Borel Determinacy

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Alternations of quantifier

versus

Games
Alternations of quantifier

\[ F(\vec{z}) \equiv \exists x_0 \ \forall x_1 \ \exists x_2 \ \forall x_3 \ \exists x_4 \ P(x_0, \ldots, x_4, \vec{z}) \]

The human mind seems limited in its ability to understand and visualize beyond four or five alternations of quantifier. Indeed, it can be argued that the inventions, subtheories and central lemmas of various parts of mathematics are devices for assisting the mind in dealing with one or two additional alternations of quantifier.

Hartley Rogers

“Theory of recursive functions and effective computability”

(1967) (cf. page 322 §14.7)

Another (partial) explanation:

\[ \text{complexity} \geq \Sigma^0_4(\omega^\omega) \sim \text{Higher set theory!!!} \]
Alternations of quantifier versus Games

\[ F(\vec{z}) \equiv \exists x_0 \ \forall x_1 \ \exists x_2 \ \forall x_3 \ \exists x_4 \ P(x_0, \ldots, x_4, \vec{z}) \]

Roland Fraïssé’s idea (1954)

Relate \( F(\vec{z}) \) to a game

Two players

I

and

II

Who wins?

I wins iff \( P(x_0, \ldots, x_4, \vec{z}) \) holds

\[ F(\vec{z}) \iff \text{player I has a winning strategy} \]
### Strategies

<table>
<thead>
<tr>
<th>Move</th>
<th>Player</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>I</td>
<td>plays some $x_0$</td>
</tr>
<tr>
<td>1</td>
<td>II</td>
<td>plays some $x_1$</td>
</tr>
<tr>
<td>2</td>
<td>I</td>
<td>plays some $x_2$</td>
</tr>
<tr>
<td>3</td>
<td>II</td>
<td>plays some $x_3$</td>
</tr>
<tr>
<td>4</td>
<td>I</td>
<td>plays some $x_4$</td>
</tr>
</tbody>
</table>

The $x_i$’s in $X$ $\Rightarrow$ I wins $\iff P(x_0, \ldots, x_4, \vec{z})$

Strategy for $I = \sigma_I : \{nil\} \cup X \cup X^2 \rightarrow X$

Strategy for $II = \sigma_I : X \cup X^2 \rightarrow X$

I follows strategy $\sigma_I$ if

$$
\begin{align*}
&x_0 = \sigma_I(nil) \\
&x_2 = \sigma_I(x_1) \\
&x_4 = \sigma_I(x_1, x_3)
\end{align*}
$$

II follows strategy $\sigma_{II}$ if

$$
\begin{align*}
&x_1 = \sigma_{II}(x_0) \\
&x_3 = \sigma_{II}(x_0, x_1)
\end{align*}
$$

Winning strategy: ALWAYS wins
Alternations of quantifier and games

\[ F(\vec{z}) \equiv \exists x_0 \ \forall x_1 \ \exists x_2 \ \forall x_3 \ \exists x_4 \ P(x_0, \ldots, x_4, \vec{z}) \]
\[ \equiv \text{player I has a winning strategy} \]
\[ \text{for the game where I wins} \]
\[ \text{if } (x_0, \ldots, x_4) \in P \]

\[ \neg F(\vec{z}) \equiv \forall x_0 \ \exists x_1 \ \forall x_2 \ \exists x_3 \ \forall x_4 \ \neg P(x_0, \ldots, x_4, \vec{z}) \]
\[ \equiv \text{player II has a winning strategy} \]
\[ \text{for the game where I wins} \]
\[ \text{if } (x_0, \ldots, x_4) \in P \]

Law of Excluded Middle: either \( F(\vec{z}) \) or \( \neg F(\vec{z}) \)

Hence either \( I \) has a winning strategy
or \( II \) has a winning strategy
Infinitely many alternations of quantifier

\[ \exists x_0 \, \forall x_1 \, \exists x_2 \, \forall x_3 \ldots \]

Moschovakis’ game quantifier \( \exists \alpha \, P(\alpha, \vec{z}) \)

\[ \forall x_0 \, \exists x_1 \, \forall x_2 \, \exists x_3 \ldots \]

\[ \neg P((x_i)_{i \in \mathbb{N}}, \vec{z}) \]

What does this mean? Infinite game

Two players I and II

<table>
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<tr>
<th>Move 2i</th>
<th>I plays ( x_{2i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Move 2i + 1</td>
<td>II plays ( x_{2i+1} )</td>
</tr>
</tbody>
</table>

Rule

I wins iff \( (x_i)_{i \in \mathbb{N}} \in A \)

where \( A = \{ (x_i)_{i \in \mathbb{N}} \mid P((x_i)_{i \in \mathbb{N}}, \vec{z}) \} \)
Two players I and II

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Rule

\(I\) wins iff

\[P((x_i)_{i \in \mathbb{N}}, \vec{z})\]

Strategy for I

\[\sigma_I : X^{< \omega} \rightarrow X\]

Strategy for II

\[\sigma_{II} : (X^{< \omega} \setminus \{nil\}) \rightarrow X\]

I follows \(\sigma_I\) if \(\forall i \in \mathbb{N} \ x_{2i} = \sigma_I((x_{2j+1})_{j < i})\)

II follows \(\sigma_{II}\) if \(\forall i \in \mathbb{N} \ x_{2i+1} = \sigma_{II}((x_{2j})_{j \leq i})\)

Winning strategy: ALWAYS wins

\[\exists x_0 \ \forall x_1 \ \exists x_2 \ \forall x_3 \ldots \ P((x_i)_{i \in \mathbb{N}}, \vec{z})\]

\[\equiv I \text{ has a winning strategy}\]

\[\forall x_0 \ \exists x_1 \ \forall x_2 \ \exists x_3 \ldots \ \neg P((x_i)_{i \in \mathbb{N}}, \vec{z})\]

\[\equiv II \text{ has a winning strategy}\]

Need \(X\) well-ordered or Axiom of dependent choices
Excluded middle and determinacy

\[ \exists x_0 \ \forall x_1 \ \exists x_2 \ \forall x_3 \ldots \ P((x_i)_{i \in \mathbb{N}}, \vec{z}) \]
\[ \equiv \ I \ has \ a \ winning \ strategy \]

\[ \forall x_0 \ \exists x_1 \ \forall x_2 \ \exists x_3 \ldots \ \neg P((x_i)_{i \in \mathbb{N}}, \vec{z}) \]
\[ \equiv \ II \ has \ a \ winning \ strategy \]

**Fact.**
\[
\begin{cases}
\neg (\exists x_0 \ \forall x_1 \ \exists x_2 \ \forall x_3 \ldots P((x_i)_{i \in \mathbb{N}}, \vec{z})) \\
\forall x_0 \ \exists x_1 \ \forall x_2 \ \exists x_3 \ldots \neg P((x_i)_{i \in \mathbb{N}}, \vec{z})
\end{cases}
\]

is equivalent to

if and only the game is determined

(one of the players has a winning strategy)
Which sets are determined?
Countable sets are determined

Infinite game $G(A)$

Two players $I$ and $II$

| move $2i$ : $I$ plays $x_{2i}$ | Rule |
| move $2i + 1$ : $II$ plays $x_{2i+1}$ | $I$ wins iff $(x_i)_{i \in \mathbb{N}} \in A$ |

Fact. If $A \subset X^\omega$ is countable then $II$ has a winning strategy in $G(A)$

Proof. Diagonal argument. If $A = \{ f_i \mid i \in \mathbb{N} \}$, $2i + 1$ player $II$ plays $x_{2i+1} \neq f_i(2i + 1)$

Are all sets determined?

