Borel Determinacy

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Alternations of quantifier

versus

Games

Alternations of quantifier

$$F(\vec{z}) \quad \equiv \quad \exists x_0\ \forall x_1\ \exists x_2\ \forall x_3\ \exists x_4\ P(x_0, \ldots, x_4, \vec{z})$$

The human mind seems limited in its ability to understand and vizualize beyond four or five alternations of quantifier. Indeed, it can be argued that the inventions, subtheories and central lemmas of various parts of mathematics are devices for assisting the mind in dealing with one or two additional alternations of quantifier.

Hartley Rogers
"Theory of recursive functions and effective computability" (1967) (cf. page 322 §14.7)

Another (partial) explanation:

$\boxed{\textit{complexity} \geq \Sigma^0_4(\omega^\omega) \rightsquigarrow \textit{Higher set theory!!!}}$

Alternations of quantifier versus Games

$$F(\vec{z}) \quad \equiv \quad \exists x_0\ \forall x_1\ \exists x_2\ \forall x_3\ \exists x_4\ P(x_0, \ldots, x_4, \vec{z})$$

Roland Fraïssé's idea (1954)

Relate $F(\vec{z})$ to a game

Two players

$\mathtt{I}$

and

$\mathtt{II}$

move 0 :	$\mathtt{I}$	plays some x_0
move 1 :	$\mathtt{II}$	plays some x_1
move 2 :	$\mathtt{I}$	plays some x_2
move 3 :	$\mathtt{II}$	plays some x_3
move 4 :	$\mathtt{I}$	plays some x_4

Who wins?

$\mathtt{I}$ wins iff $P(x_0, \ldots, x_4, \vec{z})$ holds

$$\boxed{F(\vec{z}) \iff \text{player } \mathtt{I} \text{ has a winning strategy}}$$

Strategies

move 0 :	$\mathtt{I}$ plays some x_0	
move 1 :	$\mathtt{II}$ plays some x_1	
move 2 :	$\mathtt{I}$ plays some x_2	
move 3 :	$\mathtt{II}$ plays some x_3	
move 4 :	$\mathtt{I}$ plays some x_4	

The x_i's in X

$\mathtt{I}$ wins
iff
$P(x_0, \ldots, x_4, \vec{z})$

Strategy for $\mathtt{I} = \sigma_{\mathtt{I}} : \{nil\} \cup X \cup X^2 \longrightarrow X$
Strategy for $\mathtt{II} = \sigma_{\mathtt{I}} : X \cup X^2 \longrightarrow X$

$$\mathtt{I} \text{ follows strategy } \sigma_{\mathtt{I}} \text{ if } \begin{cases} x_0 = \sigma_{\mathtt{I}}(nil) \\ x_2 = \sigma_{\mathtt{I}}(x_1) \\ x_4 = \sigma_{\mathtt{I}}(x_1, x_3) \end{cases}$$

$$\mathtt{II} \text{ follows strategy } \sigma_{\mathtt{II}} \text{ if } \begin{cases} x_1 & = & \sigma_{\mathtt{II}}(x_0) \\ x_3 & = & \sigma_{\mathtt{II}}(x_0, x_1) \end{cases}$$

Winning strategy: ALWAYS wins

Alternations of quantifier and games

$$
\begin{aligned}
F(\vec{z}) &\equiv \exists x_0\ \forall x_1\ \exists x_2\ \forall x_3\ \exists x_4\ P(x_0, \ldots, x_4, \vec{z}) \\
&\equiv \textit{player } \mathtt{I} \textit{ has a winning strategy} \\
&\quad\ \textit{for the game where } \mathtt{I} \textit{ wins} \\
&\quad\ \textit{if } (x_0, \ldots, x_4) \in P
\end{aligned}
$$

$$
\begin{aligned}
\neg F(\vec{z}) &\equiv \forall x_0\ \exists x_1\ \forall x_2\ \exists x_3\ \forall x_4\ \neg P(x_0, \ldots, x_4, \vec{z}) \\
&\equiv \textit{player } \mathtt{II} \textit{ has a winning strategy} \\
&\quad\ \textit{for the game where } \mathtt{I} \textit{ wins} \\
&\quad\ \textit{if } (x_0, \ldots, x_4) \in P
\end{aligned}
$$

Law of Excluded Middle: either $F(\vec{z})$ or $\neg F(\vec{z})$

Hence either I has a winning strategy
or II has a winning strategy

Infinitely many alternations of quantifier

$\exists x_0\ \forall x_1\ \exists x_2\ \forall x_3 \ldots \qquad P((x_i)_{i\in\mathbb{N}}, \vec{z})$

Moschovakis' game quantifier $\Game\alpha\ P(\alpha, \vec{z})$

$\forall x_0\ \exists x_1\ \forall x_2\ \exists x_3 \ldots \qquad \neg P((x_i)_{i\in\mathbb{N}}, \vec{z})$

What does this mean? **Infinite game**

Two players I and II		Rule
move $2i$:	I plays x_{2i}	I wins iff
move $2i+1$:	II plays x_{2i+1}	$(x_i)_{i\in\mathbb{N}} \in A$

where $A = \{(x_i)_{i\in\mathbb{N}} \mid P((x_i)_{i\in\mathbb{N}}, \vec{z})\}$

Two players $\mathtt{I}$ and $\mathtt{II}$	Rule
move $2i$: $\mathtt{I}$ plays x_{2i} move $2i+1$: $\mathtt{II}$ plays x_{2i+1}	$\mathtt{I}$ wins iff $P((x_i)_{i\in\mathbb{N}}, \vec{z})$

$$\text{Strategy for } \mathtt{I} \;=\; \sigma_{\mathtt{I}} : X^{<\omega} \longrightarrow X$$

$$\text{Strategy for } \mathtt{II} \;=\; \sigma_{\mathtt{II}} : (X^{<\omega} \setminus \{nil\}) \longrightarrow X$$

$\mathtt{I}$ follows $\sigma_{\mathtt{I}}$ if $\forall i \in \mathbb{N}\ x_{2i} = \sigma_{\mathtt{I}}((x_{2j+1})_{j<i})$

$\mathtt{II}$ follows $\sigma_{\mathtt{II}}$ if $\forall i \in \mathbb{N}\ x_{2i+1} = \sigma_{\mathtt{II}}((x_{2j})_{j\leq i})$

Winning strategy: ALWAYS wins

$$\exists x_0\ \forall x_1\ \exists x_2\ \forall x_3 \ldots \quad P((x_i)_{i\in\mathbb{N}}, \vec{z})$$

$\equiv$ *$\mathtt{I}$ has a winning strategy*

$$\forall x_0\ \exists x_1\ \forall x_2\ \exists x_3 \ldots \quad \neg P((x_i)_{i\in\mathbb{N}}, \vec{z})$$

$\equiv$ *$\mathtt{II}$ has a winning strategy*

Need X well-ordered or Axiom of dependent choices

Excluded middle and determinacy

$$\exists x_0\ \forall x_1\ \exists x_2\ \forall x_3 \ldots \quad P((x_i)_{i\in\mathbb{N}}, \vec{z})$$

$$\equiv \quad \textit{I has a winning strategy}$$

$$\forall x_0\ \exists x_1\ \forall x_2\ \exists x_3 \ldots \ \neg P((x_i)_{i\in\mathbb{N}}, \vec{z})$$

$$\equiv \quad \textit{II has a winning strategy}$$

Fact.

$$\left\{ \begin{array}{c} \neg\,(\exists x_0\ \forall x_1\ \exists x_2\ \forall x_3 \ldots P((x_i)_{i\in\mathbb{N}}, \vec{z})) \\ \textit{is equivalent to} \\ \forall x_0\ \exists x_1\ \forall x_2\ \exists x_3 \ldots \neg P((x_i)_{i\in\mathbb{N}}, \vec{z}) \end{array} \right.$$

if and only if the game is determined

(one of the players has a winning strategy)

Which sets are determined?

Countable sets are determined

Infinite game $\mathcal{G}(A)$

Two players I and II			Rule
move $2i$:	I plays x_{2i}		I wins iff
move $2i+1$:	II plays x_{2i+1}		$(x_i)_{i\in\mathbb{N}} \in A$

Fact. If $A \subset X^\omega$ is countable then II has a winning strategy in $\mathcal{G}(A)$

Proof. Diagonal argument. If $A = \{f_i \mid i \in \mathbb{N}\}$, $2i+1$ player II plays $x_{2i+1} \neq f_i(2i+1)$

Are all sets determined?

NO (requires the axiom of choice)

Borel subsets of X^ω

Discrete topology on X Product topology on X^ω

metrics $d(f,g) = 2^{-\max\{n \mid \forall i<n \ f(i)=g(i)\}}$

Basis of clopen sets: the uX^ω for $u \in X^{<\omega}$

Care. if X uncountable, open sets may be unions of uncountably many clopen sets

But metrizability implies closed set are $\underset{\sim}{\mathbf{G}}_\delta$ in $X^{<\omega}$

($\underset{\sim}{\mathbf{G}}_\delta$ = intersection of countably many open sets)

This allows for the usual definition of Borel sets

$$\begin{aligned}
\underset{\sim}{\mathbf{\Sigma}}^0_1(X^\omega) &= \textit{open sets} \\
\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X^\omega) &= \textit{countable unions of sets in } \bigcup_{\beta<\alpha} \underset{\sim}{\mathbf{\Pi}}^0_\beta(X^\omega) \\
\underset{\sim}{\mathbf{\Pi}}^0_\alpha(X^\omega) &= \textit{complements of sets in } \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X^\omega)
\end{aligned}$$

Borel determinacy

Theorem. (Donald Martin, 1975)
All Borel subsets of X^ω are determined
(whatever big is X)

Find *simple* winning strategies in $\mathcal{G}(A)$?

