
2. If X is recursive then these prefix-free sets can be taken recursive.
3. If X is r.e. then these prefix-free sets can be taken r.e. (in other words, if an r.e. set is the union of k prefix-free sets then it is the union of k r.e. prefix-free sets).

Proof. All $\Leftarrow$ implications are trivial. Let's prove the $\Rightarrow$ ones.
1. Define inductively subsets $X_i \subseteq X$ as follows: $X_0 = \min(X)$, $X_{i+1} = \min(\{u \in X : \exists v \in X_i \ v \prec u\})$. It is easy to check that if X is k-bdd-chain and then $X = X_0 \cup \ldots X_{k-1}$.
2. In case X is recursive, so are the X_i's.
3. For the case X is r.e., the construction needs to be modified. Arguing by induction, it suffices to partition any k-bdd-chain set X (with $k \geq 2$), into two r.e. sets Y and Z such that Y is prefix-free and Z is $k-1$-bdd-chain. Suppose $X = range(\theta)$ where $\theta : \mathbb{N} \to \mathbb{N}$ is total recursive. We construct Y and Z by stages: $Y = \bigcup_{t\in\mathbb{N}} Y_t$ and $Z = \bigcup_{t\in\mathbb{N}} Z_t$ where

- $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$ and $Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots$,
- Y_t is finite prefix-free,
- Z_t is finite $k-1$-bdd-chain,
- $Y_t \cap Z_t = \emptyset$.
- $Y_0 = \{\theta(0)\}$ and $Z_0 = \emptyset$
- $Y_{t+1} \cup Z_{t+1} = Y_t \cup Z_t \cup \{\theta(t+1)\}$

The inductive construction puts $\theta(t+1)$ in Y_{t+1} if $Y_t \cup \{\theta(t+1)\}$ is still prefix-free. Else, $\theta(t+1)$ is put in Z_{t+1}.
It is clear that Y and Z are r.e. and that Y_t is always prefix-free. Let's prove by induction on t that Z_t satisfies the $k-1$-bounded chain condition. The case $t = 0$ is trivial. Suppose $v_1 \prec ... \prec v_k$ were a chain of elements in Z_{t+1}. Let $v_k = \theta(s)$ with $s \leq t$. The fact that, at stage $s+1$, the element v_k has been put in Z_{s+1} and not in Y_{s+1}, means that there exists $u \in Y_s$ such that u and v_k are prefix comparable.
If $v_k \prec u$ then $v_1, ..., v_k, u$ is a $k+1$ chain in X, a contradiction.
If $u \prec v_k$ then u can be inserted inside the chain $v_1, ..., v_k$ to make a $k+1$ chain $v_1, ..., v_{i-1}, u, v_i, ..., u$. Again, a contradiction.
This proves that Z satisfies the $k-1$-chain condition. □

7.2 Syntactical complexity of $U_{\bowtie}^{-1}(\mathcal{O})$ and $U_{\infty}^{-1}(\mathcal{O})$

For given sets $\mathcal{O} \subseteq \mathbf{2}^{\leq\omega}$ we study the complexity of the sets $U_{\bowtie}^{-1}(\mathcal{O}) \subseteq \mathbf{2}^*$ and $U_{\infty}^{-1}(\mathcal{O}) \subseteq \mathbf{2}^{\omega}$. As expected, they are always at least as complex as $\mathcal{O}$ in their respective spaces.

Remark 7.3. Clearly, $U_{\bowtie}^{-1}(\mathcal{O})\mathbf{2}^{\omega} \subseteq U_{\infty}^{-1}(\mathcal{O})$. In fact, $U_{\bowtie}^{-1}(\mathcal{O})\mathbf{2}^{\omega}$ is the subset of sequences $\alpha \in U_{\infty}^{-1}(\mathcal{O})$ such that U reads only a finite part of α during its possibly infinite computation.

Theorem 7.4. *Table 1 summarizes the syntactical complexity of $U_{\bowtie}^{-1}(\mathcal{O})$ and $U_{\infty}^{-1}(\mathcal{O})$ for $\mathcal{O}$ in some particular classes. For each complexity (up to the second level) it also gives the simplest and hardest possible $\mathcal{O}$'s.*

Note 7.5. The optimal character of the results in Table 1 can be shown using Wadge hard sets for semicomputable reductions, cf. [3].