NO (requires the axiom of choice)
Borel subsets of $X^\omega$

Discrete topology on $X$  
Product topology on $X^\omega$
metrics $d(f, g) = 2^{-\max\{n | \forall i < n \ f(i) = g(i)\}}$

Basis of clopen sets: the $uX^\omega$ for $u \in X^{<\omega}$

Care. if $X$ uncountable, open sets may be unions of uncountably many clopen sets

But metrizability implies closed set are $G_\delta$ in $X^{<\omega}$

($G_\delta = \text{intersection of countably many open sets}$)

This allows for the usual definition of Borel sets

$\Sigma_1^0(X^\omega) = \text{open sets}$

$\Sigma_\alpha^0(X^\omega) = \text{countable unions of sets in } \bigcup_{\beta < \alpha} \Pi_\beta^0(X^\omega)$

$\Pi_\alpha^0(X^\omega) = \text{complements of sets in } \Sigma_\alpha^0(X^\omega)$
Borel determinacy

**Theorem.** (Donald Martin, 1975)
All Borel subsets of $X^\omega$ are determined
(whatever big is $X$)

**Find simple winning strategies in $G(A)$?**
Alas... best (general) complexity is $\Delta_{2}^{1,S}$
if $A$ is Borel with code $S$

**Upper bound proof.** The set of ws for $I$ is $\Pi_{1}^{1,S}$:
$$\sigma_{I} \text{ is ws } \equiv \forall g \sigma_{I} \star g \in A$$
and every $\Pi_{1}^{1,S}$ family contains some $\Delta_{2}^{1,S}$ set

(cf. Rogers §16.7 Coro. XLV(c), p. 430)
Determinacy in classical mathematics

• 1953, Gale & Stewart  Boolean combinations of open subsets of $X^\omega$

• 1955, Philip Wolfe   $\Sigma^0_2(X^\omega)$ and $\Pi^0_2(X^\omega)$ sets

• 1964, Morton Davis  $\Sigma^0_3(X^\omega)$ and $\Pi^0_3(X^\omega)$ sets

Results proved in 2d-order arithmetic

$\equiv$ mathematics of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$

$\equiv$ classical set theory for mathematicians

(with $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ one can encode reals, continuous functions, . . . )
Determinacy in higher set theory

- 1970, Donald Martin $\Sigma^1_1(\omega^\omega)$ in ZF + large cardinal axiom
  $\Sigma^1_1(X^\omega)$ in ZF + stronger large cardinal axiom

- 1972, Jeff B. Paris $\Sigma^0_4(X^\omega)$ and $\Pi^0_4(X^\omega)$ sets in ZF
  (set theory with cardinal $(2^{\aleph_0})^+$ is enough hence 3rd-order arithmetic is enough)

- 1975, Donald Martin Borel subsets of $X^\omega$ in ZF
- 1985, Donald Martin Much simpler proof (by far... ) in ZF

Higher set theory (in ZF) is required!!!

- 1971, Harvey Friedman
  For $\Sigma^0_5(\omega^\omega)$ and beyond, 2d-order arithmetic NOT ENOUGH
  For $\Sigma^0_{5+\alpha}(\omega^\omega)$ need $\alpha$ iterations of set exponentiation

- ~2010, Donald Martin
  For $\Sigma^0_4(\omega^\omega)$ 2d-order arithmetic NOT ENOUGH
A few simple results about determinacy and strategies
Determinacy and complementation

If $A \subseteq X^\omega$ then the shift of $A$ is

$$XA = \{(x, x_0, x_1, x_2, \ldots) \mid (x_0, x_1, x_2, \ldots) \in A\}$$

Let $\mathcal{A} \subseteq \mathcal{P}(X^\omega)$ be closed under shift:

$$A \in \mathcal{A} \implies xA \in \mathcal{A}$$

$$\forall A \in \mathcal{A} \quad A \text{ is determined}$$

$$\iff$$

$$\forall A \in \mathcal{A} \quad X^\omega \setminus A \text{ is determined}$$

I has a ws in $\mathcal{G}(XA) \implies$ II has a ws in $\mathcal{G}(X^\omega \setminus A)$

II has a ws in $\mathcal{G}(XA) \implies$ I has a ws in $\mathcal{G}(X^\omega \setminus A)$
Winning strategies viewed as trees

Strategy $\sigma_I$ for $I \equiv$ tree $S_{\sigma_I} \subseteq X^{<\omega}$ of all plays when $I$ follows $\sigma_I$

\[
\begin{align*}
  u \in S_{\sigma_I} \land |u| \text{ even} & \implies \exists! x \ ux \in S_{\sigma_I} \\
  u \in S_{\sigma_I} \land |u| \text{ odd} & \implies \forall x \ ux \in S_{\sigma_I}
\end{align*}
\]
(Thus, $I$ always has exactly one possible move and there is no constraint for $II$-moves)

Strategy $\sigma_{II}$ for $II \equiv$ tree $S_{\sigma_{II}} \subseteq X^{<\omega}$ of all plays when $II$ follows $\sigma_{II}$

\[
\begin{align*}
  u \in S_{\sigma_{II}} \land |u| \text{ odd} & \implies \exists! x \ ux \in S_{\sigma_{II}} \\
  u \in S_{\sigma_{II}} \land |u| \text{ even} & \implies \forall x \ ux \in S_{\sigma_{II}}
\end{align*}
\]
(Thus, $II$ always has exactly one possible move and there is no constraint for $I$-moves)

\[
\begin{align*}
  \sigma_I \text{ winning for } I & \iff [S_{\sigma_I}] \subseteq A \\
  \sigma_{II} \text{ winning for } II & \iff [S_{\sigma_{II}}] \subseteq X^\omega \setminus A
\end{align*}
\]
($[S] = \text{set of infinite branches of } S$)
Non deterministic winning strategies

ND strategy $\sigma_I$ for $I \equiv$ tree $S_{\sigma_I} \subseteq X^{<\omega}$ of all plays when $I$ follows $\sigma_I$

\[
\begin{align*}
S_{\sigma_I} & \text{ is pruned: } \forall u \in S_{\sigma_I} \exists x \ ux \in S_{\sigma_I} \\
u \in S_{\sigma_I} & \wedge |u| \text{ odd } \implies \forall x \ ux \in S_{\sigma_I}
\end{align*}
\]
(Thus, $I$ always has some move and there is no constraint for $I$-moves)

ND strategy $\sigma_{II}$ for $II \equiv$ tree $S_{\sigma_{II}} \subseteq X^{<\omega}$ of all plays when $II$ follows $\sigma_{II}$

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\end{align*}
\]
(Thus, $II$ always has some move and there is no constraint for $I$-moves)

$\sigma_I$ winning for $I$ $\iff$ $[S_{\sigma_I}] \subseteq A$

$\sigma_{II}$ winning for $II$ $\iff$ $[S_{\sigma_{II}}] \subseteq X^\omega \setminus A$
Winning strategies and positions

\[ u \in X^{<\omega} \quad A \subseteq X^\omega \quad A_u = A \cap \text{clopen set } uX^\omega \]

Fact. If \(|u|\) is odd (next move for II) then

II has no winning strategy in \(G(A_u)\) iff

\[ \forall x \in X \text{ II has no winning strategy in } G(A_{ux}) \]

(No “miracle” move \(x\) for player II)
Winning and Defensive strategies

\[ u \in X^{<\omega} \quad A \subseteq X^{\omega} \quad A_u = A \cap \text{clopen set } uX^{\omega} \]

**Fact.** Let \( |u| \) even (next move for player I)

II has no winning strategy in \( G(A_u) \)

iff

\[ \exists x \in X \text{ II has no winning strategy in } G(A_{ux}) \]

(\( Player \ I \ has \ a \ move \ x \ so \ that \ II \ still \ has \ no \ ws \ afterwards \))

Always choosing such an \( x = \)

**Defensive strategy for player I**

CARE: defensive strategy \( \nRightarrow \) winning strat.
Gale & Stewart’s results about $\Sigma^0_1(X^\omega)$

they contain

many core ideas of the theory
Theorem. (Gale & Stewart, 1953)

Every closed or open $A \subseteq X^\omega$ is determined

Proof. Let $A$ closed be the set of infinite branches of a pruned tree $T \subseteq X^{<\omega}$, i.e. $A = [T]$.

If player II has no ws in $G(A)$
then any DEFENSIVE strategy for player I is winning:

$\exists x_0$ (move of I) so that II has no ws in $G(A_{x_0})$
$\forall x_1$ (move of II) II has no ws in $G(A_{x_0 x_1})$
$\exists x_2$ (move of I) II has no ws in $G(A_{x_0 x_1 x_2})$
$\forall x_3$ (move of II) II has no ws in $G(A_{x_0 x_1 x_2 x_3})$
...

