Decision problems among the main subfamilies of rational relations

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Abstract

We consider the four families of recognizable, synchronous, deterministic rational and rational subsets of a direct product of free monoids. They form a strict hierarchy and we investigate the following decision problem: given a relation in one of the family, does it belong to a smaller family? We settle the problem entirely when all monoids have a unique generator and fill some gaps in the general case. In particular, adapting a proof of Stearns, we show that it is recursively decidable whether or not a deterministic subset of an arbitrary number of free monoids is recognizable. Also we exhibit a single exponential algorithm for determining if a synchronous relation is recognizable.

1 Introduction

The rational relations are the subsets of a direct product of free monoids accepted by multi-tape automata, historically introduced by Rabin and Scott in their deterministic version in the late fifties, see [12]. The nondeterministic model which is nowadays considered as the right generalization, was very shortly proposed by Elgot and Mezei [6] who gave a far reaching account of their closure properties. Concerning the decision problems, the most general undecidable results were discovered by Fischer and Rosenberg [7] shortly thereafter. A decade later, these results were refined under specific conditions on the direct products by Lisovik, [11] and Ibarra, [10], almost at the same time, though independently, yet another consequence of the division of the world between the West and the East. There are very few nontrivial decidable properties, essentially Stearns [14] (though expressed in a different

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framework, see paragraph 3.2 below, and therefore overlooked) and Bertoni [2].

The present work focuses on specific decision properties. Indeed, let us recall that for a monoid of the form $M = A_1^* \times \cdots \times A_k^*$ with $k > 1$, there exists a strict hierarchy of families of subsets (their definition is given in section 2),

$$\mathcal{F}_0 = \mathrm{Rec}(M) \subsetneq \mathcal{F}_1 = \mathrm{Sync}(M) \subsetneq \mathcal{F}_2 = \mathrm{DRat}(M) \subsetneq \mathcal{F}_3 = \mathrm{Rat}(M) \quad (1)$$

respectively known in increasing order, as the recognizable, synchronous, deterministic rational and rational families. A natural question is therefore the following. Given $0 \leq i < j \leq 3$ and a subset of M belonging to $\mathcal{F}_j$, is is decidable whether or not it belongs to $\mathcal{F}_i$?

For the general class of rational subsets, i.e., for $\mathcal{F}_3$, the question has long been settled by Fischer and Rosenberg who proved that it is undecidable whether or not a rational relation is deterministic, see [7, Theorem 9]. This result requires however at least two free monoids with at least two generators. Lisovik [11] strengthened this result by showing none of the proper subclasses of the $\mathrm{Rat}(M)$ to be decidable, even in the special case of the direct product of a two generator and a one generator free monoids, see column 1 of Table 1. In his textbook, J. Sakarovitch raises the question for the three remaining nontrivial cases when $k = 2$, [13, p. 632 and 659]. It just happens that in this case, a strong result due to Stearns implicitly provides a decision procedure for the question whether a deterministic relation is recognizable or not. The complexity of Stearns' procedure was further improved by Valiant, as explained in paragraph 3.1.

Let us now discuss the three main results of this contribution. First, adapting Stearns' result, we prove that it is recursively decidable whether or not a determinisitic relation over an arbitrary product of free monoids is recognizable. Second, we show that there exists a single exponential algorithm deciding whether or not a synchronous relation is recognizable whatever the number of free monoids in the direct product. Third, we settle completely the special case where all alphabets A_i's consist of a unique letter or, equivalently, where the product $A_1^* \times \cdots \times A_k^*$ is commutative. Indeed, under this hypothesis, all problems can be decided by resorting to the decidability of the arithmetics of Presburger. Provided that rational relations are given as Presburger formulas, the complexity of any of the above decision problem is that of Presburger arithmetics (up to a linear factor). Consequently, there remains open the problem of deciding whether or not a deterministic relation is synchronous when $k > 1$.

	Rat(M)	DRat(M)	Sync(M)
DRat(M)	undecidable (1)		
Sync(M)	undecidable (2)	open	
Rec(M)	undecidable (3)	decidable (4)	decidable (5) in exponential time

Table 1: Decision status in $M = A_1^* \times \cdots \times A_k^*$

Credits for Table 1:

(1) Case $k \geq 2$ and $|A_1|, |A_2| \geq 2$: Fischer & Rosenberg [7], 1967.

$(1, 2, 3)$ Case $k = 2$ and $|A_1| = 1$, $|A_2| = 2$: Lisovik [11], 1979.

(4) $\begin{cases} k = 2, \text{ Stearns [14], 1967 (triple exponential time)} \\ \text{Valiant [15], 1975, (double exponential time)} \\ \text{Arbitrary } k : \text{ this paper.} \end{cases}$

(5) This paper.

2 Preliminaries

Given an *alphabet* A, we denote by A^* the free monoid it generates, i.e., the set of *words* written on the alphabet A, by 1 the *empty* word and by A^+ the set of nonempty words. The *length* of a word $u \in A^*$ is denoted by $|u|$. All alphabets considered here are finite and non-empty. The purpose of this paper is to study some decision properties of the product monoid $A_1^* \times \cdots \times A_k^*$. The componentwise concatenation of the direct product extends to subsets: if $R, S \subseteq A_1^* \times \cdots \times A_k^*$, then $RS = \{(x_1y_1, \ldots, x_ky_k) \mid (x_1, \ldots, x_k) \in R, (y_1, \ldots, y_k) \in S\}$. Observe that such a monoid is commutative if and only if all alphabets have one generator, in which case it is isomorphic to $\mathbb{N}^k$.

We assume some familiarity of the reader with the theory of k-tape automata and of rational subsets of free commutative monoids. The standard references are the handbooks of J. Berstel [1], S. Eilenberg [3] and J. Sakarovitch [13] for the former and the article [5] for the latter. We take for granted all closure properties of synchronous, deterministic and rational relations as well as the characterization of rational subsets of $\mathbb{N}^k$ as finite union of linear sets.

2.1 Nondeterministic and deterministic k-tape automata

The k-tape automaton is the direct generalization of the ordinary one tape. There are possible variations in the definition, leading to equivalent notions.

We choose the one which is convenient in this work.

With minor technical differences, a k-tape automaton is a k-tape, one-way, read only Turing machine meant to accept k-tuples of words. It is provided with a finite memory and k input tapes divided into cells each containing a symbol. At the beginning of the computation read-only heads are positioned on the leftmost cell of each tape. Based on the current state, one and only one of the symbols is read and the corresponding head moves one step to the right and a transition to a new state is performed. There exists a nondeterministic and a deterministic versions of these devices. We start with the first one.

It is convenient, given k alphabets $A_1, A_2, \ldots, A_k$, to denote by H_i the set of all k-tuples such that the i-th component is the unique which is not the empty word. We denote by H the union of the H_i's.

$$H_i = \{1\}^{i-1} \times A_i \times \{1\}^{k-i} \qquad , \qquad H = \bigcup_{i=1}^{k} H_i$$

Definition 2.1. *A* k-tape automaton $\mathcal{A}$ *is a tuple* $(A_1, A_2, \ldots, A_k, Q, E, I, T)$ *where:*

i) $A_1, A_2, \ldots A_k$ *are finite non-empty alphabets,*

ii) Q *is the finite set of states,*

iii) $I \subseteq Q$ *is the set of initial states,*

iv) $T \subseteq Q$ *is the set of final states,*

v) $E \subseteq Q \times H \times Q$ *is the set of transitions.*

Given $(q, h, p) \in E$, q is the *current* state, p is the *next* state and h is the *label* of the transition. A *path* from q_0 to q_n in $\mathcal{A}$, where $q_0, q_n \in Q$, is a sequence of transitions of E of the form

$$(q_0, h_1, q_1)(q_1, h_2, q_2) \ldots (q_{n-1}, h_n, q_n)$$

also written

$$q_0 \xrightarrow{h_1} q_1 \xrightarrow{h_2} q_2 \cdots q_{n-1} \xrightarrow{h_n} q_n \tag{2}$$

The path is said to be a *successful* if and only if $q_0 \in I$ and $q_n \in T$. The *label* of the path is the componentwise concatenation of the labels of the successive transitions, namely, the k-tuple $h \in A_1^* \times \cdots \times A_k^*$ where $h = h_1 h_2 \ldots h_n$. We shall use the simpler notation $q_0 \xrightarrow{h_1 h_2 \ldots h_n} q_n$.

