Kolmogorov Complexity and Set theoretical Representations of Integers

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Abstract

We reconsider some classical natural semantics of integers (namely iterators of functions, cardinals of sets, index of equivalence relations) in the perspective of Kolmogorov complexity. To each such semantics one can attach a simple representation of integers that we suitably effectivize in order to develop an associated Kolmogorov theory. Such effectivizations are particular instances of a general notion of "self-enumerated system" that we introduce in this paper. Our main result asserts that, with such effectivizations, Kolmogorov theory allows to quantitatively distinguish the underlying semantics. We characterize the families obtained by such effectivizations and prove that the associated Kolmogorov complexities constitute a hierarchy which coincides with that of Kolmogorov complexities defined via jump oracles and/or infinite computations (cf. [6]). This contrasts with the well-known fact that usual Kolmogorov complexity does not depend (up to a constant) on the chosen arithmetic representation of integers, let it be in any base $n \geq 2$ or in unary. Also, in a conceptual point of view, our result can be seen as a mean to measure the degree of abstraction of these diverse semantics.

1 Introduction

1.1 Notations

Notation 1.1. Equality, inequality and strict inequality up to a constant between total functions $D \to \mathbb{N}$, where D is any set, are denoted as follows:

$$
\begin{array}{rcl}
f \ \leq_{\mathrm{ct}} \ g & \Leftrightarrow & \exists c \in \mathbb{N}\ \forall x \in D\ f(x) \leq g(x) + c \\
f \ =_{\mathrm{ct}} \ g & \Leftrightarrow & f \leq_{\mathrm{ct}} g\ \wedge\ g \leq_{\mathrm{ct}} f \\
 & \Leftrightarrow & \exists c \in \mathbb{N}\ \forall x \in D\ |f(x) - g(x)| \leq c \\
f \ <_{\mathrm{ct}} \ g & \Leftrightarrow & f \leq_{\mathrm{ct}} g\ \wedge\ \neg(g \leq_{\mathrm{ct}} f) \\
 & \Leftrightarrow & f \leq_{\mathrm{ct}} g\ \wedge\ \forall c \in \mathbb{N}\ \exists x \in D\ g(x) > f(x) + c
\end{array}
$$

As we shall consider $\mathbb{N}$-valued partial functions with domain $\mathbb{N}$, $\mathbb{Z}$, $\mathbf{2}^*$, $\mathbb{N}^2$,..., the following definition is convenient.

Definition 1.2. A basic set $\mathbb{X}$ is any non empty finite product of sets among $\mathbb{N}, \mathbb{Z}$ or the set $\mathbf{2}^*$ of finite binary words or the set Σ^* of finite words in some finite or countable alphabet Σ.

Let's also introduce some notations for partial recursive functions.

Notation 1.3. We denote by Y^X (resp. $X \to Y$) the set of total (resp. partial) functions from X to Y.
Let $\mathbb{X}, \mathbb{Y}$ be basic sets. We denote by $PR^{\mathbb{X}\to\mathbb{Y}}$ and $Rec^{\mathbb{X}\to\mathbb{Y}}$ (resp. $PR^{A,\mathbb{X}\to\mathbb{Y}}$ and $Rec^{A,\mathbb{X}\to\mathbb{Y}}$) the family of partial and total recursive (resp. A-recursive) functions $\mathbb{X} \to \mathbb{Y}$. In case $\mathbb{X} = \mathbb{Y} = \mathbb{N}$,we simply write PR, Rec and PR^A, Rec^A.

1.2 Kolmogorov complexity and representations of $\mathbb{N}$, $\mathbb{Z}$

Kolmogorov complexity $K : \mathbb{N} \to \mathbb{N}$ maps an integer n onto the length of any shortest binary program $\mathtt{p} \in \mathbf{2}^*$ which outputs n. The invariance theorem asserts that, up to an additive constant, K does not depend on the program semantics $\mathtt{p} \mapsto n$, provided it is a universal partial recursive function.
As a straightforward corollary of the invariance theorem, K does not depend (again up to a constant) on the representation of integers, i.e. whether the program output n is really in $\mathbb{N}$ or is a word in some alphabet $\{1\}$ or $\{0, ..., k-1\}$, for some $k \geq 2$, which gives the unary or base k representation of n. A result which is easily extended to all partial recursive representations of integers, cf. Thm.2.6.

In this paper, we show that this is no more the case with (suitably effectivized) representations associated to classical set theoretical semantics of integers.
We shall particularly consider the following semantics which stress some role of integers:

- integers as iterators (Church [4], 1933). Restricting to injective functions, one can consider positive and negative iterations which is a semantics for relative integers.

- integers as cardinals of sets (Russell [16] §IX, 1908, cf. [21] p.178),

- integers as cardinals of quotient sets, i.e. indexes of equivalence relations,

- relative integers as differences of integers. A notion also used for non negative integers in order to measure how much bigger some set (resp. iterator) is relative to some other one.

Programs are at the core of Kolmogorov theory. They do not work on abstract entities but require formal representations of objects. Thus, we have to define effectivizations of the above abstract set theoretical notions in order to allow their elements to be computed by programs. To do so, we use computable functions and functionals and recursively enumerable sets.

Effectivized representations of integers constitute particular instances of *self-enumerated systems* (Def.3.1). This is a notion of family $\mathcal{F}$ of partial functions from $\mathbf{2}^*$ to some fixed set D for which an invariance theorem can be proved using straightforward adaptation of original Kolmogorov's proof (Thm.3.11). Which leads to a notion of Kolmogorov complexity $K^D_{\mathcal{F}} : D \to \mathbb{N}$ (Def.3.12). The ones considered in this paper are

$$K^{\mathbb{N}}_{Church}\ ,\ K^{\mathbb{Z}}_{Church}\ ,\ K^{\mathbb{Z}}_{\Delta Church}\ ,\ K^{\mathbb{N}}_{card}\ ,\ K^{\mathbb{Z}}_{\Delta card}\ ,\ K^{\mathbb{N}}_{index}\ ,\ K^{\mathbb{Z}}_{\Delta index}$$

associated to the systems obtained by effectivization of the Church, cardinal and "quotient cardinal" representations of $\mathbb{N}$ and the passage to differences of integers as outlined above.

The main result of this paper states that the above Kolmogorov complexities coincide (up to an additive constant) with those obtained via oracles and infinite computations (see §6) as introduced in [1], [2], and our paper [6].

Theorem 1.4 (Main result).

$$\begin{array}{rclcrcl}
K^{\mathbb{N}}_{Church} & =_{\rm ct} & K^{\mathbb{Z}}_{Church}\restriction\mathbb{N} & =_{\rm ct} & K^{\mathbb{Z}}_{\Delta Church}\restriction\mathbb{N} & =_{\rm ct} & K \\
K^{\mathbb{N}}_{card} & =_{\rm ct} & K_{\max} & & K^{\mathbb{Z}}_{\Delta card}\restriction\mathbb{N} & =_{\rm ct} & K^{\emptyset'} \\
K^{\mathbb{N}}_{index} & =_{\rm ct} & K^{\emptyset'}_{\max} & & K^{\mathbb{Z}}_{\Delta index}\restriction\mathbb{N} & =_{\rm ct} & K^{\emptyset''}
\end{array}$$

Thm.1.4 gathers the contents of Thms. 7.5, 7.6, 8.9, 8.10, Cor.9.15 and §9.10.

A preliminary "light" version of this result was presented in [5], 2002.

The strict ordering result $K >_{\rm ct} K_{\max} >_{\rm ct} K^{\emptyset'}$ (cf. Notations 1.1) proved in [1, 6] and its obvious relativization (cf. Prop.6.9) yield the following hierarchy theorem.

Theorem 1.5.

$$\begin{array}{llllllllll} & K^{\mathbb{N}}_{Church} & & & & & & & & \\ & =_{\mathrm{ct}} & & & & & & & & \\ \log >_{\mathrm{ct}} & K^{\mathbb{Z}}_{Church}\upharpoonright\mathbb{N} & >_{\mathrm{ct}} & K^{\mathbb{N}}_{card} & >_{\mathrm{ct}} & K^{\mathbb{Z}}_{\Delta card}\upharpoonright\mathbb{N} & >_{\mathrm{ct}} & K^{\mathbb{N}}_{index} & >_{\mathrm{ct}} & K^{\mathbb{Z}}_{\Delta index}\upharpoonright\mathbb{N} \\ & =_{\mathrm{ct}} & & & & & & & & \\ & K^{\mathbb{Z}}_{\Delta Church}\upharpoonright\mathbb{N} & & & & & & & & \end{array}$$

This hierarchy result for set theoretical representations somewhat reflects the *degrees of abstraction* of the underlying semantics.

Though Church representation via iteration functionals can be considered as somewhat complex, we see that, surprisingly, the associated Kolmogorov complexities collapse to the simplest possible one.

Also, it turns out that, for cardinal and index representations, the passage from $\mathbb{N}$ to $\mathbb{Z}$, i.e. from $K^{\mathbb{N}}_{card}$ to $K^{\mathbb{Z}}_{\Delta card}$ and from $K^{\mathbb{N}}_{index}$ to $K^{\mathbb{Z}}_{\Delta index}$ does add complexity. However, for Church iterators, the passage to $\mathbb{Z}$ does not modify Kolmogorov complexity, let it be via the Δ operation (for $K^{\mathbb{Z}}_{\Delta Church}$) or restricting iterators to injective functions (for $K^{\mathbb{Z}}_{Church}$).

The results about the $\Delta card$ and $\Delta index$ classes are corollaries of those about the *card* and *index* classes and of the following result (Thm.6.10) which gives a simple normal form for functions computable relative to a jump oracle, and is interesting on its own.

Theorem 1.6. *Let $A \subseteq \mathbb{N}$. A function $G : \mathbf{2}^* \to \mathbb{Z}$ is partial A'-recursive if and only if there exist total A-recursive functions $f, g : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ such that, for all* $\mathtt{p}$*,*

$$G(\mathtt{p}) = \max\{f(\mathtt{p}, t) : t \in \mathbb{N}\} - \max\{g(\mathtt{p}, t) : t \in \mathbb{N}\}$$

(in particular, $G(\mathtt{p})$ is defined if and only if both max*'s are finite).*

1.3 Systems associated to representations of $\mathbb{N}$, $\mathbb{Z}$

The equalities and inequalities (up to a constant) in Theorems 1.4, 1.5 are, in fact, corollaries of equalities and inequalities between families of functions $\mathbf{2}^* \to \mathbb{N}$ (namely, the associated self-enumerated systems, cf. §3.1) which are interesting on their own. Consideration of such families allows to distinguish the usual recursive representations (in base $n \geq 2$ or in unary) from Church representation though $K =_{\mathrm{ct}} K_{Church}$ (cf. point 1 of Thm1.7).
The following result gathers Thms.7.5, 7.6, 8.5, 8.10, 9.14 and §9.10),

Theorem 1.7. *Denote by $X \to Y$ the class of* partial *functions from X to Y.*
Let card : $P(\mathbb{N}) \to \mathbb{N}$ be the cardinal function defined on finite sets.
Let index : $P(\mathbb{N}^2) \to \mathbb{N}$ be defined on equivalence relations with finitely many

classes and give the index (i.e. the number of equivalence classes).
Let Church : $(\mathbb{N} \to \mathbb{N})^{(\mathbb{N}\to\mathbb{N})} \to \mathbb{N}$ *be the functional such that*

$$Church(\Psi) = \begin{cases} n & \text{if } \Psi \text{ is the iterator } f \mapsto f^{(n)} \\ \text{undefined} & \text{otherwise} \end{cases}$$

1. *A function* $F : \mathbf{2}^* \to \mathbb{N}$ *is of the form* $F = Church \circ \Phi$ *where* $\Phi : \mathbf{2}^* \to (\mathbb{N} \to \mathbb{N})^{(\mathbb{N}\to\mathbb{N})}$ *is a computable functional, if and only if* F *is the restriction to a* Π^0_2 *set of a partial recursive function.*

2. *A function* $F : \mathbf{2}^* \to \mathbb{N}$ *is of the form* $\mathtt{p} \mapsto card(W^{\mathbb{N}}_{\varphi(\mathtt{p})})$ *for some total (resp. partial) recursive* $\varphi : \mathbf{2}^* \to \mathbb{N}$ *if and only if* F *is the* max *of a total (resp. partial) recursive sequence of functions (cf. Def.6.1).*

3 *A function* $F : \mathbf{2}^* \to \mathbb{N}$ *is of the form* $\mathtt{p} \mapsto index(W^{\mathbb{N}^2}_{\varphi(\mathtt{p})})$ *for some total (resp. partial) recursive* $\varphi : \mathbf{2}^* \to \mathbb{N}$ *if and only if* F *is the* max *of a total (resp. partial)* $\emptyset'$*-recursive sequence of functions and satisfies* $F^{-1}(0)$ *is* Π^0_1 *(resp.* $\Sigma^0_1 \wedge \Pi^0_1$ *with the* Σ^0_1 *part containing domain*(F)*).*

4. *A function* $F : \mathbf{2}^* \to \mathbb{N}$ *is of the form* $\mathtt{p} \mapsto card(W^{\mathbb{N}}_{\varphi_1(\mathtt{p})}) - card(W^{\mathbb{N}}_{\varphi_2(\mathtt{p})})$ *(resp.* $\mathtt{p} \mapsto index(W^{\mathbb{N}^2}_{\varphi_1(\mathtt{p})}) - index(W^{\mathbb{N}^2}_{\varphi_2(\mathtt{p})})$*)) for some total recursive* $\varphi_1, \varphi_2 : \mathbf{2}^* \to \mathbb{N}$ *if and only if* F *is partial* $\emptyset'$*-recursive (resp.* $\emptyset''$*-recursive).*

1.4 Road map of the paper

§2 introduces abstract representations and their effectivizations.
§3 is devoted to the notion of self-enumerated system with its associated Kolmogorov complexity. Simple operations on self-enumerated systems are studied in §4. The self-enumerated system for the set of recursively enumerable subsets of $\mathbb{N}$ is defined in §5.
§6 recalls material from Becher & Chaitin & Daicz, 2001 [1] and our paper [6], 2004, about some extensions of Kolmogorov complexity involving infinite computations. This is to make the paper self-contained.
§7, 8, 9 develop the effectivizations of the set-theoretical semantics mentioned in §1.2 and prove all the mentioned theorems.

2 Abstract representations and effectivizations

2.1 Some arithmetical representations of $\mathbb{N}$

Abstract entities such as numbers can be represented in many different ways. In fact, each representation illuminates some particular role and/or property, i.e. some possible semantics chosen in order to efficiently access special operations or stress special properties of integers.

Usual arithmetical representations of $\mathbb{N}$ using words on a digit alphabet can be looked at as (total) surjective (non necessarily injective) functions $R : C \to \mathbb{N}$ where C is some simple free algebra or a quotient of some free algebra. Such representations are the "degree zero" of abstraction for representations and, as is well-known, their associated Kolmogorov complexities all coincide (Thm.2.6 below).

Example 2.1 (Base k representations).
1. Integers in unary representation correspond to elements of the free algebra built up from one generator and one unary function, namely 0 and the successor function $x \mapsto x+1$. The associated function $R : 1^* \to \mathbb{N}$ is simply the length function.

2. The various base k (with $k \geq 2$) representations of integers also involve term algebras, not necessarily free. They differ by the set $A \subset \mathbb{N}$ of digits they use but all are based on the usual interpretation $R : A^* \to \mathbb{N}$ such that $R(a_n \ldots a_1 a_0) = \sum_{i=0,\ldots,n} a_i k^i$. Which, written à la Hörner,

$$k(k(\ldots k(ka_n + a_{n-1}) + a_{n-2}) \ldots) + a_1) + a_0$$

is a composition of applications $S_{a_0} \circ S_{a_1} \circ \ldots \circ S_{a_n}(0)$ where $S_a : x \mapsto kx + a$. If a representation uses digits $a \in A$ then it corresponds to the algebra generated by 0 and the S_a's where $a \in A$.

i. The k-adic representation uses digits $1, 2, \ldots, k$ and corresponds to a free algebra built up from one generator and k unary functions.

ii. The usual k-ary representation uses digits $0, 1, \ldots, k-1$ and corresponds to the quotient of a free algebra built up from one generator and k unary functions, namely 0 and the S_a's where $a = 0, 2, \ldots, k-1$, by the relation $S_0(0) = 0$.

iii. Avizienis base k representation uses digits $-k+1, \ldots, -1, 0, 1, \ldots, k-1$ (it is a much redundant representation used in computers to perform additions without carry propagation) and corresponds to the quotient of the free algebra built up from one generator and $2k-1$ unary functions, namely 0 and the S_a's where $a = -k+1, \ldots, -1, 0, 1, \ldots, k-1$, by the relations $\forall x\ (S_{-k+i} \circ S_{j+1}(x) = S_i \circ S_j(x))$ where $-k < j < k-1$ and $0 < i < k$.

Somewhat exotic representations of integers can also be associated to deep results in number theory.

Example 2.2.
1. $R : \mathbb{N}^4 \to \mathbb{N}$ such that $R(x, y, z, t) = x^2 + y^2 + z^2 + t^2$ is a representation based on Lagrange's four squares theorem.

2. $R : (Prime \cup \{0\})^7 \to \mathbb{N}$ such that $R(x_1, \ldots, x_7) = x_1 + \ldots + x_7$ is a representation based on Schnirelman's theorem (1931) in its last improved version obtained by Ramaré, 1995 [13], which insures that every even number is the sum of at most 6 prime numbers (hence every number is the sum of at most 7 primes).

Such representations appear in the study of the expressional power of some weak arithmetics. For instance, the representation as sums of 7 primes allows for a very simple proof of the definability of multiplication with addition and the divisibility predicate (a result valid in fact with successor and divisibility, (Julia Robinson, 1948 [14])).

2.2 Abstract representations

Foundational questions, going back to Russell, [16] 1908, and Church, [4] 1933, lead to quite different representations of $\mathbb{N}$: set theoretical representations involving abstract sets and functionals much more complex than the integers they represent.

We shall consider the following simple and general notion.

Definition 2.3 (Abstract representations).
A representation of an infinite set E is a pair (C, R) where C is some (necessarily infinite) set and $R : C \to E$ is a *surjective partial* function.

Remark 2.4.
1. Though R really operates on the sole subset $domain(R)$, the underlying set C is quite significant in the effectivization process which is necessary to get some Kolmogorov complexity.
2. We shall consider representations with arbitrarily complex domains in the Post hierarchy (cf. Prop.7.4, 8.3, 9.14). In fact, the sole cases in this paper where R is a total function are the usual recursive representations.
3. Representations can also involve a proper class C. However, we shall stick to the case C is a set.