Alas… best (general) complexity is $\Delta_2^{1,S}$
if A is Borel with code S

Upper bound proof. The set of ws for $\mathtt{I}$ is $\Pi_1^{1,S}$:

$$\sigma_{\mathtt{I}} \textit{ is ws} \equiv \forall g \; \sigma_{\mathtt{I}} \star g \in A$$

and every $\Pi_1^{1,S}$ family contains some $\Delta_2^{1,S}$ set

(cf. Rogers §16.7 Coro. XLV(c), p. 430)

Determinacy in classical mathematics

- 1953, Gale & Stewart Boolean combinations of open subsets of X^ω
- 1955, Philip Wolfe $\utilde{\mathbf{\Sigma}}^0_2(X^\omega)$ and $\utilde{\mathbf{\Pi}}^0_2(X^\omega)$ sets
- 1964, Morton Davis $\utilde{\mathbf{\Sigma}}^0_3(X^\omega)$ and $\utilde{\mathbf{\Pi}}^0_3(X^\omega)$ sets

Results proved in $\boxed{\boxed{\text{2d-order arithmetic}}}$

$\equiv$ mathematics of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$

$\equiv$ $\boxed{\boxed{\text{classical set theory for mathematicians}}}$

(with $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ one can encode reals, continuous functions, ...)

Determinacy in higher set theory

- 1970, Donald Martin $\underset{\sim}{\boldsymbol{\Sigma}}^1_1(\omega^\omega)$ in ZF + large cardinal axiom

  $\underset{\sim}{\boldsymbol{\Sigma}}^1_1(X^\omega)$ in ZF + stronger large cardinal axiom

- 1972, Jeff B. Paris $\underset{\sim}{\boldsymbol{\Sigma}}^0_4(X^\omega)$ and $\underset{\sim}{\boldsymbol{\Pi}}^0_4(X^\omega)$ sets in ZF

  (set theory with cardinal $(2^{\aleph_0})^+$ is enough
  hence 3rd-order arithmetic is enough)

- 1975, Donald Martin Borel subsets of X^ω in ZF
- 1985, Donald Martin Much simpler proof (by far...) in ZF

Higher set theory (in ZF) is required!!!

- 1971, Harvey Friedman

For $\underset{\sim}{\boldsymbol{\Sigma}}^0_5(\omega^\omega)$ and beyond, 2d-order arithmetic NOT ENOUGH

For $\underset{\sim}{\boldsymbol{\Sigma}}^0_{5+\alpha}(\omega^\omega)$ need α iterations of set exponentiation

- ∼2010, Donald Martin

For $\underset{\sim}{\boldsymbol{\Sigma}}^0_4(\omega^\omega)$ 2d-order arithmetic NOT ENOUGH

A few simple results about determinacy and strategies

Determinacy and complementation

If $A \subseteq X^\omega$ then the shift of A is
$XA = \{(x, x_0, x_1, x_2, \ldots) \mid (x_0, x_1, x_2, \ldots) \in A)\}$

Let $\mathcal{A} \subseteq \mathcal{P}(X^\omega)$ be closed under shift:

$$A \in \mathcal{A} \Longrightarrow xA \in \mathcal{A}$$

$$\boxed{\begin{array}{c} \forall A \in \mathcal{A} \quad A \textit{ is determined} \\ \Longleftrightarrow \\ \forall A \in \mathcal{A} \quad X^\omega \setminus A \textit{ is determined} \end{array}}$$

$\mathtt{I}$ has a ws in $\mathcal{G}(XA)$	$\Longrightarrow$	$\mathtt{II}$ has a ws in $\mathcal{G}(X^\omega \setminus A)$
$\mathtt{II}$ has a ws in $\mathcal{G}(XA)$	$\Longrightarrow$	$\mathtt{I}$ has a ws in $\mathcal{G}(X^\omega \setminus A)$

Winning strategies viewed as trees

Strategy $\sigma_{\mathtt{I}}$ for $\mathtt{I}$ $\equiv$ tree $S_{\sigma_{\mathtt{I}}} \subseteq X^{<\omega}$ of

all plays when $\mathtt{I}$ follows $\sigma_{\mathtt{I}}$

$$\begin{cases} u \in S_{\sigma_{\mathtt{I}}} \wedge |u| \textit{ even} \implies \exists! x \; ux \in S_{\sigma_{\mathtt{I}}} \\ u \in S_{\sigma_{\mathtt{I}}} \wedge |u| \textit{ odd} \implies \forall x \; ux \in S_{\sigma_{\mathtt{I}}} \end{cases}$$

(Thus, $\mathtt{I}$ always has exactly one possible move and there is no constraint for $\mathtt{II}$-moves)

Strategy $\sigma_{\mathtt{II}}$ for $\mathtt{II}$ $\equiv$ tree $S_{\sigma_{\mathtt{II}}} \subseteq X^{<\omega}$ of

all plays when $\mathtt{II}$ follows $\sigma_{\mathtt{II}}$

$$\begin{cases} u \in S_{\sigma_{\mathtt{II}}} \wedge |u| \textit{ odd} \implies \exists! x \; ux \in S_{\sigma_{\mathtt{II}}} \\ u \in S_{\sigma_{\mathtt{II}}} \wedge |u| \textit{ even} \implies \forall x \; ux \in S_{\sigma_{\mathtt{II}}} \end{cases}$$

(Thus, $\mathtt{II}$ always has exactly one possible move and there is no constraint for $\mathtt{I}$-moves)

$\sigma_{\mathtt{I}}$ winning for $\mathtt{I}$ $\iff$ $[S_{\sigma_{\mathtt{I}}}] \subseteq A$

$\sigma_{\mathtt{II}}$ winning for $\mathtt{II}$ $\iff$ $[S_{\sigma_{\mathtt{II}}}] \subseteq X^{\omega} \setminus A$

($[S]$ = set of infinite branches of S)

Non deterministic winning strategies

ND strategy $\sigma_{\mathtt{I}}$ for $\mathtt{I}$ $\equiv$ tree $S_{\sigma_{\mathtt{I}}} \subseteq X^{<\omega}$ of all plays when $\mathtt{I}$ follows $\sigma_{\mathtt{I}}$

$$\begin{cases} S_{\sigma_{\mathtt{I}}} \textit{ is pruned: } \forall u \in S_{\sigma_{\mathtt{I}}} \; \exists x \; ux \in S_{\sigma_{\mathtt{I}}} \\ u \in S_{\sigma_{\mathtt{I}}} \wedge |u| \textit{ odd} \implies \forall x \; ux \in S_{\sigma_{\mathtt{I}}} \end{cases}$$

(Thus, $\mathtt{I}$ always has some move and there is no constraint for $\mathtt{II}$-moves)

ND strategy $\sigma_{\mathtt{II}}$ for $\mathtt{II}$ $\equiv$ tree $S_{\sigma_{\mathtt{II}}} \subseteq X^{<\omega}$ of all plays when $\mathtt{II}$ follows $\sigma_{\mathtt{II}}$

$$\begin{cases} S_{\sigma_{\mathtt{I}}} \textit{ is pruned: } \forall u \in S_{\sigma_{\mathtt{II}}} \; \exists x \; ux \in S_{\sigma_{\mathtt{I}}} \\ u \in S_{\sigma_{\mathtt{II}}} \wedge |u| \textit{ even} \implies \forall x \; ux \in S_{\sigma_{\mathtt{II}}} \end{cases}$$

(Thus, $\mathtt{II}$ always has some move and there is no constraint for $\mathtt{I}$-moves)

$$\sigma_{\mathtt{I}} \text{ winning for } \mathtt{I} \iff [S_{\sigma_{\mathtt{I}}}] \subseteq A$$

$$\sigma_{\mathtt{II}} \text{ winning for } \mathtt{II} \iff [S_{\sigma_{\mathtt{II}}}] \subseteq X^{\omega} \setminus A$$

Winning strategies and positions

$u \in X^{<\omega}$ $A \subseteq X^{\omega}$ $A_u = A \cap \textit{clopen set } uX^{\omega}$

Fact. If $|u|$ is odd (next move for $\texttt{II}$) then

$$\texttt{II} \text{ has no winning strategy in } \mathcal{G}(A_u)$$

iff

$$\forall x \in X \ \texttt{II} \text{ has no winning strategy in } \mathcal{G}(A_{ux})$$

(No "miracle" move x for player $\mathtt{II}$)

Winning and Defensive strategies

$u \in X^{<\omega}$ $A \subseteq X^{\omega}$ $A_u = A \cap$ *clopen set* uX^{ω}

Fact. Let $|u|$ even (next move for player $\mathtt{I}$)

$\mathtt{II}$ has no winning strategy in $\mathcal{G}(A_u)$

iff

$\exists x \in X$ $\mathtt{II}$ has no winning strategy in $\mathcal{G}(A_{ux})$

(Player $\mathtt{I}$ has a move x so that $\mathtt{II}$ still has no ws afterwards)

Always choosing such an x =

Defensive strategy for player $\mathtt{I}$

CARE: defensive strategy $\not\Rightarrow$ winning strat.