The relation $R \subseteq A_1^* \times \cdots \times A_k^*$ *accepted* by the automaton $\mathcal{A}$ is the set of labels of successful paths of $\mathcal{A}$. A relation accepted by some k-tape automaton is called *rational* and $\mathrm{Rat}(A_1^* \times \cdots \times A_k^*)$ denotes the family of all rational relations.

The deterministic version of multi-tape automata imposes two restrictions. The current state determines which tape to read from, independently of the actual contents of the cells scanned. Secondly, given the state and the letter on the corresponding tape, there is at most one possible next state. Furthermore, in order to increase the recognition power, the device is allowed to sense the end of the input, i.e., to scan the empty cell to the right of the last letter of the input. Technically, the input on each tape is provided with an endmarker #.

Definition 2.2. *1. A k-tape automaton $\mathcal{A}$ is* deterministic *if the set of states is partitioned as $Q = Q_1 \cup Q_2 \cup \cdots \cup Q_k$ and the set of transitions E is subject to the following conditions*

i) the set I of initial states is reduced to a unique element q_-,

ii) $E \subseteq \bigcup_{i=1}^{k} (Q_i \times H_i \times Q)$

iii) for all $(q_i, h_i) \in Q_i \times H_i$, there exists at most one transition (q_i, h_i, p) in E.

2. Let # be a new symbol not belonging to any of the alphabets A_i. A relation $R \subseteq A_1^* \times \cdots \times A_k^*$ is deterministic rational *if the relation*

$$\{(w_1\#, \ldots, w_k\#) \mid (w_1, \ldots, w_k) \in R\}$$

is accepted by some deterministic k-tape automaton. The family of deterministic rational relations is denoted by $\mathrm{DRat}(A_1^* \times \cdots \times A_k^*)$.

2.2 Synchronous relations

In the previous paragraphs, the input tapes are processed at different variable speeds. The idea with the synchronous relations is to oblige the read heads to move simultaneously. This seems to imply that the input has the same length on each tape. In order to overcome this too severe restriction, all shortest components of the input are padded with occurrences of an extra dummy symbol # not belonging to any of the alphabets A_i, i.e., $(w_1, w_2, \ldots, w_k)$ is transformed into

$$\begin{array}{l} w^{\#} = (w_1\#^{e_1}, w_2\#^{e_2}, \ldots, w_n\#^{e_n}) \\ \text{with } e_i = -|w_i| + \max_{1 \le j \le n} |w_j|, i = 1, \ldots, n \end{array} \tag{3}$$

For example, with the triple $w = (ab, cdab, 1)$ we get $w^{\#} = (ab\#\#, cdab, \#\#\#\#)$. We extend this notation to subsets $R \subseteq A_1^* \times \cdots \times A_k^*$ in the natural way by writing $R^{\#}$ for the result of this operation. Observe that $R^{\#}$ can be viewed as a subset of the free monoid generated by $(A_1 \cup \{\#\}) \times \cdots \times (A_k \cup \{\#\})$. In particular, the above triple can be viewed as a word of length 4.

Definition 2.3. *A relation $R \subseteq A_1^* \times \cdots \times A_k^*$ is* synchronous*, if the relation $R^{\#}$, viewed as a subset of the free monoid generated by $(A_1 \cup \{\#\}) \times \cdots \times (A_k \cup \{\#\})$, is recognizable by a finite automaton. The family of synchronous relations is denoted by* $\mathrm{Sync}(A_1^* \times \cdots \times A_k^*)$.

An important case are the so-called length-preserving synchronous relations all the tuples of which satisfy the condition that all their non-empty components have the same length. Call support of such a relation the subset of indices whose components are non-empty. The following can be easily established by resorting to standard automata-theoretic methods.

Proposition 2.4. *Each synchronous relation on $A_1^* \times \cdots \times A_k^*$ is a finite union of subsets of the form $R_1 R_2 \dots R_n$ for some $n > 0$, where the R_i's are synchronous length-preserving relations of decreasing supports.*

2.3 Recognizable relations

This family has the weakest expressive power of the four families that we consider. Contrarily to the model of k-tape automaton, it does not assume a common memory for all tapes. Instead, each tape has its own memory and may work separately. More formally, with the notations of Definition 2.1, we have

Definition 2.5. *A relation $R \subseteq A_1^* \times A_2^* \times \cdots \times A_k^*$ is* recognizable *if it is accepted by a k-tape automaton satisfying the following conditions*

(i) $Q = Q_1 \times Q_2 \times \cdots \times Q_k$,

(ii) the set of transitions $E \subseteq Q \times H \times Q$ satisfies the following condition

$$(q_1, \dots, q_k, h, p_1, \dots, p_k) \in E \ \wedge \ h \in E_i \ \Rightarrow \ \bigwedge_{j \neq i} p_j = q_j$$

The family of recognizable relations is denoted by $\mathrm{Rec}(A_1^* \times \cdots \times A_k^*)$.

The following result attributed to Elgot and Mezei is a useful characterization of the recognizable subsets of a direct product of free monoids.

Theorem 2.6. *A subset $R \subseteq A_1^* \times \cdots \times A_k^*$ is recognizable if and only if it is a finite union of direct products of the form $X_1 \times \cdots \times X_k$ where X_i is a recognizable subset of A_i^*.*

3 The general case

In this section we show that for the direct product of two free monoids, the decidability of the family of recognizable relations in the family deterministic rational relations was implicit in a result of Stearns. We explain how this can be seen. In the last paragraph, using Stearns' method, we extend this result to deterministic relations of arbitrary arity. Then we give a direct procedure of lower complexity, which solves recognizability in the family of synchronous relations. This answers the question posed in [13] by J. Sakarovitch's handbook.

3.1 Interpretation of relations as languages

Binary rational relations can be viewed as particular context-free languages. This very simple observation has some happy consequences. Indeed, polynomial time decidable or more generally decidable properties for context-free languages carry over to rational relations. Conversely, undecidable results for rational relations can be extended to context-free languages.

To each relation $R \subseteq A_1^* \times A_2^*$ we associate the language $\mathcal{L}(R) = \{u^\rho \# v \in A_1^* \# A_2^* \mid (u,v) \in R\}$ where u^ρ is the *mirror* image of u ($1^\rho = 1$ and $(a_1 \cdots a_n)^\rho = a_n \cdots a_1$). The following will be of little surprise for most readers.

Proposition 3.1. *The relation R is rational if and only if the language $\mathcal{L}(R)$ is a linear language where the symbol $\#$ can only be produced by a terminal rule of the form $X \to \#$. Furthermore, $\mathcal{L}(R)$ is a rational language if and only if R is a recognizable relation. Finally, if R is a deterministic rational relation then $\mathcal{L}(R)$ is a deterministic context-free language.*

Proof. The construction is an immediate extension of that yielding a left-linear grammar from a finite automaton. More precisely, consider a two-tape automaton (A_1, A_2, Q, E, I, T) recognizing the relation R. Without loss of generality we may assume that $I = \{i\}$ and $T = \{t\}$. Then a linear grammar generating $\mathcal{L}(R)$ is obtained by taking Q as set of nonterminal symbols, t as axiom, i as the symbol generating the marker $\#$ and the following as set of production rules: to each $(q,a,b,p) \in E$ assign the rule $p \to aqb$. We leave it to the reader to check that the grammar is correct. The converse is also clear. Observe that the condition on the production of the marker $\#$ is necessary, see e.g., the grammar $S \to U\#$, $U \to aUb|1$ does not generate a rational relation.

If the relation is recognizable then it a finite union of direct products $X_1 \times X_2$ with $X_1 \in \mathrm{Rat}(A_1^*)$ and $X_2 \in \mathrm{Rat}(A_2^*)$. Because of the closure property of rational languages under union, it suffices to observe that equality $\mathcal{L}(X_1 \times X_2) = X_1^\rho \# X_2$ holds. Conversely, assume the language $\mathcal{L}(R)$ is recognized by a finite deterministic automaton with set of states Q. For all $q \in Q$ define Pref_q as the subset of words taking the initial state to q and Suff_q as the subset of words taking q to a final state. Then R is the union of $\mathrm{Pref}_q^\rho \times \mathrm{Suff}_p$ for all pairs (q,p) for which $(q,\#,p)$ is a transition of the automaton.