$\forall n x_0 \ldots x_n \in T$ else the play enters the open set $X^\omega \setminus A$
so that any strategy for II in $G(A_{x_0 \ldots x_n})$ is winning. But II has no ws in $G(A_{x_0 x_1 x_2 x_3})$. Contradiction!

$\forall n x_0 \ldots x_n \in T \implies$ the infinite play $\in$ closed set $A = [T]$
Pruned subtree $S$ of $X^{<\omega}$  
$[S] = \text{infinite branches of } S$

$G_S(A)$ or $G_F(A)$: New rule for the two players:  
The play stays in the subtree $S$

$\equiv$ replace $A$ by $A \cap [S]$,  
$X^\omega \setminus A$ by $(X^\omega \setminus A) \cap [S]$

Trivial example: Game $G(A_s)$ at position $s$  
reduces to the subgame $G_{sX^\omega}(A)$
Determinacy of $\text{BOOL}(\Sigma^0_1(X^\omega))$

**Theorem.** (Gale & Stewart, 1953)
If every subgame of $\mathcal{G}(A)$ is determined then $\mathcal{G}(A \cup U)$ is determined for all $U$ open

**Proof.** We show that there is a particular subgame $F$ st

- if $\mathcal{G}_F(A)$ is determined then so is $\mathcal{G}(A)
- S = \{s \mid I \text{ has a ws in } \mathcal{G}((A \cup U)_s)\}$ (S may not be a tree)
- $F = X^\omega \setminus SX^\omega = [T]$ for some pruned tree $T$ disjoint from $S$
- $U \cap F = \emptyset$ : if $sX \subseteq U$ then $s \in S$ trivially
I has a ws in $\mathcal{G}_F(A) \implies$ I has a ws in $\mathcal{G}(A \cup U)$
Playing in $\mathcal{G}(A \cup U)$, I follows his ws for $\mathcal{G}_F(A)$ while II stays in $T$. If II leaves $T$ then the play gets into $S$ and I uses a ws for $\mathcal{G}((A \cup U)_s)$

II has a ws in $\mathcal{G}_F(A) \Rightarrow$ II has a ws in $\mathcal{G}(A \cup U)$
Playing in $\mathcal{G}(A \cup U)$, II follows his ws for $\mathcal{G}_F(A)$ while I stays in $T$. I cannot leave $T$ : else, if I leaves $T$ at $s$ then $s \in S$ and I gets a ws for $\mathcal{G}((A \cup U)_s)$, contradicting Fact page 20
Determinacy of BOOL($\Sigma^0_1(X^\omega)$)

**Theorem.** (Gale & Stewart, 1953)
If every subgame of $G(A)$ is determined then $G(A \cap U)$ is determined for all $U$ open

*Proof.* $S = \{s \mid sX^\omega \subseteq U \text{ and } I \text{ has a ws in } G(A_s)\}$
$T = \{s \mid sX^\omega \subseteq U \text{ and } II \text{ has a ws in } G(A_s)\}$

I has a ws in $G(SX^\omega) \implies$ I has a ws in $G(A \cap U)$
I follows a ws for $G(SX^\omega)$ until the play enters $S$
Then he uses a ws for $G(A_s)$

II has a ws in $G(SX^\omega) \implies$ II has a ws in $G(A \cap U)$
II follows ws for $G(SX^\omega)$. If output $\notin U$ II wins
Else the play enters $S$ or $T$. Cannot enter $S$ else I could win.
If it enters $T$ then II uses ws for $G(A_s)$
Corollary. (Gale & Stewart, 1953)
Every Boolean combination of open subsets of $X^\omega$ is determined

Proof.
Extend closed determinacy to subgames
Apply closure by complementation, union and intersection with open sets
Fact. If \( I \) has a winning strategy for a closed game then it has a largest non deterministic one

Proof. \( S \) a tree, \( \Theta(S) \subseteq S \), \( \Lambda(S) \subseteq S \)

\[
\begin{align*}
\Theta(S) & = \{ u \in S \mid \forall v \leq_{\text{pref}} u (|v| \text{ odd } \Rightarrow \forall x \text{ } vx \in S) \} \\
\Lambda(S) & = \{ u \in S \mid \exists x \in X \text{ } ux \in S \} \\
\end{align*}
\]

To prune a tree one has to transfinitely iterate \( \Lambda \) (cf. page 31)

Suppose \( I \) has a ws for \( G(F) \)

\[
F = [T] \quad T \text{ tree } \subseteq X^{<\omega}
\]

\[
T(0) = T \quad T^{\alpha+1} = \Lambda(\Theta(T^{(\alpha)})) \quad T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)}
\]

Fact: \( \exists \xi < \aleph_1 \text{ st } T^{(\xi)} = T^{(\xi+1)} = \Lambda(T^{(\xi)}) = \Theta(T^{(\xi)}) \)

Fact: 1) Every ND ws for \( I \) (viewed as a tree) is \( \subseteq T^{(\alpha)} \)

2) If \( I \) has a ws for \( G([T]) \) then \( T^{(\xi)} \neq \emptyset \)

3) \( T^{(\xi)} \) is a non deterministic ws for \( I \) and is the largest one

1) Proof by induction over \( \alpha \). 2) Obvious from 1)

3) Closure under pruning and \( \Theta \) insures \( T^{(\xi)} \) is a strategy for \( I \) (if non empty). It is winning since \( T^{(\xi)} \subseteq T \) and \( F = [T] \)
Winning strategies may be quite complex even for closed games!

**Fact.** There exists a computable tree $T \subseteq \omega^{<\omega}$ st
- I has a ws in the closed game $G([T])$
- I has no $\Delta^1_1$ ws in $G([T])$

**Proof.** Recall Kleene’s result (cf. Rogers §16.7 Coro. XLII(b), p. 419):

**Fact.** There exists a computable tree $S \subseteq \omega^{<\omega}$ which has an infinite branch but no $\Delta^1_1$ one.

Let $\theta : \omega^{<\omega} \to \omega^{<\omega}$ suppress all odd rank letters of a finite sequence: for instance, $\theta(abcde) = \theta(abcdef) = ace$

Let $T = \theta^{-1}(S)$ $T$ is a computable tree

Player I has a ws in $G([T])$: do not care about II moves
- play a fixed infinite branch of $S$

If $\sigma$ is a ws for I in $G([T])$ and II plays $0^\omega$
- then $\sigma \star 0^\omega \in [T]$ hence $f = \theta(\sigma \star 0^\omega) \in [S]$

If $\sigma$ were $\Delta^1_1$ then $f$ would be $\Delta^1_1$ branch of $S$. Contradiction!
Why so complex ws for closed games? Because pruning a tree may require iterations beyond recursive ordinals!

\[ T^{(0)} = T \quad T^{\alpha+1} = \Lambda(\Theta(T^{(2\alpha)})) \quad T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)} \]

\[ T^{(\xi)} = T^{(\xi+1)} \text{ largest ND ws for I in } G([T]) \]

\( R \) order on \( \mathbb{N} \) of type \( \eta > \xi \) \quad \nu : \mathbb{N} \to \eta \) isomorphism

**Fact.** \( T^{(\xi)} \) is \( \Delta^1_1, T, R \) hence so is its leftmost infinite branch

**Proof.** \( \Phi_T(Z^{(k)}, Z^{(\ell)}) \equiv (\theta(k) = 0 \Rightarrow Z^{(k)} = T) \)

\[ \land (\theta(\ell) = \theta(k) + 1 \Rightarrow Z^{(\ell)} = \Lambda(\Theta(Z^{(k)}))) \]

\[ \land (\theta(k) \text{ limit } \Rightarrow Z^{(k)} = \bigcap\{Z^{(p)} \mid \theta(p) < \theta(k)\}) \]

\( u \in T^{(\xi)} \equiv \exists(Z^{(n)})_{n \in \mathbb{N}} \forall k, \ell (u \in Z^{(k)} \land \Phi_T(Z^{(k)}, Z^{(\ell)})) \)

\[ \equiv \forall(Z^{(n)})_{n \in \mathbb{N}} (\forall k, \ell \Phi_T(Z^{(k)}, Z^{(\ell)})) \Rightarrow (\forall n u \in Z^{(n)}) \]

**Fact.** Let \( T = \theta^{-1}(S) \) with \( S \) a computable tree with an infinite branch but no \( \Delta^1_1 \) one. Then the ordinal \( \xi \) is not \( \Delta^1_1 \)