Proof. Let $out : \mathbf{2}^* \times \mathbb{N} \to \mathbf{2}^*$ be the total recursive map such that $out(p,t)$ is the current output at computation step t of the universal machine U on input p, no matter if U has halted or overread p (the problem of self-delimitation of p is to be considered separately). Observe that out is monotone increasing in its second argument with respect to the prefix ordering. Also, in case of an infinite input α, at step t at most t symbols have been read, so that the current output is exactly $out(\alpha \upharpoonright t, t)$.
Recall that $domain(U_{\bowtie}) \subseteq \mathbf{2}^*$ has complexity $\Sigma_1^0 \wedge \Pi_1^0$ (cf. Prop.4.9).
We now consider the different cases from Table 1 and express $U_{\bowtie}^{-1}(\mathcal{O})$ and $U_{\infty}^{-1}(\mathcal{O})$ by formulas having the stated syntactical complexities.

Table line 1a. Suppose $Y \subseteq \mathbf{2}^*$ is Σ_1^0. Then,

Table 1

	$\mathcal{O}$	$X, Y \subseteq \mathbf{2}^*$	$U_{\bowtie}^{-1}(\mathcal{O})$	$U_{\infty}^{-1}(\mathcal{O})$
1a	$Y\mathbf{2}^{\leq\omega}$	Y Σ^0_1 ,	$\Sigma^0_1 \wedge \Pi^0_1$	Σ^0_1
1b	X	X Σ^0_1 prefix-free	$\Sigma^0_1 \wedge \Pi^0_1$	$\Sigma^0_1 \wedge \Pi^0_1$
2a	$\{u, uv\}$	$u, v \in \mathbf{2}^*$, $v \neq \lambda$	$bool(\Sigma^0_1)$	$bool(\Sigma^0_1)$
2b	$X \cup Y\mathbf{2}^{\leq\omega}$	X, Y Σ^0_1 and X bdd-chain	$bool(\Sigma^0_1)$	$bool(\Sigma^0_1)$
3a	X	X Σ^0_1	Σ^0_2	Σ^0_2
3b	X	X Π^0_1	Σ^0_2	Σ^0_2
3c	$Y\mathbf{2}^{\leq\omega}$	Y Π^0_1	Σ^0_2	Σ^0_2
3d	$X \cup Y\mathbf{2}^{\leq\omega}$	X, Y Σ^0_2	Σ^0_2	Σ^0_2
4a	$\mathbf{2}^{\leq\omega} \setminus (X \cup Y\mathbf{2}^{\leq\omega})$	X, Y Σ^0_1	Π^0_2	Π^0_2
4b	$\mathbf{2}^{\leq\omega} \setminus (X \cup Y\mathbf{2}^{\leq\omega})$	X, Y Σ^0_2	Π^0_2	Π^0_2
4c	$\mathcal{O} \subseteq \mathbf{2}^\omega$ is $\Pi^0_2(\mathbf{2}^\omega)$		Π^0_2	Π^0_2
4d	$X \cup Y\mathbf{2}^{\leq\omega} \cup \mathcal{Z}$	X, Y Σ^0_1, X bdd-chain and $\mathcal{Z} \subseteq \mathbf{2}^\omega$ is Π^0_2	Π^0_2	Π^0_2
5a	$\mathcal{O}$ is $\Sigma^0_2(\mathbf{2}^{\leq\omega})$		$bool(\Sigma^0_2)$	$bool(\Sigma^0_2)$
5b	$\mathcal{O}$ is $\Pi^0_2(\mathbf{2}^{\leq\omega})$		$bool(\Sigma^0_2)$	$bool(\Sigma^0_2)$

(for the definition of bdd-chain see Def.7.1)

$$\begin{aligned} p \in {U_{\bowtie}}^{-1}(Y\mathbf{2}^{\leq\omega}) &\Leftrightarrow p \in domain(U_{\bowtie}) \ \wedge\ \exists y \in Y\ \exists t\ y \preceq out(p,t) \\ \alpha \in {U_{\infty}}^{-1}(Y\mathbf{2}^{\leq\omega}) &\Leftrightarrow \exists y \in Y\ \exists t\ y \preceq out(\alpha\restriction t, t) \end{aligned}$$

Table line 1b. Suppose $X \subseteq \mathbf{2}^*$ is Σ^0_1 prefix-free. Then

$$\begin{aligned} p \in {U_{\bowtie}}^{-1}(X) &\Leftrightarrow \exists t\ out(p,t) \in X \ \wedge\ \forall t\ \forall x \in X\ \neg(x \prec out(p,t)) \\ \alpha \in {U_{\infty}}^{-1}(X) &\Leftrightarrow \exists t\ out(\alpha\restriction t, t) \in X \wedge\ \forall t\ \forall x \in X\ \neg(x \prec out(\alpha\restriction t, t)) \end{aligned}$$

Table line 2a–b. It clearly suffices to prove 2b. Using Prop.7.2, we have $X = X_1 \cup \ldots X_k$ for some k, where the X_i's are r.e. prefix-free. Apply Table lines 1a, 1b to the X_i's and $Y\mathbf{2}^{\leq\omega}$.