If the relation is deterministic then the language is recognized by a deterministic pushdown automaton working in two step. First, given a pair

(u, v) it pushes the word u^ρ onto the stack (the top of the stack being the first letter of u) then it alternatively consumes the word v or pops the stack according to whether the transition is in $Q_1 \times A_1 \times \{1\} \times Q_1$ or in $Q_2 \times \{1\} \times A_2 \times Q_2$. $\square$

3.2 Deciding $\mathbf{Rec}(A_1^* \times \ldots \times A_k^*)$ in $\mathbf{DRat}(A_1^* \times \ldots \times A_k^*)$

Stearns proved in 1967 [14] that given a deterministic pushdown context-free language it is recursively decidable whether or not it is recognizable by a automaton. Valiant lowered the complexity to a double exponential. With the above considerations this yields an algorithm working in double exponential time, which decides whether or not a deterministic relation $R \subseteq A_1^* \times A_2^*$ is recognizable. The purpose of this paragraph is to extend this decidability result to arbitrary k-ary deterministic relations.

Before tackling the actual proof which is an elaboration on Stearns's proof and which proceeds by induction on the integer k, we observe that we can somehow simplify the deterministic automaton given as instance. Indeed, a deterministic relation $R \subseteq A_1^* \times \cdots \times A_k^*$ can be uniquely decomposed as $R \cap A_1^+ \times \cdots \times A_k^*$ and $R \cap (\{1\} \times A_2^* \cdots \times A_k^*)$. The second relation can be identified with a deterministic relation on a product of $k-1$ free monoids. Concerning the first relation, we leave it to the reader to verify that without loss of generality, we may assume, first, that the initial state q_- belongs to Q_1 and second that the set of terminal states is reduced to a unique state q_+ which also belongs to Q_1 and which is the source of no transition.

In order to make the connection between the general k-ary case and the binary case, we will have A_1^* and $A_2^* \times \cdots \times A_k^*$ play a dissymmetric role. Given a pair $(u, x) \in A_1^* \times (A_2^* \times \cdots \times A_k^*)$, we say that u is the *input* and x the *output*. Finally, given $q \in Q_1$ and $u \in A_1^*$ we define

$$R_q(u) = \{x \in A_2^* \times \cdots \times A_k^* \mid q \xrightarrow{(u,x)} q_+\}$$

This notation is reminiscent of the right contexts associated with the state of a one tape deterministic automaton.

The following two crucial definitions are adapted from Stearns' original paper. It uses the following notion. A path as in (2) is an N-*path* if there exist at most N times where the visited state passes from Q_1 to $Q_2 \cup \cdots \cup Q_k$ or conversely (i.e., $q_i \in Q_1$ if and only if $q_{i+1} \notin Q_1$). Somewhat incorrectly we denote by 1 the unit of the product monoid $A_2^* \times \cdots \times A_k^*$ instead of the more rigorous (and awkward) $(\overbrace{1, \ldots, 1}^{k-1 \text{ times}})$.

Definition 3.2. *Given an integer N and two input words $u, v \in A_1^*$, the word v is* N-invisible in the context of u *if the following holds for all $x \in A_1^*$*

for all states $q \in Q$ and all N-paths $q \xrightarrow{(u,x)} p$ there exists a path $p \xrightarrow{(v,1)} p$.

Stearns was working with pushdown automata. Our first component (the u and the v) plays the role of the top of his stack. Our second component plays the role of his input word. Saying that the top of the stack is invisible means that it can be popped without consuming the input word. We kept the same terminology for easier reference to the original paper. The justification of this notion can be seen as follows. Consider for simplicity a binary relation $R \subseteq A_1^* \times A_2^*$. With each $x \in A_2^*$ in the image of R associate a word $u \in A_1^*$ of minimal length satisfying $(u, x) \in R$. If R is recognizable, there exist only finitely many such words $u \in A_1^*$. Fix one such u and consider the regular set containing all $x \in A_2^*$ associated to it. In the above definition, an invisible word is simply an idempotent in the transition monoid of this regular set.

Definition 3.3. *A nonempty input word $u \in A_1^*$ is* null-transparent *if for all states $q, p \in Q_1$, the condition that $q \xrightarrow{(u,1)} p$ is a path implies that $p \xrightarrow{(u,1)} p$ is also a path.*

Here again, we did not modify the original definition. The intuition is the following. Consider two integers $0 \leq n < m$ and a word $w \in A^*$. If inequality $R_q(u^n v) \neq R_q(u^m v)$ holds then an element $x = (x_2, \cdots, x_k) \in A_2^* \times \cdots \times A_k^*$ belonging to the symmetric difference satisfies the condition $|x_2| + \cdots + |x_k| \geq n$. Indeed, if this condition is not satisfied, consider for example the case where $q \xrightarrow{(u^n v, x)} q_+$ holds. Because the labels of the transitions have all empty components except one which is a letter of a subalphabet, for some $0 \leq i < n$ the path is of the following form

$$q \xrightarrow{(u^i, x')} p \xrightarrow{(u,1)} r \xrightarrow{(u^{n-i-1}v, x'')} q_+$$

for $x'x'' = x$ and $p, r \in Q$. Because the existence of the path $r \xrightarrow{(u,1)} r$ is guaranteed by the property, there exists a path of the form

$$q \xrightarrow{(u^i, x')} p \xrightarrow{(u,1)} r \xrightarrow{(u^{m-n}, 1)} r \xrightarrow{(u^{n-i-1}v, x'')} q_+$$

leading to a contradiction. The first bound of the following Lemma is Theorem 3 in [14], the second is Lemma 3 in [15] (and is an improvement of Theorem 4 in[14]). We denote by K the cardinality of the set of states.

Lemma 3.4. *Let $\mathcal{A}$ be a deterministic n-tape automaton with K states and let $u_1 \cdots u_\ell$ be a product of ℓ nonempty words in A^**

(i) If $\ell > K!$ then there exists $1 \leq i < j \leq \ell$ such that $u_{i+1} \cdots u_j$ is null-transparent.

(ii) If $\ell > 2(NK)^K$ then there exists $1 \leq i < j \leq \ell$ such that $u_{i+1} \cdots u_j$ is N-invisible in the context of $u_1 \cdots u_i$.

The following is, in our setting, Theorem 4 of [14] in the version of Theorem 5 of [15]. We pose $f(N) = 2(NK)^K$.

Lemma 3.5. *Let $R \subseteq A_1^* \times A_2^* \times \cdots \times A_k^*$ be a deterministic relation accepted by an automaton with K states. The set $\{R(u) \mid u \in A_1^*\}$ is finite if and only if it is equal to the set $\{R(u) \mid u \in A_1^*, |u| \leq f(KK!)\}$.*

Proof. Given $q \in Q_1$ and $u, v \in A_1^*$, we write $u \sim_q v$ whenever $R_q(u) = R_q(v)$. The Lemma asserts that the equivalence relation $\sim_{q_-}$ has finite index.

By contradiction, assume there exists a word in A_1^* of length greater than $f(KK!)$ which is not $\sim_{q_-}$ equivalent to any word of length less than or equal to $f(KK!)$. By Lemma 3.4 (ii) this word factorizes as uvw such that v is $KK!$-invisible relative to u, which by minimality of the length of uvw implies that the word u is greater than $KK!$ (otherwise equality $uw \sim_{q_-} uvw$ would hold). We prove that the condition $uw \not\sim_{q_-} uvw$, leads to a contradiction. Take $t \in A_2^* \times \cdots A_k^*$ which is an evidence of this non equivalence with minimal sum of lengths of the $k-1$ components. Factor the prefix u as $\alpha_1 \cdots \alpha_{\ell+1}$ in the following way

$$q_- \xrightarrow{(\alpha_1,\tau_1)} q \xrightarrow{(\alpha_2,\tau_2)} q \ldots q \xrightarrow{(\alpha_\ell,\tau_\ell)} q \xrightarrow{(\alpha_{\ell+1},\tau_{\ell+1})} p$$

where $\tau_1 \cdots \tau_{\ell+1}$ is a componentwise prefix of t. By the length of u there exists such a factorization for which $\ell > K!$ holds and all τ_i's are nonempty. Apply Lemma 3.4 (i). There exits $1 \leq i < j \leq \ell$ such that $\alpha_{i+1} \cdots \alpha_j$ is null-transparent in the context of $\alpha_1 \cdots \alpha_i$. Set $u_1 = \alpha_1 \cdots \alpha_i$, $u_2 = \alpha_{i+1} \cdots \alpha_j$ and $u_3 = \alpha_{j+1} \cdots \alpha_{\ell+1}$, i.e., $u = u_1 u_2 u_3$. We show that the hypothesis $uw \not\sim_{q_-} uvw$ implies that all $u_1(u_2)^i u_3 vw$, $i \geq 0$ are pairwise nonequivalent relative to $\sim_{q_-}$. There exists a factorization $t_1 t_2 t_3 = t$ such that

$$\begin{array}{l} q_- \xrightarrow{(u_1,t_1)} q \xrightarrow{(u_2,t_2)} q \xrightarrow{(u_3vw,t_3)} q_+ \text{ (resp. } r \text{ with } r \neq q_+) \\ \text{and} \\ q_- \xrightarrow{(u_1,t_1)} q \xrightarrow{(u_2,t_2)} q \xrightarrow{(u_3w,t_3)} p \text{ with } p \neq q_+ \text{(resp. } q_+) \end{array} \qquad (4)$$