Gale & Stewart's results about $\underset{\sim}{\boldsymbol{\Sigma}}^0_1(X^\omega)$

they contain
many core ideas of the theory

Determinacy of $\mathbf{\Sigma}^0_1(X^\omega)$ and $\mathbf{\Pi}^0_1(X^\omega)$

Theorem. (Gale & Stewart, 1953)

$\boxed{\text{Every closed or open } A \subseteq X^\omega \text{ is determined}}$

Proof. Let A closed be the set of infinite branches of a pruned tree $T \subseteq X^{<\omega}$, i.e. $A = [T]$

If player $\mathtt{II}$ has no ws in $\mathcal{G}(A)$

then any DEFENSIVE strategy for player $\mathtt{I}$ is winning:

$\exists x_0$ (move of $\mathtt{I}$) so that $\mathtt{II}$ has no ws in $\mathcal{G}(A_{x_0})$

$\forall x_1$ (move of $\mathtt{II}$) $\mathtt{II}$ has no ws in $\mathcal{G}(A_{x_0 x_1})$

$\exists x_2$ (move of $\mathtt{I}$) $\mathtt{II}$ has no ws in $\mathcal{G}(A_{x_0 x_1 x_2})$

$\forall x_3$ (move of $\mathtt{II}$) $\mathtt{II}$ has no ws in $\mathcal{G}(A_{x_0 x_1 x_2 x_3})$

...

$\forall n\ x_0 \dots x_n \in T$ else the play enters the open set $X^\omega \setminus A$ so that any strategy for $\mathtt{II}$ in $\mathcal{G}(A_{x_0 \dots x_n})$ is winning. But $\mathtt{II}$ has no ws in $\mathcal{G}(A_{x_0 x_1 x_2 x_3})$. Contradiction!

$\forall n\ x_0 \dots x_n \in T \ \Rightarrow$ the infinite play $\in$ closed set $A = [T]$

Subgames

Pruned subtree S of $X^{<\omega}$
$[S]$ = infinite branches of S

$\mathcal{G}_S(A)$ or $\mathcal{G}_F(A)$: New rule for the two players:
The play stays in the subtree S

$\equiv$ replace A by $A \cap [S]$, $\quad X^\omega \setminus A$ by $(X^\omega \setminus A) \cap [S]$

Trivial example: Game $\mathcal{G}(A_s)$ at position s
reduces to the subgame $\mathcal{G}_{sX^\omega}(A)$

Determinacy of BOOL($\underset{\sim}{\mathbf{\Sigma}}^0_1(X^\omega)$)

Theorem. (Gale & Stewart, 1953)

If every subgame of $\mathcal{G}(A)$ is determined then $\mathcal{G}(A \cup U)$ is determined for all U open

Proof. We show that there is a particular subgame F st

if $\mathcal{G}_F(A)$ is determined then so is $\mathcal{G}(A)$

$S = \{s \mid \mathtt{I}\ \textit{has a ws in}\ \mathcal{G}((A \cup U)_s)\}$ (S may not be a tree)

$F = X^\omega \setminus SX^\omega = [T]$ for some pruned tree T disjoint from S

$U \cap F = \emptyset$: if $sX \subseteq U$ then $s \in S$ trivially

$\mathtt{I}$ has a ws in $\mathcal{G}_F(A) \Longrightarrow \mathtt{I}$ has a ws in $\mathcal{G}(A \cup U)$

Playing in $\mathcal{G}(A \cup U)$, $\mathtt{I}$ follows his ws for $\mathcal{G}_F(A)$ while $\mathtt{II}$ stays in T. If $\mathtt{II}$ leaves T then the play gets into S and $\mathtt{I}$ uses a ws for $\mathcal{G}((A \cup U)_s)$

$\mathtt{II}$ has a ws in $\mathcal{G}_F(A) \Rightarrow \mathtt{II}$ has a ws in $\mathcal{G}(A \cup U)$

Playing in $\mathcal{G}(A \cup U)$, $\mathtt{II}$ follows his ws for $\mathcal{G}_F(A)$ while $\mathtt{I}$ stays in T. $\mathtt{I}$ cannot leave T : else, if $\mathtt{I}$ leaves T at s then $s \in S$ and $\mathtt{I}$ gets a ws for $\mathcal{G}((A \cup U)_s$, contradicting Fact page 20

Determinacy of BOOL($\underset{\sim}{\mathbf{\Sigma}}^0_1(X^\omega)$)

Theorem. (Gale & Stewart, 1953)
If every subgame of $\mathcal{G}(A)$ is determined then $\mathcal{G}(A \cap U)$ is determined for all U open

Proof. $S = \{s \mid \ sX^\omega \subseteq U$ *and* $\mathtt{I}$ *has a ws in* $\mathcal{G}(A_s\}$
$T = \{s \mid \ sX^\omega \subseteq U$ *and* $\mathtt{II}$ *has a ws in* $\mathcal{G}(A_s\}$

$\mathtt{I}$ has a ws in $\mathcal{G}(SX^\omega) \Longrightarrow \mathtt{I}$ has a ws in $\mathcal{G}(A \cap U)$
$\mathtt{I}$ follows a ws for $\mathcal{G}(SX^\omega)$ until the play enters S
Then he uses a ws for $\mathcal{G}(A_s)$

$\mathtt{II}$ has a ws in $\mathcal{G}(SX^\omega) \Rightarrow \mathtt{II}$ has a ws in $\mathcal{G}(A \cap U)$
$\mathtt{II}$ follows ws for $\mathcal{G}(SX^\omega)$. If output $\notin U$ $\mathtt{II}$ wins
Else the play enters S ot T. Cannot enter S else $\mathtt{I}$ could win.
If it enters T then $\mathtt{II}$ uses ws for $\mathcal{G}(A_s)$

Determinacy of BOOL($\underset{\sim}{\mathbf{\Sigma}}^0_1(X^\omega)$)

Corollary. (Gale & Stewart, 1953)
Every Boolean combination of open subsets of X^ω is determined

Proof.
Extend closed determinacy to subgames
Apply closure by complementation,
union and intersection with open sets

Closed sets and largest non deterministic ws

Fact. If $\mathtt{I}$ has a winning strategy for a closed game then it has a largest non deterministic one

Proof. S a tree, $\Theta(S) \subseteq S$, $\Lambda(S) \subseteq S$

$$\begin{cases} \Theta(S) & = & \{u \in S \mid \forall v \leq_{\text{pref}} u \ (|v| \ odd \Rightarrow \forall x \ vx \in S)\} \\ \Lambda(S) & = & \{u \in S \mid \exists x \in X \ ux \in S)\} \end{cases}$$

To prune a tree one has to transfinitely iterate Λ *(cf. page 31)*

Suppose $\mathtt{I}$ has a ws for $\mathcal{G}(F)$ $\quad F = [T]$ $\ T$ tree $\subseteq X^{<\omega}$

$T^{(0)} = T \quad T^{\alpha+1} = \Lambda(\Theta(T^{(\alpha)})) \quad T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)}$

Fact: $\exists \xi < \aleph_1$ st $T^{(\xi)} = T^{(\xi+1)} = \Lambda(T^{(\xi)}) = \Theta(T^{(\xi)})$

Fact: 1) Every ND ws for $\mathtt{I}$ (viewed as a tree) is $\subseteq T^{(\alpha)}$

2) If $\mathtt{I}$ has a ws for $\mathcal{G}([T])$ then $T^{(\xi)} \neq \emptyset$

3) $T^{(\xi)}$ is a non deterministic ws for $\mathtt{I}$ and is the largest one

1) Proof by induction over α. 2) Obvious from 1)

3) Closure under pruning and Θ insures $T^{(\xi)}$ is a strategy for $\mathtt{I}$ (if non empty). It is winning since $T^{(\xi)} \subseteq T$ and $F = [T]$

Winning strategies may be quite complex even for closed games!

Fact. There exists a computable tree $T \subseteq \omega^{<\omega}$ st

- $\mathtt{I}$ has a ws in the closed game $\mathcal{G}([T])$
- $\mathtt{I}$ has no Δ^1_1 ws in $\mathcal{G}([T])$

Proof. Recall Kleene's result (cf. Rogers §16.7 Coro. XLI(b), p. 419):

Fact. There exists a computable tree $S \subseteq \omega^{<\omega}$ which has an infinite branch but no Δ^1_1 one

Let $\theta : \omega^{<\omega} \to \omega^{<\omega}$ suppress all odd rank letters of a finite sequence: for instance, $\theta(abcde) = \theta(abcdef) = ace$

Let $T = \theta^{-1}(S)$ $\quad$ T is a computable tree

Player $\mathtt{I}$ has a ws in $\mathcal{G}([T])$: do not care about $\mathtt{II}$ moves, play a fixed infinite branch of S

If σ is a ws for $\mathtt{I}$ in $\mathcal{G}([T])$ and $\mathtt{II}$ plays 0^ω then $\sigma \star 0^\omega \in [T]$ hence $f = \theta(\sigma \star 0^\omega) \in [S]$

If σ were Δ^1_1 then f would be Δ^1_1 branch of S. Contradiction!

Why so complex ws for closed games? Because pruning a tree may require iterations beyond recursive ordinals!