2.3 Effectivizing representations: why?

Turning to a computer science (or recursion theoretic) point of view, there are some objections to the consideration of abstract sets, functions and functionals as we did in §2.2:

- We cannot apprehend abstract sets, functions and functionals but solely programs to compute them (if they are computable in some sense).

- Moreover, programs dealing with sets, functions and functionals have to go through some intensional representation of these objects in order to be able to compute with such objects.

To get effectiveness, we turn from set theory to computability theory. Abstract sets, functions and functionals will be "effectivized" via recursively enumerable sets, partial recursive functions or max of total or partial recursive functions, and partial computable functionals.

2.4 Effectivizing of representations: how

A formal representation of an integer n is a finite object (in general a word) which describes some characteristic property of n or of some abstract object which characterizes n. To effectivize a representation $R : C \to E$, we shall process as follows:

1. Restrict the set C to a subfamily D of elements which, in some sense, are computable or partial computable. Of course, we want the restriction of R to D to be still surjective.

2. Consider a natural notion of computability $\mathbf{2}^* \to D$ and the family of such "computable" functions $\mathbf{2}^* \to D$.

3. The effectivizations of $R : C \to E$ are all functions $R \circ \phi : \mathbf{2}^* \to E$ where ϕ is a "computable" function $\mathbf{2}^* \to D$.

Remark 2.5. Whereas abstract representations are quite natural and conceptually simple, their effectivizations $R \circ \phi$ may be quite complex. In the examples we shall consider, their domains involve levels 2 or 3 of the arithmetical hierarchy (cf. Prop.7.4, 8.3). In particular, such representations are not Turing reducible one to the other.

2.5 Partial recursive representations

We already mentioned in §2.1 that all usual arithmetic representations lead to the same Kolmogorov complexity (up to an additive constant). The reason for this fact is the following simple result which insures that point 3 of the process described in §2.4 is trivial for for all partial recursive representations R.

Theorem 2.6. *If C, E are basic sets and $R : C \to E$ is partial recursive and surjective then*

$$R \circ PR^{\mathbf{2}^* \to C} = PR^{\mathbf{2}^* \to E}$$

Proof. Inclusion $R \circ PR^{\mathbf{2}^* \to C} \subseteq PR^{\mathbf{2}^* \to E}$ is trivial. For the other inclusion, using the fact that $R : C \to E$ is surjective, we can define a total recursive right inverse $S : E \to C$ of R such that, for $x \in E$, $S(\mathtt{x})$ is the element of $R^{-1}(x)$ which appears first in a recursive enumeration of the graph of R. Using the trivial inclusion $S \circ PR^{\mathbf{2}^* \to E} \subseteq PR^{\mathbf{2}^* \to C}$ we get

$$PR^{\mathbf{2}^* \to E} = R \circ S \circ PR^{\mathbf{2}^* \to E} \subseteq R \circ PR^{\mathbf{2}^* \to C}$$

□

3 An abstract setting for Kolmogorov complexity: self-enumerated systems

Effectivizations of abstract representations based on classical set theoretical semantics of integers will be done in §7–9. In each case, the family of functions $\mathbf{2}^* \to \mathbb{N}$ so obtained (cf. point 3 of §2.4) has a simple self-enumerability property which suffices for a straightforward adaptation of the usual proof of Kolmogorov invariance theorem.
We now introduce the underlying abstract setting for the definition of Kolmogorov complexity: *self-enumerated systems*. This setting allows to unify the multiple variations of the invariance theorem, the proofs of which repeat, mutatis mutandis, the same classical proof due to Kolmogorov (cf. Li & Vitanyi's textbook [9] p.97).
This abstract setting also leads to a study of operations on self-enumerated systems, for instance that presented in 5.

3.1 Self-enumerated systems

Some intuition for the next definition is given in Note 3.2 and Rk.3.3.

Definition 3.1 (Self-enumerated systems).
1. A self-enumerated system is a pair $(D, \mathcal{F})$ where D is a set (the domain of the system) and $\mathcal{F}$ is a family of partial functions $\mathbf{2}^* \to D$ satisfying the following conditions:

i. $D = \bigcup_{F \in \mathcal{F}} Range(F)$, i.e. every element of D appears in the range of some function $F \in \mathcal{F}$.

ii. If $\varphi : \mathbf{2}^* \to \mathbf{2}^*$ is a recursive total function and $F \in \mathcal{F}$ then $F \circ \varphi \in \mathcal{F}$.

iii. There exists $U \in \mathcal{F}$ (called a universal function for $\mathcal{F}$) and a total recursive function $comp_U : \mathbb{N} \times \mathbf{2}^* \to \mathbf{2}^*$ such that

$$\forall F \in \mathcal{F} \ \ \exists \mathtt{e} \in \mathbb{N} \ \ \forall \mathtt{p} \in \mathbf{2}^* \ \ F(\mathtt{p}) = U(comp_U(\mathtt{e}, \mathtt{p}))$$

In other words, letting $U_{\mathtt{e}}(\mathtt{p}) = U(comp_U(\mathtt{e}, \mathtt{p}))$, the sequence of functions $(U_{\mathtt{e}})_{\mathtt{e} \in \mathbb{N}}$ is an enumeration of $\mathcal{F}$.

2. (Full systems) $(D, \mathcal{F})$ is a *full system* if condition ii holds for all *partial recursive functions* φ.

3. (Good universal functions) A universal function U for $\mathcal{F}$ is good if its associated *comp* function satisfies the condition

$$\forall \mathtt{e} \ \exists c_{\mathtt{e}} \ \forall \mathtt{p} \ |comp_U(\mathtt{e}, \mathtt{p})|) \leq |\mathtt{p}| + c_{\mathtt{e}}$$

i.e. for all $\mathtt{e}$, we have $(\mathtt{p} \mapsto |comp_U(\mathtt{e}, \mathtt{p})|) \leq_{\mathrm{ct}} |\mathtt{p}|$ (cf. Notation 1.1).

Note 3.2 (Intuition).
1. The set $\mathbf{2}^*$ is seen as a family of programs to get elements of D. The choice of binary programs is a fairness condition in view of the definition of Kolmogorov complexity (cf. Def.3.12) based on the length of programs: larger the alphabet, shorter the programs.

2. Each $F \in \mathcal{F}$ is seen as a programming language with programs in $\mathbf{2}^*$. Special restrictions: no input, outputs are elements of D.

3. Denomination *comp* stands for "compiler" since it maps a program $\mathtt{p}$ from "language" F (with code $\mathtt{p}$) to its U-compiled form $comp_U(\mathtt{e}, \mathtt{p})$ in the "language" U.

4. "Compilation" with a good universal function does not increase the length of programs but for some additive constant which depends only on the language, namely on the sole code e.

Remark 3.3. In view of the enumerability condition *iii* and since there is no recursive enumeration of total recursive functions, one would a priori rather require condition *ii* to be true for all partial recursive functions $\varphi : \mathbf{2}^* \to \mathbf{2}^*$, i.e. consider the sole full systems.
However, there are interesting self-enumerated systems which are not full systems. The simplest one is Max_{Rec}, cf. Prop.6.2. Other examples we shall deal with involve higher order domains consisting of infinite objects, for instance the domain $RE(\mathbb{N})$ of all recursively enumerable subsets of $\mathbb{N}$, cf. §5. *The partial character of computability is already inherent to the objects in the domain or to the particular notion of computability and an enumeration theorem does hold for such a family $\mathcal{F}$ of total functions.*

From conditions i and iii of Def.3.1, we immediately see that

Proposition 3.4. *Let $(D, \mathcal{F})$ be a self-enumerated system. Then D and $\mathcal{F}$ are countable and any universal function for $\mathcal{F}$ is surjective.*

3.2 Good universal functions always exist

Notation 3.5. Let $\gamma : \mathbb{N} \times \mathbf{2}^* \to \mathbf{2}^*$ be the recursive total injective map such that $\gamma(\mathtt{e}, \mathtt{p}) = 0^{\mathtt{e}}1p$. Let $\delta_1 : \mathbf{2}^* \to \mathbb{N}$, $\delta_2 : \mathbf{2}^* \to \mathbf{2}^*$ be the total recursive maps such that $(\delta_1, \delta_2)(\gamma(\mathtt{e}, \mathtt{p})) = (\mathtt{e}, \mathtt{p})$ and $(\delta_1, \delta_2)(\mathtt{q}) = (0, \lambda)$ if $\mathtt{q} \notin range(\gamma)$ (where λ is the empty word).

Proposition 3.6 (Existence of good universal functions). *Every self-enumerated system contains a universal function with γ as associated comp function and which is therefore good universal.*

Proof. Let U and $comp_U$ be as in Def.3.1 and set

$$U_{opt} = U \circ comp_U \circ (\delta_1, \delta_2)$$

Since $comp_U \circ (\delta_1, \delta_2) : \mathbf{2}^* \to \mathbf{2}^*$ is total recursive, condition ii of Def.3.1 insures that $U_{opt} \in \mathcal{F}$. Now, we have

$$U_{opt}(\gamma(\mathtt{e}, \mathtt{p})) = U(comp_U((\delta_1, \delta_2)(\gamma(\mathtt{e}, \mathtt{p})))) = U(comp_U(\mathtt{e}, \mathtt{p}))$$

so that U_{opt} is universal with γ as associated *comp* function. □

3.3 Relativization of self-enumerated systems

Def.3.1 can be obviously relativized to any oracle A. However, contrary to what can be a priori expected, this is no generalization but particularization. The main reason is Prop.3.6: there always exists a universal function with γ as associated *comp* function.

Definition 3.7. Let $A \subseteq \mathbb{N}$. A self-enumerated A-system is a pair $(D, \mathcal{F})$ where $\mathcal{F}$ is a family of partial functions $\mathbf{2}^* \to D$ satisfying condition i of Def.3.1 and the variants of conditions ii and iii where recursiveness is replaced by A-recursiveness.

Example 3.8. If $\mathbb{X}$ is a basic set then $(\mathbb{X}, PR^{A,\mathbf{2}^* \to \mathbb{X}})$ is obviously a self-enumerated A-system.

Proposition 3.9. *Every self-enumerated A-system contains a universal function with γ as associated comp function.*
In particular, every such system is also a self-enumerated system. Thus, $(\mathbb{X}, PR^{A,\mathbf{2}^* \to \mathbb{X}})$ is a self-enumerated system.

Proof. Repeat the same easy argument used for Prop.3.6. □

3.4 The Invariance Theorem

Definition 3.10. Let $F : \mathbf{2}^* \to D$ be any partial function. The Kolmogorov complexity $K^D_F : D \to \mathbb{N} \cup \{+\infty\}$ associated to F is the function defined as follows (convention: $\min \emptyset = +\infty$):

$$K^D_F(x) = \min\{|\mathtt{p}| : F(\mathtt{p}) = x\}$$

Thanks to Prop.3.6, the usual Invariance Theorem can be extended to any self-enumerated system, which allows to define Kolmogorov complexity for such a system.

Theorem 3.11 (Invariance Theorem, Kolmogorov, 1965 [7]).
Let $(D, \mathcal{F})$ be a self-enumerated system.

1. *When F varies in the family $\mathcal{F}$, there is a least K^D_F, up to an additive constant (cf. Notation 1.1):*

$$\exists F \in \mathcal{F} \ \ \forall G \in \mathcal{F} \quad K^D_F \leq_{\rm ct} K^D_G$$

Such F's are said to optimal in $\mathcal{F}$.

2. *Every good universal function for $\mathcal{F}$ is optimal.*

Proof. It suffices to prove 2. The usual proof works. Consider a good universal enumeration U of $\mathcal{F}$. Let $F \in \mathcal{F}$ and let $\mathtt{e}$ be such that

$$U(comp_U(\mathtt{e}, \mathtt{p})) = F(\mathtt{p}) \text{ for all } p \in \mathbf{2}^*$$

First, since U is surjective (Prop.3.4), all values of K_U^D are finite. Thus, $K_U^D(x) < K_F^D(x)$ for $x \notin Range(F)$ (since then $K_F^D(x) = +\infty$).
For every $x \in Range(F)$, let $\mathtt{p}_x$ be a smallest program such that $F(\mathtt{p}_x) = x$, i.e. $K_F^D(x) = |\mathtt{p}_x|$. Then $x = F(\mathtt{p}_x) = U(comp_U(\mathtt{e}, \mathtt{p}_x))$ and, since U is good,

$$K_U^D(x) \leq |comp_U(e, \mathtt{p}_x)| \leq |\mathtt{p}_x| + c_{\mathtt{e}} = K_F^D(x) + c_{\mathtt{e}}$$

and therefore $K_U^D \leq_{\mathrm{ct}} K_F^D$. □

As usual, Theorem 3.11 allows for an intrinsic definition of the Kolmogorov complexity associated to the self-enumerated system $(D, \mathcal{F})$.

Definition 3.12 (Kolmogorov complexity of a self-enumerated system). Let $(D, \mathcal{F})$ be a self-enumerated system.
The Kolmogorov complexity $K_{\mathcal{F}}^D : D \to \mathbb{N}$ is the function K_U^D where U is some fixed *good universal enumeration* in $\mathcal{F}$.
Up to an additive constant, this definition is independent of the particular choice of U.

The following straightforward result, based on Example 3.8, insures that Def.3.12 is compatible with the usual Kolmogorov complexity and its relativizations.

Proposition 3.13. *Let $A \subseteq \mathbb{N}$ and let $D = \mathbb{X}$ be a basic set. The Kolmogorov complexities $K^{\mathbb{X}}_{PR^{\mathbf{2}^* \to \mathbb{X}}}$ and $K^{\mathbb{X}}_{PR^{A, \mathbf{2}^* \to \mathbb{X}}}$ defined above are exactly the usual Kolmogorov complexity $K_{\mathbb{X}} : \mathbb{X} \to \mathbb{N}$ and its relativization $K^A_{\mathbb{X}}$.*

4 Some operations on self-enumerated systems

4.1 The composition lemma

The following easy fact is a convenient tool to effectivize representations (cf. §2.3, 2.4). We shall also use it in §4.3 to go from systems with domain $\mathbb{N}$ to ones with domain $\mathbb{Z}$.

Lemma 4.1 (The composition lemma).
Let $(D, \mathcal{F})$ be a self-enumerated system and $\varphi : D \to E$ be a surjective partial function. Set $\varphi \circ \mathcal{F} = \{\varphi \circ F : F \in \mathcal{F}\}$.

1. *$(E, \varphi \circ \mathcal{F})$ is also a self-enumerated system. Moreover, if U is universal or good universal for $\mathcal{F}$ then so is $\varphi \circ U$ for $\varphi \circ \mathcal{F}$.*

2. *For every $x \in E$,*

$$K^E_{\varphi\circ\mathcal{F}}(x) =_{\rm ct} \min\{K^D_{\mathcal{F}}(y) : \varphi(y) = x\}$$

In particular, $K^E_{\varphi\circ\mathcal{F}} \circ \varphi \ \leq_{\rm ct} \ K^D_{\mathcal{F}}$ and if $\varphi : D \to E$ is a total bijection from D to E then $K^E_{\varphi\circ\mathcal{F}} \circ \varphi \ =_{\rm ct} \ K^D_{\mathcal{F}}$.

Proof. Point 1 is straightforward. As for point 2, let $U : \mathbf{2}^* \to D$ be some universal function for $\mathcal{F}$ and observe that, for $x \in E$,

$$\begin{array}{rcl} K^E_{\varphi\circ\mathcal{F}}(x) & = & \min\{|\mathtt{p}| : \mathtt{p} \text{ such that } \varphi(U(\mathtt{p})) = x\} \\ & = & \min\{\min\{|\mathtt{p}| : \mathtt{p} \text{ s.t. } U(\mathtt{p}) = y\} : y \text{ s.t. } \varphi(y) = x\} \\ & = & \min\{K^D_{\mathcal{F}}(y) : y \text{ s.t. } \varphi(y) = x\} \end{array}$$

In particular, taking $x = \varphi(z)$, we get $K^E_{\varphi\circ\mathcal{F}}(\varphi(z)) \leq_{\rm ct} \ K^D_{\mathcal{F}}(z)$.
Finally, observe that if φ is bijective then z is the unique y such that $\varphi(y) = x$, so that the above min reduces to $K^D_{\mathcal{F}}(z)$. □

4.2 Product of self-enumerated systems

We shall need a notion of product of self-enumerated systems.

Theorem 4.2. *Let $(D_1, \mathcal{F}_1)$ and $(D_2, \mathcal{F}_2)$ be self-enumerated systems.*
We identify a pair $(F_1, F_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ with the function $\mathbf{2}^* \to D_1 \times D_2$ which maps $\mathtt{p}$ to $(F_1(\mathtt{p}), F_2(\mathtt{p}))$.
Then $(D_1 \times D_2, \mathcal{F}_1 \times \mathcal{F}_2)$ is also a self-enumerated system.
If $(D_1, \mathcal{F}_1)$ and $(D_2, \mathcal{F}_2)$ are full systems then so is $(D_1 \times D_2, \mathcal{F}_1 \times \mathcal{F}_2)$.

Proof. Let $c : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ be the injective map such that $c(\mathtt{p}, \mathtt{q}) = 0^{|\mathtt{p}|}1\mathtt{pq}$.
Let $\pi_1, \pi_2 : \mathbf{2}^* \to \mathbf{2}^*$ be such that $(\pi_1, \pi_2)(c(\mathtt{p}, \mathtt{q})) = (\mathtt{p}, \mathtt{q})$ and $(\pi_1, \pi_2)(\mathtt{r}) = (\lambda, \lambda)$ if $\mathtt{r} \notin range(c)$. Clearly, c, π_1, π_2 are total recursive.
Condition ii in Def.3.1 is obvious.

Condition i. Let $(d_1, d_2) \in D_1 \times D_2$. Applying condition i to $(D_1, \mathcal{F}_1)$ and to $(D_2, \mathcal{F}_2)$, we get $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ and $\mathtt{p}_1, \mathtt{p}_2 \in \mathbf{2}^*$ such that $d_1 = F_1(\mathtt{p}_1)$ and $d_2 = F_2(\mathtt{p}_2)$. Therefore $(d_1, d_2) = (F_1 \circ \pi_1, F_2 \circ \pi_2)(c(\mathtt{p}_1, \mathtt{p}_2))$. Observe finally that $(F_1 \circ \pi_1, F_2 \circ \pi_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ (condition ii for $(D_1, \mathcal{F}_1), (D_2, \mathcal{F}_2)$).