Observe that we have $u_2 u_3 vw \not\sim_q u_3 vw$ or $u_2 u_3 w \not\sim_q u_3 w$. Indeed, if this were not the case, then because t_2 is non-empty, we would have $t_1 t_3 \in R(uw) \Leftrightarrow t_1 t_3 \notin R(uvw)$ which would violate the minimality of t. Assume without loss of generality that there exists $z \in A_2^* \times \cdots \times A_k^*$ such that $z \in R_q(u_2 u_3 vw) \Leftrightarrow z \notin R_q(u_3 vw)$ holds. Then, because the automaton is deterministic, for all integers $i \geq 0$, we have $t_2^i s \in R_q(u_2^{i+1} u_3 vw) \Leftrightarrow t_2^i s \notin R_q(u_2^i u_3 vw)$. Assume we have $u_2^{i+K} u_3 vw \sim_q u_2^i u_3 vw$ for some $i \geq 0$ and $K > 0$. This would imply for all integers $\lambda \geq 0$, $u_2^{i+\lambda K} u_3 vw \sim_q u_2^i u_3 vw$ and $u_2^{i+1+\lambda K} u_3 vw \sim_q u_2^{i+1} u_3 vw$ and therefore $t_2^i s \in R_q(u_2^{i+1+\lambda K} u_3 vw) \Leftrightarrow t_2^i s \notin R_q(u_2^{i+\lambda K} u_3 vw)$. For sufficiently large values of λ, this contradicts the fact that u_2 is null transparent as observed after Definition 3.3 and completes the proof.

$\square$

Proposition 3.6. *A rational relation $R \subseteq A_1^* \times \cdots \times A_k^*$, is recognizable if and only if there exist a finite set of words $x_1, \ldots, x_m \in A_1^*$ satisfying the following conditions*

(i) for all $1 \leq i \leq m$, $R(x_i) \in \mathrm{Rec}(A_2^* \times \cdots \times A_k^*)$

(ii) for all $x \in Dom(R)$ there exists an integer $1 \leq i \leq n$ such that $R(x) = R(x_i)$ holds.

Proof. The condition is necessary. Indeed, assume R is of the form

$$\bigcup_{1 \leq j \leq m} X_1^{(j)} \times \cdots \times X_k^{(j)}$$

where $X_i^{(j)} \in \mathrm{Rec}(A_i^*)$. We may assume without loss of generality that for each fixed $1 \leq i \leq k$ and all $1 \leq j \leq j' \leq m$ the condition $X_i^{(j)} \cap X_i^{(j')} \neq \emptyset$ implies $X_i^{(j)} = X_i^{(j')}$. Pick an element in $X_1^{(j)}$. The finite set of these elements satisfy the two conditions of the statement. The converse is clear. $\square$

We are now in a position to prove the main result of this section.

Theorem 3.7. *It is recursively decidable whether or not a deterministic relation $R \subseteq A_1^* \times \cdots \times A_k^*$ is recognizable.*

Proof. The proof is by induction on k. By the previous two results, it suffices to prove that it is possible to determine given an integer λ (1) whether or not for all $y \in A_1^*$ there exists $x \in A_1^*$, $|x| \leq \lambda$ such that $R(x) = R(y)$ and (2) whether or not each $R(x)$, $|x| \leq \lambda$, is recognizable. By induction, this last condition is decidable since $R(x)$ is deterministic. The first condition can be tested as follows: let $(Y_p)_{p \in P}$ be the coarsest refinement of all $R(x)$, $|x| \leq \lambda$. Since the Y_p's are recognizable, the subsets $X_p = R^{-1}(Y_p)$, $p \in P$ are rational in A_1^*. Let $Z_x \subseteq A_1^*$ be the union of all X_p's such that Y_p is a subset of $R(x)$. Then the statement is true if and only if R equals $S = \bigcup_{|x| \leq \lambda} Z_x \times R(x)$. Since R is deterministic and S is recognizable, equality holds if and only if the rational relation $(R - S) \cup (S - R)$ is empty, which is recursively decidable.

$\square$

3.3 Deciding $\mathrm{Rec}(A_1^* \times \cdots \times A_k^*)$ in $\mathrm{Sync}(A_1^* \times \cdots \times A_k^*)$

Here we prove a direct, elementary decision procedure of low complexity. We show that it is decidable in simple exponential time whether or not a synchronous relation over a direct product of an arbitrary number of free monoids, is recognizable. To our knowledge, the membership problem for the class DRat relative to the class Sync is still open. Also, determining whether or not an k-ary deterministic relation, $k > 2$, is recognizable does not seem to be covered by Stearns' result.

As a preliminary result, we state the following consequence of Proposition 3.6 whose proof by induction is left to the reader.

Proposition 3.8. *A rational relation $R \subseteq A_1^* \times \cdots \times A_k^*$ is recognizable if and only if for any integer j there exist only finitely many different restrictions $R_{|u_1,\ldots,u_j}$.*

We are now in a position to prove the existence of our exponential time decision procedure.

Proposition 3.9. *There exists an exponential time algorithm which decides, given a synchronous relation, whether or not it is recognizable.*

Proof. Given a synchronous relation $R \subseteq A_1^* \times \cdots \times A_k^*$ and an integer $1 \leq j \leq k$, we must check whether the collection of restrictions of the form $R_{|u_1,\ldots,u_j}$ is finite when the vector $(u_1, \ldots, u_j)$ ranges over the direct product $A_1^* \times \cdots \times A_j^*$. Define the equivalence relation $S \subseteq (A_1^* \times \cdots \times A_j^*)^2$ by setting $(x, y) \in S$ if and only if $(x, z) \in R \iff (y, z) \in R$ holds for all $z \in A_{j+1}^* \times \cdots \times A_k^*$. If R is synchronous, then S is again synchronous. Indeed, let R' be the (again synchronous) relation obtained from R by exchanging the j first and the $k - j$ last components: $R' = \{(x_{j+1}, \ldots, x_k, x_1, \ldots, x_j) \mid (x_1, \ldots, x_k) \in R\}$. We have

$$\begin{gathered}(x, y) \notin S \iff \exists z \in A_{j+1}^* \times \cdots \times A_k^* \\ ((x, z) \in R \wedge (z, y) \notin R') \vee ((x, z) \notin R \wedge (z, y) \in R')\end{gathered} \tag{5}$$

Thus S is the complement of the relation $R \circ \overline{R'} \cup \overline{R} \circ R'$ which is synchronous because of the closure properties of synchronous relations under complement, union and composition.

Now we are left with testing whether the equivalence relation S has finite index. The idea is to assign to each j-tuple $(x_1, \ldots, x_j)$ a canonical representative for the class it belongs to. This could be done via the characterization of the synchronous relations as established in [4] but this result does not lead in an obvious way to a polynomial upper bound. Therefore, we use a different, automaton driven approach which takes advantage of the notion of *hierarchical ordering* on a free monoid A^* which we recall briefly. Choose an arbitrary linear ordering $<$ on A and extend it to the free monoid A^* by setting $u < v$ if $|u| < |v|$ or if $|u| = |v|$ and $u <_{\rm lex} v$ (there exist $w, u', v' \in A^*$ and $a, b \in A$, such that $u = wau'$, $v = wbv'$ and $a < b$ holds). Given the alphabets $A_1, \ldots, A_j$ we extend the individual hierarchical orderings to the lexicographical ordering on the direct product $A_1^* \times \cdots \times A_j^*$ in the usual way (if the first component of the vector x is smaller than that of the vector y then claim x is lexicographically smaller than y, else compare recursively the next components of the two vectors). Denote by $<_{\rm hl}$ this combination of hierarchical and lexicographical orders on $A_1^* \times \cdots \times A_j^*$.