$T^{(0)} = T \quad T^{\alpha+1} = \Lambda(\Theta(T^{(2\alpha)})) \quad T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)}$

$T^{(\xi)} = T^{(\xi+1)}$ largest ND ws for $\mathtt{I}$ in $\mathcal{G}([T])$

R order on $\mathbb{N}$ of type $\eta > \xi$ $\qquad \iota : \mathbb{N} \to \eta$ isomorphism

Fact. $T^{(\xi)}$ is $\Delta_1^{1,T,R}$ hence so is its leftmost infinite branch

Proof. $\Phi_T(Z^{(k)}, Z^{(\ell)}) \equiv \big(\theta(k) = 0 \Rightarrow Z^{(k)} = T\big)$

$$\wedge \big(\theta(\ell) = \theta(k) + 1 \Rightarrow Z^{(\ell)} = \Lambda(\Theta(Z^{(k)}))\big)$$

$$\wedge \big(\theta(k) \textit{ limit} \Rightarrow Z^{(k)} = \bigcap\{Z^{(p)} \mid \theta(p) < \theta(k)\}\big)$$

$$u \in T^{(\xi)} \equiv \exists (Z^{(n)})_{n\in\mathbb{N}} \, \forall k, \ell \, (u \in Z^{(k)} \wedge \Phi_T(Z^{(k)}, Z^{(\ell)}))$$

$$\equiv \forall (Z^{(n)})_{n\in\mathbb{N}} \, (\forall k, \ell \, \Phi_T(Z^{(k)}, Z^{(\ell)})) \Rightarrow (\forall n \, u \in Z^{(n)}))$$

Fact. Let $T = \theta^{-1}(S)$ with S a computable tree with an infinite branch but no Δ_1^1 one. Then the ordinal ξ is not Δ_1^1

Proof. $S \ \Delta_1^1 \Rightarrow T \ \Delta_1^1$ and $T, R \ \Delta_1^1 \Rightarrow \Delta_1^{1,T,R} = \Delta_1^1$

(cf. Rogers §16.6 Thm XXXIV p. 412)

How many iterations to prune a tree?

$$\left\{\begin{array}{rcl} S^{(0)} & = & S \\ S^{\alpha+1} & = & \Lambda(S^{(\alpha)}) \\ S^{(\lambda)} & = & \bigcap_{\alpha<\lambda} S^{(\alpha)} \end{array}\right. \quad \begin{array}{|l} \Lambda(S) = \{u \in S \mid \\ \qquad\qquad \exists x \in X \ ux \in S)\} \\ S^{(\xi)} = S^{(\xi+1)} \textit{ is pruned} \end{array}$$

When S is well-founded, $\xi =$ ordinal rank of S, $\xi < \omega_1^{\mathsf{CK}}$ hence ξ is computable

(Spector, 1955: computable ordinals=Δ_1^1 ordinals, cf. Rogers §16.6 Coro XXXVI p. 415)

In general, when S not well-founded, $\xi \geq \omega_1^{\mathsf{CK}}$ (nevertheless, ξ is Δ_2^1) Example: cf. page 30

Other example: $S = \{(e, u, t) \mid e, t \in \mathbb{N},\ u \in \omega^{<\omega}$ *and* the current output of φ_e at time t is $u\}$

Order on S :

$(e, u, t) \leq (e', u', t') \Leftrightarrow e = e' \wedge (u <_{\mathrm{pref}} u' \vee (u = u' \wedge t \leq t'))$

S is a computable tree which contains every well-founded computable tree Hence the ξ associated to S is $\geq \omega_1^{CK}$

Wolfe's results about $\underset{\sim}{\mathbf{\Sigma}}^0_2(X^\omega)$ and $\underset{\sim}{\mathbf{\Pi}}^0_2(X^\omega)$

$$\equiv \underset{\sim}{\mathbf{F}_\sigma}(X^\omega) \textit{ and } \underset{\sim}{\mathbf{G}_\delta}(X^\omega)$$

(countable unions of closed sets
and
countable intersections of open sets)

Determinacy of $\underset{\sim}{\mathbf{\Sigma}}^0_2(X^\omega)$ and $\underset{\sim}{\mathbf{\Pi}}^0_2(X^\omega)$

Theorem. (Philip Wolfe, 1955)
Every $\underset{\sim}{\mathbf{F}_\sigma}$ or $\underset{\sim}{\mathbf{G}_\delta}$ set $A \subseteq X^\omega$ is determined

Proof.(cf. Moschovakis) $A = \bigcup_{i\in\mathbb{N}}[T_i]$ an $\underset{\sim}{\mathbf{F}_\sigma}$ set T_i pruned tree

Set W of *sure winning positions* for I in $\mathcal{G}(A)$

$$\begin{array}{rcl}
u \in W_0 & \iff & \exists i \quad I \textit{ wins } \mathcal{G}([T_i]_u) \\
u \in H_{\alpha,i} & \iff & \forall v \leq u \ (|v| \textit{even} \Rightarrow v \in T_i \cup \bigcup_{\beta<\alpha} W_\beta) \\
u \in W_\alpha & \iff & \exists i \quad I \textit{ wins } \mathcal{G}([H_{\alpha,i}]_u)
\end{array}$$

$W = \bigcup_\alpha W_\alpha = \bigcup_{\alpha\leq\xi} W_\alpha$ $\quad \xi$ countable ordinal, $W_\xi = W_{\xi+1}$

Induction on ordinal α : $\begin{cases} u \in W_\alpha & \Rightarrow \quad \mathtt{I} \textit{ wins } \mathcal{G}(A_u) \\ u \notin W_\alpha & \Rightarrow \quad \mathtt{II} \textit{ wins } \mathcal{G}(A_u) \end{cases}$

• Induction on ordinal α : $u \in W_\alpha \Rightarrow \mathtt{I}$ *wins* $\mathcal{G}(A_u)$
$\mathtt{I}$ follows a ws for $\mathcal{G}([H_{\alpha,i}]_u)$

If the play enters $\bigcup_{\beta<\alpha} W_\beta$ at uu' then $\mathtt{I}$ switches to aws for $\mathcal{G}(A_{uu'})$. By induction hypothesis, the infinite play is in A

Else the play stays in T_i hence the infinite play is in A

• If nil $\in W$ then $\mathtt{I}$ has a ws for $\mathcal{G}(A)$

• Else here is a ws for $\mathtt{II}$ in $\mathcal{G}(A)$:
nil $\notin W_{\xi+1}$ hence for all i, $\mathtt{I}$ has no ws for $\mathcal{G}([H_{\xi+1,i}])$
hence $\mathtt{II}$ has a ws for $\mathcal{G}([H_{\xi+1,i}])$ (closed games being determined)

$\mathtt{II}$ follows his ws for $\mathcal{G}([H_{\xi+1,0}])$ until the play leaves $W_\xi \cup T_0$
at some u_0 $\quad u_0 \notin W_\xi$ and $u_0 \notin T_0$

$u_0 \notin W_\xi = W_{\xi+1}$ hence $\forall i$ $\mathtt{I}$ has no ws for $\mathcal{G}([H_{\xi+1,i}]_{u_0})$ hence $\mathtt{II}$ has a ws for $\mathcal{G}([H_{\xi+1,i}]_{u_0})$ (since closed games are determined)

$\mathtt{II}$ follows his ws until the play leaves $W_\xi \cup T_0$ at $u_0 u_1 \ldots$

The final play $\notin [T_0]$, $\notin [T_1] \ldots$ hence $\notin A$
Thus, $\mathtt{II}$ has a ws in $\mathcal{G}(A)$ $\qquad$ A is determined

Morton Davis' results about $\undertilde{\mathbf{\Sigma}}_3^0(X^\omega)$ and $\undertilde{\mathbf{\Pi}}_3^0(X^\omega)$

$$\equiv \undertilde{\mathbf{F}_{\sigma\delta}}(X^\omega) \textit{ and } \undertilde{\mathbf{G}_{\delta\sigma}}(X^\omega)$$

(countable intersections of countable unions of closed sets
and
countable unions of countable intersections of open sets)

Determinacy of $\underset{\sim}{\mathbf{\Sigma}}^0_3(X^\omega)$ and $\underset{\sim}{\mathbf{\Pi}}^0_3(X^\omega)$

Theorem. (Morton Davis, 1964)
Every $\underset{\sim}{\mathbf{F}_{\sigma\delta}}$ or $\underset{\sim}{\mathbf{G}_{\delta\sigma}}$ set $A \subseteq X^\omega$ is determined

Strategies (non necessarily winning) are trees: `II`-strategy S

$$\begin{cases} \forall u \in S\ (|u|\ odd\ \Rightarrow \exists x\ ux \in S) & \textit{(II can stay in S)} \\ \forall u \in S\ (|u|\ even\ \Rightarrow \forall x\ ux \in S) & \textit{(I cannot leave S)} \end{cases}$$

`II`-strategy S relative to a subgame T

$$\begin{cases} \forall u \in S\ (|u|\ odd\ \Rightarrow \exists x\ ux \in S) & \textit{(II can stay in S)} \\ \forall u \in S\ (|u|\ even\ \Rightarrow \forall x\ (ux \in T \Rightarrow ux \in S)) & \textit{(I cannot leave S} \\ & \textit{except if it leaves T)} \end{cases}$$

Determinacy of $\underset{\sim}{\mathbf{\Sigma}}^0_3(X^\omega)$ and $\underset{\sim}{\mathbf{\Pi}}^0_3(X^\omega)$

Strategies (non necessarily winning) are trees: $\mathtt{II}$-strategy S

$$\begin{cases} \forall u \in S\ (|u|\ odd\ \Rightarrow \exists x\ ux \in S) & \textit{(}\mathtt{II}\textit{ can stay in }S\textit{)} \\ \forall u \in S\ (|u|\ even\ \Rightarrow \forall x\ ux \in S) & \textit{(}\mathtt{I}\textit{ cannot leave }S\textit{)} \end{cases}$$

Lemma 1. Suppose $\begin{cases} \mathtt{I} \text{ has no ws for } \mathcal{G}(A) \\ U \subseteq A \text{ is open} \end{cases}$

Then there is a $\mathtt{II}$-strategy S such that

(1) $\mathtt{I}$ has no ws for $\mathcal{G}_S(A)$ and (2) $U \cap [S] = \emptyset$

($\mathtt{II}$ has a non-catastrophic strategy to avoid any fixed open set U
catastrophic = $\mathtt{I}$ has a ws in the associated subgame)

Lemma 1'. Suppose $\begin{cases} \mathtt{I} \text{ has no ws for } \mathcal{G}_T(A) & T \text{ subgame} \\ U \subseteq A \text{ is open} \end{cases}$

Then there is a $\mathtt{II}$-strategy S relative to the subgame T st

(1) $\mathtt{I}$ has no ws for $\mathcal{G}_S(A)$ and (2) $U \cap [S] = \emptyset$

(3) If U contains no clopen sX^ω with $|s| \leq n$

then one can require $S \cap X^{\leq n} = T \cap X^{\leq n}$

(variant of Lemma 1: subgame relativized + slightly improved)

Determinacy of $\utilde{\mathbf{\Sigma}}^0_3(X^\omega)$ and $\utilde{\mathbf{\Pi}}^0_3(X^\omega)$

Lemma 1. Suppose $\begin{cases} \mathtt{I} \text{ has no ws for } \mathcal{G}(A) \\ U \subseteq A \text{ is open} \end{cases}$

Then there is a $\mathtt{II}$-strategy S such that

(1) $\mathtt{I}$ has no ws for $\mathcal{G}_S(A)$ and (2) $U \cap [S] = \emptyset$

($\mathtt{II}$ has a non-catastrophic strategy to avoid any fixed open set U)

Proof of Lemma 1.