Table line 3a–d. It clearly suffices to prove 3d. Suppose $\mathcal{O} = X \cup Y\mathbf{2}^{\leq\omega}$ where $X, Y \subseteq \mathbf{2}^*$ are Σ^0_2. Then

$$\begin{aligned} p \in U_{\bowtie}^{-1}(\mathcal{O}) \Leftrightarrow\ & p \in domain(U_{\bowtie}) \\ & \wedge\ (U_{\bowtie}(p) \text{ is in } X \text{ or extends an element of } Y) \\ \Leftrightarrow\ & p \in domain(U_{\bowtie}) \\ & \wedge\ [\exists y\ \exists t\ (y \in Y \ \wedge\ y \preceq out(p,t)) \\ & \quad \vee\ \exists t\ (out(p,t) \in X \ \wedge\ \forall t' > t\ out(p,t) = out(p,t'))] \end{aligned}$$

$\alpha \in U_{\infty}^{-1}(\mathcal{O})$ can be expressed similarly: forget the first condition about the domain and replace p by $\alpha\restriction t$.

Table lines 4a, 4b. Direct corollaries of Table line 3d.

Table line 4c. Suppose $\mathcal{O} \subseteq \mathbf{2}^\omega$ is defined as follows:

$$\alpha \in \mathcal{O} \quad \Leftrightarrow \ \forall i\ \exists j \geq i\ R(i, j, \alpha \restriction j)$$

where R is recursive. Then

$$\begin{array}{rcl} p \in U_{\bowtie}^{-1}(\mathcal{O}) & \Leftrightarrow & p \in domain(U_{\bowtie}) \\ & & \quad \wedge\ \forall i\ \exists j \geq i\ \exists t\ \exists u\ (out(p,t) = u \wedge |u| = j \wedge R(i,j,u)) \\ \alpha \in U_{\infty}^{-1}(\mathcal{O}) & \Leftrightarrow & \forall i\ \exists j \geq i\ \exists t\ \exists u\ (out(\alpha \restriction t, t) = u \wedge |u| = j \wedge R(i,j,u)) \end{array}$$

Table line 4d. Apply Table lines 2b, 4c to $X \cup Y\mathbf{2}^{\leq\omega}$ and $\mathcal{Z}$.

Table line 5a. Using Prop.2.5, let $\mathcal{O} = \bigcup_{i \in \mathbb{N}} \mathbf{2}^{\leq\omega} \setminus (X_i \cup Y_i\mathbf{2}^{\leq\omega})$ where $X, Y \subseteq \mathbb{N} \times \mathbf{2}^*$ are r.e. Then,

$$\begin{array}{rcl} p \in U_{\bowtie}^{-1}(\mathcal{O}) & \Leftrightarrow & p \in domain(U_{\bowtie}) \\ & & \wedge(U_{\bowtie}(p) \text{ is finite in } \mathcal{O} \text{ or infinite in } \mathcal{O}) \\ & \Leftrightarrow & p \in domain(U_{\bowtie})\ \wedge \\ & & \{[\exists i \exists t \forall t' > t (out(p,t) = out(p,t') \wedge out(p,t) \notin X_i \cup Y_i\mathbf{2}^*)] \\ & & \vee[(\forall t \exists t' > t\ out(p,t) \prec out(p,t')) \\ & & \qquad \wedge\ \exists i \forall y \forall t (y \in Y_i \Rightarrow \neg(y \preceq out(p,t)))]\} \end{array}$$

$\alpha \in U_{\infty}^{-1}(\mathcal{O})$ can be expressed similarly: forget the first condition about the domain and replace p by $\alpha \restriction t$. □

7.3 Proof of Corollaries 1.17, 1.19, 1.20 and Prop.1.12

Corollary 1.17 : use line 3d of Table 1 (Thm.7.4) and Thm.1.10. Corollaries 1.19, 1.20 : use lines 3d, 4d of Table 1 and Thm.1.11. For the particular case $\mathcal{Z} = \mathbf{2}^\omega \setminus \mathbf{2}^*0^\omega$ stated in Cor.1.20, use Prop.5.8. Prop.1.12 : line 2b of Table 1 insures that $U_{\infty}^{-1}(\mathcal{O})$ is Δ_2^0 and Prop.3.3 insures that $\mu(U_{\infty}^{-1}(\mathcal{O}))$ is computable in $\emptyset'$, hence not random in $\emptyset'$.

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