Now we modify the synchronous automaton recognizing S in such a way as a to select the $2k$-tuples $(x_1, \ldots, x_j, y_1, \ldots, y_j) \in S$ such that the relation $(x_1, \ldots, x_j) <_{\rm hl} (y_1, \ldots, y_j)$ holds. This is achieved in the following way. For each integer $1 \leq i \leq j$, add two components to each state of the automaton. The first of these two components records whether x_i has length less than, equal to or greater than the length of y_i. This can be done easily because the automaton is synchronous. The second component records, when x_i and y_i are not prefix of one another, whether or not the leftmost letters, say a and b respectively for which x_i and y_i disagree, satisfy $a < b$. More technically, this means that we have the conditions $x_i = uax'_i$ and $y_i = uby'_i$ where $u, x'_i, y'_i \in A_i^*$ and $a, b \in A_i$. Based on these pieces of information, it is easy, upon termination of the run, to determine whether or not the vector $(x_1, \ldots, x_j)$ is smaller than the vector $(y_1, \ldots, y_j)$ in the above ordering $<_{\rm hl}$. Let $L \subseteq A_1^* \times \cdots \times A_j^*$ be the set of such vectors $(y_1, \ldots, y_j)$. Then S has finite index if and only if the subset $\text{range}(R) - L$ is finite. The complexity claim is a direct consequence that all constructions involved in the proof can be achieved in polynomial time except for the computation of the complement of relations which can be achieved in exponential time. □

4 The commutative case

When all $k > 1$ free monoids are generated by a single element, and only in this case, do the rational subsets of the direct product form a Boolean algebra, [8, 5]. It is convenient to assume that the unique generator of these monoids is the symbol 1 and to denote by 0 the empty word. Then the direct product is isomorphic to additive monoid $\mathbb{N}^k$ through the mapping which assigns to the unary representation of k numbers, these numbers themselves. Said differently, we shall identify the element $(1^{n_1}, \cdots, 1^{n_k})$ with the k-tuple of integers $(n_1, \cdots, n_k)$ and we shall denote by $(0, \cdots, 0)$ the unit of the monoid.

The general tool for deciding properties under this hypothesis is a combination of the decidability of the theory of the integers with the addition only, due to Presburger, and of Ginsburg and Spanier's characterization of the rational subsets of $\mathbb{N}^k$ as the family of sets of tuples expressible in this theory, [8, Theorem 1.3.]. More precisely, given a formula of the Presburger arithmetics with the free variables $x_1, \ldots, x_k$, the set of k-tuples $(a_1, \ldots, a_k) \in \mathbb{N}^k$ for which the formula is true once the a_i's are substituted for the variables x_i's, is a rational relation and conversely, all rational relations can be obtained this way.

We prove decidability results corresponding to column 1 of Table 1 in the commutative case, and reserve the last paragraph for a precise estimate of the complexity issues.

4.1 Deciding $\mathrm{Rec}(\mathbb{N}^k)$ in $\mathrm{Rat}(\mathbb{N}^k)$

Ginsburg and Spanier showed that it is possible, given a rational subset of $\mathbb{N}^k$, to express in Presburger arithmetics the fact that it is recognizable.

Theorem 4.1 (Ginsburg and Spanier, [8] 1966)**.** *Given a rational relation in $\mathbb{N}^k$, it is recursively decidable whether or not it is recognizable.*

We shall return to the proof of this result in paragraph 4.4 when dealing with complexity issues.

4.2 Deciding $\mathrm{DRat}(\mathbb{N}^k)$ in $\mathrm{Rat}(\mathbb{N}^k)$

Given two vectors $u, v \in \mathbb{N}^k$ we write $u \leq v$ if u is componentwise smaller than or equal to v and $u < v$ if $u \leq v$ and $u \neq v$ holds. If u, v are two vectors, we denote by $\min(u, v)$ their greatest lower bound, i.e., the vector w satisfying $w_i = \min\{u_i, v_i\}$ for all $i = 1, \ldots, k$.

The following further notations are useful. Given $u \in \mathbb{N}^k$ we define the set of indices on which it has nonzero, resp. zero, components.

$$\mathrm{Supp}(u) = \{i \in \{1, \ldots, k\} \mid u_i \neq 0\} \quad \mathrm{Null}(u) = \{i \in \{1, \ldots, k\} \mid u_i = 0\}$$

Given $I \subseteq \{1, \ldots, k\}$ we denote by $\mathbb{N}^k_I$ the subsets of vectors whose entries in I are equal to 0.

$$\mathbb{N}^k_I = \{u \in \mathbb{N}^k \mid \mathrm{Supp}(u) \subseteq I\}$$

Finally, if R, S are subsets of $\mathbb{N}^k$ then we set $R - S \;=\; \{t \in \mathbb{N}^k \mid \exists s \in S,\ s + t \in R\}$.

Theorem 4.2. *Given $R \subseteq \mathbb{N}^k$, the following conditions are equivalent*

(i) R is deterministic rational

(ii) there exist $\mu, \pi \in \mathbb{N}^k$ such that, for $u < \mu$ and $v < \pi$, the relations

$$S_u = (R - u) \cap \mathbb{N}^k_{\mathrm{Null}(\mu - u)}\ ,\ T_v = (R - \mu - v) \cap \mathbb{N}^k_{\mathrm{Null}(\pi - v)}$$

are deterministic rational and the following equality holds

$$R \;=\; (\bigcup_{u<\mu} u + S_u) \;\cup\; \left(\pi^* + \bigcup_{v<\pi} (\mu + v + T_v)\right) \tag{6}$$

In the case where $k = 2$, the situation is depicted in Figure 1.

Proof. We first show that condition (i) implies condition (ii). Indeed, consider a deterministic k-tape automaton. Without loss of generality, we can suppose that it is never stuck. The labels of the transitions are of the from $(\overbrace{0, \ldots, 0}^{j \text{ times}}, a, \overbrace{0, \ldots, 0}^{k-j-1 \text{ times}})$, where $0 \leq j \leq k$ and $a \in \{1, \#\}$. Because of Definition 2.2 of deterministic automata there exists a unique infinite path starting form the initial state and labeled by the vectors for which $a = 1$. Call μ the label of the path before reaching a cycle (the "initial mess") and π the label of the cycle (the "period"). In particular, for every $v \in \mathbb{N}^k$ and $j \in \mathbb{N}$, we have

$$\mu + v \in R \;\Leftrightarrow\; \mu + j\pi + v \in R$$

which yields equality

$$R - \mu - v \;=\; R - \mu - \pi - v = \; R - \mu - \pi^* - v \tag{7}$$

This equality shows that the right handside of (6) is included in the left handside. We now prove the opposite inclusion. Consider an input $x \in R$. If $x \not\geq \mu$, then we have $x \in u + S_u$ where $u = \min(x, \mu)$. If, on the contrary $x \geq \mu$, let $j \in \mathbb{N}$ be the maximum integer such that $x - \mu - j\pi$ is not greater than or equal to π. Setting $v = \min(\pi, x - \mu - j\pi)$ we get $x - \mu - j\pi \in T_v$ as claimed.

It now suffices to observe that S_u and T_v are deterministic rational. Indeed, since R is deterministic rational, so are $R - u$ and $R - \mu - v$. Because $\mathbb{N}^k_{\mathrm{Null}(\mu-u)}$ and $\mathbb{N}^k_{\mathrm{Null}(\pi-v)}$ are recognizable, so are the intersections S_u and T_v.

We now turn to the proof that (ii) implies (i). Let $\mathcal{S}_u$ and $\mathcal{T}_v$ be deterministic automata accepting S_u and T_v for $u < \mu$ and $v < \pi$. A deterministic automaton $\mathcal{A}$ accepting R works informally as follows. Given an input $x \in \mathbb{N}^k$, it determines whether or not $x \geq \mu$ holds. If it does not, then the computation proceeds by simulating $\mathcal{S}_u$ where let $u = \min(x, \mu)$. Otherwise, it computes the maximum integer j such that $x \geq \mu + j\pi$ holds. Then it simulates the automaton $\mathcal{T}_v$ where $v = \min(x, \mu + j\pi)$. More precisely, the condition $x \geq \mu$ is tested by determining, in increasing order of i, if $x_i \geq \mu_i$ holds. The set of values i for which it fails determines the vector u. The reader will easily be convinced that the case where $x \geq \mu$ holds, can be treated similarly. $\square$

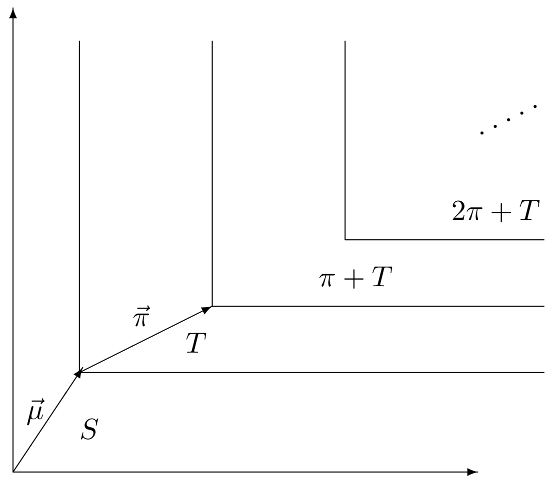

Figure 1: Illustration in $\mathbb{N}^2$, with $S = \bigcup_{u<\mu} (u + S_u)$, $T = \bigcup_{v<\pi} (\mu + v + T_v)$

Theorem 4.3. *It is recursively decidable whether or not a rational subset of $\mathbb{N}^k$ is deterministic.*

Proof. By induction on k, we show that one can recursively associate to any Presburger formula $\theta(x; y)$, where the k-tuple a will act as variables and the ℓ-tuple b as parameters, a Presburger formula $\Psi_\theta(y)$ satisfying the following property:

> *For any $b \in \mathbb{N}^\ell$, the relation $\{x \in \mathbb{N}^k \mid \theta(x; b)\}$ is deterministic rational if and only if $\Psi_\theta(b)$ holds true.*

Since Presburger arithmetics is decidable, this gives the decision procedure asserted by the theorem.