$S =$ defensive $\mathtt{II}$-strategy

$S = \{s \in X^{<\omega} \mid \mathtt{I} \textit{ has no ws in } \mathcal{G}(A_s)\}$

(1): cf. slide 20

(2): Else there is some $s \in T$ such that $sX^\omega \subseteq U$
and any strategy for $\mathtt{I}$ is trivially winning in $\mathcal{G}(A_s)$

Remark. Lemma 1 reproves open determinacy. If A open let $U = A$, the non catastrophic strategy for $\mathtt{II}$ is a winning one

Determinacy of $\underset{\sim}{\mathbf{\Sigma}}^0_3(X^\omega)$ and $\underset{\sim}{\mathbf{\Pi}}^0_3(X^\omega)$

Lemma 2. Suppose $\begin{cases} \mathtt{I} \text{ has no ws for } \mathcal{G}(A) \\ H \subseteq A \text{ is } \underset{\sim}{\mathbf{G}}_\delta \end{cases}$

Then there is a $\mathtt{II}$-strategy S such that
(1) $\mathtt{I}$ has no ws for $\mathcal{G}_S(A)$ and (2) $H \cap [S] = \emptyset$
($\mathtt{II}$ has a non-catastrophic strategy to avoid any fixed $\underset{\sim}{\mathbf{G}}_\delta$ set U)

Remark. Lemma 2 reproves $\underset{\sim}{\mathbf{G}}_\delta$ determinacy. If A is $\underset{\sim}{\mathbf{G}}_\delta$ let

$H = A$, the non catastrophic strategy for $\mathtt{II}$ is a winning one

Lemma 2'. Suppose $\begin{cases} \mathtt{I} \text{ has no ws for } \mathcal{G}_T(A) \quad T \text{ subgame} \\ H \subseteq A \text{ is } \underset{\sim}{\mathbf{G}}_\delta \end{cases}$

Then there is a $\mathtt{II}$-strategy S relative to the subgame T st
(1) $\mathtt{I}$ has no ws for $\mathcal{G}_S(A)$ and (2) $H \cap [S] = \emptyset$
(3) If H contains no clopen sX^ω with $|s| \leq n$
then one can require $S \cap X^{\leq n} = T \cap X^{\leq n}$
(variant of Lemma 2: subgame relativized + slightly improved)

Determinacy of $\underset{\sim}{\mathbf{\Sigma}}^0_3(X^\omega)$ and $\underset{\sim}{\mathbf{\Pi}}^0_3(X^\omega)$

Lemma 2. If $\mathtt{I}$ has no ws for $\mathcal{G}(A)$, $H \subseteq A$ is $\underset{\sim}{\mathbf{G}}_\delta$
Then there is a $\mathtt{II}$-strategy S such that
(1) $\mathtt{I}$ has no ws for $\mathcal{G}_S(A)$ and (2) $H \cap [S] = \emptyset$
($\mathtt{II}$ has a non-catastrophic strategy to avoid any fixed $\underset{\sim}{\mathbf{G}}_\delta$ set U)

Proof of Lemma 2. $H = \bigcap_{i \in \mathbb{N}} C_i X^\omega$
the C_i's antichains of $X^{<\omega}$, $C_{i+1} \subseteq C_i X^{<\omega}$, $C_0 = \{nil\}$

$Z = \{u \in X^{<\omega} \mid \exists$ *$\mathtt{II}$-strategy* $T^{(u)}$ *relative to subgame* $uX^{<\omega}$
st $H \cap [T^{(u)}] = \emptyset$ and $\mathtt{I}$ has no ws for $\mathcal{G}_{T^{(u)}}(A)\}$

We prove
$(*)_i \qquad u \in C_i \setminus Z \Rightarrow$ *I has ws in* $\mathcal{G}((A \cup (C_{i+1} \setminus Z)X^\omega)_u)$

$(*)_i \qquad u \in C_i \setminus Z \Rightarrow I$ *has ws in* $\mathcal{G}((A \cup (C_{i+1} \setminus Z)X^\omega)_u)$

Suppose $(*)_i$ false. Let $u \in C_i \setminus Z$ be st I has ws in $\mathcal{G}((A \cup (C_{i+1} \setminus Z)X^\omega)_u)$.

L.1' yields a II-strategy S relative to the subgame $uX^{<\omega}$ such that $[S] \cap (C_{i+1} \setminus Z)X^\omega = \emptyset$ and I has no ws in $\mathcal{G}_S((A \cup (C_{i+1} \setminus Z)X^\omega)_u)$.

To get a contradiction, we describe a ws strategy for II relative to the subgame $uX^{<\omega}$.

First, II follows S. Since $[S]$ is disjoint from the open set $(C_{i+1} \setminus Z)X^\omega$, while II follows S it does not meet $C_{i+1} \setminus Z$.

If some $v \in C_{i+1}$ is reached then $v \in Z \cap C_{i+1}$ and II switches to its strategy $T^{(v)}$ (cf. definition of Z) st I has no ws in $\mathcal{G}_{T^{(v)}}(A)$ and $[T^{(v)}] \cap H = \emptyset$.
The resulting infinite play either avoids C_{i+1} hence $\notin H$ or meets $Z \cap C_{i+1}$ hence is given by some $T^{(v)}$ and $\notin H$.

Thus, II has a ws relative to the subgame $uX^{<\omega}$. In particular, $u \in Z$, contradicting the hypothesis $u \in C_i \setminus Z$.

To conclude the proof of Lemma 2, we show that nil $\in Z$.

Else, nil $\in X^{<\omega} \setminus Z = C_0 \setminus Z$. We define a strategy for $\mathtt{I}$. Using $(*)_0$ with $u = nil$, $\mathtt{I}$ follows a ws in $\mathcal{G}(A \cup (C_1 \setminus Z)X^\omega)$. If and when the play enters $C_1 \setminus Z$ at u_1 then, applying $(*)_1$ with $u = u_1$, $\mathtt{I}$ switches to a ws in $\mathcal{G}((A \cup (C_2 \setminus Z)X^\omega)_{u_1})$. And so on...
The resulting infinite play
• either does not meet some $(C_{i+1} \setminus Z)X^\omega$ and is given by a ws for $\mathtt{I}$ in $\mathcal{G}((A \cup (C_{i+1} \setminus Z)X^\omega)_{u_i})$. Then it is in A and $\mathtt{I}$ wins.
• or it does meet all $C_{i+1} \setminus Z$ hence all C_j's and is in H hence in A.

Thus, we have obtained a ws for $\mathtt{I}$ in $\mathcal{G}(A)$. Contradicting the hypothesis of Lemma 2.

Proof of $\underset{\sim}{\Sigma}^0_3(X^\omega)$ determinacy

$A = \bigcup_{i\in\mathbb{N}} H_i \quad H_i = \bigcap_{j\in\mathbb{N}} C_{i,j}X^\omega \qquad C_{i,j} \subseteq X^{<\omega}$
$H_i \subseteq H_{i+1}$

$\boxed{\text{Case } X \text{ finite}}$ Suppose $\mathtt{I}$ has no ws in $\mathcal{G}(A)$
We inductively define a decreasing sequence $(T_n)_{n\in\mathbb{N}}$ of non-catastrophic $\mathtt{II}$-strategies which avoid the H_i's

By Lemma 2, get a $\mathtt{II}$-strategy T_0 st
(1) $\mathtt{I}$ has no ws for $\mathcal{G}_{T_0}(A)$ and (2) $H_0 \cap [T_0] = \emptyset$

By Lemma 2', get $\mathtt{II}$-strategy T_1 relative to subgame T_0 st
(1) $\mathtt{I}$ has no ws for $\mathcal{G}_{T_1}(A)$ and (2) $H_2 \cap [T_2] = \emptyset$
And so on...

X being finite, X^ω is compact hence $[T] = \bigcap_{i\in\mathbb{N}}[T_i] \neq \emptyset$
T is a $\mathtt{II}$-strategy st
(1) $\mathtt{I}$ has no ws for $\mathcal{G}_T(A)$ (obvious since $T \subseteq T_0$)
(2) $(\bigcup_{i\in\mathbb{N}} H_i) \cap [T] = \emptyset$

Thus, the intersection T of the T_i's is a ws for $\mathtt{II}$ in $\mathcal{G}_T(A)$

Proof of $\underset{\sim}{\mathbf{\Sigma}}^0_3(X^\omega)$ determinacy

$\boxed{\text{Case } X \text{ is infinite}}$

Consider the interior U of A.