**Proof.** \( S \Delta^1_1 \Rightarrow T \Delta^1_1 \) and \( T, R \Delta^1_1 \Rightarrow \Delta^1_1, T, R = \Delta^1_1 \)

(cf. Rogers §16.6 Thm XXXIV p. 412)
How many iterations to prune a tree?

\[
\begin{align*}
S^{(0)} &= S \\
S^{\alpha+1} &= \Lambda(S^{(\alpha)}) \\
S^{(\lambda)} &= \bigcap_{\alpha < \lambda} S^{(\alpha)}
\end{align*}
\]

\[\Lambda(S) = \{ u \in S \mid \exists x \in X \text{ } ux \in S \}\]

When \( S \) is well-founded, \( \xi = \text{ordinal rank of } S \),
\( \xi < \omega_1^{\text{CK}} \) hence \( \xi \) is computable

(Spector, 1955: computable ordinals=\( \Delta^1_1 \) ordinals, cf.
Rogers §16.6 Coro XXXVI p. 415)

In general, when \( S \) not well-founded, \( \xi \geq \omega_1^{\text{CK}} \)
(nevertheless, \( \xi \) is \( \Delta^1_2 \))

Example: cf. page 30

Other example: \( S = \{(e, u, t) \mid e, t \in \mathbb{N}, u \in \omega^{<\omega} \text{ and}
\text{the current output of } \varphi_e \text{ at time } t \text{ is } u\}\)

Order on \( S \):
\( (e, u, t) \leq (e', u', t') \iff e = e' \land (u <_{\text{pref}} u' \lor (u = u' \land t \leq t')) \)

\( S \) is a computable tree which contains every well-founded
computable tree

Hence the \( \xi \) associated to \( S \) is \( \geq \omega_1^{\text{CK}} \)
Wolfe’s results about

\[ \Sigma^0_2(X^\omega) \text{ and } \Pi^0_2(X^\omega) \]

\[ \equiv \mathcal{F}_\sigma(X^\omega) \text{ and } \mathcal{G}_\delta(X^\omega) \]

(countable unions of closed sets and countable intersections of open sets)
Determinacy of $\Sigma^0_2(X^\omega)$ and $\Pi^0_2(X^\omega)$

**Theorem.** (Philip Wolfe, 1955)
Every $F_\sigma$ or $G_\delta$ set $A \subseteq X^\omega$ is determined

**Proof.** (cf. Moschovakis)
\[ A = \bigcup_{i \in \mathbb{N}} [T_i] \] an $F_\sigma$ set $T_i$ pruned tree

Set $W$ of sure winning positions for $I$ in $G(A)$

- $u \in W_0 \iff \exists i \ I \text{ wins } G([T_i]_u)$
- $u \in H_{\alpha,i} \iff \forall v \leq u \ (|v| \text{ even } \Rightarrow v \in T_i \cup \bigcup_{\beta < \alpha} W_\beta)$
- $u \in W_\alpha \iff \exists i \ I \text{ wins } G([H_{\alpha,i}]_u)$

$W = \bigcup_\alpha W_\alpha = \bigcup_{\alpha \leq \xi} W_\alpha \quad \xi \text{ countable ordinal, } W_\xi = W_{\xi + 1}$

Induction on ordinal $\alpha$:
\[
\begin{cases}
  u \in W_\alpha & \Rightarrow \ I \text{ wins } G(A_u) \\
  u \notin W_\alpha & \Rightarrow \ II \text{ wins } G(A_u)
\end{cases}
\]
• Induction on ordinal $\alpha$: $u \in W_\alpha \Rightarrow I$ wins $G(A_u)$

I follows a ws for $G([H_\alpha, i]_u)$

If the play enters $\bigcup_{\beta < \alpha} W_\beta$ at $uu'$ then I switches to aws for $G(A_{uu'})$. By induction hypothesis, the infinite play is in $A$

Else the play stays in $T_i$; hence the infinite play is in $A$

• If nil $\in W$ then I has a ws for $G(A)$

• Else here is a ws for II in $G(A)$:
  
  nil $\notin W_{\xi+1}$ hence for all $i$, I has no ws for $G([H_{\xi+1}, i])$ hence II has a ws for $G([H_{\xi+1}, i])$ (closed games being determined)

II follows his ws for $G([H_{\xi+1}, 0])$ until the play leaves $W_{\xi} \cup T_0$ at some $u_0$ where $u_0 \notin W_{\xi}$ and $u_0 \notin T_0$

$u_0 \notin W_{\xi} = W_{\xi+1}$ hence $\forall i$ I has no ws for $G([H_{\xi+1}, i]_{u_0})$ hence II has a ws for $G([H_{\xi+1}, i]_{u_0})$ (since closed games are determined)

II follows his ws until the play leaves $W_{\xi} \cup T_0$ at $u_0u_1\ldots$

The final play $\notin [T_0], \notin [T_1]\ldots$ hence $\notin A$

Thus, II has a ws in $G(A)$

A is determined
Morton Davis’ results about

\[ \Sigma^0_3(X^\omega) \text{ and } \Pi^0_3(X^\omega) \]

\[ \equiv F_{\sigma\delta}(X^\omega) \text{ and } G_{\delta\sigma}(X^\omega) \]

(countable intersections of countable unions of closed sets and countable unions of countable intersections of open sets)
Determinacy of $\Sigma^0_3(X^\omega)$ and $\Pi^0_3(X^\omega)$

**Theorem.** (Morton Davis, 1964)
Every $\mathbf{F}_{\sigma\delta}$ or $\mathbf{G}_{\delta\sigma}$ set $A \subseteq X^\omega$ is determined

**Strategies (non necessarily winning) are trees:** II-strategy $S$
\[
\begin{cases}
\forall u \in S (|u| \text{ odd } \Rightarrow \exists x \ ux \in S) \quad (\text{II can stay in } S) \\
\forall u \in S (|u| \text{ even } \Rightarrow \forall x \ ux \in S) \quad (I \text{ cannot leave } S)
\end{cases}
\]

III-strategy $S$ relative to a subgame $T$
\[
\begin{cases}
\forall u \in S (|u| \text{ odd } \Rightarrow \exists x \ ux \in S) \\
\forall u \in S (|u| \text{ even } \Rightarrow \forall x \ (ux \in T \Rightarrow ux \in S)) \quad (I \text{ cannot leave } S \text{ except if it leaves } T)
\end{cases}
\]
Determinacy of $\Sigma^0_3(X^\omega)$ and $\Pi^0_3(X^\omega)$

Strategies (non necessarily winning) are trees: II-strategy $S$

\[
\begin{cases}
\forall u \in S \ (|u| \text{ odd } \Rightarrow \exists x \ ux \in S) \quad (\text{II can stay in } S) \\
\forall u \in S \ (|u| \text{ even } \Rightarrow \forall x \ ux \in S) \quad (\text{I cannot leave } S)
\end{cases}
\]

Lemma 1. Suppose \[\begin{cases}
I \text{ has no ws for } G(A) \\
U \subseteq A \text{ is open}
\end{cases}\]

Then there is a II-strategy $S$ such that
(1) I has no ws for $G_S(A)$ and (2) $U \cap [S] = \emptyset$

(II has a non-catastrophic strategy to avoid any fixed open set $U$
catastrophic $= I$ has a ws in the associated subgame)

Lemma 1’. Suppose \[\begin{cases}
I \text{ has no ws for } G_T(A) \\
U \subseteq A \text{ is open}
\end{cases}\]

Then there is a II-strategy $S$ relative to the subgame $T$
st
(1) I has no ws for $G_S(A)$ and (2) $U \cap [S] = \emptyset$
(3) If $U$ contains no clopen $sX^\omega$ with $|s| \leq n$
then one can require $S \cap X^{\leq n} = T \cap X^{\leq n}$
(variant of Lemma 1: subgame relativized $+$ slightly improved)
Determinacy of $\Sigma^0_3(X^\omega)$ and $\Pi^0_3(X^\omega)$