For $k = 1$ it suffices to take as $\Psi_\theta(y)$ any tautology since all rational subsets of $\mathbb{N}$ are deterministic.

Assume now the property holds for $1, \ldots, k-1$ to k (where $k \geq 2$). Let $\theta(x; y)$ and $b \in \mathbb{N}^\ell$ define the rational relation $R = \{x \in \mathbb{N}^k \mid \theta(x; b)\}$. From θ we can construct a formula $\theta'(x; y, z, t)$ such that, for all $\mu, \pi, u, v \in \mathbb{N}^k$ the following holds

$$\begin{array}{lllll} S_u & = & (R - u) \cap \mathbb{N}^k_{\mathrm{Null}(\mu - u)} & = & \{x \in \mathbb{N}^k \mid \theta'(x; b, \mu, u)\} \\ T_v & = & (R - \mu - v) \cap \mathbb{N}^k_{\mathrm{Null}(\pi - v)} & = & \{x \in \mathbb{N}^k \mid \theta'(x; b, \mu + \pi, \mu + v)\} \end{array}$$

Now, for $I \subset \{1, \ldots, k\}$, $(R - w) \cap \mathbb{N}^k_I$ is deterministic rational if and only if so is its projection on $\mathbb{N}^I$. Denoting x_I the subtuple of x which retains the sole variables indexed by I, there is a formula $\theta'_I(x_I; y, z, t)$ which defines this projection. Finally, observe that if the cardinal of I is strictly less than k, we can apply the induction hypothesis to formula θ'_I. Using Theorem 4.2, this leads to the following formula $\Psi_\theta(b)$ to express that $R = \{x \in \mathbb{N}^k \mid \theta(x;b)\}$ is deterministic:

$$\begin{aligned}\Psi_\theta(b) \;=\; \exists\mu\;\exists\pi\;(A(b,\mu,\pi)\;\wedge\;\forall u<\mu\;\forall v<\pi\;\textstyle\bigwedge_{\emptyset\neq I\subseteq\{1,\ldots,k\}}\\ (\mathrm{Null}(\mu-u)=I\;\Rightarrow\;\Psi_{\theta'_I}(b,\mu,u))\\ \wedge\;(\mathrm{Null}(\pi-v)=I\;\Rightarrow\;\Psi_{\theta'_I}(b,\mu+\pi,\mu+v)))\end{aligned}$$

where $A(b,\mu,\pi)$ expresses equality (6). □

4.3 Deciding $\mathrm{Sync}(\mathbb{N}^k)$ in $\mathrm{Rat}(\mathbb{N}^k)$

With every subset subset $\emptyset \neq I \subseteq \{1,\ldots,k\}$ we associate the vector e_I which has all entries equal to 0 except those positions in I which are equal to 1. We recall that given an element a in a monoid, the expression a^+ denotes the subset consisting of all a^i with $i > 0$. Also $\mathbb{N}_+$ denotes the set of strictly positive integers.

Theorem 4.4. *Given a subset $X \subseteq \mathbb{N}^k$, the following are equivalent*

(i) X is rational (resp. synchronous)

(ii) for all strictly decreasing sequences of subsets $\{1,\ldots,k\} \supseteq I_1 \supsetneq \cdots \supsetneq I_p \supsetneq \emptyset$ the following subset of $\mathbb{N}^p$ is rational (resp. recognizable).

$$\{(x_1,\ldots,x_p) \in \mathbb{N}_+^p \mid x_1e_{I_1} + \cdots + x_pe_{I_p} \in X\} \tag{8}$$

Proof. Observe that $\mathbb{N}^k - \{0\}^k$ is the (finite) union of all subsets of the form

$$e_{I_1}^+ + \cdots + e_{I_p}^+ \tag{9}$$

for all possible sequences $\{1,\ldots,k\} \supseteq I_1 \supsetneq \cdots \supsetneq I_p \supsetneq \emptyset$. Thus a subset $X \subseteq \mathbb{N}^k$ is equal to the union of all intersections of the form $e_{I_1}^+ + \cdots + e_{I_p}^+ \cap X$. We show that if the set (8) is rational, i.e., by Ginsburg and Spanier's characterization if it is definable by a formula $\theta(x_1,\cdots,x_p)$, then so is $e_{I_1}^+ + \cdots + e_{I_p}^+ \cap X$ and therefore that the set X is rational. But this intersection is expressed by the formula

$$\psi(y_1,\cdots,y_k) = \exists x_1>0\cdots\exists x_p>0 \quad \theta(x_1,\cdots,x_p)\bigwedge_{i\in I_1}(y_i=x_1)$$
$$\bigwedge_{1<r\leq p}\;\bigwedge_{i\in I_r-I_{r+1}}(y_i = x_1+\ldots+x_r)$$

Conversely, is X is rational, then its intersection with a subset of the form (9) is rational, therefore expressible by a formula $\theta(y_1,\ldots,y_k)$. For $i=1,\ldots,p$ let z_i be any variable y_j where $j \in I_i$. Then (8) is expressed by formula

$$\exists y_1\ldots\exists y_k\;\theta(y_1,\ldots,y_k)\bigwedge(x_1=z_1)\bigwedge_{1<i\leq p}(x_i=z_i-z_{i-1}) \tag{10}$$

which completes the case when X is rational.

Assume now that X is synchronous. By Proposition 2.4 it is a finite union of subsets of the form

$$E_1 e_{J_1} + \cdots + E_q e_{J_q} \tag{11}$$

where $J_1 \supsetneq \cdots \supsetneq J_q$ and $E_1, \cdots, E_q$ are rational subsets of $\mathbb{N} \setminus \{0\}$. Observe that the unique subset of the form (9) which has a nonempty intersection with $E_1 e_{J_1} + \cdots + E_q e_{J_q}$ is $e^+_{J_1} + \cdots + e^+_{J_q}$. Therefore it suffices to prove that for a synchronous relation such as (11), the subset

$$\{(x_1, \ldots, x_q) \in \mathbb{N}^q_+ \mid x_1 e_{J_1} + \cdots + x_q e_{J_q} \in X\} \tag{12}$$

is recognizable. To that purpose, observe that an automaton recognizing the set (11), when appropriately transformed as in paragraph 2.2, has a set of states decomposed into q disjoint subsets, $Q = Q_1 \cup \cdots \cup Q_q$. Indeed, consider a computation on the input $x_1 e_{J_1} + \cdots + x_q e_{J_q}$. The successful path is divided into q successive subpaths, respectively labeled by $x_1 e_{J_1}$, then $x_2 e_{J_2}$, …, finally $x_q e_{J_q}$. These subpaths visit states in Q_1 then in Q_2, …, finally in Q_q. Thus we may decompose the set (12) into finitely many subsets

$$E^{(i)}_1 e_{J_1} + \cdots + E^{(i)}_q e_{J_q}, \quad i = 1, \ldots, N \tag{13}$$

where $E^{(i)}_1$, …, $E^{(i)}_q$ are recognizable such that the transitions between successive subsets of states are fixed. This shows that (12) is precisely the recognizable subset

$$\bigcup_{1 \le i \le N} E^{(i)}_1 \times \cdots \times E^{(i)}_q \subseteq \mathbb{N}^q$$

Conversely, assume each subset (8) is recognizable, say it is a finite union of subsets of the form $E_1 \times \cdots \times E_p$ where each E_i is a rational subset of $\mathbb{N}$. Then $(e^+_{I_1} + \cdots + e^+_{I_p}) \cap X$ is a finite union of synchronous relations $E_1 e^+_{I_1} + \cdots + E_p e^+_{I_p}$ thus X is also a finite union of synchronous relations and consequently it is itself synchronous.