Then $A = U \cup B$ where B is also $\underset{\sim}{\mathbf{G}}_{\delta\sigma}$

B contains no open set.

$B = \bigcup_{i \in \mathbb{N}} H_i$ with H_i $\underset{\sim}{\mathbf{G}}_{\delta}$

Up to subsequence extraction, can suppose H_i contains no sX^ω with $|s| \leq i$

Use condition (3) in Lemma 2' to get T_{i+1} such that $T_{i+1} \cap X^{\leq i} = T_i \cap X^{\leq i}$.

Then $\bigcap_{i \in \mathbb{N}} [T_i] \neq \emptyset$

Donald Martin's proof of Borel determinacy

Main idea of the proof

In usual reduction theories one looks for a *hard* set A which reduces every set Z in a particular family $\mathcal{F}$:

$$\forall Z \in \mathcal{F} \quad Z = f^{-1}(A) \quad \text{for some } f$$

(f computable or polytime or continuous)

In Martin's proof, the association is reversed

For every Borel set $A \subseteq [S] \subseteq X^\omega$ $\quad$ S pruned tree

Martin's proof looks for

> a space Y^ω, a pruned tree $T \subseteq X^{<\omega}$ (possibly huge Y)
> a clopen subset C of $[T] \subseteq Y^\omega$ (very simple set)
> a continuous surjective map $\pi : [T] \to [S]$ (the reduction map)

such that

1. $C = \pi^{-1}(A)$ (A is Borel whereas C is clopen)
2. every winning strategy in $\mathcal{G}_T(C)$ yields a ws in $\mathcal{G}_S(A)$

No direct extension beyond Borel sets

Case $X = \omega^\omega$

Suppose $C = \pi^{-1}(A)$, π continuous and C clopen
Then $A = \pi(C)$ and $\omega^\omega \setminus A = \pi(F \setminus C)$

In general, $\pi(\mathit{closed})$ has descriptive complexity $\boldsymbol{\Sigma}^1_1$
Thus, $\pi(C)$ and $\pi(F \setminus C)$ are $\boldsymbol{\Sigma}^1_1$
Thus, A and $\omega^\omega \setminus A$ are $\boldsymbol{\Sigma}^1_1$,
i.e. A is $\boldsymbol{\Delta}^1_1$ hence is Borel (Suslin's theorem, 1917)

À la Martin reductions in pure topology

(Almost) forget strategies. Topological problem

For every Borel set $A \subseteq [S] \subseteq X^\omega$ S pruned tree

$$\text{look for} \left| \begin{array}{l} \text{a topological space } Y^\omega, \text{ a tree } T \subseteq Y^{<\omega} \\ \text{a clopen } C \subseteq [T] \qquad ([T] = \text{set of infinite branches}) \\ \text{a continuous surjective map } \pi : [T] \to X^\omega \end{array} \right.$$

such that $C = \pi^{-1}(A)$

- Trivial if we do not ask for a topological space Y^ω :

  Set $Y = X$ and increase the topology so that A is clopen

- Trivial if surjectivity is omitted: (case $A \neq \emptyset, X^\omega$)

$$\text{let} \begin{cases} a \in A \\ b \in F \setminus A \end{cases}, \ [T] \text{ clopen} \neq \emptyset, Y^\omega, \ \pi(x) = \begin{cases} a \textit{ if } x \in C \\ b \textit{ if } x \notin C \end{cases}$$

- If X is finite, Y has to be infinite else $\pi(\textit{clopen})$ is compact

- If $X = \omega$, **true by Wadge hardness theorem**

A Borel not $\utilde{\boldsymbol{\Sigma}}^0_\xi$ implies every $\utilde{\boldsymbol{\Sigma}}^0_\xi$ in ω^ω is $\pi^{-1}(A)$ for some π

Vicious circle: Wadge theory relies on Borel determinacy!!!

A combinatorico-topological problem: *drills before entering Martin's proof*

Extend the problem to allow inductive constructions and future strategy requirements

For every tree $S \subseteq X^{<\omega}$, ($[S]=$ set of infinite branches)

for every $k \in \mathbb{N}$ (technical point: k is a trick to cope with non compactness in inductive constructions)

for every Borel set $A \subseteq [S] \subseteq X^{\omega}$ look for

> a topological space Y^{ω} where $Y \supseteq X$
> a tree $T \subseteq Y^{<\omega}$ such that $T \cap Y^{\leq k} = S \cap X^{\leq k}$
> a clopen $C \subseteq [T]$ ($[T]=$ set of infinite branches)
> a monotone length preserving surjective map $\pi : T \to S$
> (alphabetical transduction)

such that $\boxed{C = \pi^{-1}(A)}$

where $\pi : [T] \to [S]$ obvious extension of π

Topological problem: A open in $[S]$, $k = 0$

A open in $[S]$ hence $A = \bigcup_{u\in\tau} uX^\omega \cap [S] \qquad \tau \subseteq X^{<\omega} \setminus \{nil\}$

- Add elements representing the u's: $Y = X \cup \{\ulcorner u \urcorner \mid u \in \tau\}$
  To preserve length, a new element is always the first one
  if $u = x_0 x_1 \ldots x_{n-1}$ then $\ulcorner u \urcorner$ has unique length $< n$ successors

$$\begin{aligned}
\widetilde{\tau} &= \{\ulcorner u \urcorner x_1 \ldots x_{n-1} \mid u = x_0 x_1 \ldots x_{n-1} \in \tau\} \quad \textit{antichain of } Y^{<\omega} \\
T &= (S \setminus \tau X^{<\omega}) \cup \{\ulcorner u \urcorner x_1 \ldots x_i \mid u = x_0 x_1 \ldots x_{n-1} \in \tau \\
&\qquad \wedge (u \leq_{\text{pref}} x_0 \ldots x_i \textit{ or } x_0 \ldots x_i \leq_{\text{pref}} u) \\
&\qquad \wedge x_0 \ldots x_i \in S\} \\
C &= [\widetilde{\tau} X^{<\omega}] = \{(y_i)_{i\in\mathbb{N}} \in [T] \mid y_0 \in Y \setminus X\} \quad \boxed{[C] \textit{ clopen in } [T]}
\end{aligned}$$

(a condition on the sole first component defines a clopen set)

- $\pi : T \to S$ st $\quad \pi(s) = s \qquad$ if $s \in S \setminus \tau X^{<\omega}$
  $\pi(\ulcorner u \urcorner x_1 \ldots x_i) = x_0 x_1 \ldots x_i \qquad$ if $u = x_0 x_1 \ldots x_{n-1} \in \tau$

$\pi : T \to S$	alphabetical
$\pi : [T] \to [S]$	bijective, continuous but not homeomorphism
$\pi^{-1}(A) = [C]$	Y has the cardinality of X

Topological problem: A open in $[S]$, any k

A open in $[S]$ hence $A = \bigcup_{u \in \tau} uX^\omega \cap [S]$

Choose antichain $\tau \subseteq X^{<\omega}$
st every u in τ has length $> k$ and set

$$\begin{array}{rcl} \widetilde{\tau} & = & \{x_0 \ldots x_{k-1} \ulcorner u \urcorner x_{k+1} \ldots x_n \mid u = x_k \ldots x_n \in \tau\} \\ T & = & (S \setminus \tau X^{<\omega}) \\ & & \cup \; \{x_0 \ldots x_{k-1} \ulcorner u \urcorner x_{k+1} \ldots x_i \mid u = x_0 \ldots x_n \in \tau \\ & & \qquad \wedge \; (u \leq_{\text{pref}} x_{k+1} \ldots x_i \textit{ or } x_{k+1} \ldots x_i \leq_{\text{pref}} u)\} \\ & & \qquad \wedge \; x_0 \ldots x_{k-1} x_k x_{k+1} \ldots x_n \in S\} \\ C & = & [\widetilde{\tau} X^{<\omega}] = \{(y_i)_{i \in \mathbb{N}} \in [T] \mid y_k \in Y \setminus X\} \quad \boxed{[C] \textit{ clopen in } [T]} \end{array}$$

(a condition on the sole k-th component defines a clopen set)

Then argue as in the case $k = 0$

The topological problem: induction step

Suppose the problem has positive answer for all levels $< \xi$ of the Borel hierarchy over X^ω. We get positive answer for level ξ

Let $A = \bigcup_{n\in\mathbb{N}} A_n$ where the A_n's have Borel ranks $< \xi$

- Applying the induction hypothesis with k
$C_0 = \pi_0^{-1}(A_0)$ for $Y_0 \supseteq X$, $T_0 \subseteq Y_0^{<\omega}$, clopen C_0 of $[T_0]$,
$\pi_0 : [T_0] \to [S]$, $\quad T_0 \cap Y_0^{\leq k} = S \cap X^{\leq k}$

- In $[T_0]$, Borel rank of $\pi_0^{-1}(A_1)$ is $\leq$ rank A_1 in $[S]$ hence $< \xi$
Applying the induction hypothesis with $k+1$
$C_1 = \pi_1^{-1}(\pi_0^{-1}(A_1))$ for $Y_1 \supseteq Y_0$, $T_1 \subseteq Y_1^{<\omega}$, clopen C_1 of $[T_1]$, $\pi_1 : [T_1] \to [T_0]$, $\quad T_1 \cap Y_1^{\leq k+1} = T_0 \cap Y_0^{\leq k+1}$