Condition iii. Let U_1, U_2 be universal for $\mathcal{F}_1, \mathcal{F}_2$ with $comp_1, comp_2 : \mathbb{N} \times \mathbf{2}^* \to \mathbf{2}^*$ as associated *comp* functions. Set

$$U = (U_1 \circ \pi_1, U_2 \circ \pi_2) \ , \ comp(\mathtt{e}, \mathtt{p}) = c(comp_1(\sigma_1(\mathtt{e}), \mathtt{p}), comp_2(\sigma_2(\mathtt{e}), \mathtt{p}))$$

where $\sigma : \mathbb{N}^2 \to \mathbb{N}$ is Cantor polynomial bijection and $(\sigma_1, \sigma_2) = \sigma^{-1}$.
We show that U is universal for $\mathcal{F}_1 \times \mathcal{F}_2$ with associated *comp* function.

For every $(F_1, F_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ there exist $\mathtt{a}, \mathtt{b} \in \mathbf{2}^*$ such that $F_1(\mathtt{p}) = U_1(comp_1(\mathtt{a}, \mathtt{p}))$ and $F_2(\mathtt{p}) = U_2(comp_2(\mathtt{b}, \mathtt{p}))$. Therefore

$$\begin{aligned}(F_1, F_2)(\mathtt{p}) &= (U_1(comp_1(\mathtt{a}, \mathtt{p})), U_2(comp_2(\mathtt{b}, \mathtt{p}))) \\ &= (U_1 \circ \pi_1, U_2 \circ \pi_2)(c(comp_1(\mathtt{a}, \mathtt{p}), comp_2(\mathtt{b}, \mathtt{p}))) \\ &= U(comp(\sigma(\mathtt{a}, \mathtt{b}), \mathtt{p}))\end{aligned}$$

which proves that U is universal for the product system $\mathcal{F}_1 \times \mathcal{F}_2$. □

4.3 From domain $\mathbb{N}$ to domain $\mathbb{Z}$: the Δ operation

Definition 4.3 (The Δ operation). Let $diff : \mathbb{N}^2 \to \mathbb{Z}$ be the function $(m, n) \mapsto m - n$. If $(\mathbb{N}, \mathcal{F})$ is a self-enumerated system with domain $\mathbb{N}$, using notations from Lemma 4.1 and Thm.4.2, we let $(\mathbb{Z}, \Delta\mathcal{F})$ be the system

$$(\mathbb{Z}, diff \circ (\mathcal{F} \times \mathcal{F}))$$

As a direct corollary of Lemma 4.1 and Thm.4.2, we have

Proposition 4.4. *If $(\mathbb{N}, \mathcal{F})$ is a self-enumerated system (resp. full system) with domain $\mathbb{N}$ then so is $(\mathbb{Z}, \Delta\mathcal{F})$.*

The following propositions collect some easy facts about self-enumerated systems with domain $\mathbb{Z}$ and their associated Kolmogorov complexities.

Proposition 4.5. *Let $(\mathbb{Z}, \mathcal{G})$ be a self-enumerated system.*

1. *Let $\mathcal{F} = \{G \restriction G^{-1}(\mathbb{N}) : G \in \mathcal{G}\}$. Then $(\mathbb{N}, \mathcal{F})$ is also a self-enumerated system and $K^{\mathbb{N}}_{\mathcal{F}} = K^{\mathbb{Z}}_{\mathcal{G}} \restriction \mathbb{N}$.*

2. *Denote by $opp : \mathbb{Z} \to \mathbb{Z}$ the function $n \mapsto -n$. If $\mathcal{G} \circ opp = \mathcal{G}$ then $K^{\mathbb{Z}}_{\mathcal{G}} =_{\rm ct} K^{\mathbb{Z}}_{\mathcal{G}} \circ opp$.*

Proof. 1. Conditions i-ii of Def.3.1 are obvious. As for iii, observe that if $U \in \mathcal{G}$ is universal for $\mathcal{G}$ then $U \restriction U^{-1}(\mathbb{N})$ is in $\mathcal{F}$ and is universal for $\mathcal{F}$ with the same associated *comp* function. Now, $K_{U \restriction U^{-1}(\mathbb{N})} = K_U \restriction \mathbb{N}$. Whence $K^{\mathbb{N}}_{\mathcal{F}} = K^{\mathbb{Z}}_{\mathcal{G}} \restriction \mathbb{N}$.

2. Observe that if $\varphi, F \in \mathcal{G}$ and $K_\varphi \leq_{\rm ct} K_F$ then $K_{\varphi \circ opp} \leq_{\rm ct} K_{F \circ opp}$. Since $\mathcal{G} \circ opp = \mathcal{G}$, we see that if φ is optimal then so is $\varphi \circ opp$. Whence $K_\varphi =_{\rm ct} K_{\varphi \circ opp}$, and therefore $K^{\mathbb{Z}}_{\mathcal{G}} =_{\rm ct} K^{\mathbb{Z}}_{\mathcal{G}} \circ opp$. □

Proposition 4.6. *Let $A \subseteq \mathbb{N}$.*

1. $PR^{A,\mathbf{2}^* \to \mathbb{N}} = PR^{A,\mathbf{2}^* \to \mathbb{Z}} \cap (\mathbb{N} \to \mathbb{N}) = \{G \restriction G^{-1}(\mathbb{N}) : G \in PR^{A,\mathbf{2}^* \to \mathbb{Z}}\}$.
In particular, $K^{A,\mathbb{Z}} \restriction \mathbb{N} =_{\rm ct} K^{A,\mathbb{N}}$.

2. $PR^{A,\mathbf{2}^* \to \mathbb{Z}} = PR^{A,\mathbf{2}^* \to \mathbb{Z}} \circ opp = \Delta PR^{A,\mathbf{2}^* \to \mathbb{N}}$.
In particular, $K^{A,\mathbb{Z}} =_{\rm ct} K^{A,\mathbb{Z}} \circ opp$.

5 Self-enumerated systems for r.e. sets

We now come to an example of self-enumerated systems of a somewhat different kind, which will be used in the effectivization of set theoretical representations of integers. We shall use the classical notion of acceptable enumeration of the family $RE(\mathbb{X})$ of recursively enumerable subsets of a basic set $\mathbb{X}$ and Rogers' theorem (cf. Odifreddi [12] p.219).

Theorem 5.1 (Rogers' theorem). *Let* $(W'_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ *and* $(W''_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ *be two acceptable enumerations of* $RE(\mathbb{X})$*. Then there exists a recursive bijection* $\theta : \mathbb{N} \to \mathbb{N}$ *such that* $W''_{\mathtt{e}} = W'_{\theta(\mathtt{e})}$ *for all* $\mathtt{e} \in \mathbf{2}^*$*.*

Rogers' theorem allows to get a natural intrinsic notion of "partial computable" map $\mathbf{2}^* \to RE(\mathbb{X})$.

Notation 5.2. Let $\mathcal{W} = (W_{\mathtt{e}})_{\mathtt{e}\in\mathbb{N}}$ be some fixed acceptable enumeration of $RE(\mathbb{X})$. If $f : \mathbf{2}^* \to \mathbb{N}$ is partial recursive then $G^{\mathcal{W}}_f : \mathbf{2}^* \to RE(\mathbb{X})$ denotes the function such that, for all $\mathtt{p} \in \mathbf{2}^*$,

$$G^{\mathcal{W}}_f(\mathtt{p}) = \begin{cases} W_{f(\mathtt{p})} & \text{if } f(\mathtt{p}) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Proposition 5.3. *The families of functions*

$$\begin{aligned} \mathcal{P}F^{RE(\mathbb{X})}_{\mathcal{W}} &= \{G^{\mathcal{W}}_f : \mathbf{2}^* \to RE(\mathbb{X}) : f \in PR^{\mathbf{2}^*\to\mathbb{N}}\} \\ \mathcal{F}^{RE(\mathbb{X})}_{\mathcal{W}} &= \{G^{\mathcal{W}}_f : \mathbf{2}^* \to RE(\mathbb{X}) : f \in Rec^{\mathbf{2}^*\to\mathbb{N}}\} \end{aligned}$$

do not depend on the considered acceptable enumeration of $RE(\mathbb{X})$*.*
We shall therefore omit the subscript $\mathcal{W}$ *in the sequel.*

Proof. Applying Thm.5.1 to acceptable enumerations $(W'_{\mathtt{e}})_{\mathtt{e}\in\mathbb{N}}$ and $(W''_{\mathtt{e}})_{\mathtt{e}\in\mathbb{N}}$, we get $W''_{f(\mathtt{p})} = W'_{\theta(f(\mathtt{p}))}$ and $W'_{f(\mathtt{p})} = W'_{\theta^{-1}(f(\mathtt{p}))}$. To conclude, observe that $\theta \circ f$ and $\theta^{-1} \circ f$ are both total or partial recursive as is f. □

We shall need the following proposition in order to prove that $\mathcal{F}^{RE(\mathbb{X})}$ is a self-enumerated system.

Proposition 5.4. *There exists a total recursive function* $\sigma : \mathbb{N} \times \mathbf{2}^* \to \mathbb{N}$ *such that, for any total function* $\rho : \mathbf{2}^* \to RE(\mathbb{X})$*, the following conditions are equivalent:*

a. ρ *is of the form* $\mathtt{p} \mapsto W_{\sigma(\mathtt{e},\mathtt{p})}$ *for some* $\mathtt{e} \in \mathbf{2}^*$

b. $\rho \in \mathcal{F}^{RE(\mathbb{X})}$

c. For some $g \in PR^{\mathbf{2}^*\to\mathbb{N}}$*, for all* $\mathtt{p}$*,* $\rho(\mathtt{p}) = \begin{cases} W_{g(\mathtt{p})} & \text{if } g(\mathtt{p}) \text{ is defined} \\ \emptyset & \text{otherwise} \end{cases}$.

Proof. Since $a \Rightarrow b \Rightarrow c$ is trivial whatever be the total recursive function σ, it remains to define σ such that $c \Rightarrow a$ holds.
Let $(\psi_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ be an enumeration of partial recursive functions $\mathbf{2}^* \to \mathbf{2}^*$ The parameter theorem insures that there exists a total recursive function $s : \mathbb{N} \times \mathbf{2}^* \to \mathbb{N}$ such that $W_{\psi_{\mathtt{e}}(\mathtt{p})} = W_{\sigma(\mathtt{e},\mathtt{p})}$. An equality also valid when $\psi_{\mathtt{e}}(\mathtt{p})$ is undefined, in the sense that both sets are empty.
Let ρ, g be as in c. Let $\mathtt{e}$ be such that $g = \psi_{\mathtt{e}}$. Then, $W_{g(\mathtt{p})} = W_{\psi_{\mathtt{e}}(\mathtt{p})} = W_{\sigma(\mathtt{e},\mathtt{p})}$ an equality valid also if $g(\mathtt{p})$ is undefined, in the sense that all sets are empty. This proves $c \Rightarrow a$. $\square$

We can now come to the notion of self-enumerated systems for r.e. sets.

Theorem 5.5 (Self-enumerated systems for r.e. sets).
$(RE(\mathbb{X}), \mathcal{F}^{RE(\mathbb{X})})$ and $(RE(\mathbb{X}), \mathcal{P}F^{RE(\mathbb{X})})$ are self-enumerated systems.

Proof. Conditions i, ii of Def.3.1 are obvious for both systems.
If U satisfies iii for $PR^{\mathbf{2}^*\to\mathbb{N}}$ then G_U (cf. Notation 5.2) satisfies iii for $\mathcal{P}F^{RE(\mathbb{X})}$ with the same associated *comp* function. Also, Prop.5.4 proves that the function $\mathtt{p} \mapsto W_{U(\mathtt{p})}$ satisfies condition iii for $\mathcal{F}^{RE(\mathbb{X})}$ with σ as *comp* function. $\square$

6 Infinite computations

Chaitin, 1976 [3], and Solovay, 1977 [19], considered infinite computations producing infinite objects (namely recursively enumerable sets) so as to define Kolmogorov complexity of such infinite objects.
Following the idea of possibly infinite computations leading to finite output (i.e. remove the halting condition), Becher & Chaitin & Daicz, 2001 [1] (see also [2], 2005) introduced a variant K^∞ of Kolmogorov complexity.
In our paper [6], 2004, we introduced two variants $K_{\max}$, $K_{\min}$ of Kolmogorov complexity and proved that $K^\infty = K_{\max}$. These variants are based on two self-enumerated systems, namely the classes of max and min of partial recursive sequences of partial recursive functions.

6.1 Self-enumerated systems of max of partial recursive functions

Notation 6.1. Let $A \subseteq \mathbb{N}$. Let $\mathbb{X}$ be $\mathbb{N}$ or $\mathbb{Z}$. If $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{X}$, we denote by $\max f : \mathbf{2}^* \to \mathbb{X}$ the function such that $(\max f)(\mathtt{p}) = \max\{f(\mathtt{p}, t) : t \in \mathbb{N}\}$ (with the convention that $\max X$ is undefined if X is empty or infinite).
We define the families of functions

$$\begin{aligned} Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A} &= \{\max f : f \in PR^{A,\mathbf{2}^*\times\mathbb{N}\to\mathbb{X}}\} \\ Max^{\mathbf{2}^*\to\mathbb{X}}_{Rec^A} &= \{\max f : f \in Rec^{A,\mathbf{2}^*\times\mathbb{N}\to\mathbb{X}}\} \end{aligned}$$

Proposition 6.2. *Let $A \subseteq \mathbb{N}$. Then*

$$(\mathbb{N}, Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}) \quad , \quad (\mathbb{Z}, Max^{\mathbf{2}^*\to\mathbb{Z}}_{PR^A}) \quad , \quad (\mathbb{N}, Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$$

are self-enumerated systems.

Proof. First consider the no oracle case (i.e. $A = \emptyset$). Conditions i-ii in Def.3.1 are trivial. The classical enumeration theorem easily extends to $Max^{\mathbf{2}^*\to\mathbb{X}}_{PR}$ (cf. [6], Thm.3.2), proving condition iii for $(\mathbb{X}, Max^{\mathbf{2}^*\to\mathbb{X}}_{PR})$ where $\mathbb{X}$ is $\mathbb{N}$ or $\mathbb{Z}$.
It remains to show condition iii for $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$. We use the following straightforward fact:

Fact 6.3. *If $f \in PR^{\mathbf{2}^*\times\mathbb{N}\to\mathbb{N}}$ and*

$$g(\mathtt{p}, t) = \max(\{0\} \cup \{f(\mathtt{p}, i) : i \leq t \ \wedge \ f(\mathtt{p}, i) \textit{converges in at most } t \textit{ steps}\})$$

then $g \in Rec^{\mathbf{2}^*\times\mathbb{N}\to\mathbb{N}}$ and $\max g$ is an extension of $\max f$ with value 0 on $domain(\max g) \setminus domain(\max f)$ (which is the set of n's such that $f(n,t)$ is defined for no t).

Let $U \in Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ be universal for $Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ and let V be an extension of U in $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$ given by the above fact. If $F \in Rec^{\mathbf{2}^*\to\mathbb{N}}$ then it is in $PR^{\mathbf{2}^*\to\mathbb{N}}$ and there exists $\mathtt{e}$ such that $F(\mathtt{p}) = U(comp_U(\mathtt{e}, \mathtt{p}))$ for all $\mathtt{p} \in \mathbf{2}^*$. Since V extends U and F is total, we also have $F(\mathtt{p}) = V(comp_U(\mathtt{e}, \mathtt{p}))$. Thus, V is universal for $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$ with the same *comp* function.

Relativization to oracle A proves conditions ii^A, iii^A, (cf. Def.3.7) for $(\mathbb{X}, Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A})$ and $(\mathbb{N}, Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec})$. We conclude using Prop.3.9. □

Remark 6.4. The system $(\mathbb{Z}, Max^{\mathbf{2}^*\to\mathbb{Z}}_{Rec^A})$ is not self-enumerated. In fact, it does not satisfy the invariance theorem (cf. [6], Thm.4.8).

6.2 Kolmogorov complexities $K_{\max}, K^{\emptyset'}_{\max}, \ldots$

We apply Def.3.12 to the self-enumerated systems considered in §6.1.

Definition 6.5 (Kolmogorov complexities). Let $\mathbb{X}$ be $\mathbb{N}$ or $\mathbb{Z}$. We denote by $K^{A,\mathbb{X}}_{\max} : \mathbb{X} \to \mathbb{N}$ the Kolmogorov complexity of the self-enumerated system $(\mathbb{X}, Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A})$.
In case $\mathbb{X} = \mathbb{N}$, we omit the superscript $\mathbb{N}$. In case $\mathbb{X} = \mathbb{N}$ and $A = \emptyset$ we simply write $K_{\max}$.

Using Fact 6.3 and the fact that $K^{\mathbb{N}}_G \leq K^{\mathbb{N}}_F$ whenever $F, G : \mathbf{2}^* \to \mathbb{N}$ and F is a restriction of G, it is not hard to prove the following result (cf. [6], Prop.4.8).

Proposition 6.6. *Let $A \subseteq \mathbb{N}$. Then $K^A_{\max}$ is also the Kolmogorov complexity of the self-enumerated system $(\mathbb{N}, Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$. I.e.*

$$K^{\mathbb{N}}_{Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}} = K^{\mathbb{N}}_{Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}}$$

Remark 6.7. The above proposition has no analog with $\mathbb{Z}$ since $Max^{\mathbf{2}^*\to\mathbb{Z}}_{Rec^A}$ is not self-enumerated.

6.3 $Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ and the jump

The following proposition is easy.

Proposition 6.8. *Let $A \subseteq \mathbb{N}$ and let $\mathbb{X}$ be $\mathbb{N}$ or $\mathbb{Z}$. Then*

$$Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A} \subset PR^{A',\mathbf{2}^*\to\mathbb{X}}$$

Proof. 1. Let $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{X}$ be partial A-recursive. A partial A'-recursive definition of $(\max f)(\mathtt{p})$ is as follows:

i. First, check whether there exists t such that $f(\mathtt{p}, t)$ is defined.
If the check is negative then $(\max f)(\mathtt{p})$ is undefined.

ii. If check i is positive then start successive steps of the following process.
- At step t, check whether $f(\mathtt{p}, t)$ is defined,
- if defined, compute its value and check whether there exists $u > t$ such that $f(\mathtt{p}, u)$ is greater than the maximum value computed up to that step.

iii. If at some step the last check in ii is negative then halt and output the maximum value computed up to now.

Clearly, oracle A' allows for the checks in i and ii. Also, the above process halts if and only if $f(\mathtt{p}, t)$ is defined for some t and $\{f(\mathtt{p}, t) : t \in \mathbb{N}\}$ is bounded, i.e. if and only if $(\max f)(\mathtt{p})$ is defined. In that case it outputs exactly $(\max f)(\mathtt{p})$.

2. To see that the inclusion is strict, observe that the graph of any function in $Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A}$ is $\Sigma^{0,A}_1 \wedge \Pi^{0,A}_1$ since

$$y = (\max f)(\mathtt{p}) \;\Leftrightarrow\; ((\exists t\ f(\mathtt{p}, t) = y) \;\wedge\; \neg(\exists u\ \exists z > y\ f(\mathtt{p}, u) = z))$$

Whereas the graph of functions in $PR^{A',\mathbf{2}^*\to\mathbb{X}}$ can be $\Sigma^{0,A'}_1$ and not $\Delta^{0,A'}_1$, i.e. $\Sigma^{0,A}_2$ and not $\Delta^{0,A}_2$. □

In the vein of Prop.6.8, let's mention the following result, cf. [1] and [6].