**Lemma 1.** Suppose $\begin{cases} I \text{ has no ws for } G(A) \\ U \subseteq A \text{ is open} \end{cases}$

Then there is a II-strategy $S$ such that

1. $I$ has no ws for $G_S(A)$ and
2. $U \cap [S] = \emptyset$

($II$ has a non-catastrophic strategy to avoid any fixed open set $U$)

**Proof of Lemma 1.**

$S =$ defensive II-strategy

$S = \{ s \in X^{<\omega} \mid I \text{ has no ws in } G(A_s) \}$

(1): cf. slide 20

(2): Else there is some $s \in T$ such that $sX^\omega \subseteq U$
and any strategy for $I$ is trivially winning in $G(A_s)$

**Remark:** Lemma 1 reproves open determinacy. If $A$ open let $U = A$, the non catastrophic strategy for II is a winning one.
Determinacy of $\Sigma^0_3(X^\omega)$ and $\Pi^0_3(X^\omega)$

**Lemma 2.** Suppose \[
\begin{cases}
\text{I has no ws for } G(A) \\
H \subseteq A \text{ is } G_\delta
\end{cases}
\]
Then there is a II-strategy $S$ such that
(1) I has no ws for $G_S(A)$ and (2) $H \cap [S] = \emptyset$
(II has a non-catastrophic strategy to avoid any fixed $G_\delta$ set $U$)

Remark. Lemma 2 reproves $G_\delta$ determinacy. If $A$ is $G_\delta$ let $H = A$, the non-catastrophic strategy for II is a winning one

**Lemma 2’.** Suppose \[
\begin{cases}
\text{I has no ws for } G_T(A) \\
H \subseteq A \text{ is } G_\delta
\end{cases}
\]
Then there is a II-strategy $S$ relative to the subgame $T$ st
(1) I has no ws for $G_S(A)$ and (2) $H \cap [S] = \emptyset$
(3) If $H$ contains no clopen $sX^\omega$ with $|s| \leq n$
then one can require $S \cap X^{\leq n} = T \cap X^{\leq n}$
(variant of Lemma 2: subgame relativized + slightly improved)
Lemma 2. If $I$ has no ws for $G(A)$, $H \subseteq A$ is $G_{\delta}$
Then there is a II-strategy $S$ such that
(1) $I$ has no ws for $G_S(A)$ and (2) $H \cap [S] = \emptyset$
($II$ has a non-catastrophic strategy to avoid any fixed $G_{\delta}$ set $U$)

Proof of Lemma 2. $H = \bigcap_{i \in \mathbb{N}} C_i X^\omega$
the $C_i$'s antichains of $X^{<\omega}$, $C_{i+1} \subseteq C_i X^{<\omega}$, $C_0 = \{nil\}$

$Z = \{u \in X^{<\omega} \mid \exists$ II-strategy $T(u)$ relative to subgame $uX^{<\omega}$
st $H \cap [T(u)] = \emptyset$ and I has no ws for $G_{T(u)}(A)\}$

We prove
$(\ast)_i \quad u \in C_i \setminus Z \Rightarrow I$ has ws in $G((A \cup (C_{i+1} \setminus Z) X^\omega)_u)$
\((\ast)_i\) \quad u \in C_i \setminus Z \Rightarrow I \text{ has ws in } G((A \cup (C_{i+1} \setminus Z)X^\omega)_u)

Suppose \((\ast)_i\) false. Let \(u \in C_i \setminus Z\) be st \(I\) has ws in 
\(G((A \cup (C_{i+1} \setminus Z)X^\omega)_u)\).

L.1' yields a II-strategy \(S\) relative to the subgame \(uX^{<\omega}\) such that 
\([S] \cap (C_{i+1} \setminus Z)X^\omega = \emptyset\) and \(I\) has no ws in
\(G_S((A \cup (C_{i+1} \setminus Z)X^\omega)_u)\).

To get a contradiction, we describe a ws strategy for II relative to the subgame \(uX^{<\omega}\).

First, II follows \(S\). Since \([S]\) is disjoint from the open set 
\((C_{i+1} \setminus Z)X^\omega\), while II follows \(S\) it does not meet \(C_{i+1} \setminus Z\).

If some \(v \in C_{i+1}\) is reached then \(v \in Z \cap C_{i+1}\) and II switches to its strategy \(T^{(v)}\) (cf. definition of \(Z\)) st \(I\) has no 
ws in \(G_{T^{(v)}}(A)\) and \([T^{(v)}] \cap H = \emptyset\).

The resulting infinite play either avoids \(C_{i+1}\) hence \(\notin H\) or 
meets \(Z \cap C_{i+1}\) hence is given by some \(T^{(v)}\) and \(\notin H\).

Thus, II has a ws relative to the subgame \(uX^{<\omega}\). In particular, \(u \in Z\), contradicting the hypothesis \(u \in C_i \setminus Z\).
To conclude the proof of Lemma 2, we show that nil \in Z.

Else, nil \in X^{<\omega} \setminus Z = C_0 \setminus Z. We define a strategy for I. Using (\ast)_0 with u = nil, I follows a ws in \mathcal{G}(A \cup (C_1 \setminus Z)X^\omega).

If and when the play enters C_1 \setminus Z at u_1 then, applying (\ast)_1 with u = u_1, I switches to a ws in \mathcal{G}((A \cup (C_2 \setminus Z)X^\omega)_{u_1}).

And so on...

The resulting infinite play

- either does not meet some (C_{i+1} \setminus Z)X^\omega and is given by a ws for I in \mathcal{G}((A \cup (C_{i+1} \setminus Z)X^\omega)_{u_i}). Then it is in A and I wins.
- or it does meet all C_{i+1} \setminus Z hence all C_j’s and is in H hence in A.

Thus, we have obtained a ws for I in \mathcal{G}(A). Contradicting the hypothesis of Lemma 2.
Proof of $\Sigma_3^0(X^\omega)$ determinacy

$$A = \bigcup_{i \in \mathbb{N}} H_i \quad H_i = \bigcap_{j \in \mathbb{N}} C_{i,j}X^\omega \quad C_{i,j} \subseteq X^{<\omega}$$

Case $X$ finite: Suppose $I$ has no ws in $G(A)$

We inductively define a decreasing sequence $(T_n)_{n \in \mathbb{N}}$ of non-catastrophic $\Sigma^1_1$-strategies which avoid the $H_i$’s

By Lemma 2, get a $\Sigma^1_1$-strategy $T_0$ st

(1) $I$ has no ws for $G_{T_0}(A)$ and (2) $H_0 \cap [T_0] = \emptyset$

By Lemma 2’, get $\Sigma^1_1$-strategy $T_1$ relative to subgame $T_0$ st

(1) $I$ has no ws for $G_{T_1}(A)$ and (2) $H_2 \cap [T_2] = \emptyset$

And so on...

$X$ being finite, $X^\omega$ is compact hence $[T] = \bigcap_{i \in \mathbb{N}} [T_i] \neq \emptyset$

$T$ is a $\Sigma^1_1$-strategy st

(1) $I$ has no ws for $G_T(A)$ (obvious since $T \subseteq T_0$)

(2) $(\bigcup_{i \in \mathbb{N}} H_i) \cap [T] = \emptyset$

Thus, the intersection $T$ of the $T_i$’s is a ws for $\Sigma^1_1$ in $G_T(A)$
Proof of $\Sigma^0_3(X^\omega)$ determinacy

**Case $X$ is infinite**

Consider the interior $U$ of $A$.

Then $A = U \cup B$ where $B$ is also $G^\delta_\sigma$.

$B$ contains no open set.

$$B = \bigcup_{i \in \mathbb{N}} H_i \text{ with } H_i \sim^\delta$$

Up to subsequence extraction, can suppose $H_i$ contains no
$sX^\omega$ with $|s| \leq i$

Use condition (3) in Lemma 2’ to get $T_{i+1}$ such that
$T_{i+1} \cap X^{\leq i} = T_i \cap X^{\leq i}$.