- In $[T_1]$, Borel rank of $(\pi_0 \circ \pi_1)^{-1}(A_2)$ is $\leq$ rank A_2 in $[S] < \xi$
Applying the induction hypothesis with $k+2$
$C_2 = \pi_2^{-1}((\pi_1 \circ \pi_0)^{-1}(A_2))$ for $Y_2 \supseteq Y_1$, $T_2 \subseteq Y_2^{<\omega}$, clopen C_2 of $[T_2]$, $\pi_1 : [T_2] \to [T_1]$, $\quad T_2 \cap Y_2^{\leq k+2} = T_1 \cap Y_1^{\leq k+2}$

- and so on $\cdots$

The topological problem: induction step

$$\cdots \longrightarrow [T_{i+1}] \xrightarrow{\pi_{i+1}} [T_i] \longrightarrow \cdots T_1 \xrightarrow{\pi_1} [T_0] \xrightarrow{\pi_0} [S]$$

$$T_{i+1} \cap Y_{i+1}^{\leq k+i+1} = T_i \cap Y_i^{\leq k+i+1} \qquad T_0 \cap Y_0^{\leq k} = S \cap X^{\leq k}$$

Consider the inverse limit (No cardinal explosion here)

$$\overleftarrow{Y} = \bigcup_{i \in \mathbb{N}} Y_i \qquad \overleftarrow{T} = \{u \mid u \in T_i \textit{ for all } i \geq |u|\}$$

$$\overleftarrow{\pi_i} : [\overleftarrow{T}] \to [T_i] \qquad \overleftarrow{\pi} : [\overleftarrow{T}] \to [S]$$

where $\overleftarrow{\pi_i} \restriction T_j = \pi_{j+1} \circ \cdots \circ \pi_i$ for $j > i$
$\pi_{i+1} \circ \overleftarrow{\pi_{i+1}} = \overleftarrow{\pi_i}$ and $\pi_0 \circ \overleftarrow{\pi_0} = \overleftarrow{\pi}$

Since $\pi_i^{-1}(A_i)$ is clopen in $[T_i]$, $\overleftarrow{\pi}^{-1}(A_i)$ is clopen in $[\overleftarrow{T}]$

Thus, $\overleftarrow{\pi}^{-1}(A) = \bigcup_{i \in \mathbb{N}} \overleftarrow{\pi}^{-1}(A_i)$ is open in $[\overleftarrow{T}]$.

Apply the open case to get
- set Y_ω, tree $S_\omega \subseteq Y_\omega^{<\omega}$
- clopen subset C_ω of $[T_\omega]$
- onto map $\pi_\omega : [T_\omega] \to [\overleftarrow{T}]$

st $C_\omega = \pi_\omega^{-1}(\overleftarrow{\pi}^{-1}(A))$. Finally, $C_\omega = (\overleftarrow{\pi} \circ \pi_\omega)^{-1}(A)$

The topological problem: end of proof

The family of sets $A \subseteq X^\omega$ for which the problem has a solution (Y, S, C, π)

- contains the open subsets of X^ω
- is closed under countable unions
- is (trivially) closed under complementation

$$\Downarrow$$

Topological problem solved

for all Borel subsets of X^ω

There is no cardinal explosion: Y has cardinality of X

Martin's proof: Covering of a pruned tree

$\mathit{Strat}_{\mathtt{I}}(S)$ is the set of non deterministic $\mathtt{I}$-strategies where both players have to stay in the pruned tree S

k-covering of a pruned tree $S \subseteq X^{<\omega}$

$$
= \left|\begin{array}{ll}
\text{pruned tree} & T \subseteq Y^{<\omega} \\
\text{monotone length preserving surjective map} & \pi : T \to S \\
\text{map} & \phi_{\mathtt{I}} : \mathit{Strat}_{\mathtt{I}}(T) \to \mathit{Strat}_{\mathtt{I}}(S) \\
\text{map} & \phi_{\mathtt{I}} : \mathit{Strat}_{\mathtt{II}}(T) \to \mathit{Strat}_{\mathtt{II}}(S)
\end{array}\right.
$$

such that

(1) $Y^{\leq 2k} \cap T = X^{\leq 2k} \cap S$ $\quad$ ($2k = k$ moves of $\mathtt{I}$ $+$ k moves of $\mathtt{II}$)

(2) $\phi_{\mathtt{I}} : \mathit{Strat}_{\mathtt{I}}(T) \to \mathit{Strat}_{\mathtt{I}}(S)$ and $\phi_{\mathtt{II}}$ are local:

$$\forall \beta \in \mathit{Strat}_{\mathtt{I}}(T)\ \forall u \in S \quad \phi_{\mathtt{I}}(\beta)(u) \text{ depends on } \beta \restriction \{v \mid |v| \leq |u|\}$$

(3) Plays in S where $\mathtt{I}$ follows $\phi_{\mathtt{I}}(\beta)$ can be lifted
to plays in T where $\mathtt{I}$ follows β *Idem with $\mathtt{II}$ and $\phi_{\mathtt{II}}$*

$$\forall \beta \in \mathit{Strat}_{\mathtt{I}}(T)\ \forall f \in [S]\ \exists g \in [T]$$
$$(f \in [\phi_{\mathtt{I}}(\beta)] \implies (g \in [\beta] \ \wedge\ \pi(g) = f)$$

Martin's proof: unravelling and determinacy

$S \subseteq X^{<\omega}$ pruned tree and $A \subseteq [S]$

k-covering $(Y, T, \pi, \phi_{\mathtt{I}}, \phi_{\mathtt{II}})$ of S
unravels $A \subseteq [S]$
if $\pi^{-1}(A)$ is clopen in $[T]$

Fact. If some covering unravels $A \subseteq [S]$
then the game $\mathcal{G}_S(A)$ is determined

Proof. The clopen game $\mathcal{G}_T(\pi^{-1}(A))$ is determined
Let β be a ws for $\mathtt{I}$ (same argument with a ws for $\mathtt{II}$)
Lift any infinite play f in the S-game where $\mathtt{I}$ follows $\phi_{\mathtt{I}}(\beta)$
to an infinite play g in the T-game where $\mathtt{I}$ follows β
Since β is a ws for $\mathtt{I}$ in $\mathcal{G}_T(\pi^{-1}(A))$, we have $g \in \pi^{-1}(A)$
Since $\pi(g) = f$ we have $f \in A$.
Thus, $\phi_{\mathtt{I}}(\beta)$ is a ws for $\mathtt{I}$ in $\mathcal{G}_S(A)$

Martin's proof: unravelling closed sets

Space where we play: pruned tree $S \subseteq X^{<\omega}$

Game $\mathcal{G}_T(A)$ closed set $A = [J] \subseteq [S]$ J pruned subtree of S

Copy of the set $(X^2)^{<\omega}$: $E = \{\ulcorner u \urcorner \mid u \in X^{<\omega}, |u| \textit{ even}\}$

$\boxed{k\text{-covering to unravel } A}$

$$Y = X \cup Y_{\mathrm{I}} \cup Y_{\mathrm{II}}^{+} \cup Y_{\mathrm{II}}^{-}$$

$$\begin{cases} Y_{\mathrm{I}} & = X \times \times \mathit{Strat}_{\mathrm{I}}(S) \\ Y_{\mathrm{II}}^{+} & = \bigcup_{\alpha \in \mathit{Strat}_{\mathrm{I}}(S)} X \times \mathit{Strat}_{\mathrm{II}}(\alpha) \\ Y_{\mathrm{II}}^{-} & = X \times E \end{cases}$$

$\widetilde{T} = \textit{prefixes of } \; X^{2k} \times Y_{\mathrm{I}} \times (Y_{\mathrm{II}}^{+} \cup Y_{\mathrm{II}}^{-}) \times X^{<\omega}$

(Only moves y_{2k} and y_{2k+1} are not in X)

$T = \textit{sequences in } \widetilde{T} \textit{ such that}\ldots$ *(see next slide)*

$\boxed{k\text{-covering to unravel } A}$ $\quad A = [J]$ with J subtree of S

$$Y = X \cup Y_{\mathrm{I}} \cup Y_{\mathrm{II}}^{+} \cup Y_{\mathrm{II}}^{-} \quad \begin{cases} Y_{\mathrm{I}} & = & X \times \times Strat_{\mathrm{I}}(S) \\ Y_{\mathrm{II}}^{+} & = & \bigcup_{\alpha} X \times Strat_{\mathrm{II}}(\alpha) \\ Y_{\mathrm{II}}^{-} & = & X \times E \end{cases}$$

$$\widetilde{T} = \textit{prefixes of} \quad X^{2k} \times Y_{\mathrm{I}} \times (Y_{\mathrm{II}}^{+} \cup Y_{\mathrm{II}}^{-}) \times X^{<\omega}$$

(Only moves y_{2k} and y_{2k+1} are not in X)

$$T = \textit{sequences in } \widetilde{T} \textit{ such that}$$

(1) *if I chooses (x_{2k}, σ_I) then I follows σ_I afterwards*

(2) *if II chooses (x_{2k+1}, σ_{II}) then σ_{II} is a subtree of J and of σ_I and II follows σ_{II} afterwards*

(3) *if II chooses $(x_{2k+1}, \ulcorner u \urcorner)$ then $|u|$ even, $x_0 \ldots x_{2k+1} u \in \sigma_I \setminus J$ and every extension of $x_0 \ldots x_{2k+1}$ in T is compatible with $x_0 \ldots x_{2k+1} u$ (thus, T forces the players to play u after $x_0 \ldots x_{2k+1}$)*

(The infinite play is in A in case (2) and outside A in case (3))

$\pi : T \to S$ is the obvious map such that

$$\pi\left(x_0 \ldots x_{2k-1}(x_{2k}, \sigma_{\mathrm{I}})(x_{2k+1}, \sigma_{\mathrm{II}} \text{ or } \ulcorner u \urcorner)\, x_{2k+2} \ldots x_n\right) = x_0 \ldots x_n$$

ϕ_{I} *and* ϕ_{II} … *see next slides*

$\boxed{\phi_{\mathtt{I}}}$ $\quad \beta$ strategy for $\mathtt{I}$ in $T \quad \rightsquigarrow \quad \phi_{\mathtt{I}}(\beta)$ strategy for $\mathtt{I}$ in S

Suppose $\mathtt{II}$ plays $x_1, x_3, \ldots$ in the S-game

• For its k first moves $x_0, x_2, \ldots, x_{2k-2}$ in the S-game, $\phi_{\mathtt{I}}(\beta)$ tells $\mathtt{I}$ to follow what strategy β does in the T-game.