Proposition 6.9. *Let $A \subseteq \mathbb{N}$.*

1. *K^A and $K^A_{\max}$ are recursive in A'.*

2. *$K^A >_{\mathrm{ct}} K^A_{\max} >_{\mathrm{ct}} K^{A'}$.*

6.4 The Δ operation on $Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ and the jump

The following variant of Prop.6.8 is a normal form for partial A'-recursive $\mathbb{Z}$-valued functions. We shall use it in §7-8.

Theorem 6.10. *Let $A \subseteq \mathbb{N}$. Then*

$$PR^{A',\mathbf{2}^*\to\mathbb{Z}} = \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}) = \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$$

I.e., every partial A'-recursive function is the difference of two functions in Max_{Rec^A} (cf. Notation 6.1).

Before entering the proof of Thm.6.10, let's recall two well-known facts about oracular computation and approximation of the jump.

Lemma 6.11. *Let $(B_t)_{t\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ which converges pointwise to $B \subseteq \mathbb{N}$, i.e.*

$$\forall n \;\; \exists t_n \;\; \forall t \geq t_n \quad B_t \cap \{0,1,...,n\} = B \cap \{0,1,...,n\}$$

Let $\mathbb{X}, \mathbb{Y}$ be basic sets and let $\psi : \mathbb{X} \to \mathbb{Y}$ be a partial B-recursive function computed by some oracle Turing machine $\mathcal{M}$ with oracle B. Let $\mathtt{x} \in \mathbb{X}$. Then, $\psi(\mathtt{x})$ is defined if and only if there exists $t_{\mathtt{x}}$ such that

i. the computation of $\mathcal{M}$ on input $\mathtt{x}$ with oracle $B_{t_{\mathtt{x}}}$ halts in at most $t_{\mathtt{x}}$ steps,

ii. for all $t \geq t_{\mathtt{x}}$ the computation of $\mathcal{M}$ on input $\mathtt{x}$ with oracle B_t is step by step exactly the same as that with oracle $B_{t_{\mathtt{x}}}$ (in particular, it asks the same questions to the oracle, gets the same answers and halts at the same computation step $\leq t_{\mathtt{x}}$).

Lemma 6.12. *Let $A \subseteq \mathbb{N}$ and let $A' \subseteq \mathbb{N}$ be the jump of A. There exists a total A-recursive sequence $(Approx(A',t))_{t\in\mathbb{N}}$ of subsets of $\mathbb{N}$ which is monotone increasing with respect to set inclusion and which has union A'. In particular, this sequence converges pointwise to A'.*

We can now prove Thm.6.10.

Proof of Thm.6.10. Using Prop.6.8, we get

$$\Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}) \subseteq \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}) \subseteq \Delta(PR^{A',\mathbf{2}^*\to\mathbb{N}}) = PR^{A',\mathbf{2}^*\to\mathbb{Z}}$$

Thus, to get the wanted equalities, it suffices to prove inclusion

$$PR^{A',\mathbf{2}^*\to\mathbb{N}} \subseteq \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$$

Let $\mathcal{M}$ be an oracle Turing machine with inputs in $\mathbf{2}^*$, which, with oracle A', computes the partial A'-recursive function $\varphi^{A'} : \mathbf{2}^* \to \mathbb{N}$.

To prove that $\varphi^{A'}$ is in $\Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$, we define total A-recursive functions $f, g : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ which are (non strictly) monotone increasing and such that $\varphi^{A'} = \max f - \max g$.

The idea to get f, g is as follows. We consider A-recursive approximations of oracle A' (as given by Lemma 6.12) and use them as fake oracles. Function f is obtained by letting $\mathcal{M}$ run with the fake oracles and restart its computation each time some better approximation of A' shows the previous fake oracle has given an incorrect answer. Function g collects all the outputs of the computations which have been recognized as incorrect in the computing process for f.

We now formally define f, g.
First, since we do not care about computation time and space, we can suppose without loss of generality, that, at any step t, $\mathcal{M}$ asks to the oracle about the integer t and writes down the oracle answer on the t-th cell of some dedicated tape.
Consider $t+1$ steps of the computation of $\mathcal{M}$ on input $\mathtt{p}$ with oracle $Approx(A', t)$ (cf. Lemma 6.12). We denote by $\mathcal{C}_{\mathtt{p},t+1}$ this limited computation. We say that $\mathcal{C}_{\mathtt{p},t+1}$ halts if $\mathcal{M}$ (with that fake oracle) halts in at most $t+1$ steps.
We denote by $output(\mathcal{C}_{\mathtt{p},t})$ the current value (which is in $\mathbb{Z}$) of the output tape after step t. The A-recursive definition of f, g is as follows.

i. $f(\mathtt{p}, 0) = g(\mathtt{p}, t) = 0$

ii. Suppose $Approx(A', t+1) \cap \{0, ..., t\} = Approx(A', t) \cap \{0, ..., t\}$. Then, up to the halting step of $\mathcal{C}_{\mathtt{p},t}$ or up to step t in case $\mathcal{C}_{\mathtt{p},t}$ does not halt, both computations $\mathcal{C}_{\mathtt{p},t}, \mathcal{C}_{\mathtt{p},t+1}$ are stepwise identical.

    (a) If $\mathcal{C}_{\mathtt{p},t}$ halts then so does $\mathcal{C}_{\mathtt{p},t+1}$ at the same step. And both computations have the same output.
    In that case, we set $f(\mathtt{p}, t+1) = f(\mathtt{p}, t)$, $g(\mathtt{p}, t+1) = g(\mathtt{p}, t)$.

    (b) If $\mathcal{C}_{\mathtt{p},t}$ does not halt then let $\delta_{t+1} = output(\mathcal{C}_{\mathtt{p},t+1}) - output(\mathcal{C}_{\mathtt{p},t})$, and set

$$\begin{array}{rcl} f(\mathtt{p}, t+1) & = & f(\mathtt{p}, t) + 1 + \max(0, \delta_{t+1}) \\ g(\mathtt{p}, t+1) & = & g(\mathtt{p}, t) + 1 + \max(0, -\delta_{t+1}) \end{array}$$

    i.e. we add $|\delta_{t+1}|$ to f or g according to the sign of δ_{t+1}.

iii. Suppose $Approx(A', t+1) \cap \{0, ..., t\} \neq Approx(A', t) \cap \{0, ..., t\}$. Since these approximations are monotone increasing, we necessarily have $Approx(A', t) \cap \{0, ..., t\} \neq A' \cap \{0, ..., t+1\}$.
Thus, the fake oracle in $\mathcal{C}_{\mathtt{p},t}$ has given answers which are not compatible with A'. In that case, we set

$$\begin{array}{rcl} f(\mathtt{p}, t+1) & = & f(\mathtt{p}, t) + g(\mathtt{p}, t) + 1 + \max(0, output(\mathcal{C}_{\mathtt{p},t+1})) \\ g(\mathtt{p}, t+1) & = & f(\mathtt{p}, t) + g(\mathtt{p}, t) + 1 + \max(0, -output(\mathcal{C}_{\mathtt{p},t+1})) \end{array}$$

i.e. we uprise f, g to a common value (namely $f(\mathtt{p},t)+g(\mathtt{p},t)$) and then add $|output(\mathcal{C}_{\mathtt{p},t+1})|$ to f or g according to the sign of $output(\mathcal{C}_{\mathtt{p},t+1})$.

From the above inductive definition, we see that, for each $t > 0$,

$$f(\mathtt{p},t) - g(\mathtt{p},t) = output(\mathcal{C}_{\mathtt{p},t})$$

Suppose $\varphi^{A'}(\mathtt{p})$ is defined.
Applying Lemmas 6.11, 6.12, we see that there exist $s_{\mathtt{p}} \leq t_{\mathtt{p}}$ such that
- $\mathcal{M}$, on input $\mathtt{p}$, with oracle A', halts in $s_{\mathtt{p}}$ steps,
- $Approx(A', t_{\mathtt{p}}) \cap \{0, ..., t_{\mathtt{p}}\} = A' \cap \{0, ..., t_{\mathtt{p}}\}$.

Thus, for all $t \geq t_{\mathtt{p}}$, $f_{\mathtt{p},t} = f_{\mathtt{p},t_{\mathtt{p}}}$ and $g_{\mathtt{p},t} = g_{\mathtt{p},t_{\mathtt{p}}}$ and $f_{\mathtt{p},t} - g_{\mathtt{p},t} = \varphi^{A'}(\mathtt{p})$.

Suppose $\varphi^{A'}(\mathtt{p})$ is not defined.
Observe that, each time the "fake" computation $\mathcal{C}_{\mathtt{p},t}$ with oracle $Approx(A', t)$ does not halt or appears not to be the "right" one with oracle A' (because $Approx(A', t+1) \cap \{0, ..., t\}$ differs from $Approx(A', t) \cap \{0, ..., t\}$), we strictly increase both f, g (this is why we put $+1$ in the equations of iib and iii). Applying Lemmas 6.11, 6.12, we see that, if $\varphi^{A'}(\mathtt{p})$ is not defined then $\mathcal{C}_{\mathtt{p},t}$ does not halt for infinitely many t's, so that $f(\mathtt{p},t)$ and $g(\mathtt{p},t)$ increase infinitely often. Therefore, $(\max f)(\mathtt{p})$ and $(\max g)(\mathtt{p})$ are both undefined, and so is their difference.

This proves that $\varphi^{A'} = \max f - \max g$. Since the sequence $(Approx(A', t))_{t \in \mathbb{N}}$ is A-recursive, so are f, g. Thus, $\max f, \max g$ are in $Max^{\mathbf{2}^* \to \mathbb{N}}_{Rec^A}$ and their difference $\varphi^{A'}$ is in $\Delta(Max^{\mathbf{2}^* \to \mathbb{N}}_{Rec^A})$. □

7 Cardinal representations of $\mathbb{N}$

7.1 Basic cardinal representation and its effectivizations

Among the conceptual representations of integers, the most basic one goes back to Russell, [16] 1908 (cf. [21] p.178), and considers non negative integers as equivalence classes of sets relative to cardinal comparison.

Definition 7.1 (Cardinal representation of $\mathbb{N}$). Let $card(Y)$ denote the cardinal of Y, i.e. the number of its elements.
The cardinal representation of $\mathbb{N}$ relative to an infinite set X is the partial function $P(X) \to \mathbb{N}$ with domain $P_{<\omega}(X)$ (the family of all finite subsets of X) which maps Z to $card(Z)$.

Definition 7.2 (Effectivizing the cardinal representation of $\mathbb{N}$). We effectivize the cardinal representation by replacing $P(X)$ by $RE(\mathbb{X})$ where $\mathbb{X}$ is some basic set. Two kinds of self-enumerated systems can be naturally associated (cf. §5 and Lemma 4.1):

$$(\mathbb{N}, card \circ \mathcal{F}^{RE(\mathbb{X})}) \quad , \quad (\mathbb{N}, card \circ \mathcal{P}F^{RE(\mathbb{X})})$$

Remark 7.3. One can also consider the total representation obtained by restriction to the set $P_{<\omega}(X)$ of all finite subsets of X. But this amounts to a partial recursive representation and is relevant to §2.5.

7.2 Syntactical complexity of cardinal representations

The well-known Σ^0_2 completeness of the set $\{n : W_n \text{ is finite}\}$ yields the following result.

Proposition 7.4. *The family $\{domain(\varphi) : \varphi \in card \circ \mathcal{F}^{RE(\mathbb{X})}\}$ is exactly the family of Σ^0_2 subsets of $\mathbf{2}^*$. Idem with $card \circ \mathcal{P}F^{RE(\mathbb{X})}$.*

7.3 Characterization of the *card* self-enumerated systems

Theorem 7.5. *For any basic set $\mathbb{X}$,*

1i. $card \circ \mathcal{F}^{RE(\mathbb{X})} \;=\; Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$

ii. $card \circ \mathcal{P}F^{RE(\mathbb{X})} \;=\; Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$

2. $K^{\mathbb{N}}_{card\circ\mathcal{F}^{RE(\mathbb{X})}} \;=_{\mathrm{ct}}\; K^{\mathbb{N}}_{card\circ\mathcal{P}F^{RE(\mathbb{X})}} \;=_{\mathrm{ct}}\; K_{\max}$

We shall simply write $K^{\mathbb{N}}_{card}$ for the Kolmogorov complexity of the card systems.

Proof. Point 2 is a direct corollary of Point 1 and Prop.6.6. Let's prove point 1.

1i. *Inclusion $\subseteq$.* Let $g : \mathbf{2}^* \to \mathbb{N}$ be total recursive. We define a total recursive function $u : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ such that

$$(*) \quad \{u(\mathtt{p}, t) : t \in \mathbb{N}\} = \begin{cases} \{0, ..., n\} & \text{if } W_{g(\mathtt{p})} \text{ contains exactly } n \text{ points} \\ \mathbb{N} & \text{if } W_{g(\mathtt{p})} \text{ is infinite} \end{cases}$$

The definition is as follows. First, set $u(\mathtt{p}, 0) = 0$ for all $\mathtt{p}$. Consider a recursive enumeration of $W_{g(\mathtt{p})}$. If at step t, some new point is enumerated then set $u(\mathtt{p}, t+1) = u(\mathtt{p}, t) + 1$, else set $u(\mathtt{p}, t+1) = u(\mathtt{p}, t)$.
From $(*)$ we get $card(W_{\mathtt{p}}) = (\max f)(\mathtt{p})$, so that $\mathtt{p} \mapsto card(W_{g(\mathtt{p})})$ is in $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$.

1ii. *Inclusion $\subseteq$.* Now g is partial recursive and we define u as above with the extra condition that $\{u(\mathtt{p}, t) : t \in \mathbb{N}\} = \emptyset$ if $g(\mathtt{p})$ is undefined.

1i. *Inclusion $\supseteq$.* Let $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ be total recursive. The idea to prove that $\max f : \mathbf{2}^* \to \mathbb{N}$ is in $card \circ \mathcal{F}^{RE(\mathbb{X})}$ is quite simple. For every $\mathtt{p}$, we define an r.e. subset of $\mathbb{X}$ which collects some new elements each time $f(\mathtt{p}, t)$ gets greater than $\max\{f(\mathtt{p}, t') : t' < t\}$.
Formally, let $\psi : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ be the partial recursive function such that

$$\psi(\mathtt{p}, t) = \begin{cases} 0 & \text{if } \exists u \; f(\mathtt{p}, u) > t \\ \text{undefined} & \text{otherwise} \end{cases}$$

Letting $\psi_{\mathtt{p}}(t) = \psi(\mathtt{p}, t)$, we have

$$domain(\psi_{\mathtt{p}}) = \begin{cases} \{t : 0 \leq t < (\max f)(\mathtt{p})\} & \text{if } (\max f)(\mathtt{p}) \text{ is defined} \\ \mathbb{N} & \text{otherwise} \end{cases}$$

The parameter property yields a recursive function $g : \mathbf{2}^* \to \mathbb{N}$ such that $W_{g(\mathtt{p})} = domain(\psi_{\mathtt{p}})$. Thus, $card(W_{g(\mathtt{p})}) = card(domain(\psi_{\mathtt{p}})) = (\max f)(\mathtt{p})$. Which proves that $\max f$ is in $card \circ \mathcal{F}^{RE(\mathbb{X})}$.

1ii. *Inclusion* $\supseteq$. Now, f is partial recursive and we define ψ as above with the extra condition that $\psi(\mathtt{p})$ is undefined if $f(\mathtt{p}, t)$ is defined for no t. Again, the parameter property yields a total recursive function $g : \mathbf{2}^* \to \mathbb{N}$ such that $W_{g(\mathtt{p})} = domain(\psi_{\mathtt{p}})$. Let h be the restriction of g to $\{\mathtt{p} : \exists t \;\; f(\mathtt{p}, t) \text{ is defined}\}$. Then $card(W_{h(\mathtt{p})}) = (\max f)(\mathtt{p})$, which proves that $\max f$ is in $card \circ \mathcal{P}F^{RE^A(\mathbb{X})}$. $\square$

7.4 Characterization of the $\Delta card$ system

We now look at the self-delimited system with domain $\mathbb{Z}$ obtained from $card \circ \mathcal{F}^{RE(\mathbb{X})}$ by the operation Δ introduced in §4.3.

Theorem 7.6. $\Delta(card \circ \mathcal{F}^{RE(\mathbb{X})}) = \Delta(card \circ \mathcal{P}F^{RE(\mathbb{X})}) = PR^{\emptyset', \mathbf{2}^* \to \mathbb{Z}}$. *Hence* $K^{\mathbb{Z}}_{\Delta(card \circ \mathcal{F}^{RE(\mathbb{X})})} =_{\mathrm{ct}} K^{\emptyset', \mathbb{Z}}$.

We shall simply write write $K^{\mathbb{Z}}_{\Delta card}$ *and* $K^{\mathbb{N}}_{\Delta card}$ *for the Kolmogorov complexity of the* $\Delta card$ *system and its restriction to* $\mathbb{N}$.

Proof. The equalities about the self-enumerated systems is a direct corollary of Thm.7.5 and Thm.6.10. The equalities about Kolmogorov complexities are trivial corollaries of those about self-enumerated systems. $\square$

8 Index representations of $\mathbb{N}$

8.1 Basic index representation and its effectivizations

A variant of the cardinal representation considers indexes of equivalence relations. More precisely, it views an integer as an equivalence class of equivalence relations relative to index comparison.

Definition 8.1 (Index representation). The index representation of $\mathbb{N}$ relative to an infinite set X is the partial function $index_X : P(X^2) \to \mathbb{N}$ with domain the family of equivalence relations on subsets of X which have finite index, and which associates to such a relation its index.

Definition 8.2 (Effectivizing index representation of $\mathbb{N}$). We effectivize the index representation by replacing $P(X^2)$ by $RE(\mathbb{X}^2)$ where $\mathbb{X}$

is some basic set. Two kinds of self-enumerated systems can be naturally associated (cf. §5 and Lemma 4.1):

$$(\mathbb{N}, index \circ \mathcal{F}^{RE(\mathbb{X}^2)}) \quad , \quad (\mathbb{N}, index \circ \mathcal{P}F^{RE(\mathbb{X}^2)})$$

8.2 Syntactical complexity of index representations

The following proposition gives the syntactical complexity of the above effectivizations of the index representations. It is a straightforward corollary of Theorem 8.5 proved below.