Then $\bigcap_{i \in \mathbb{N}} [T_i] \neq \emptyset$
Donald Martin’s proof of Borel determinacy
Main idea of the proof

In usual reduction theories one looks for a hard set $A$ which reduces every set $Z$ in a particular family $\mathcal{F}$:

$$\forall Z \in \mathcal{F} \quad Z = f^{-1}(A) \quad \text{for some } f$$

($f$ computable or polytime or continuous)

In Martin’s proof, the association is reversed

For every Borel set $A \subseteq [S] \subseteq X^\omega$ $S$ pruned tree

Martin’s proof looks for

- a space $Y^\omega$, a pruned tree $T \subseteq X^{<\omega}$ (possibly huge $Y$)
- a clopen subset $C$ of $[T] \subseteq Y^\omega$ (very simple set)
- a continuous surjective map $\pi : [T] \rightarrow [S]$ (the reduction map)

such that

1. $C = \pi^{-1}(A)$ ($A$ is Borel whereas $C$ is clopen)
2. every winning strategy in $G_T(C)$ yields a ws in $G_S(A)$
No direct extension beyond Borel sets

Case $X = \omega^\omega$

Suppose $C = \pi^{-1}(A)$, $\pi$ continuous and $C$ clopen
Then $A = \pi(C)$ and $\omega^\omega \setminus A = \pi(F \setminus C)$

In general, $\pi(\text{closed})$ has descriptive complexity $\Sigma_1^1$
Thus, $\pi(C)$ and $\pi(F \setminus C)$ are $\Sigma_1^1$
Thus, $A$ and $\omega^\omega \setminus A$ are $\Sigma_1^1$,
i.e. $A$ is $\Delta_1^1$ hence is Borel (Suslin’s theorem, 1917)
À la Martin reductions in pure topology

(Almost) forget strategies. Topological problem

For every Borel set \( A \subseteq [S] \subseteq X^\omega \) \( S \) pruned tree

\[
\text{a topological space } Y^\omega, \text{ a tree } T \subseteq Y^{<\omega}
\]

look for

\[
\text{a clopen } C \subseteq [T] \quad ([T]= \text{ set of infinite branches})
\]

\[
\text{a continuous surjective map } \pi : [T] \to X^\omega
\]

such that \( C = \pi^{-1}(A) \)

- Trivial if we do not ask for a topological space \( Y^\omega \):
  
  Set \( Y = X \) and increase the topology so that \( A \) is clopen

- Trivial if surjectivity is omitted: (case \( A \neq \emptyset, X^\omega \))

\[
\begin{align*}
\{ a & \in A, \quad [T] \text{ clopen } \neq \emptyset, \quad Y^\omega, \quad \pi(x) = \begin{cases} 
  a \text{ if } x \in C \\
  b \text{ if } x \notin C
\end{cases} 
\end{align*}
\]

- If \( X \) is finite, \( Y \) has to be infinite else \( \pi(\text{clopen}) \) is compact

- If \( X = \omega \), true by Wadge hardness theorem

A Borel not \( \Sigma^0_\xi \) implies every \( \Sigma^0_\xi \) in \( \omega^\omega \) is \( \pi^{-1}(A) \) for some \( \pi \)

Vicious circle: Wadge theory relies on Borel determinacy!!!
A combinatorico-topological problem: 

*drills before entering Martin’s proof*

Extend the problem to allow inductive constructions and future strategy requirements

For every tree $S \subseteq X^{<\omega}$, $([S]=\text{set of infinite branches})$

for every $k \in \mathbb{N}$ (technical point: $k$ is a trick to cope with non compactness in inductive constructions)

for every Borel set $A \subseteq [S] \subseteq X^\omega$ look for

- a topological space $Y^\omega$ where $Y \supseteq X$
- a tree $T \subseteq Y^{<\omega}$ such that $T \cap Y^{\leq k} = S \cap X^{\leq k}$
- a clopen $C \subseteq [T]$ $([T]=\text{set of infinite branches})$
- a monotone length preserving surjective map $\pi : T \rightarrow S$ (alphabetical transduction)

such that $C = \pi^{-1}(A)$

where $\pi : [T] \rightarrow [S]$ obvious extension of $\pi$
Topological problem: A open in $[S]$, $k = 0$

A open in $[S]$ hence $A = \bigcup_{u \in \tau} uX^\omega \cap [S]$ \hspace{1cm} $\tau \subseteq X^{<\omega} \setminus \{\text{nil}\}$

- Add elements representing the $u$'s: $Y = X \cup \{\lceil u \rceil | u \in \tau\}$

To preserve length, a new element is always the first one

if $u = x_0x_1 \ldots x_{n-1}$ then $\lceil u \rceil$ has unique length $< n$ successors

$\tilde{\tau} = \{\lceil u \rceil x_1 \ldots x_{n-1} | u = x_0x_1 \ldots x_{n-1} \in \tau\}$ \hspace{1cm} antichain of $Y^{<\omega}$

$T = (S \setminus \tau X^{<\omega}) \cup \{\lceil u \rceil x_1 \ldots x_i | u = x_0x_1 \ldots x_{n-1} \in \tau$

$\wedge (u \leq_{\text{pref}} x_0 \ldots x_i \text{ or } x_0 \ldots x_i \leq_{\text{pref}} u)\}$

$\wedge x_0 \ldots x_i \in S\}$

$C = [\tilde{\tau} X^{<\omega}] = \{(y_i)_{i \in \mathbb{N}} \in [T] | y_0 \in Y \setminus X\}$ \hspace{1cm} $[C]$ clopen in $[T]$

(a condition on the sole first component defines a clopen set)

- $\pi: T \to S$ \hspace{1cm} $\pi(s) = s$ \hspace{1cm} if $s \in S \setminus \tau X^{<\omega}$

$\pi(\lceil u \rceil x_1 \ldots x_i) = x_0x_1 \ldots x_i$ \hspace{1cm} if $u = x_0x_1 \ldots x_{n-1} \in \tau$

$\pi: T \to S$ \hspace{1cm} alphabetical

$\pi: [T] \to [S]$ \hspace{1cm} bijective, continuous but not homeomorphism

$\pi^{-1}(A) = [C]$ \hspace{1cm} $Y$ has the cardinality of $X$
Topological problem: A open in $[S]$, any $k$

$A$ open in $[S]$ hence $A = \bigcup_{u \in \tau} uX^\omega \cap [S]$

Choose antichain $\tau \subseteq X^{<\omega}$
st every $u$ in $\tau$ has length $> k$ and set

$$\tilde{\tau} = \{x_0 \ldots x_{k-1} \upharpoonright u \upharpoonright x_{k+1} \ldots x_n \mid u = x_k \ldots x_n \in \tau\}$$

$$T = (S \setminus \tau X^{<\omega})$$

$$\cup \{x_0 \ldots x_{k-1} \upharpoonright u \upharpoonright x_{k+1} \ldots x_i \mid u = x_0 \ldots x_n \in \tau$$

$$\wedge (u \leq \text{pref } x_{k+1} \ldots x_i \text{ or } x_{k+1} \ldots x_i \leq \text{pref } u)\}$$

$$\wedge x_0 \ldots x_{k-1} x_k x_{k+1} \ldots x_n \in S\}$$

$$C = [\tilde{\tau} X^{<\omega}] = \{(y_i)_{i \in \mathbb{N}} \in [T] \mid y_k \in Y \setminus X\} \quad [C \text{ clopen in } [T]]$$

(a condition on the sole $k$-th component defines a clopen set)

Then argue as in the case $k = 0$
The topological problem: induction step

Suppose the problem has positive answer for all levels < ξ of the Borel hierarchy over X^ω. We get positive answer for level ξ

Let A = \bigcup_{n \in \mathbb{N}} A_n where the A_n’s have Borel ranks < ξ

- Applying the induction hypothesis with k
  C_0 = \pi_0^{-1}(A_0) for Y_0 \supseteq X, T_0 \subseteq Y_0^{<\omega}, clopen C_0 of [T_0],
  \pi_0 : [T_0] \rightarrow [S],
  T_0 \cap Y_0^{\leq k} = S \cap X^{\leq k}

- In [T_0], Borel rank of \pi_0^{-1}(A_1) is \leq \text{rank } A_1 in [S] hence < ξ
  Applying the induction hypothesis with k + 1
  C_1 = \pi_1^{-1}(\pi_0^{-1}(A_1)) for Y_1 \supseteq Y_0, T_1 \subseteq Y_1^{<\omega}, clopen C_1 of [T_1],
  \pi_1 : [T_1] \rightarrow [T_0],
  T_1 \cap Y_1^{\leq k+1} = T_0 \cap Y_0^{\leq k+1}