• If strategy β in the T-game tells $\mathtt{I}$ to play $(x_{2k}, \sigma_{\mathtt{I}})$ then strategy $\phi_{\mathtt{I}}(\beta)$ in the S-game tells $\mathtt{I}$ to play x_{2k}

• After $\mathtt{II}$ has played x_{2k+1} in the S-game, player $\mathtt{I}$ has to imagine a corresponding move $(x_{2k+1}, ?)$ in the T-game

$\boxed{\phi_{\mathtt{I}} \text{ Case 1. } I \text{ has a ws } \alpha \text{ in } \mathcal{G}_{\widetilde{\sigma_I}}([\sigma_I] \setminus A)}$

$$\widetilde{\sigma}_{\mathtt{I}} = \{v \in \sigma_{\mathtt{I}} \mid x \textit{ compatible with } x_0 \cdots x_{2k+1}\}$$

$\phi_{\mathtt{I}}(\beta)$ tells $\mathtt{I}$ to follow this strategy α

At some step the play is $x_0 \cdots x_{2k+1} u$ in the open set $[\sigma_{\mathtt{I}}] \setminus A$,

Then $L = x_0 \cdots x_{2k-1}(x_{2k}, \sigma_{\mathtt{I}})(x_{2k+1}, \ulcorner u \urcorner) u \in T$

From now on, $\phi_{\mathtt{I}}(\beta)$ in the S-game tells $\mathtt{I}$ to follow what β tells for a play extending L (in the T-game)

The lifting property holds

$\boxed{\phi_{\mathtt{I}} \text{ } \textit{Case 2. II has a ws in } \mathcal{G}_{\widetilde{\sigma_I}}([\sigma_I] \setminus A)}$

$$\widetilde{\sigma_{\mathtt{I}}} = \{v \in \sigma_{\mathtt{I}} \mid x \textit{ compatible with } x_0 \cdots x_{2k+1}\}$$

Let δ be the defensive strategy of $\mathtt{II}$ which allows him to stay in the closed set $[\sigma_{\mathtt{I}}] \cap A$

As long as $\mathtt{II}$ plays in δ, strategy $\phi_{\mathtt{I}}(\beta)$ tells $\mathtt{I}$ to follow strategy β assuming that $\mathtt{I}$ has played $(x_{2k}, \sigma_{\mathtt{I}})$ and $\mathtt{II}$ has played (x_{2k+1}, δ) in the T-game

If $\mathtt{II}$ leaves his defensive strategy δ at play $v = x_0 \cdots x_n$ then $\mathtt{I}$ gets a ws (for the subtree of $\sigma_{\mathtt{I}}$ of sequences compatible with v) and we can argue as in Case 1.

The lifting property holds

$\boxed{\phi_{\mathtt{II}}}$ $\quad \beta$ strat. for $\mathtt{II}$ in $T \quad \rightsquigarrow \quad \phi_{\mathtt{II}}(\beta)$ strat. for $\mathtt{II}$ in S

Suppose $\mathtt{I}$ plays $x_0, x_1, \ldots$ in the S-game

• For its k first moves $x_1, x_3, \ldots, x_{2k-1}$ in the S-game, $\phi_{\mathtt{II}}(\beta)$ tells $\mathtt{II}$ to follow what strategy β tells in the T-game

• After $\mathtt{I}$ has played x_{2k} in the S-game. Player $\mathtt{II}$ has to imagine a corresponding move $(x_{2k}, \sigma_{\mathtt{I}})$ in the T-game

$Z =$ set of $x_{2k+1}u$ st $|u|$ even and there is $\mathtt{I}$-strat. $\sigma_{\mathtt{I}}$ in the S-game st β tells $\mathtt{II}$ to play $(x_{2k+1}, \ulcorner u \urcorner)$ if $\mathtt{I}$ plays $(x_{2k}, \sigma_{\mathtt{I}})$

Consider the $(S \cap (x_1 \cdots x_{2k})X^{<\omega})$-game where $\mathtt{II}$ wins if the infinite play is in $U = S \cap (ZX^{<\omega})$

$\boxed{\phi_{\mathtt{II}} \textit{ Case 1. II has a ws in this game}}$

$\phi_{\mathtt{II}}(\beta)$ tells $\mathtt{II}$ to follow this strategy until the play enters U, say at u. Let $\sigma_{\mathtt{I}}$ witness that $u \in U$.
Afterwards, $\phi_{\mathtt{II}}(\beta)$ tells $\mathtt{II}$ to follow β on the T-game where the special moves are $(x_{2k}, \sigma_{\mathtt{I}})$, $(x_{2k+1}, \ulcorner u \urcorner)$
The lifting property holds

$\boxed{\phi_{\mathtt{II}}\ \textit{Case 2. I has a ws in this game}}$

Let δ be the defensive strategy of $\mathtt{I}$ which allows him to put the play in the closed set
If $\mathtt{I}$ plays (x_{2k}, δ) then β cannot ask $\mathtt{II}$ to play $(x_{2k+1}, \ulcorner u \urcorner)$.
Else $x_0 \cdots x_{2k+1} u \in U$ contradicting the fact that $\mathtt{II}$ cannot leave the defensive $\mathtt{I}$-strategy δ
Thus, β asks $\mathtt{II}$ to play some $(x_{2k+1}, \sigma_{\mathtt{II}})$ in the T-game

As long as $\mathtt{I}$ plays in $\sigma_{\mathtt{II}}$, strategy $\phi_{\mathtt{II}}(\beta)$ tells $\mathtt{II}$ to follow strategy β assuming that $\mathtt{I}$ has played (x_{2k}, δ)

If $\mathtt{I}$ leaves his defensive strategy δ then $\mathtt{II}$ has a ws and we can argue as in Case 1.

The lifting property holds

Martin's proof: inverse limits of coverings

Fact. If $(T_{i+1}, \pi_{i+1}, \phi^{\mathtt{I}}_{i+1}, \phi^{\mathtt{II}}_{i+1})$ is a $(k+i)$-covering of T_i for $i \in \mathbb{N}$

Then there are $\left|\begin{array}{l}\text{a pruned tree } T_\infty \\ \text{maps } \pi_{\infty,i},\ \phi^{\mathtt{I}}_{\infty,i},\ \phi^{\mathtt{II}}_{\infty,i}\end{array}\right.$

such that

$$\begin{cases} (T_\infty, \pi_{\infty,i}, \phi^{\mathtt{I}}_{\infty,i}, \phi^{\mathtt{II}}_{\infty,i}) \text{ is a } (k+i)\text{-covering of } T_i \\ \pi_{i+1} \circ \pi_{\infty,i+1} = \pi_{\infty,i} \\ \phi^{\mathtt{I}}_{i+1} \circ \phi^{\mathtt{I}}_{\infty,i+1} = \phi^{\mathtt{I}}_{\infty,i} \\ \phi^{\mathtt{II}}_{i+1} \circ \phi^{\mathtt{II}}_{\infty,i+1} = \phi^{\mathtt{II}}_{\infty,i} \end{cases}$$

Proof.

$T_\infty =$ the u's such that $u \in T_i$ for all i st $|u| \le 2(k+i)$

If $|u| \le 2(k+i)$ then $\pi_{\infty,i}(u) = u$ else

$\pi_{\infty,i}(u) = \pi_{i+1} \circ \cdots \circ \pi_j(u)$ for any j st $|u| \le 2(k+j)$

$\phi^{\mathtt{I}}_{\infty,i}$, $\phi^{\mathtt{II}}_{\infty,i}$: Similar because $\phi^{\mathtt{I}}_i$ is local:

$\phi^{\mathtt{I}}_i(\beta) \restriction (S \cap X^{\le i})$ depends only on $\beta \restriction (T \cap Y^{\le i})$

Lifting property Suppose $\left| \begin{array}{l} \beta_\infty \text{ I-strategy in the } T_\infty\text{-game} \\ f \in [\phi^{\mathrm{I}}_{\infty,i}(\beta_\infty)] \subseteq [T_i] \end{array} \right.$

Lift f to f_{i+1} with π_{i+1}, then to f_{i+2} with π_{i+2}, and so on...

Since $f_j \restriction 2(k+i) = f_i \restriction 2(k+i)$ for $j \geq i$
the f_i's converge to f_∞ such that $f_\infty \restriction 2(k+i) = f_i \restriction 2(k+i)$

$\pi_{\infty,i}(f_\infty) = f_i$

Martin's proof completed

Closed sets are unravelled

Unravelling is closed under complementation

Unravelling is closed under countable unions
(use i-unravelling for the i-th set)

Conclusion: every Borel set can be unravelled
hence is determined (cf. page 56)

Thank you for your attention