Proposition 8.3. *The family* $\{domain(F) : F \in index \circ \mathcal{F}^{RE(\mathbb{X})}\}$ *is exactly the family of* Σ^0_3 *subsets of* $\mathbf{2}^*$. *Idem with* $index \circ \mathcal{P}F^{RE(\mathbb{X})}$.

8.3 Characterization of the *index* self-enumerated systems

We now come to the characterization of the index self-enumerated families. It turns out that these two families are almost equal to $Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{Rec}$, almost meaning here "up to an extra condition on the inverse image of 0".

The following simple result illustrates the significance of Thm.8.5.

Proposition 8.4. *Let* $F : \mathbf{2}^* \to \mathbb{N}$. *The following table gives the syntactical complexity of* $F^{-1}(k)$ *in case* F *is in the mentioned class.*

F *is in*	$k = 0$	$k = 1$	$k \geq 2$
$index \circ \mathcal{F}^{RE(\mathbb{X})}$	Π^0_1	Π^0_2	$\Sigma^0_2 \wedge \Pi^0_2$
$index \circ \mathcal{P}F^{RE(\mathbb{X})}$	$(\Sigma^0_1 \wedge \Pi^0_1)^{sp}$	Π^0_2	$\Sigma^0_2 \wedge \Pi^0_2$
$Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{Rec}$	Π^0_2	$\Sigma^0_2 \wedge \Pi^0_2$	$\Sigma^0_2 \wedge \Pi^0_2$

where $(\Sigma^0_1 \wedge \Pi^0_1)^{sp}$ *means that* $F^{-1}(0) = A \cap B$ *and* A *is* Σ^0_1 *and* B *is* Π^0_1 *and* $A \supseteq domain(F)$.
In case $F \in Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{Rec}$ *and* $F^{-1}(0)$ *is* $\Sigma^0_1 \wedge \Pi^0_1$ *then* $F^{-1}(1)$ *is* Π^0_2.

Proof. 1. Suppose $F \in index \circ \mathcal{F}^{RE(\mathbb{X})}$ and $F(\mathtt{p}) = index(W^{\mathbb{X}^2}_{f(\mathtt{p})})$ where $f : \mathbf{2}^* \to \mathbb{N}$ is total recursive. Then

$$\begin{array}{lll}
F(\mathtt{p}) = 0 & \Leftrightarrow & W_{f(\mathtt{p})} = \emptyset \\
F(\mathtt{p}) = 1 & \Leftrightarrow & W_{f(\mathtt{p})} \text{ is an equivalence relation} \\
& & \wedge\ \exists \mathtt{x}\ (\mathtt{x}, \mathtt{x}) \in W_{f(\mathtt{p})} \\
& & \wedge\ \forall \mathtt{x}\ \forall \mathtt{y}\ (\{(\mathtt{x}, \mathtt{x}), (\mathtt{y}, \mathtt{y})\} \subseteq W_{f(\mathtt{p})} \Rightarrow (\mathtt{x}, \mathtt{y}) \in W_{f(\mathtt{p})}) \\
F(\mathtt{p}) = k & \Leftrightarrow & W_{f(\mathtt{p})} \text{ is an equivalence relation} \\
& & \wedge\ \exists \mathtt{x}_1 ... \exists \mathtt{x}_k\ ((\bigwedge_{1 \leq i \leq k} (\mathtt{x}_i, \mathtt{x}_i) \in W_{f(\mathtt{p})}) \wedge \bigwedge_{1 \leq i < j \leq k} (\mathtt{x}_i, \mathtt{x}_j) \notin W_{f(\mathtt{p})}) \\
& & \wedge\ \forall \mathtt{x}_1 ... \forall \mathtt{x}_{k+1} \bigvee_{1 \leq i < j \leq k+1} (\mathtt{x}_i, \mathtt{x}_j) \in W_{f(\mathtt{p})}
\end{array}$$

Observe that $W_{f(\mathtt{p})}$ is an equivalence relation if and only if

$$\begin{array}{l}\forall \mathtt{x}\ \forall \mathtt{y}\ (((\mathtt{x},\mathtt{y}) \in W_{f(\mathtt{p})} \ \Rightarrow\ \{(\mathtt{x},\mathtt{x}),(\mathtt{y},\mathtt{x})\} \subseteq W_{f(\mathtt{p})}) \\ \wedge\ \forall \mathtt{x}\ \forall \mathtt{y}\ \forall \mathtt{z}\ (\{(\mathtt{x},\mathtt{y}),(\mathtt{y},\mathtt{z})\} \subseteq W_{f(\mathtt{p})} \Rightarrow\ (\mathtt{x},\mathtt{z}) \in W_{f(\mathtt{p})}))\end{array}$$

This gives the syntactical complexities of line 1 of the table.

2. If $F \in index \circ \mathcal{P}F^{RE(\mathbb{X})}$ then f is partial recursive and we have to add in the above formulas the Σ^0_1 condition expressing that $f(\mathtt{p})$ is defined.

3. Suppose $F \in Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{Rec}$ and $F(\mathtt{p}) = \max\{f(\mathtt{p},t) : t \in \mathbb{N}\}$ where $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ is total recursive in $\emptyset'$, hence has Δ^0_2 graph. Then

$$\begin{array}{lcl}F(\mathtt{p}) = 0 & \Leftrightarrow & \forall t\ f(\mathtt{p},t) = 0 \\ F(\mathtt{p}) = 1 & \Leftrightarrow & F(\mathtt{p}) \neq 0\ \wedge\ \forall t\ f(\mathtt{p},t) \leq 1 \\ F(\mathtt{p}) = k & \Leftrightarrow & \exists t\ f(\mathtt{p},t) = k\ \wedge\ \forall t\ f(\mathtt{p},t) \leq k\end{array}$$

This proves line 3 of the table and the last assertion of the proposition. □

Theorem 8.5. *For any basic set* $\mathbb{X}$*,*

i. $index \circ \mathcal{F}^{RE(\mathbb{X})} = \{F \in Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{Rec} : F^{-1}(0)$ *is* $\Pi^0_1\}$

ii. $index \circ \mathcal{P}F^{RE(\mathbb{X})} = \{F \in Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{Rec} : F^{-1}(0)$ *is* $\Sigma^0_1 \wedge \Pi^0_1\}$

Proof. **Proof of inclusions $\subseteq$ in i–ii.**
Using Prop.8.4, it suffices to prove that $index \circ \mathcal{P}F^{RE(\mathbb{X}^2)} \subseteq Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{Rec}$.
Let $G \in index \circ \mathcal{P}F^{RE(\mathbb{X}^2)}$ and let $g : \mathbf{2}^* \to \mathbb{N}$ be partial recursive such that

$$G(\mathtt{p}) = \begin{cases} index(W^{\mathbb{X}^2}_{g(\mathtt{p})}) & \text{if } g(\mathtt{p}) \text{ is defined and } W^{\mathbb{X}^2}_{g(\mathtt{p})} \text{ is an} \\ & \text{equivalence relation with finite index} \\ \text{undefined} & \text{otherwise}\end{cases}$$

We define a total $\emptyset'$-recursive function $u : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ such that

$$(*)\quad \{u(\mathtt{p},t) : t \in \mathbb{N}\} = \begin{cases} \{0,...,n\} & \text{if } G(\mathtt{p}) \text{ is defined and } G(\mathtt{p}) = n \\ \mathbb{N} & \text{if } G(\mathtt{p}) \text{ is undefined}\end{cases}$$

The definition is as follows. Since g is partial recursive and we look for an $\emptyset'$-recursive definition of $u(\mathtt{p},t)$, we can use oracle $\emptyset'$ to check if $g(\mathtt{p})$ is defined.
If $g(\mathtt{p})$ is undefined then we let $u(\mathtt{p},t) = t$ for all t. Which insures $(*)$.
Suppose now that $g(\mathtt{p})$ is defined. First, set $u(\mathtt{p},0) = 0$.
Consider a recursive enumeration of $W^{\mathbb{X}^2}_{g(\mathtt{p})}$. Let R_t be the set of pairs enumerated at steps $< t$ and D_t be the set of $\mathtt{x} \in \mathbb{X}$ which appear in pairs in R_t (so that R_0 and D_0 are empty). Since at most one new pair is enumerated at each step, the set R_t contains at most t pairs and D_t contains at most $2\,t$ points.
At step $t+1$, use oracle $\emptyset'$ to check the following properties:

α_t. For every $\mathtt{x} \in D_{t+1}$ the pair $(\mathtt{x},\mathtt{x})$ is in $W^{\mathbb{X}^2}_{g(\mathtt{p})}$.

β_t. For every pair $(\mathtt{x},\mathtt{y}) \in R_{t+1}$ the pair $(\mathtt{y},\mathtt{x})$ is in $W^{\mathbb{X}^2}_{g(\mathtt{p})}$.

γ_t. For every pairs $(\mathtt{x},\mathtt{y}),(\mathtt{y},\mathtt{z}) \in R_{t+1}$ the pair $(\mathtt{x},\mathtt{z})$ is in $W^{\mathbb{X}^2}_{g(\mathtt{p})}$.

δ_t. For every $\mathtt{x} \in D_{t+1}$ there exists $\mathtt{y} \in D_t$ such that the pair $(\mathtt{x},\mathtt{y})$ is in $W^{\mathbb{X}^2}_{g(\mathtt{p})}$.

Since R_{t+1}, D_{t+1} are finite, all these properties α_t-δ_t are *finite* conjunctions of Σ^0_1 statements. Hence oracle $\emptyset'$ can decide them all.

Observe that if $W^{\mathbb{X}^2}_{g(\mathtt{p})}$ is an equivalence relation then answers to α_t-γ_t are positive for all t. And if $W^{\mathbb{X}^2}_{g(\mathtt{p})}$ is not an equivalence relation then, for some $\pi \in \{\alpha,\beta,\gamma\}$, answers to π_t are negative for all t large enough .
Also, if $W^{\mathbb{X}^2}_{g(\mathtt{p})}$ is an equivalence relation then a new equivalence class is witnessed each time δ_t is false. And every equivalence class is so witnessed.

Thus, in case $g(\mathtt{p})$ is defined, we insure $(*)$ by letting

$$u(\mathtt{p},t+1) = \begin{cases} u(\mathtt{p},t) & \text{if all answers to } \alpha_t\text{-}\delta_t \text{ are positive} \\ u(\mathtt{p},t)+1 & \text{otherwise} \end{cases}$$

From $(*)$, we get $G = \max u$. Since u is total $\emptyset'$-recursive, this proves that G is in $Max^{\emptyset',\mathbf{2}^*\to\mathbb{X}}_{Rec}$.

Proof of inclusion $\supseteq$ in i.
We reduce to the case $\mathbb{X} = \mathbb{N}$. Let $F \in Max^{\emptyset',\mathbf{2}^*\to\mathbb{N}}_{PR}$ be such that $F^{-1}(0)$ is Π^0_1.
Let $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ be total recursive in $\emptyset'$ such that $F(\mathtt{p}) = \max\{f(\mathtt{p},t) : t \in \mathbb{N}\}$. With no loss of generality, we can suppose that f is monotone increasing in its second argument t. Using Thm.6.10, there are total recursive $g,h : \mathbf{2}^* \times \mathbb{N}^2 \to \mathbb{N}$ such that $f(\mathtt{p},t) = (\max_{u\in\mathbb{N}} g(\mathtt{p},t,u)) - (\max_{u\in\mathbb{N}} h(\mathtt{p},t,u))$ hence

$$(*) \quad F(\mathtt{p}) = \max_{t\in\mathbb{N}} f(\mathtt{p},t) = \max_{t\in\mathbb{N}}[(\max_{u\in\mathbb{N}} g(\mathtt{p},t,u)) - (\max_{u\in\mathbb{N}} h(\mathtt{p},t,u))]$$

For $t \leq s$, we consider the following approximations of $f(\mathtt{p},t)$:

$$\begin{array}{rcl} \widetilde{f}(\mathtt{p},t,s) & = & (\max_{u\leq s} g(\mathtt{p},t,u)) - (\max_{u\leq s} h(\mathtt{p},t,u)) \\ \widehat{f}(\mathtt{p},t,s) & = & \max(0, \max_{i\leq t} \widetilde{f}(\mathtt{p},i,s)) \end{array}$$

Clearly, $\widehat{f}(\mathtt{p},t,s)$ is monotone increasing with respect to t and s.
Also, for every t there exists σ_t such that, for $s \geq \sigma_t$, we have $f(\mathtt{p},t) = \widetilde{f}(\mathtt{p},t,s)$, so that (since f is increasing in t), for $s \geq \widehat{\sigma_t} = \max_{i\leq t} s_i$, we have $f(\mathtt{p},t) = \widehat{f}(\mathtt{p},t,s)$ and $F(\mathtt{p}) = \max_{t\in\mathbb{N}} \widehat{f}(\mathtt{p},t,\widehat{\sigma_t})$.

To prove that F is in $index \circ \mathcal{F}^{RE(\mathbb{N}^2)}$, given $\mathtt{p} \in \mathbf{2}^*$, we construct an r.e. equivalence relation $\rho_{\mathtt{p}}$ on a subset of $\mathbb{N}$ by the following inductive process:

i. $\rho_{\mathtt{p}} = \bigcup_{t\in\mathbb{N}} \rho_{\mathtt{p},s}$ where $\rho_{\mathtt{p},s}$ is a finite equivalence relation.
Also, $\rho_{\mathtt{p},s+1}$ contains $\rho_{\mathtt{p},s}$ for all s.

ii. *Preliminary phase (which may last forever) before the induction.*
Let the Π^0_1 set $F^{-1}(0)$ be of the form $F^{-1}(0) = \{\mathtt{p} : \forall t\ R(\mathtt{p},t)\}$ where $R \subseteq \mathbf{2}^* \times \mathbb{N}$ is a recursive relation. Let $\zeta = \sup\{s : \forall t \leq s\ R(\mathtt{p},t)\}$.
If $s \leq \zeta$ then $\rho_{\mathtt{p},s} = \emptyset$.
If $s > \zeta$ then $\rho_{\mathtt{p},s}$ is a finite equivalence relation on $\mathbb{N}$.
This insures that $index(\rho_{\mathtt{p}}) = 0$ if and only if $\zeta = +\infty$ if and only if $F(\mathtt{p}) = 0$.

iii. *Inductive invariant property.*
If $s > \zeta$ then the finite equivalence relation $\rho_{\mathtt{p},s}$ consists of one non empty class Z^s containing 0 and finitely many (maybe zero) *singleton classes* of elements in a (possibly empty) set D^s. Also, $D^s = \bigcup_{t\leq s} D^s_t$ where $D^s_0, ..., D^s_s$ are pairwise disjoint.
- $D^s_t = \emptyset$ if $\widehat{f}(\mathtt{p},i) \leq 1$ for all $i \leq t$,
- D^s_t has $\widehat{f}(\mathtt{p},t,s) - 1$ elements if $t \leq s$ is least such that $\widehat{f}(\mathtt{p},t,s) \geq 2$,
- D^s_t has $\widehat{f}(\mathtt{p},t,s) - \widehat{f}(\mathtt{p},t-1,s)$ elements if $\widehat{f}(\mathtt{p},t,s) \geq 2$ and t is not least such.
Intuition. We would like to have the above equalities with the $f(\mathtt{p},t)$'s for $t \leq s$, since this would imply that $index(\rho_{\mathtt{p},t}) = f(\mathtt{p},t)$. But we can only deal with their approximations $\widehat{f}(\mathtt{p},t,s)$.

iv. *Initial steps of the induction (which may last forever).*
Let $\xi = \sup\{s : \widehat{f}(\mathtt{p},s,s) \leq 1\}$ and $\alpha = \max(\zeta+1, \xi+1)$.
For $\zeta < s \leq \alpha$ we let $Z^s = \{0\}$ and $D^s = \emptyset$. Also, $D^s_t = \emptyset$ for all $t \leq s$.
Intuition. Since $s > \zeta$, we know that $F(\mathtt{p}) \neq 0$, so that we start filling $\rho_{\mathtt{p}}$ by putting 0 in its domain. We wait until $f(\mathtt{p},s) > 1$ to start putting other classes. Of course, as we cannot compute f we use its approximation $\widehat{f}$.

v. *Inductive step.* Suppose α is finite and let $s > \alpha$. Then
Case $\forall t \leq s\ \widehat{f}(\mathtt{p},t,s+1) = \widehat{f}(\mathtt{p},t,s)$. Then we set $Z^{s+1} = Z^s$ and $D^{s+1}_t = D^s_t$ for $t \leq s$ and we define $D^{s+1}_{s+1} \subset \mathbb{N}$ as a set of $\widehat{f}(\mathtt{p},s+1,s+1) - \widehat{f}(\mathtt{p},s,s+1)$ integers which is disjoint from $Z^s \cup D^s_s$ (hence from all the D^{s+1}_t's for $t \leq s$).
Case $\exists t \leq s\ \widehat{f}(\mathtt{p},t,s+1) \neq \widehat{f}(\mathtt{p},t,s)$. Let τ be the least such t. Then, $\widehat{f}(\mathtt{p},t,s+1) = \widehat{f}(\mathtt{p},t,s)$ for $t < \tau$. We let
- $Z^{s+1} = Z^s \cup \bigcup_{\tau\leq t\leq s} D^s_t$,
- $D^{s+1}_t = D^s_t$ for $t < \tau$,
- $D^{s+1}_\tau, ..., D^{s+1}_{s+1}$ are new sets, pairwise disjoint and disjoint from $Z^{s+1} \cup \bigcup_{t\leq\tau} D^s_t$, which contain as many elements as required by iii.
Intuition. In the second case, we know that $s < s_\tau$ so that we annihilate the singleton classes of the elements in the D^s_t's for $\tau \leq t \leq s$ by

aggregating them to the class of 0.
If $s < \widehat{\sigma_t}$ then there will be some $s' \geq s$ for which we shall be in the second case.
If $s \geq \widehat{\sigma_t}$ then $f(\mathtt{p}, t) = \widetilde{f}(\mathtt{p}, t, s) = \widehat{f}(\mathtt{p}, t, s)$ for all $t \leq s$. Hence we are necessarily in the first case.

For $s \geq \widehat{\sigma_t}$ we have $D_i^s = D_i^{\widehat{\sigma_t}}$ for all $i \leq t$. Thus, for each t, the D_t^s class gets constant for s big enough. Also, for $s < \widehat{\sigma_t}$ the class D_t^s gets aggregated to the class of 0. This shows that the refinement $\rho_{\mathtt{p}}$ of the $\rho'_{\mathtt{p},s}s$ is an equivalence relation. Also, due to equalities in iii, we have $\mathit{index}(\rho_{\mathtt{p}}) = \max_{t\in\mathbb{N}} f(\mathtt{p}, t) = F(\mathtt{p})$ (with value $+\infty$ on the left in case $F(\mathtt{p})$ is undefined). Finally, the parameter theorem gives some total recursive $\gamma : \mathbf{2}^* \to \mathbb{N}$ such that $\rho_{\mathtt{p}} = W_{\gamma(\mathtt{p})}$. Thus, F is in $\mathit{index} \circ \mathcal{F}^{RE(\mathbb{X})}$.