□

Corollary 4.5. *It is recursively decidable whether or not a rational subset X of $\mathbb{N}^n$ is synchronous.*

Proof. Given a rational subset X we proceed as follows. For all sequences $\{1, \ldots, k\} \supseteq I_1 \supsetneq \cdots \supsetneq I_p$ we construct the intersection $(e^+_{I_1} + \cdots + e^+_{I_p}) \cap X$ obtaining thus a formula of the form as in (10) . Checking whether or not it defines a recognizable subset is done by using Theorem 4.1.

□

4.4 Complexity of the decision procedures

We now reduce the complexity of all the decision procedures described in paragraphs 4.1, 4.2, 4.3 to that of Presburger arithmetics.

Definition 4.6. *We denote by $P(n, \alpha)$ the complexity of the decision problem of Presburger formulas of length less than or equal to n and quantifier alternation less than or equal to α.*

Remark 4.7. *It is known (Grädel, 1988 [9]) that $P(n, \alpha)$ is bounded by alternating time $O(1)^{n^{\alpha+O(1)}}$.*

4.4.1 Complexity of deciding $\mathrm{Rec}(\mathbb{N}^k)$ in $\mathrm{Rat}(\mathbb{N}^k)$

Theorem 4.8. *Given a Presburger formula θ with k free variables, length n and quantifier alternation α, there exists an algorithm with complexity $P(2n + O(k \log k), \alpha + 4)$ which decides whether the rational relation $\{a \in \mathbb{N}^k \mid \theta(a)\}$ is recognizable or not.*

Proof. We restate Ginsburg and Spanier's decision procedure [8]and look at its complexity. Let $R = \{a \in \mathbb{N}^k \mid \theta(a)\}$. Consider the congruence on $\mathbb{N}^k$ associated to R, namely

$$a \sim_R b \;\Leftrightarrow\; \forall c \; (a + c \in R \Leftrightarrow b + c \in R)$$

As is well-known, R is recognizable if and only if $\sim_R$ has finite index. This is expressible in Presburger arithmetics via the closed formula

$$\exists N \in \mathbb{N} \; \forall a \in \mathbb{N}^k \; \exists b \in \mathbb{N}^k \; \Big(\bigwedge_{1 \le i \le k} b_i \le N \;\wedge\; \forall c \in \mathbb{N}^k \; (\theta(a + c) \Leftrightarrow \theta(b + c))\Big)$$

which has length $2n + O(k \log k)$ (the $\log k$ term appears when counting the length in binary of the indices of variables) and quantifier alternation $\le \alpha + 4$. ☐

4.4.2 Complexity of deciding $\mathrm{DRat}(\mathbb{N}^k)$ in $\mathrm{Rat}(\mathbb{N}^k)$

The analysis of complexity of the decision procedure requires some involved technique which basically consists of developing the recursive definition of Theorem 4.2 in order to get an equivalent iterative expression.

The following obvious result will be useful in the proof of Theorem 4.10.

Proposition 4.9. *Let $R, S \subseteq \mathbb{N}^k$ and $I, J \subseteq \{1, \cdots, k\}$. Then*

$$((R \cap \mathbb{N}^k_I) - S) \cap \mathbb{N}^k_J \subseteq \mathbb{N}^k_{I \cup J}$$

Theorem 4.10. *Given a Presburger formula θ with k free variables, length n and quantifier alternation α, there exists an algorithm with complexity $P(n + O(k 2^k \log k), \alpha + 2k + 1)$ which decides whether the rational relation $\{a \in \mathbb{N}^k \mid \theta(a)\}$ is deterministic or not.*

Proof. Let $R = \{x \in \mathbb{N}^k \mid \theta(x)\}$. The inclusion of the right handside in the left handside of equality (6) of Theorem 4.2 always holds. Concerning the opposite inclusion, observe that, if $x \in R$ then $x \in u + (R - u) \;\Leftrightarrow\; x \geq u$ and (using (7)) $x \in \mu + j\pi + v + (R - \mu - v) \;\Leftrightarrow\; x \geq \mu + j\pi + v$. Thus,

$$\begin{array}{lll} x \in R & \Rightarrow & (x \in u + S_u \Leftrightarrow (x \geq \mu + v \wedge x - \mu - v \in \mathbb{N}^k_{\mathrm{Null}(\pi - v)})) \\ x \in R & \Rightarrow & (x \in \mu + j\pi + v + T_v \Leftrightarrow (x \geq \mu + j\pi + v \\ & & \qquad\qquad \wedge x - \mu - j\pi - v \in \mathbb{N}^k_{\mathrm{Null}(\pi - v)})) \end{array}$$

Thus, equality (6) can be written

$$\forall x \; (x \in R \Rightarrow \exists u \; \exists v \; A(x, \mu, \pi, u, v))$$

where

$$\begin{array}{rcl} A(x, \mu, \pi, u, v) & = & (u < \mu \wedge v < \pi) \;\wedge\; ((x \geq u \;\wedge\; x - u \in \mathbb{N}^k_{\mathrm{Null}(\mu - u)}) \\ & & \vee \; \exists j \; (x \geq \mu + j\pi + v \;\wedge\; x - \mu - j\pi - v \in \mathbb{N}^k_{\mathrm{Null}(\pi - v)})) \end{array}$$

To avoid confusion in the iteration, let's write $S_{\mu,u}(R)$ and $T_{\mu,\pi,v}(R)$ in place of S_u and T_v. Renaming variables and reorganizing quantifications, we see that Theorem 4.2 can be restated as follows: R is deterministic rational if and only if

$$\begin{array}{l} \exists \mu^{(0)} \; \exists \pi^{(0)} \; \forall u^{(0)} < \mu^{(0)} \; \forall v^{(0)} < \pi^{(0)} \\ \quad [\forall x \; (x \in R \;\Rightarrow\; \exists u \; \exists v \; A(x, \mu^{(0)}, \pi^{(0)}, u, v)) \\ \quad \wedge \; (S_{\mu^{(0)}, u^{(0)}}(R), T_{\mu^{(0)}, \pi^{(0)}, v^{(0)}}(R) \text{ are deterministic rational})] \end{array} \tag{14}$$

Iteratively applying ℓ times (14) to itself (i.e. to the last part about relations having to be deterministic rational), we see that R is deterministic rational if and only if

$$\begin{array}{l} \exists \mu^{(0)} \; \exists \pi^{(0)} \; \forall u^{(0)} < \mu^{(0)} \; \forall v^{(0)} < \pi^{(0)} \\ \quad \cdots \quad \exists \mu^{(\ell)} \; \exists \pi^{(\ell)} \; \forall u^{(\ell)} < \mu^{(\ell)} \; \forall v^{(\ell)} < \pi^{(\ell)} \\ \bigwedge_{i=0,\cdots,\ell} \bigwedge_{f \in \{0,1\}^i} \forall x \; (x \in W_f \;\Rightarrow\; \exists u \; \exists v \; A(x, \mu^{(i)}, \pi^{(i)}, u, v)) \\ \wedge \; \bigwedge_{f \in \{0,1\}^\ell} W_f \text{ is deterministic rational}] \end{array} \tag{15}$$

where

- $W_\varepsilon = R$ if ε is the empty word,
- $W_{f0} = S_{\mu^{(i)}, u^{(i)}}(W_f)$ if $f \in \{0,1\}^i$
- $W_{f1} = T_{\mu^{(i)}, \pi^{(i)}, v^{(\ell)}}(W_f)$ if $f \in \{0,1\}^i$

Now, for $\ell = k - 1$, the last conjunct asserting that some relations are deterministic is trivially true since, applying Proposition 4.9, these relations are included in N_I^k where $I \subseteq \{1, \cdots, k\}$ contains at least $k-1$ elements. Thus, R is deterministic if and only if the following formula holds:

$$\begin{array}{l} \exists\mu^{(0)}\ \exists\pi^{(0)}\ \forall u^{(0)} < \mu^{(0)}\ \forall v^{(0)} < \pi^{(0)} \\ \quad \cdots \quad \exists\mu^{(k-1)}\ \exists\pi^{(k-1)}\ \forall u^{(k-1)} < \mu^{(k-1)}\ \forall v^{(k-1)} < \pi^{(k-1)} \\ \forall x\ \bigwedge_{i=0,\cdots,k-1} \bigwedge_{f\in\{0,1\}^i} (x \in W_f\ \Rightarrow\ \exists u\ \exists v\ A(x, \mu^{(i)}, \pi^{(i)}, u, v)) \end{array} \tag{16}$$