- In [T_1], Borel rank of (\pi_0 \circ \pi_1)^{-1}(A_2) is \leq \text{rank } A_2 in [S] < ξ
  Applying the induction hypothesis with k + 2
  C_2 = \pi_2^{-1}((\pi_1 \circ \pi_0)^{-1}(A_2)) for Y_2 \supseteq Y_1, T_2 \subseteq Y_2^{<\omega}, clopen C_2 of [T_2],
  \pi_1 : [T_2] \rightarrow [T_1],
  T_2 \cap Y_2^{\leq k+2} = T_1 \cap Y_1^{\leq k+2}

- and so on ...
The topological problem: induction step

\[ \cdots \rightarrow [T_{i+1}] \xrightarrow{\pi_{i+1}} [T_i] \rightarrow \cdots \rightarrow T_1 \xrightarrow{\pi_1} [T_0] \xrightarrow{\pi_0} [S] \]

\[ T_{i+1} \cap Y_{i+1}^{\leq k+i+1} = T_i \cap Y_i^{\leq k+i+1} \]

\[ T_0 \cap Y_0^{\leq k} = S \cap X^{\leq k} \]

Consider the inverse limit (No cardinal explosion here)

\[ \leftarrow Y = \bigcup_{i \in \mathbb{N}} Y_i \quad \quad \leftarrow T = \{ u \mid u \in T_i \text{ for all } i \geq |u| \} \]

\[ \leftarrow \pi_i : \leftarrow T \rightarrow [T_i] \quad \quad \leftarrow \pi : \leftarrow T \rightarrow [S] \]

where \( \leftarrow \pi_i \upharpoonright T_j = \pi_{j+1} \circ \cdots \circ \pi_i \) for \( j > i \)

\( \pi_{i+1} \circ \leftarrow \pi_{i+1} = \leftarrow \pi_i \quad \text{and} \quad \pi_0 \circ \leftarrow \pi_0 = \leftarrow \pi \)

Since \( \pi_i^{-1}(A_i) \) is clopen in \([T_i]\), \( \leftarrow \pi_i^{-1}(A_i) \) is clopen in \( \leftarrow T \)

Thus, \( \leftarrow \pi^{-1}(A) = \bigcup_{i \in \mathbb{N}} \leftarrow \pi_i^{-1}(A_i) \) is open in \( \leftarrow T \).

Apply the open case to get

set \( Y_\omega \), tree \( S_\omega \subseteq Y_\omega^{<\omega} \)

clopen subset \( C_\omega \) of \( [T_\omega] \) st

onto map \( \pi_\omega : [T_\omega] \rightarrow \leftarrow T \)

\[ C_\omega = \pi_\omega^{-1}(\leftarrow \pi^{-1}(A)) \]

Finally,

\[ C_\omega = (\leftarrow \pi \circ \pi_\omega)^{-1}(A) \]
The topological problem: end of proof

The family of sets $A \subseteq X^\omega$ for which the problem has a solution $(Y, S, C, \pi)$

- contains the open subsets of $X^\omega$
- is closed under countable unions
- is (trivially) closed under complementation

\[\downarrow\]

Topological problem solved

for all Borel subsets of $X^\omega$

There is no cardinal explosion: $Y$ has cardinality of $X$
Martin’s proof: Covering of a pruned tree

$Strat_I(S)$ is the set of non deterministic I-strategies where both players have to stay in the pruned tree $S$

$k$-covering of a pruned tree $S \subseteq X^{<\omega}$

\[
\begin{align*}
\text{pruned tree} & \quad T \subseteq Y^{<\omega} \\
\text{monotone length preserving preserving surjective map} & \quad \pi : T \rightarrow S \\
\text{map} & \quad \phi_I : Strat_I(T) \rightarrow Strat_I(S) \\
\text{map} & \quad \phi_{II} : Strat_{II}(T) \rightarrow Strat_{II}(S)
\end{align*}
\]

such that

1. $Y^{\leq 2k} \cap T = X^{\leq 2k} \cap S$ \hspace{1cm} (2$k = k$ moves of $I + k$ moves of $II$)
2. $\mathbf{\phi_I : Strat_I(T) \rightarrow Strat_I(S)}$ and $\phi_{II}$ are local:

$$\forall \beta \in Strat_I(T) \forall u \in S \quad \phi_I(\beta)(u) \text{ depends on } \beta \upharpoonright \{v \mid |v| \leq |u|\}$$

3. Plays in $S$ where $I$ follows $\phi_I(\beta)$ can be lifted to plays in $T$ where $I$ follows $\beta$. \hspace{1cm} Idem with $II$ and $\phi_{II}$

$$\forall \beta \in Strat_I(T) \forall f \in [S] \exists g \in [T] \quad (f \in [\phi_I(\beta)] \implies (g \in [\beta] \land \pi(g) = f))$$
Martin’s proof: unravelling and determinacy

\[ S \subseteq X^{<\omega} \text{ pruned tree and } A \subseteq [S] \]

\[ k\text{-covering } (Y, T, \pi, \phi_I, \phi_{II}) \text{ of } S \]

unravels \[ A \subseteq [S] \]

if \[ \pi^{-1}(A) \text{ is clopen in } [T] \]

**Fact.** If some covering unravels \[ A \subseteq [S] \]

then the game \( \mathcal{G}_S(A) \) is determined

**Proof.** The clopen game \( \mathcal{G}_T(\pi^{-1}(A)) \) is determined

Let \( \beta \) be a ws for I (same argument with a ws for \( \text{II} \))

Lift any infinite play \( f \) in the \( S \)-game where I follows \( \phi_I(\beta) \)

to an infinite play \( g \) in the \( T \)-game where I follows \( \beta \)

Since \( \beta \) is a ws for I in \( \mathcal{G}_T(\pi^{-1}(A)) \), we have \( g \in \pi^{-1}(A) \)

Since \( \pi(g) = f \) we have \( f \in A \).

Thus, \( \phi_I(\beta) \) is a ws for I in \( \mathcal{G}_S(A) \)
Martin’s proof: unravelling closed sets

Space where we play: pruned tree $S \subseteq X^{<\omega}$

Game $\mathcal{G}_T(A)$ closed set $A = [J] \subseteq [S]$ $J$ pruned subtree of $S$

Copy of the set $(X^2)^{<\omega}$: $E = \{\ulcorner u \urcorner | u \in X^{<\omega}, |u| \text{ even} \}$

$k$-covering to unravel $A$

\[
\begin{align*}
Y &= X \cup Y_I \cup Y_{II}^+ \cup Y_{II}^- \\
Y_I &= X \times \times \text{Strat}_I(S) \\
Y_{II}^+ &= \bigcup_{\alpha \in \text{Strat}_I(S)} X \times \text{Strat}_{II}(\alpha) \\
Y_{II}^- &= X \times E
\end{align*}
\]

$\tilde{T} = \text{prefixes of } X^{2k} \times Y_I \times (Y_{II}^+ \cup Y_{II}^-) \times X^{<\omega}$

(Only moves $y_{2k}$ and $y_{2k+1}$ are not in $X$)

$T = \text{sequences in } \tilde{T} \text{ such that... }$ (see next slide)
$k$-covering to unravel $A$

$A = [J]$ with $J$ subtree of $S$

$$Y = X \cup Y_I \cup Y_{II}^+ \cup Y_{II}^-$$

$$\{\begin{array}{l}
Y_I = X \times \times Strat_I(S) \\
Y_{II}^+ = \bigcup_{\alpha} X \times Strat_{II}(\alpha) \\
Y_{II}^- = X \times E
\end{array}$$

$$\tilde{T} = \text{prefixes of } X^{2k} \times Y_I \times (Y_{II}^+ \cup Y_{II}^-) \times X^{<\omega}$$

(Only moves $y_{2k}$ and $y_{2k+1}$ are not in $X$)