Proof of inclusion $\supseteq$ in ii.
Now, $F^{-1}(0)$ is $(\Sigma^0_1 \wedge \Pi^0_1)^{sp}$, i.e. $F^{-1}(0) = A \cap B$ where A is Σ^0_1 and B is Π^0_1 and $A \supseteq \mathit{domain}(F)$.
We add to the above construction of γ another phase before the preliminary phase ii. In this phase, we wait for $\mathtt{p}$ to appear in A. If and when $\mathtt{p}$ appears in A, we start the above construction of $\rho_{\mathtt{p}}$ with B in place of $F^{-1}(0)$.
Case $\mathtt{p} \notin A$. Then $\gamma(\mathtt{p})$ is undefined. This is OK since the inclusion $A \supseteq \mathit{domain}(F)$ insures that $F(\mathtt{p})$ is undefined.
Case $\mathtt{p} \in A$. Then $\gamma(\mathtt{p})$ is defined. Since $\mathtt{p} \in A$, we know that $F(\mathtt{p}) = 0$ if and only if $\mathtt{p} \in B$, hence the construction of $\rho_{\mathtt{p}}$ and γ is such that $\mathit{index}(\rho_{\mathtt{p}}) = F(\mathtt{p})$ (with left member $+\infty$ in case $F(\mathtt{p})$ is undefined). This proves that F is in $\mathit{index} \circ \mathcal{P}F^{RE(\mathbb{X})}$. □

8.4 Adding 1 to a system or maximizing it with 1

In order to get the Kolmogorov complexity of *index* systems, and also to characterize the $\Delta\mathit{index}$ systems, we need a simple auxiliary result.

Notation 8.6. If $\mathcal{G}$ is a family of functions $\mathbf{2}^* \to \mathbb{N}$, we let

$$1 + \mathcal{G} = \{1 + f : f \in \mathcal{G}\} \quad , \quad \max(1, \mathcal{G}) = \{\max(1, f) : f \in \mathcal{G}\}$$

Proposition 8.7. *1.*

$$\begin{array}{rcl}
\max(1, \mathit{index} \circ \mathcal{F}^{RE(\mathbb{X})}) &=& (\mathbf{2}^* \to \mathbb{N} \setminus \{0\}) \cap (\mathit{index} \circ \mathcal{F}^{RE(\mathbb{X})}) \\
\max(1, \mathit{index} \circ \mathcal{P}F^{RE(\mathbb{X})}) &=& (\mathbf{2}^* \to \mathbb{N} \setminus \{0\}) \cap (\mathit{index} \circ \mathcal{P}F^{RE(\mathbb{X})}) \\
\max(1, \mathit{Max}^{\emptyset', \mathbf{2}^* \to \mathbb{N}}_{\mathit{Rec}}) &=& (\mathbf{2}^* \to \mathbb{N} \setminus \{0\}) \cap \mathit{Max}^{\emptyset', \mathbf{2}^* \to \mathbb{N}}_{\mathit{Rec}} \\
1 + \mathit{Max}^{\emptyset', \mathbf{2}^* \to \mathbb{N}}_{\mathit{Rec}} &=& (\mathbf{2}^* \to \mathbb{N} \setminus \{0\}) \cap \mathit{Max}^{\emptyset', \mathbf{2}^* \to \mathbb{N}}_{\mathit{Rec}}
\end{array}$$

2. The four systems $\max(1, \mathit{index} \circ \mathcal{F}^{RE(\mathbb{X})})$, $\max(1, \mathit{index} \circ \mathcal{P}F^{RE(\mathbb{X})})$, $\max(1, \mathit{Max}^{\emptyset', \mathbf{2}^* \to \mathbb{N}}_{\mathit{Rec}})$ *and* $1 + \mathit{Max}^{\emptyset', \mathbf{2}^* \to \mathbb{N}}_{\mathit{Rec}}$ *coincide.*

Proof. Let $\delta : \mathbb{N} \to \mathbb{N}$ be total recursive such that

$$W_{\delta(n)} = \begin{cases} \{(0,0)\} \cup \{(E_{\phi(n)}(x), E_{\phi(n)}(y)) : (x,y) \in W_n\} & \text{if } W_n \neq \emptyset \\ \{(0,0)\} & \text{if } W_n = \emptyset \end{cases}$$

where $\phi(n)$ is the integer i such that (i,i) appears first in the enumeration of W_n, and $E_i : \mathbb{N} \to \mathbb{N}$ is the function such that $E_i(i) = 0$, $E_i(0) = i$ and $E_i(x) = x$ for $x \notin \{0, i\}$.
Observe that $\mathit{index}(W_{\delta(n)}) = \max(1, \mathit{index}(W_n))$.
Also, $\max_{t\in\mathbb{N}} \max(1, f(\mathtt{p}, t)) = \max(1, \max_{t\in\mathbb{N}} f(\mathtt{p}, t))$. This proves point 1.
Point 2 is a straightforward corollary of Point 1 and Thm.8.5. □

8.5 Kolmogorov complexity of the *index* systems

The following result is straightforward.

Proposition 8.8. *Let* $(\mathbb{N}, \mathcal{F})$ *be a self-enumerated system with* U *as a good universal function. Then* $(\mathbb{N}\setminus\{0\}, \max(1, \mathcal{F}))$ *(resp.* $(\mathbb{N}\setminus\{0\}, 1+\mathcal{F})$*) is also a self-enumerated system with* $\max(1, U)$ *(resp.* $1 + U$*) as a good universal function. In particular,*

$$K^{\mathbb{N}}_{\mathcal{F}} \restriction \mathbb{N} \setminus \{0\} = K^{\mathbb{N}\setminus\{0\}}_{\max(1,\mathcal{F})} = K^{\mathbb{N}\setminus\{0\}}_{1+\mathcal{F}}$$

We can now get the Kolmogorov complexity of the *index* systems.

Theorem 8.9. $K^{\mathbb{N}}_{\mathit{index}\circ\mathcal{F}^{RE(\mathbb{X}^2)}} =_{\mathrm{ct}} K^{\mathbb{N}}_{\mathit{index}\circ\mathcal{P}F^{RE(\mathbb{X}^2)}} =_{\mathrm{ct}} K^{\emptyset'}_{\max}$.

We shall write $K^{\mathbb{N}}_{\mathit{index}}$ *for the Kolmogorov complexity of the index systems.*

Proof. Propositions 8.7 and 8.8 yield

$$K^{\mathbb{N}}_{\mathit{index}\circ\mathcal{F}^{RE(\mathbb{X}^2)}} \restriction \mathbb{N} \setminus \{0\} =_{\mathrm{ct}} K^{\mathbb{N}}_{\mathit{index}\circ\mathcal{P}F^{RE(\mathbb{X}^2)}} \restriction \mathbb{N} \setminus \{0\} =_{\mathrm{ct}} K^{\emptyset'}_{\max} \restriction \mathbb{N} \setminus \{0\}$$

Increasing the constant in these $=_{\mathrm{ct}}$ equalities to deal with the values at 0, we get the equalities of the theorem. □

8.6 Characterization of the Δ*index* self-enumerated systems

Theorem 8.10.
1. $\Delta(\mathit{index} \circ \mathcal{F}^{RE(\mathbb{X})})) = \Delta(\mathit{index} \circ \mathcal{P}F^{RE(\mathbb{X})})) = PR^{\emptyset'',\mathbf{2}^*\to\mathbb{Z}}$

2. $K^{\mathbb{Z}}_{\Delta(\mathit{index}\circ\mathcal{F}^{RE(\mathbb{X})})} =_{\mathrm{ct}} K^{\mathbb{Z}}_{\Delta(\mathit{index}\circ\mathcal{P}F^{RE(\mathbb{X})})} =_{\mathrm{ct}} K^{\emptyset'',\mathbb{Z}}$.

We shall simply write write $K^{\mathbb{Z}}_{\Delta\mathit{index}}$ *and* $K^{\mathbb{N}}_{\Delta\mathit{index}}$ *for the Kolmogorov complexity of the* Δ*index system and its restriction to* $\mathbb{N}$.

Proof. Point 2 is a direct corollary of Point 1. Let's prove point 1. Prop.8.7 and Thm.8.5 respectively insure the following inclusions:

$$\Delta(1 + Max_{Rec}^{\emptyset',\mathbf{2}^*\to\mathbb{N}}) = \Delta(\max(1, index \circ \mathcal{F}^{RE(\mathbb{X}^2)})) \subseteq \Delta(index \circ \mathcal{F}^{RE(\mathbb{X}^2)})$$

$$\Delta(index \circ \mathcal{F}^{RE(\mathbb{X}^2)}) \subseteq \Delta(index \circ \mathcal{P}F^{RE(\mathbb{X}^2)}) \subseteq \Delta(Max_{Rec}^{\emptyset',\mathbf{2}^*\to\mathbb{N}})$$

Since, trivially, $\Delta(1 + Max_{Rec}^{\emptyset',\mathbf{2}^*\to\mathbb{N}}) = \Delta(Max_{Rec}^{\emptyset',\mathbf{2}^*\to\mathbb{N}})$, we see that all these systems are equal. We conclude with Thm.6.10. □

9 Functional representations of $\mathbb{N}$

9.1 Basic Church representation of $\mathbb{N}$

First, let's introduce some simple notions related to function iteration.

Definition 9.1 (Iteration).
1. Let Id_X the identity function over X. If $f : X \to X$ is a partial function, we inductively define for $n \in \mathbb{N}$ the n-th iterate $f^{(n)} : X \to X$ of f as the partial function such that $f^{(0)} = Id_X$ and $f^{(n+1)} = f^{(n)} \circ f$.

2. Let $\mathcal{P}$ be an infinite subset of $X \to X$ which is closed under composition. We denote by $It_{\mathcal{P}}^{(n)} : \mathcal{P} \to \mathcal{P}$ the total functional $f \mapsto f^{(n)}$.
We denote by $It_{\mathcal{P}}^{\mathbb{N}} : \mathbb{N} \to \mathcal{P}^{\mathcal{P}}$ the total functional $n \mapsto It_{\mathcal{P}}^{(n)}$.

We can now come to Church's functional representation of integers.

Definition 9.2 (Church representation of $\mathbb{N}$).
1. *(Church, 1933 [4])* If X is an infinite set, the Church representation of $\mathbb{N}$ relative to X is the function

$$Church_{X\to X}^{\mathbb{N}} : (X \to X)^{(X\to X)} \to \mathbb{N}$$

which is the unique left inverse of $It_{X\to X}^{\mathbb{N}}$ with domain $Range(It_{X\to X}^{\mathbb{N}}) = \{It_{X\to X}^{(n)} : n \in \mathbb{N}\}$, i.e. $Church_{X\to X}^{\mathbb{N}} \circ It_{X\to X}^{\mathbb{N}} = Id_{\mathbb{N}}$ and

$$Church_{X\to X}^{\mathbb{N}}(F) = \begin{cases} n & \text{if } F = It_{X\to X}^{(n)} \\ \text{undefined} & \text{if } \forall n \in \mathbb{N}\ F \neq It_{X\to X}^{(n)} \end{cases}$$

2. Let $\mathcal{P}$ be an infinite subset of $X \to X$ which is closed under composition and such that $It_{\mathcal{P}}^{\mathbb{N}}$ is injective (which is obviously the case if $\mathcal{P} = X \to X$). The Church representation of $\mathbb{N}$ relative to $\mathcal{P}$ is the function

$$Church_{\mathcal{P}}^{\mathbb{N}} : \mathcal{P}^{\mathcal{P}} \to \mathbb{N}$$

which is defined as above with $It_{\mathcal{P}}^{\mathbb{N}}$ and $It_{\mathcal{P}}^{(n)}$ in place of $It_{X\to X}^{\mathbb{N}}$ and $It_{X\to X}^{(n)}$.

9.2 General Church self-enumerated systems

As is well-known, there are several natural notions of computable functionals. Hence several ways to effectivize $(X \to X)^{(X \to X)}$, cf. §9.5, 9.6, 9.7. Nevertheless, we shall prove that all these effectivizations lead to the same self-enumerated system, cf. Thm.9.14, and that this system satisfies the conditions of Def.9.3 below.
Before entering the technicalities of effectivization of functionals, we take an axiomatic approach to give the simple argument which proves that Church semantics leads to the usual Kolmogorov complexity.

Definition 9.3 (Church self-enumerated systems). Let $\mathbb{X}$ be some basic set, $\mathcal{P} \subseteq X \to X$ and $\mathcal{D} \subset \mathcal{P}^{\mathcal{P}}$ and $\mathcal{F} \subseteq \mathbf{2}^* \to \mathcal{D}$ be such that

1. $(\mathcal{D}, \mathcal{F})$ is a full self-enumerated system (cf. Def.3.1),
2. $\mathcal{P}$ contains the successor function $Suc : \mathbb{N} \to \mathbb{N}$ and $\mathcal{D}$ contains all functionals $It_{\mathcal{P}}^{(n)}$ for $n \in \mathbb{N}$,
3. there exists a total recursive function $f : \mathbb{N} \to \mathbf{2}^*$ and a function $\Phi \in \mathcal{F}$ such that $Church_{\mathcal{P}}^{\mathbb{N}} \circ \Phi \circ f = Id_{\mathbb{N}}$,
4. if $\Phi \in \mathcal{F}$ then $\mathtt{p} \mapsto \Phi(\mathtt{p})(Suc)(0)$ is a partial recursive function $\mathbf{2}^* \to \mathbb{N}$.

Then we call $(\mathbb{N}, Church_{\mathcal{P}}^{\mathbb{N}} \circ \mathcal{F})$ a Church self-enumerated system.

9.3 Kolmogorov complexity of general Church systems

Theorem 9.4. *Denote by $Restrict(PR^{\mathbf{2}^* \to \mathbb{N}})$ the family of restrictions of functions in $PR^{\mathbf{2}^* \to \mathbb{N}}$.*
1. If $(\mathbb{N}, Church_{\mathcal{P}}^{\mathbb{N}} \circ \mathcal{F})$ is a Church self-enumerated system then

$$PR^{\mathbf{2}^* \to \mathbb{N}} \subseteq Church_{\mathcal{P}}^{\mathbb{N}} \circ \mathcal{F} \subseteq Restrict(PR^{\mathbf{2}^* \to \mathbb{N}})$$

2. $K^{\mathbb{N}}_{Church_{\mathcal{P}}^{\mathbb{N}} \circ \mathcal{F}} =_{\mathrm{ct}} K$

Proof. First inclusion of 1. Let f, Φ be as in condition 3 of Def.9.3. Let $\phi : \mathbf{2}^* \to \mathbb{N}$ be any partial recursive function. Then

$$\phi = Id_{\mathbb{N}} \circ \phi = (Church_{\mathcal{P}}^{\mathbb{N}} \circ \Phi \circ f) \circ \phi = Church_{\mathcal{P}}^{\mathbb{N}} \circ (\Phi \circ (f \circ \phi))$$

Now, since $(\mathcal{D}, \mathcal{F})$ is a full self-enumerated system and $\Phi \in \mathcal{F}$ and $f \circ \phi$ is partial recursive, $\Phi \circ (f \circ \phi)$ is also in $\mathcal{F}$. Thus ϕ is in $Church_{\mathcal{P}}^{\mathbb{N}} \circ \mathcal{F}$.

Second inclusion of 1. Let $\Phi \in \mathcal{F}$ and $\mathtt{p} \in \mathbf{2}^*$. If $(Church_{\mathcal{P}}^{\mathbb{N}} \circ \Phi)(\mathtt{p})$ is defined and has value $n \in \mathbb{N}$ then $\Phi(\mathtt{p}) = It_{\mathcal{P}}^{(n)}$. Since $Suc \in \mathcal{P}$, we have $\Phi(\mathtt{p})(Suc)(0) = It_{\mathcal{P}}^{(n)}(Suc)(0) = n$. Thus, when $(Church_{\mathcal{P}}^{\mathbb{N}} \circ \Phi)(\mathtt{p})$ is defined, we have $(Church_{\mathcal{P}}^{\mathbb{N}} \circ \Phi)(\mathtt{p}) = \Phi(\mathtt{p})(Suc)(0)$. Which proves that $Church_{\mathcal{P}}^{\mathbb{N}} \circ \Phi$

is a restriction of $\mathtt{p} \rightarrow \Phi(\mathtt{p})(Suc)(0)$, which is partial recursive by condition 4 of Def.9.3.

2. The first inclusion of point 1 yields $K^{\mathbb{N}}_{Church^{\mathbb{N}}_{\mathcal{P}} \circ \mathcal{F}} \leq_{\mathrm{ct}} K$. Since $K_g \leq K_h$ whenever $g, h : \mathbf{2}^* \rightarrow \mathbb{N}$ are such that h is a restriction of g, any optimal function for $PR^{\mathbf{2}^* \rightarrow \mathbb{N}}$ is optimal for $Restrict(PR^{\mathbf{2}^* \rightarrow \mathbb{N}})$. Thus, the second inclusion yields $K \leq_{\mathrm{ct}} K^{\mathbb{N}}_{Church^{\mathbb{N}}_{\mathcal{P}} \circ \mathcal{F}}$. □

9.4 Kolmogorov complexity of the Δ of general Church systems

Theorem 9.5. *1. If $(\mathbb{N}, Church^{\mathbb{N}}_{\mathcal{P}} \circ \mathcal{F})$ is a Church self-enumerated system then*

$$PR^{\mathbf{2}^* \rightarrow \mathbb{Z}} \subseteq \Delta Church^{\mathbb{N}}_{\mathcal{P}} \circ \mathcal{F} \subseteq Restrict(PR^{\mathbf{2}^* \rightarrow \mathbb{Z}})$$

2. $K^{\mathbb{Z}}_{\Delta Church^{\mathbb{N}}_{\mathcal{P}} \circ \mathcal{F}} \restriction \mathbb{N} =_{\mathrm{ct}} K$

Proof. 1. Observe that $\Delta(PR^{\mathbf{2}^* \rightarrow \mathbb{N}}) = PR^{\mathbf{2}^* \rightarrow \mathbb{Z}}$ and $\Delta(Restrict(PR^{\mathbf{2}^* \rightarrow \mathbb{N}})) = Restrict(PR^{\mathbf{2}^* \rightarrow \mathbb{N}})$ and apply Thm.9.4.
2. Argue as in point 2 of the proof of Thm.9.4. □

9.5 Computable and effectively continuous functionals

This subsection and the next ones are devoted to the construction of Church self-enumerated systems and to the characterization of these systems as restrictions of partial recursive functions to Π^0_2 sets (cf. Thm.9.14). A result which refines point 1 of Theorem 9.4.
First, we recall the two classical notions of partial computability for functionals, cf. Odifreddi's book [12] p.178, 188, 197.