Now, if $f \in \{0,1\}^i$ then $x \in W_f$ can be expressed as follows:

$$\begin{array}{rcl} x \in W_\varepsilon & \Leftrightarrow & \theta(x) \\ x \in W_{f0} & \Leftrightarrow & (x + u^{(i)} \in W_f\ \wedge\ x \in \mathbb{N}^k_{\mathrm{Null}(\mu^{(i)}-u^{(i)})}) \\ x \in W_{f1} & \Leftrightarrow & \exists j\ (x + \mu^{(i)} + j\pi^{(i)} + v^{(i)} \in W_f\ \wedge\ x \in \mathbb{N}^k_{\mathrm{Null}(\pi^{(i)}-v^{(i)})}) \end{array}$$

Let us abusively denote $\tau_f(x)$ and $B_f(x)$ the following terms and formulas which also contain some variables j_m's, μ_m's, π_m's, u_m's and v_m's:

- $\tau_\varepsilon = x$ and B_ε is any tautology
- $\tau_{f0}(x) = \tau_f(x) + u^{(i)}$
- $\tau_{f1}(x) = \tau_f(x) + \mu^{(i)} + j_i\pi^{(i)} + v^{(i)}$
- $B_{f0}(x)$ is $B_f(x + u^{(i)})\ \wedge\ x \in \mathbb{N}_{\mathrm{Null}(\mu^{(i)}-u^{(i)})}$
- $B_{f1}(x)$ is $B_f(x + \mu^{(i)} + j_i\pi^{(i)} + v^{(i)})\ \wedge\ x \in \mathbb{N}_{\mathrm{Null}(\mu^{(i)}-u^{(i)})}$

Thus, $B_f(x)$ is a conjunction of formulas of the form $x + \tau \in \mathbb{N}_I^k$ where τ is a sum of some $u^{(s)}$'s and some $\mu^{(t)} + j_i\pi^{(i)} + v^{(t)}$'s. Also,

$$x \in W_f\ \Leftrightarrow\ \exists j_0 \cdots \exists j_{i-1}\ (\theta(\tau_f(x))\ \wedge\ B_f(x))$$

and we can rewrite (16) as follows:

$$\begin{array}{l} \exists\mu^{(0)}\ \exists\pi^{(0)}\ \forall u^{(0)} < \mu^{(0)}\ \forall v^{(0)} < \pi^{(0)} \\ \quad \ldots \quad \exists\mu^{(k-1)}\ \exists\pi^{(k-1)}\ \forall u^{(k-1)} < \mu^{(k-1)}\ \forall v^{(k-1)} < \pi^{(k-1)} \\ \forall x\ \forall j_0 \ldots \forall j_{k-2}\ \bigwedge_{i=0,\ldots,k-1} \bigwedge_{f\in\{0,1\}^i} \\ (\theta(\tau_f(x)) \wedge B_f(x)\ \Rightarrow\ \exists u\ \exists v\ A(x, \mu^{(i)}, \pi^{(i)}, u, v)) \end{array} \tag{17}$$

In order to avoid repeating θ in formula (17), we rewrite its last conjunct.

$$\begin{array}{l} \exists\mu^{(0)}\ \exists\pi^{(0)}\ \forall u^{(0)} < \mu^{(0)}\ \forall v^{(0)} < \pi^{(0)} \\ \quad \ldots \quad \exists\mu^{(k-1)}\ \exists\pi^{(k-1)}\ \forall u^{(k-1)} < \mu^{(k-1)}\ \forall v^{(k-1)} < \pi^{(k-1)} \\ \forall x\ \forall j_0 \ldots \forall j_{k-2}\ \forall y\ (\theta(y)\ \Rightarrow\ \bigwedge_{i=0,\ldots,k-1} (\bigwedge_{f\in\{0,1\}^i} \\ \qquad (y = \tau_f(x) \wedge B_f(x)\ \Rightarrow\ \exists u\ \exists v\ A(x, \mu^{(i)}, \pi^{(i)}, u, v))) \end{array} \tag{18}$$

This is the formula to which we apply the decision procedure of Presburger arithmetics to get the wanted algorithm for Theorem 4.10. If θ has length n and quantifier alternation α then formula (18) has length $n + O(k2^k \log k)$ and quantifier alternation $\alpha+2k+1$. The $\log k$ term comes from the length in binary of the indices of variables. And the 2^k term comes from the conjunct over $f \in \{0,1\}^{<k}$. □

4.4.3 Complexity of deciding $\mathrm{Sync}(\mathbb{N}^k)$ in $\mathrm{Rat}(\mathbb{N}^k)$

In order to get a reasonable lower bound we avoid performing a test for all decreasing subsets as suggested in Theorem 4.4 by grouping them together. This yields the following result.

Theorem 4.11. *Given a Presburger formula θ with k free variables, length n and quantifier alternation α, there exists an algorithm with complexity $P(2n+k^{k+O(1)}, \alpha+5)$ which decides whether the rational relation $\{a \in \mathbb{N}^k \mid \theta(a)\}$ is synchronous or not.*

Proof. Let $q(k)$ be the cardinal of the set S_k of strictly decreasing sequences $\{1,\ldots,k\} \supseteq I_1 \supsetneq \ldots \supsetneq I_p \neq \emptyset$. We first prove that $q(k)$ is bounded by $(k+1)!$. Observe that S_{k+1} can be split into two parts A_{k+1} and B_{k+1} depending on whether or not I_1 is equal to $\{1,\ldots,k+1\}$. Clearly, A_{k+1} is in bijection with S_k, hence it contains exactly $q(k)$ sequences. Now, $B_{k+1} = \bigcup_{i=1,\ldots,k+1} B^i_{k+1}$ where B^i_{k+1} is the subfamily of sequences such that $i \notin I_1$. Each B^i_{k+1} is also in bijection with S_k. Thus, B_{k+1} contains at most $(k+1)q(k)$ sequences. Finally, $q(k+1) \leq (k+2)q(k)$. Since $q(1)=1$ holds we get the bound as claimed.

Now we enumerate the different such sequences $\sigma_0,\ldots,\sigma_{q(k)-1}$ and we use the following notations: $\sigma_i = (I_1^{(i)},\ldots,I_{p_i}^{(i)})$. Observe that $p_i \leq k$. Set

$$T_{\sigma_i}(X) = \{(x_1,\ldots,x_p) \in \mathbb{N}_+^p \mid x_1 e_{I_1^{(i)}} + \cdots + x_{p_i} p e_{I_{p_i}^{(i)}} \in X\}$$

Group the $T_{\sigma_i}(X)$'s into a single relation $\mathcal{T}(X) \subset \mathbb{N}^k$ as follows where $q = q(k)$

$$\mathcal{T}(X) = \bigcup_{i<q(k)} \left(q(T_{\sigma_i}(X) \times \{0\}^{k-p_i}) + (\overbrace{i,\ldots,i}^{k \text{ times}}) \right)$$

Clearly, $T_{\sigma_i}(X) = \{(r_1,\ldots,r_{p_i}) \mid (qr_1+i,\ldots,qr_{p_i}+i,i,\ldots,i) \in \mathcal{T}(X)\}$. In particular, all the $T_{\sigma_i}(X)$'s are recognizable if and only if $\mathcal{T}(X)$ is recognizable. Thus, X is synchronous rational if and only if $\mathcal{T}(X)$ is recognizable. Now, $\mathcal{T}(X)$ is Presburger definable by the following formula $\Omega(x_1,\ldots,x_k)$:

$$\Omega(x_1,\ldots,x_k) \;=\; \exists y \; (\theta(y) \;\wedge\; \bigvee_{i<q} x_1 e_{I_1^i} + \ldots + x_{p_i} e_{I_p^i} = qy + (i,\ldots,i))$$

This is the formula to which we apply the decision procedure for recognizability of Theorem 4.8. If θ has length n and quantifier alternation α then this formula has length $n + O(kq(k)\log k)$ and quantifier alternation $\alpha+1$. The $\log k$ term comes from the length in binary of the indices of variables. And the $q(k)$ term comes from the conjunct over $i < q(k)$. Since $q(k) \leq (k+1)!$, we see that the length of our formula is $n+k^{k+O(1)}$. Applying the decision procedure of Theorem 4.8 to this formula yields the required complexity.

$\square$

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