$T = \text{sequences in } \tilde{T} \text{ such that}$

1. if $I$ chooses $(x_{2k}, \sigma_I)$ then $I$ follows $\sigma_I$ afterwards
2. if $II$ chooses $(x_{2k+1}, \sigma_{II})$ then $\sigma_{II}$ is a subtree of $J$ and of $\sigma_I$
   and $II$ follows $\sigma_{II}$ afterwards
3. if $II$ chooses $(x_{2k+1}, \lceil u \rceil)$ then $|u|$ even, $x_0 \ldots x_{2k+1} u \in \sigma_I \setminus J$
   and every extension of $x_0 \ldots x_{2k+1}$ in $T$ is compatible with $x_0 \ldots x_{2k+1} u$
   (thus, $T$ forces the players to play $u$ after $x_0 \ldots x_{2k+1}$)

(The infinite play is in $A$ in case (2) and outside $A$ in case (3))

$\pi : T \to S$ is the obvious map such that

$\pi(x_0 \ldots x_{2k-1}(x_{2k}, \sigma_I)(x_{2k+1}, \sigma_{II} \text{ or } \lceil u \rceil)x_{2k+2} \ldots x_n) = x_0 \ldots x_n$

$\phi_I$ and $\phi_{II} \ldots \text{see next slides}$
\( \phi_I \)  strategy for I in \( T \)  \( \leadsto \)  \( \phi_I(\beta) \)  strategy for I in \( S \)

Suppose II plays \( x_1, x_3, \ldots \) in the \( S \)-game

- For its \( k \) first moves \( x_0, x_2, \ldots, x_{2k-2} \) in the \( S \)-game, \( \phi_I(\beta) \) tells I to follow what strategy \( \beta \) does in the \( T \)-game.

- If strategy \( \beta \) in the \( T \)-game tells I to play \( (x_{2k}, \sigma_I) \) then strategy \( \phi_I(\beta) \) in the \( S \)-game tells I to play \( x_{2k} \)

- After II has played \( x_{2k+1} \) in the \( S \)-game, player I has to imagine a corresponding move \( (x_{2k+1}, ?) \) in the \( T \)-game

\( \phi_I \)  Case 1. I has a ws \( \alpha \) in \( G_{\tilde{\sigma}_I}([\sigma_I] \setminus A) \)

\[ \tilde{\sigma}_I = \{ v \in \sigma_I \mid \text{x compatible with } x_0 \cdots x_{2k+1} \} \]

\( \phi_I(\beta) \) tells I to follow this strategy \( \alpha \)

At some step the play is \( x_0 \cdots x_{2k+1}u \) in the open set \( [\sigma_I] \setminus A \), Then \( L = x_0 \cdots x_{2k-1}(x_{2k}, \sigma_I)(x_{2k+1}, \lceil u \rceil)u \in T \)

From now on, \( \phi_I(\beta) \) in the \( S \)-game tells I to follow what \( \beta \) tells for a play extending \( L \) (in the \( T \)-game)

The lifting property holds
Case 2. II has a ws in $G_{\widetilde{\sigma}_I}([\sigma_I] \setminus A)$

$$\widetilde{\sigma}_I = \{ v \in \sigma_I \mid x \text{ compatible with } x_0 \cdots x_{2k+1} \}$$

Let $\delta$ be the defensive strategy of II which allows him to stay in the closed set $[\sigma_I] \cap A$

As long as II plays in $\delta$, strategy $\phi_I(\beta)$ tells I to follow strategy $\beta$ assuming that I has played $(x_{2k}, \sigma_I)$ and II has played $(x_{2k+1}, \delta)$ in the $T$-game

If II leaves his defensive strategy $\delta$ at play $v = x_0 \cdots x_n$ then I gets a ws (for the subtree of $\sigma_I$ of sequences compatible with $v$) and we can argue as in Case 1.

The lifting property holds
Suppose I plays $x_0, x_1, \ldots$ in the $S$-game

- For its $k$ first moves $x_1, x_3, \ldots, x_{2k-1}$ in the $S$-game, $\phi_{II}(\beta)$ tells II to follow what strategy $\beta$ tells in the $T$-game.

- After I has played $x_{2k}$ in the $S$-game, Player II has to imagine a corresponding move $(x_{2k}, \sigma_I)$ in the $T$-game.

Let $Z = \text{set of } x_{2k+1}u \text{ st } |u| \text{ even and there is I-strat. } \sigma_I \text{ in the } S\text{-game st } \beta \text{ tells II to play } (x_{2k+1}, \lceil u \rceil) \text{ if I plays } (x_{2k}, \sigma_I) \text{.}$

Consider the $(S \cap (x_1 \cdots x_{2k})X^{<\omega})$-game where II wins if the infinite play is in $U = S \cap (ZX^{<\omega})$.

**Case 1. II has a ws in this game**

$\phi_{II}(\beta)$ tells II to follow this strategy until the play enters $U$, say at $u$. Let $\sigma_I$ witness that $u \in U$.

Afterwards, $\phi_{II}(\beta)$ tells II to follow $\beta$ on the $T$-game where the special moves are $(x_{2k}, \sigma_I), (x_{2k+1}, \lceil u \rceil)$.

The lifting property holds.
\( \phi_{II} \) **Case 2. I has a ws in this game**

Let \( \delta \) be the defensive strategy of I which allows him to put the play in the closed set

If I plays \((x_{2k}, \delta)\) then \( \beta \) cannot ask II to play \((x_{2k+1}, \Gamma u^\perp)\).

Else \( x_0 \cdots x_{2k+1} u \in U \) contradicting the fact that II cannot leave the defensive I-strategy \( \delta \).

Thus, \( \beta \) asks II to play some \((x_{2k+1}, \sigma_{II})\) in the \( T \)-game

As long as I plays in \( \sigma_{II} \), strategy \( \phi_{II}(\beta) \) tells II to follow strategy \( \beta \) assuming that I has played \((x_{2k}, \delta)\).

If I leaves his defensive strategy \( \delta \) then II has a ws and we can argue as in Case 1.

**The lifting property holds**
Martin's proof: inverse limits of coverings

**Fact.** If \((T_{i+1}, \pi_{i+1}, \phi^I_{i+1}, \phi^{II}_{i+1})\) is a \((k + i)\)-covering of \(T_i\) for \(i \in \mathbb{N}\) then there are a pruned tree \(T_\infty\) maps \(\pi_\infty, i, \phi^I_\infty, i, \phi^{II}_\infty, i\) such that

\[
\begin{cases}
(T_\infty, \pi_\infty, i, \phi^I_\infty, i, \phi^{II}_\infty, i) \text{ is a } (k + i)\text{-covering of } T_i \\
\pi_{i+1} \circ \pi_\infty, i+1 = \pi_\infty, i \\
\phi^I_{i+1} \circ \phi^I_\infty, i+1 = \phi^I_\infty, i \\
\phi^{II}_{i+1} \circ \phi^{II}_\infty, i+1 = \phi^{II}_\infty, i
\end{cases}
\]

**Proof.**

\(T_\infty = \) the \(u\)'s such that \(u \in T_i\) for all \(i\) st \(|u| \leq 2(k + i)\)

If \(|u| \leq 2(k + i)\) then \(\pi_\infty, i(u) = u\) else \(\pi_\infty, i(u) = \pi_{i+1} \circ \cdots \circ \pi_j(u)\) for any \(j\) st \(|u| \leq 2(k + j)\)

\(\phi^I_\infty, i, \phi^{II}_\infty, i\) : Similar because \(\phi^I_i\) is local:

\(\phi^I_i(\beta) \upharpoonright (S \cap X_{\leq i})\) depends only on \(\beta \upharpoonright (T \cap Y_{\leq i})\)
Lifting property

Suppose $\beta_\infty$ $I$-strategy in the $T_\infty$-game $f \in [\phi^I_\infty, i(\beta_\infty)] \subseteq [T_i]$

Lift $f$ to $f_{i+1}$ with $\pi_{i+1}$, then to $f_{i+2}$ with $\pi_{i+2}$, and so on...

Since $f_j \upharpoonright 2(k + i) = f_i \upharpoonright 2(k + i)$ for $j \geq i$
the $f_i$'s converge to $f_\infty$ such that $f_\infty \upharpoonright 2(k + i) = f_i \upharpoonright 2(k + i)$

$\pi_\infty, i(f_\infty) = f_i$

**Martin’s proof completed**

Closed sets are unravelled

Unravelling is closed under complementation

Unravelling is closed under countable unions

( use $i$-unravelling for the $i$-th set)

Conclusion: every Borel set can be unravelled
hence is determined (cf. page 56)
Thank you for your attention