Definition 9.6 (Kleene partial computable functionals).
1. Let $\mathbb{X}, \mathbb{Y}, \mathbb{S}, \mathbb{T}$ be some basic spaces and fix some suitable representations of their elements by words. An $(\mathbb{X} \rightarrow \mathbb{Y})$-oracle Turing machine with inputs and outputs respectively in $\mathbb{S}, \mathbb{T}$ is a Turing machine $\mathcal{M}$ which has a special oracle tape and is allowed at certain states to ask an oracle $f \in (\mathbb{X} \rightarrow \mathbb{Y})$ what are the successive digits of the value of $f(\mathtt{q})$ where $\mathtt{q}$ is the element of $\mathbb{X}$ currently written on the oracle tape.
The functional $\Phi_{\mathcal{M}} : ((\mathbb{X} \rightarrow \mathbb{Y}) \times \mathbb{S}) \rightarrow \mathbb{T}$ associated to $\mathcal{M}$ maps the pair $(f, \mathtt{s})$ on the output (when defined) computed by $\mathcal{M}$ when f is given as the partial function oracle and $\mathtt{s}$ as the input.
If on input $\mathtt{x}$ and oracle f the computation asks the oracle its value on an element on which f is undefined then $\mathcal{M}$ gets stuck, so that $\Phi_{\mathcal{M}}(f, \mathtt{x})$ is undefined.

2. A functional $\Phi : ((\mathbb{X} \rightarrow \mathbb{Y}) \times \mathbb{S}) \rightarrow \mathbb{T}$ is partial computable (also called partial recursive) if $\Phi = \Phi_{\mathcal{M}}$ for some $\mathcal{M}$.

A functional obtained via curryfications from such a functional is also called partial computable. We denote by PC^{τ} the family of partial computable functionals with type τ.

Definition 9.7 (Uspenskii (effectively) continuous functionals). Denote by $Fin(\mathbb{X} \to \mathbb{Y})$ the class of partial functions $\mathbb{X} \to \mathbb{Y}$ with finite domains. Observe that, $\alpha, \beta \in Fin(\mathbb{X} \to \mathbb{Y})$ are compatible if and only if $\alpha \cup \beta \in Fin(\mathbb{X} \to \mathbb{Y})$.

1. Let's say that the relation $R \subseteq Fin(\mathbb{X} \to \mathbb{Y}) \times \mathbb{S} \times \mathbb{T}$ is functional if

$$\alpha \cup \beta \in Fin(\mathbb{X} \to \mathbb{Y}) \ \wedge \ (\alpha, \mathtt{s}, \mathtt{t}) \in R \ \wedge \ (\beta, \mathtt{s}, \mathtt{t}') \in R \ \Rightarrow \ \mathtt{t} = \mathtt{t}'$$

To such a functional relation R can be associated a functional

$$\Phi_R : ((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$$

such that, for every $f, \mathtt{s}, \mathtt{t}$,

$$(\dagger) \qquad \Phi(f, \mathtt{s}) = \mathtt{t} \quad \Leftrightarrow \quad \exists u \subseteq f \ R(u, \mathtt{s}, \mathtt{t})$$

2. (Uspenskii [20], Nerode [11]) A functional $\Phi : ((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$ is *continuous* if it is of the form Φ_R for some functional relation R.

Φ is *effectively continuous* if R is r.e. Effectively continuous functionals are also called recursive operators (cf. Rogers [15], Odifreddi [12]).
A functional obtained via curryfications from such a functional is also called effectively continuous. We denote by *EffCont*$^{\tau}$ the family of effectively continuous functionals with type τ.

Effective continuity is more general than partial computability (cf. [12] p.188). However, restricted to total functions, both notions coincide.

Theorem 9.8. 1. (Uspenskii [20], Nerode [11]) *Partial computable functionals are effectively continuous.*
2. (Sasso [17, 18]) *There are effectively continuous functionals which are not partial computable.*
3. *A functional $\Phi : (\mathbb{Y}^{\mathbb{X}}) \times \mathbb{S} \to \mathbb{T}$ is the restriction of a partial computable functional $((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$ if and only if it is the restriction of an effectively continuous functional.*

9.6 Computability of functionals over $PR^{\mathbb{X} \to \mathbb{Y}}$

Using indexes, one can also consider computability for functionals operating on the sole partial recursive functions.

Definition 9.9. Let $(\varphi^{\mathbb{X} \to \mathbb{Y}}_{\mathtt{e}})_{\mathtt{e} \in \mathbb{N}}$ be an acceptable enumeration of $PR^{\mathbb{X} \to \mathbb{Y}}$.
1. A functional $\Phi : PR^{\mathbb{X} \to \mathbb{Y}} \times \mathbb{S} \to \mathbb{T}$ is an effective functional on partial

recursive functions if there exists some partial recursive function $f : \mathbb{N}\times\mathbb{S} \to \mathbb{T}$ such that, for all $\mathtt{s} \in \mathbb{S}$, $\mathtt{e} \in \mathbb{N}$,

$$\Phi(\varphi_{\mathtt{e}}^{\mathbb{X}\to\mathbb{Y}}, \mathtt{s}) = f(\mathtt{e}, \mathtt{s})$$

We denote by $\mathit{Eff}^{\,PR^{\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}\to\mathbb{T}}$ the family of such functionals.

2. We denote by $\mathit{Eff}^{\,PR^{\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}_1\to PR^{\mathbb{S}_2\to\mathbb{T}}}$ the family of functionals obtained by curryfication of the above class with $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$.
An easy application of the parameter property shows that these functionals are exactly those for which there exists some partial recursive function $g : \mathbb{N} \times \mathbb{S}_1 \to \mathbb{N}$ such that, for all $\mathtt{s}_1 \in \mathbb{S}_1, \mathtt{e} \in \mathbb{N}$,

$$\Phi(\varphi_{\mathtt{e}}^{\mathbb{X}\to\mathbb{Y}}, \mathtt{s}_1) = \varphi_{g(\mathtt{e},\mathtt{s}_1)}^{\mathbb{S}_2\to\mathbb{T}}$$

Note 9.10. 1. Thanks to Rogers' theorem (cf. Thm.5.1), the above definition does not depend on the chosen acceptable enumerations.

2. The above functions f, g should have the following properties:

$$\begin{array}{lcl}
\varphi_{\mathtt{e}}^{\mathbb{X}\to\mathbb{Y}} = \varphi_{\mathtt{e}'}^{\mathbb{X}\to\mathbb{Y}} & \Rightarrow & f(\mathtt{e}, \mathtt{s}) = f(\mathtt{e}', \mathtt{s}) \\
\varphi_{\mathtt{e}}^{\mathbb{X}\to\mathbb{Y}} = \varphi_{\mathtt{e}'}^{\mathbb{X}\to\mathbb{Y}} & \Rightarrow & \varphi_{g(\mathtt{e},\mathtt{s}_1)}^{\mathbb{S}_2\to\mathbb{T}} = \varphi_{g(\mathtt{e}',\mathtt{s}_1)}^{\mathbb{S}_2\to\mathbb{T}}
\end{array}$$

As shown by the following remarkable result, such functionals essentially reduce to those of Def.9.7 (cf. Odifreddi's book [12] p.206–208).

Theorem 9.11 (Uspenskii [20], Myhill & Shepherdson [10]). *The effective functionals $PR^{\mathbb{X}\to\mathbb{Y}} \to PR^{\mathbb{S}\to\mathbb{T}}$ are exactly the restrictions to $PR^{\mathbb{X}\to\mathbb{Y}}$ of effectively continuous functionals $(\mathbb{X} \to \mathbb{Y}) \to (\mathbb{S} \to \mathbb{T})$.*

9.7 Effectivizations of *Church* representation of $\mathbb{N}$ and their characterization

Theorem 9.12. *The following systems are full self-enumerated systems (cf. notations from Def.9.6, 9.7, 9.9):*

$$(PC^{\tau}, PC^{\mathbf{2}^*\to\tau})\ ,\ (\mathit{EffCont}^{\tau}, \mathit{EffCont}^{\mathbf{2}^*\to\tau})\ ,\ (\mathit{Eff}^{\,\tau}_{PR}, \mathit{Eff}^{\,\mathbf{2}^*\to\tau_{PR}})$$

Proof. Case of PC^{τ} and $\mathit{EffCont}^{\tau}$. Points i and ii (for full systems) of Def.3.1 are trivial. As for point iii, use the classical enumeration theorem for partial computable (resp. effectively continuous) functionals.
Case of $\mathit{Eff}^{\,\tau}_{PR}$. Easy corollary of 9.11. □

Definition 9.13 (Effectivizations of Church representation of $\mathbb{N}$). We effectivize the Church representation (cf. Def.9.2) by replacing $(X \to X) \to (X \to X)$ by one of the following classes:

$$PC^{(\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X})}\ ,\ \mathit{EffCont}^{(\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X})}\ ,\ \mathit{Eff}^{\,PR^{\mathbb{X}\to\mathbb{X}}\to PR^{\mathbb{X}\to\mathbb{X}}}$$

where $\mathbb{X}$ is some basic set. This leads to three self-enumerated systems $\mathbb{N}$:

$$\begin{array}{rcl} \mathcal{S}_1 &=& (\mathbb{N}\ ,\ \mathit{Church}^{\mathbb{N}}_{\mathbb{X}\to\mathbb{X}} \circ PC^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}) \\ \mathcal{S}_2 &=& (\mathbb{N}\ ,\ \mathit{Church}^{\mathbb{N}}_{\mathbb{X}\to\mathbb{X}} \circ \mathit{EffCont}^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}) \\ \mathcal{S}_3 &=& (\mathbb{N}\ ,\ \mathit{church}^{\mathbb{N}}_{PR^{\mathbb{X}\to\mathbb{X}}} \circ \mathit{Eff}^{\mathbf{2}^*\to(PR^{\mathbb{X}\to\mathbb{X}}\to PR^{\mathbb{X}\to\mathbb{X}})}) \end{array}$$

The following result greatly simplifies the landscape. Its proof requires a lot of auxiliary results and is given in §9.9.

Theorem 9.14. *1.* $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = (\mathbb{N}, PR^{\mathbf{2}^*\to\mathbb{N}} \restriction \Pi^0_2)$ *(i.e. the family of restrictions to* Π^0_2 *sets of partial recursive functions* $\mathbf{2}^* \to \mathbb{N}$*). Moreover, this system is a Church self-enumerated system (cf. Def.9.3).*

2. $\Delta\mathcal{S}_1 = \Delta\mathcal{S}_2 = \Delta\mathcal{S}_3 = (\mathbb{Z}, PR^{\mathbf{2}^*\to\mathbb{Z}} \restriction \Pi^0_2)$.

Thms. 9.4, 9.5 yield the following corollary of the above result.

Corollary 9.15. *The Kolmogorov complexities* $K^{\mathbb{N}}_{\mathit{Church}}$ *and* $K^{\mathbb{Z}}_{\Delta\mathit{Church}} \restriction \mathbb{N}$ *associated to the systems* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ *and* $\Delta\mathcal{S}_1, \Delta\mathcal{S}_2, \Delta\mathcal{S}_3$ *both coincide with the usual Kolmogorov complexity* $K^{\mathbb{N}}$*.*

9.8 Auxiliary results for the proof of Thm.9.14

For the proof of Thm.9.14 (cf. §9.9), we need some convenient tools given in the next propositions.

Proposition 9.16 (Iterators as effectively continuous functionals). *We denote by* $\mathcal{I}_n$ *the functional relation*

$$\mathcal{I}_n = \{(\alpha, \mathtt{x}, \alpha^{(n)}(\mathtt{x})) : \mathtt{x} \in \mathbb{X}\ \wedge\ \mathit{domain}(\alpha) = \{\alpha^{(i)}(\mathtt{x}) : i < n\}\}$$

Let $R \subset \mathit{Fin}(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ *be functional. Then*

$$\Phi_R = It^{(n)}_{\mathbb{X}\to\mathbb{X}}\ \Leftrightarrow\ R \supseteq \mathcal{I}_n$$

Proof. It is straightforward to see that $\mathcal{I}_n$ is functional and $\Phi_{\mathcal{I}_n} = It^{(n)}_{\mathbb{X}\to\mathbb{X}}$.
$\Leftarrow$. Suppose R is functional and $R \supseteq \mathcal{I}_n$. Let $f : \mathbb{X} \to \mathbb{X}$ and $\mathtt{x} \in \mathbb{X}$. Clearly, $\Phi_R(f)$ extends $\Phi_{\mathcal{I}_n}(f) = f^{(n)}$.
Case 1: $f^{(n)}(\mathtt{x})$ *is defined.* Then $\Phi_R(f)(\mathtt{x}) = f^{(n)}(\mathtt{x})$.
Case 2: $f^{(n)}(\mathtt{x})$ *is undefined.* Suppose $\Phi_R(f)(\mathtt{x})$ were defined and $\Phi_R(f)(\mathtt{x}) = \mathtt{y}$. By continuity, there would exist a finite restriction α of f such that $(\alpha, \mathtt{x}, \mathtt{y}) \in R$. Since $f^{(n)}(\mathtt{x})$ is undefined so is $\alpha^{(n)}(\mathtt{x})$. Let $i \leq n$ be least such that $\alpha^{(i)}(\mathtt{x})$ is undefined. Choose distinct $\mathtt{z}_i, ..., \mathtt{z}_n$ outside $\{\mathtt{y}\} \cup \mathit{range}(\alpha)$ and let β be an extension of α such that $\beta^{(j)}(\mathtt{x}) = \mathtt{z}_j$ for $j = i, ..., n$. Since $(\alpha, \mathtt{x}, \mathtt{y}) \in R$ and α is a restriction of β, we have $\Phi_R(\beta)(\mathtt{x}) = \mathtt{y}$. Now, $\beta^{(n)}(\mathtt{x}) = \mathtt{z}_n \neq y$ hence $\Phi_{R_e}(\beta)(\mathtt{x}) \neq \beta^{(n)}(\mathtt{x})$. This contradicts Case 1.

$\Rightarrow$. Suppose $domain(\alpha) = \{\alpha^{(i)}(\mathtt{x}) : i < n\}$. We show that $(\alpha, \mathtt{x}, \alpha^{(n)}(\mathtt{x})) \in R$. Since $\Phi_R(\alpha)(\mathtt{x}) = It^{(n)}_{\mathbb{X}\to\mathbb{X}}(\alpha)(\mathtt{x}) = \alpha^{(n)}(\mathtt{x})$, there exists a restriction β of α such that $(\beta, \mathtt{x}, \alpha^{(n)}(\mathtt{x})) \in R$. Thus, $\beta^{(n)}(\mathtt{x})$ is defined and $\beta^{(n)}(\mathtt{x}) = \alpha^{(n)}(\mathtt{x}))$. Which implies that β extends $\alpha \restriction \{\alpha^{(i)}(\mathtt{x}) : i < n\}$ which is α. Thus, $\beta = \alpha$. Hence $(\alpha, \mathtt{x}, \alpha^{(n)}(\mathtt{x})) \in R$. $\square$

The following simple result is quite convenient.

Proposition 9.17. *Let $n \in \mathbb{N}$ and $\Phi \in (\mathbb{X} \to \mathbb{X})^{(\mathbb{X}\to\mathbb{X})}$. If $\Phi(f)$ is a restriction of $f^{(n)}$ for every $f : \mathbb{X} \to \mathbb{X}$ then either $\Phi = It^{(n)}_{\mathbb{X}\to\mathbb{X}}$ or Φ is not an iterator.*

Proof. We reduce to the case $\mathbb{X} = \mathbb{N}$. Let $Suc : \mathbb{N} \to \mathbb{N}$ be the successor function. Since $\Phi(Suc)$ is a restriction of $Suc^{(n)}$, either $\Phi(Suc)(0)$ is undefined or $\Phi(Suc)(0) = n$. In both cases it is different from $Suc^{(p)}(0)$ for any $p \neq n$. Which proves that $\Phi \neq It^{(p)}_{\mathbb{N}\to\mathbb{N}}$ for every $p \neq n$. Hence the proposition. $\square$

Proposition 9.18 (Going from Kleene functionals to effectively continuous ones respecting iterators).
1. *Let $(W_{\mathtt{e}})_{\mathtt{e}\in\mathbb{N}}$ be an acceptable enumeration of r.e. subsets of $Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$. There exists a total recursive function $\xi : \mathbb{N} \to \mathbb{N}$ such that, for all $\mathtt{e}$,*

- *a. $W_{\xi(\mathtt{e})} \subseteq W_{\mathtt{e}}$ and $W_{\xi(\mathtt{e})}$ is functional (cf. Def.9.7, point 1),*
- *b. $W_{\xi(\mathtt{e})} = W_{\mathtt{e}}$ whenever $W_{\mathtt{e}}$ is functional.*

2. *There exists a partial recursive function $\lambda : \mathbb{N} \to \mathbb{N}$ such that if $R_{\mathtt{e}}$ is functional and $\Phi_{R_{\mathtt{e}}}$ is an iterator then $\lambda(\mathtt{e})$ is defined and $\Phi_{R_{\mathtt{e}}} = It^{(\lambda(\mathtt{e}))}_{\mathbb{X}\to\mathbb{X}}$. (However, $\lambda(\mathtt{e})$ may be defined even if $R_{\mathtt{e}}$ is not functional or $\Phi_{R_{\mathtt{e}}}$ is not an iterator).*

3. *There exists a total recursive function $\theta : \mathbb{N} \to \mathbb{N}$ such that, for all $\mathtt{e} \in \mathbb{N}$,*

- *a. if $\Phi_{R_{\mathtt{e}}}$ is an iterator then the $(\mathbb{X} \to \mathbb{X})$-oracle Turing machine $\mathcal{M}_{\theta(\mathtt{e})}$ with code $\theta(\mathtt{e})$ (cf. Def.9.6) computes the functional $\Phi_{R_{\mathtt{e}}}$,*
- *b. if $\Phi_{R_{\mathtt{e}}}$ is not an iterator then neither is the functional computed by the $(\mathbb{X} \to \mathbb{X})$-oracle Turing machine $\mathcal{M}_{\theta(\mathtt{e})}$ with code $\theta(\mathtt{e})$.*

In other words, $Church(\Phi_{R_{\mathtt{e}}}) = Church(\Phi_{\mathcal{M}_{\theta(\mathtt{e})}})$

Proof. 1. This is the classical fact underlying the enumeration theorem for effectively continuous functionals. To get $W_{\xi(\mathtt{e})}$, enumerate $W_{\mathtt{e}}$ and retain a triple $(\alpha, \mathtt{x}, \mathtt{y})$ if and only if, together with the already retained ones, it does not contradict functionality (cf. Odifreddi's book [12] p.197).

2. We reduce to the case $\mathbb{X} = \mathbb{N}$. Suppose R is functional and $\Phi_R = It^{(n)}_{\mathbb{N}\to\mathbb{N}}$.

Prop.9.16 insures that $(Suc \restriction \{0, ..., n-1\}, 0, n) \in R$ where Suc denotes the successor function on $\mathbb{N}$.
Also, for $m \neq n$, since $Suc \restriction \{0, ..., m-1\}$ and $Suc \restriction \{0, ..., n-1\}$ are compatible and R is functional, R cannot contain $(Suc \restriction \{0, ..., m-1\}, 0, m)$.
Thus, if $\Phi_R = It^{(n)}_{\mathbb{N}\to\mathbb{N}}$ then n is the unique integer such that R contains $(Suc \restriction \{0, ..., n-1\}, 0, n)$.
This leads to the following definition of the wanted partial recursive $\lambda : \mathbb{N} \to \mathbb{N}$: enumerate $R\mathtt{e}$, if and when some triple $(Suc \restriction \{0, ..., n-1\}, 0, n)$ appears, halt and output $\lambda(\mathtt{e}) = n$.

3. Given a code $\mathtt{e}$ of a functional relation $R_{\mathtt{e}}$, we let θ be the total recursive function such that $\theta(\mathtt{e})$ is a code for the oracle Turing machine $\mathcal{M}$ which acts as follows on oracle f and input $\mathtt{x}$:

i. First, $\mathcal{M}$ computes $\lambda(\mathtt{e})$.

ii. If $\lambda(\mathtt{e})$ is defined then, on input $\mathtt{x}$ and oracle f, $\mathcal{M}$ tries to compute $It^{(\lambda(\mathtt{e}))}_{\mathbb{X}\to\mathbb{X}}(f)(\mathtt{x})$ in the obvious way: ask the oracle the values of $f^{(i)}(\mathtt{x})$ for $i \leq \lambda(\mathtt{e})$.

iii. If i and ii succeed, i.e. $\lambda(\mathtt{e})$ and $f^{(\lambda(\mathtt{e}))}(\mathtt{x})$ are both defined, then $\mathcal{M}$ start enumerating $R_{\mathtt{e}}$ until $(f \restriction \{f^{(i)}(\mathtt{x}) : i < \lambda(\mathtt{e})\}, \mathtt{x}, f^{(\lambda(\mathtt{e}))}(\mathtt{x}))$ appears.

iv. $\mathcal{M}$ halts and accepts if and only if i, ii and iii all succeed. In which case $\mathcal{M}$ outputs $f^{(\lambda(\mathtt{e}))}(\mathtt{x})$.

Case $\Phi_{R_{\mathtt{e}}}$ is an iterator. Point 2 insures that $\lambda(\mathtt{e})$ is defined and $\Phi_{R_{\mathtt{e}}} = It^{(\lambda(\mathtt{e}))}_{\mathbb{X}\to\mathbb{X}}$. Prop.9.16 insures that iii succeeds. Thus, $\mathcal{M}$ computes exactly $It^{(\lambda(\mathtt{e}))}_{\mathbb{X}\to\mathbb{X}}$, as does $\Phi_{R_{\mathtt{e}}}$.

Case $\Phi_{R_{\mathtt{e}}}$ is not an iterator and $\lambda(\mathtt{e})$ is undefined. Then $\mathcal{M}$ computes the constant functional with value the nowhere defined function. Thus, $\mathcal{M}$ does not compute an iterator.

Case $\Phi_{R_{\mathtt{e}}}$ is not an iterator and $\lambda(\mathtt{e})$ is defined. Then $\Phi_{R_{\mathtt{e}}} \neq It^{(\lambda(\mathtt{e}))}_{\mathbb{X}\to\mathbb{X}}$. Let $f : \mathbb{X} \to \mathbb{X}$ and $\mathtt{x}$ be such that $\Phi_{R_{\mathtt{e}}}(f)(\mathtt{x}) \neq f^{(\lambda(\mathtt{e}))}(\mathtt{x})$, (i.e. either both quantities are defined and distinct or one is defined while the other is not).
Subcase $f^{(\lambda(\mathtt{e}))}(\mathtt{x})$ is defined. Then iii cannot succeed and $\Phi_{\mathcal{M}}(f)$ is undefined. Hence $\Phi_{\mathcal{M}}(f)(\mathtt{x}) \neq f^{(\lambda(\mathtt{e}))}(\mathtt{x})$. Which proves that $\Phi_{\mathcal{M}} \neq It^{(\lambda(\mathtt{e}))}_{\mathbb{X}\to\mathbb{X}}$. Since $\Phi_{\mathcal{M}}(f)$ is a restriction of $f^{(\lambda(\mathtt{e}))}$ for all f, Prop.9.17 insures that $\Phi_{\mathcal{M}}$ is not an iterator.
Subcase $f^{(\lambda(\mathtt{e}))}(\mathtt{x})$ is undefined (hence $\lambda(\mathtt{e}) \geq 1$). Then $\Phi_{R_{\mathtt{e}}}(f)(\mathtt{x})$ is necessarily defined. Let $\Phi_{R_{\mathtt{e}}}(f)(\mathtt{x}) = \mathtt{y}$ and let α be a finite restriction of f such that $(\alpha, \mathtt{x}, \mathtt{y}) \in R_{\mathtt{e}}$. There exist $i < \lambda(\mathtt{e})$ and $\alpha(\alpha^{(i)}(\mathtt{x}))$ is undefined. Let i be the least such one. As in the proof of Prop.9.16 ($\Leftarrow$ direction), there is an

extension β of α such that $\beta^{(\lambda(\mathtt{e}))}(\mathtt{x})$ is defined and $\beta^{(\lambda(\mathtt{e}))}(\mathtt{x}) \neq y$. Since β extends α and $(\alpha, \mathtt{x}, \mathtt{y}) \in R_{\mathtt{e}}$, we have $\Phi_{R_{\mathtt{e}}}(\alpha)(\mathtt{x}) = \Phi_{R_{\mathtt{e}}}(\beta)(\mathtt{x}) \neq \beta^{(\lambda(\mathtt{e}))}(\mathtt{x})$. Thus, with β in place of f, iii cannot succeed and $\Phi_{\mathcal{M}}(\beta)(\mathtt{x})$ is undefined. In particular, $\Phi_{\mathcal{M}}(\beta) \neq \beta^{(\lambda(\mathtt{e}))}$ hence $\Phi_{\mathcal{M}} \neq It^{(\lambda(\mathtt{e}))}_{\mathbb{X}\to\mathbb{X}}$. Since $\Phi_{\mathcal{M}}(g)$ is always a restriction of $g^{(\lambda(\mathtt{e}))}$, Prop.9.17 insures that $\Phi_{\mathcal{M}}$ is not an iterator. $\square$

We recall the following well-known result.

Proposition 9.19 (The *Apply* functional is computable).
Let $\Phi : \mathbf{2}^* \to \mathit{EffCont}^{(\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X})}$ be effectively continuous and $\varphi : \mathbb{X} \to \mathbb{X}$ be partial recursive. Then the function $g : \mathbf{2}^* \times \mathbb{X} \to \mathbb{X}$ such that $g(\mathtt{p}, \mathtt{x}) = (\Phi(\mathtt{p})(\varphi))(\mathtt{x})$ for all $\mathtt{p} \in \mathbf{2}^*$ and $\mathtt{x} \in \mathbb{X}$, is partial recursive.

Proof. Let $R \subseteq \mathbb{Y} \times \mathit{Fin}(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ be an r.e. set such that, for all $\mathtt{p}$, $R^{(\mathtt{p})} = \{(\alpha, \mathtt{x}, \mathtt{y}) : (\mathtt{p}, \alpha, \mathtt{x}, \mathtt{y}) \in R\}$ is functional and $\Phi(\mathtt{p}) = \Phi_{R^{(\mathtt{e})}}$. By continuity, $g(\mathtt{p}, \mathtt{x})$ is defined and $g(\mathtt{p}, \mathtt{x}) = \mathtt{y}$ if and only if there exists some finite restriction α of φ such that $(\alpha, \mathtt{x}, \mathtt{y}) \in R^{(\mathtt{p})}$. Since we can effectively enumerate $R^{(\mathtt{p})}$ and the family of finite restrictions of φ, we see that g is indeed partial recursive. $\square$

We shall need the following examples of effectively continuous functionals.

Proposition 9.20 (Iterators and Π^0_2 domains). *If $\varphi : \mathbf{2}^* \to \mathbb{N}$ is partial recursive and $S \subseteq \mathbf{2}^*$ is Π^0_2 then there exists an effectively continuous functional $\Phi : \mathbf{2}^* \to (\mathbb{X} \to \mathbb{X})^{\mathbb{X}\to\mathbb{X}}$ such that $\mathit{Church}^{\mathbb{N}}_{\mathbb{X}} \circ \Phi = \varphi \restriction S$, i.e. for all $\mathtt{p}$,*

$$
\begin{array}{lrcl}
(*) & \mathtt{p} \in S \cap \mathit{domain}(\varphi) & \Rightarrow & \Phi(\mathtt{p}) = It^{(\varphi(\mathtt{p}))}_{\mathbb{X}\to\mathbb{X}} \\
(**) & \mathtt{p} \notin S \cap \mathit{domain}(\varphi) & \Rightarrow & \Phi(\mathtt{p}) \textit{ is not an iterator}
\end{array}
$$

Proof. Let $S = \{\mathtt{p} \in \mathbf{2}^* : \forall u\ \exists v\ (\mathtt{p}, u, v) \in \sigma\}$ where σ is a recursive subset of $\mathbf{2}^* \times \mathbb{N}^2$. Let

$$\mathcal{I}_n = \{(\alpha, \mathtt{x}, \alpha^{(n)}(\mathtt{x})) : \alpha \in \mathit{Fin}(\mathbb{X} \to \mathbb{X})\ \wedge\ \mathit{domain}(\alpha) = \{\alpha^{(i)}(\mathtt{x}) : i < n\}\}$$

and let $\gamma : \mathbb{N}^2 \to \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$ be a total recursive function such that, for all n, $u \mapsto \gamma(n, u)$ is a bijection $\mathbb{N} \to \mathcal{I}_n$. Set

$$R_{\mathtt{p}} = \{\gamma(\varphi(\mathtt{p}), u) : \varphi(\mathtt{p}) \text{ is defined } \wedge\ \forall u' \leq u\ \exists v\ (\mathtt{p}, u', v) \in \sigma\}$$

Case $\varphi(\mathtt{p})$ is not defined. Then $R_{\mathtt{p}} = \emptyset$ so that $\Phi_{R_{\mathtt{p}}}$ is the constant functional which maps any function to the nowhere defined function. In particular, $\Phi_{R_{\mathtt{p}}}$ is not an iterator.
Case $\varphi(\mathtt{p})$ is defined. Clearly, $R_{\mathtt{p}} \subseteq \mathcal{I}_{\varphi(\mathtt{p})}$. Since $\mathcal{I}_{\varphi(\mathtt{p})}$ is functional (cf. Prop.9.16) so is $R_{\mathtt{p}}$.
Subcase $\mathtt{p} \in S$. Then $R_{\mathtt{p}} = \mathcal{I}_{\varphi(\mathtt{p})}$ so that $\Phi_{R_{\mathtt{p}}} = It^{(\varphi(n))}_{\mathbb{X}\to\mathbb{X}}$ (cf. Prop.9.16).
Subcase $\mathtt{p} \notin S$. Then $R_{\mathtt{p}}$ is finite so that $\Phi_{R_{\mathtt{p}}}$ has finite range hence cannot be an iterator.
Letting $\Phi(\mathtt{p}) = \Phi_{R_{\mathtt{e}}}$, this proves $(*)$ and $(**)$. $\square$

9.9 Proof of Thm.9.14 characterizing the *Church* representation systems

1. *Inclusions* $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_3$. The first inclusion is a corollary of point 1 of Thm.9.8. The second one is trivial.

2. *Inclusion* $\mathcal{S}_3 \subseteq \mathcal{S}_2$. Every element of $\mathcal{S}_3$ is of the form $Church \circ \Psi$ where $\Psi \in Eff^{\mathbf{2}^* \to (PR^{\mathbb{X}\to\mathbb{X}} \to PR^{\mathbb{X}\to\mathbb{X}})}$. Thm.9.11 insures that there exists $\Phi \in EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}$ such that, for all $\mathtt{p} \in \mathbf{2}^*$, $\Psi(\mathtt{p})$ is the restriction of $\Phi(\mathtt{p})$ to $PR^{\mathbb{X}\to\mathbb{X}} \to PR^{\mathbb{X}\to\mathbb{X}}$. Now, $Church \circ \Psi(\mathtt{p}) = n$ if and only if $\forall f \in PR^{\mathbb{X}\to\mathbb{X}}\ \Psi(\mathtt{p})(f) = f^{(n)}$ if and only if $\forall f \in PR^{\mathbb{X}\to\mathbb{X}}\ \Phi(\mathtt{p})(f) = f^{(n)}$. Since $Fin(\mathbb{X} \to \mathbb{X}) \subset PR^{\mathbb{X}\to\mathbb{Y}}$, continuity of $\Phi(\mathtt{p})$ implies that this last condition is equivalent to $\forall f\ \Phi(\mathtt{p})(f) = f^{(n)}$. Which means $Church \circ \Phi(\mathtt{p}) = n$. Thus, $Church \circ \Psi = Church \circ \Phi$ and is therefore in $\in \mathcal{S}_2$.

3. *Inclusion* $\mathcal{S}_2 \subseteq \mathcal{S}_1$. Every element of $\mathcal{S}_2$ is of the form $Church \circ \Phi_R$ where $R \subseteq \mathbf{2}^* \times Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ is an r.e. set such that $R_\mathtt{p} = \{(\alpha, \mathtt{x}, \mathtt{y}) : (\mathtt{p}, \alpha, \mathtt{x}, \mathtt{y}) \in R\}$ is functional for all $\mathtt{p} \in \mathbf{2}^*$. Let $h : \mathbf{2}^* \to \mathbb{N}$ be a total recursive function such that $h(\mathtt{p})$ is an r.e. code for $R_\mathtt{p}$. Prop.9.18, point 3, gives a total recursive $\theta : \mathbb{N} \to \mathbb{N}$ such that $Church(\Phi_{R_\mathtt{p}}) = Church(\Phi_{\mathcal{M}_{\theta(h(\mathtt{p}))}})$. Now, $\mathtt{p} \mapsto \Phi_{\mathcal{M}_{\theta(h(\mathtt{p}))}}$ is in $PC^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}$. Thus $Church \circ \Phi_R \in \mathcal{S}_1$.

4. $\mathcal{S}_1$ *is a Church self-enumerated system.* Condition 1 is exactly Thm.9.12. Condition 2 is obvious. Condition 4 is an instance of Prop.9.19.
As for condition 3, consider an enumeration $(W_n)_{n\in\mathbb{N}}$ of functional r.e. subsets of $Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X}^2$, a total recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $\Phi_{W_{g(n)}} = It^{(n)}_{\mathbb{X}\to\mathbb{X}}$ and a total recursive bijection $f : \mathbb{N} \to \mathbf{2}^*$ and define $\Phi : \mathbf{2}^* \to (\mathbb{X} \to \mathbb{X})^{\mathbb{X}\to\mathbb{X}}$ as follows: $\Phi(\mathtt{p}) = \Phi_{W_{g(f^{-1}(\mathtt{p}))}}$. Then $Church^{\mathbb{N}}_{\mathbb{X}} \circ \Phi \circ f = Id_\mathbb{N}$.

5. *Inclusion* $\mathcal{S}_2 \subseteq PR^{\mathbf{2}^*\to\mathbb{N}} \upharpoonright \Pi^0_2$. Since $\mathcal{S}_2$ is a Church self-enumerated system, Thm.9.4 insures that all functions in $\mathcal{S}_2$ are restrictions of partial recursive functions. Let's show that their domains are Π^0_2. Let $R \subseteq \mathbf{2}^* \times Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ be an r.e. set such that $R_\mathtt{p}$ is functional for all $\mathtt{p} \in \mathbf{2}^*$. Then

$$domain(Church^{\mathbb{N}}_X \circ \Phi_R) = \{\mathtt{p} : \Phi_{R_\mathtt{p}} \text{ is an iterator}\}$$

Now, an r.e. code for the functional relation $R_\mathtt{p}$ is given by a total recursive function $h : \mathbf{2}^* \to \mathbf{2}^*$. Applying point 2 of Prop.9.18, the partial recursive function $\lambda \circ h$ is such that if $\Phi_{R_\mathtt{p}}$ is an iterator then $\Phi_{R_\mathtt{p}} = It^{(\lambda(h(\mathtt{p})))}_{\mathbb{X}\to\mathbb{X}}$.
Thus, using Prop.9.16, $\Phi_{R_\mathtt{p}}$ is an iterator if and only if $\lambda(h(\mathtt{p}))$ is defined and $R_\mathtt{p} \supseteq \mathcal{I}_{\lambda(h(\mathtt{p}))}$. Which is a $\Sigma^0_1 \wedge \Pi^0_2$ hence Π^0_2 condition. Thus, $domain(Church^{\mathbb{N}}_X \circ \Phi_R)$ is Π^0_2.

6. *Inclusion* $\mathcal{S}_2 \supseteq PR^{\mathbf{2}^*\to\mathbb{N}} \upharpoonright \Pi^0_2$. This is exactly Prop.9.20.

7. *Equalities* $\Delta\mathcal{S}_1 = \Delta\mathcal{S}_2 = \Delta\mathcal{S}_3 = (\mathbb{Z}, PR^{\mathbf{2}^*\to\mathbb{Z}} \upharpoonright \Pi^0_2)$. Straightforward corollary of the characterization of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$. □

9.10 Functional representations of $\mathbb{Z}$

Specific to Church representation, there is another approach for an extension to $\mathbb{Z}$: *positive and negative iterations* of injective functions over some infinite set X. Formally, I.e., letting $X \overset{1-1}{\to} X$ denote the family of injective functions, consider the $\mathbb{Z}$-iterator functional

$$It_X^{\mathbb{Z}} : \mathbb{Z} \to (X \overset{1-1}{\to} X)^{X \overset{1-1}{\to} X}$$

such that, for $n \in \mathbb{N}$, $It_X^{\mathbb{Z}}(n)(f) = f^{(n)}$ and $It_X^{\mathbb{Z}}(-n)(f) = It_X^{\mathbb{Z}}(n)(f^{-1})$. Effectivization can be done as in §9.7. All previous results, in particular Thm.9.14 and Cor.9.15 go through the $\mathbb{Z}$ context.

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