Quasi-Polish Spaces and Choquet Games

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Two classes to be unified
Life in a non Hausdorff world
Scott domains
Quasi-Polish spaces
Choquet games
Approximation spaces

Two almost disjoint classes of topological spaces to be unified

Topology in mathematical Analysis

- **Polish spaces** (~ 1930's) $\begin{cases} \textit{Countable dense subset} \\ \textit{Metrizable } (\Rightarrow \textit{Hausdorff}) \\ \textit{by complete metric} \end{cases}$

$\mathbb{R}$	$L_2(\mathbb{R})$	$2^{\mathbb{N}}$	$\mathbb{N}^{\mathbb{N}}$	$[0,1]^{\mathbb{N}}$
	Hilbert	Cantor	Baire	Hilbert cube

- Universality: Polish $\approx G_\delta$ in $[0,1]^{\mathbb{N}}$
- Universality: Polish totally discontinuous $\approx$ closed in $\mathbb{N}^{\mathbb{N}} \approx G_\delta$ in $2^{\mathbb{N}}$

Rich Descriptive Set Theory

Other topological spaces

- Lusin spaces (weaken Polish topology)
- Suslin spaces (continuous images of Polish)

Topology in Algebra, Algebraic Geometry and Computer Science

often NON Hausdorff

- Zariski on $\mathbb{C}^n$ T_1
- Spectral spaces T_0 — Stone duality; Ring spectrum (Hochster)
- Scott domains T_0 (D. Scott ~ 1970)

ω-algebraic domains
ω-continuous domains

Differences and Analogies

Polish spaces		ω-algebraic & ω-continuous domains
Hausdorff	$\neq$	T_0
complete metric	$\approx$	directed complete partial order
Countable dense subset	$\approx$	Countable approximation basis

Intersection of these two classes
= discrete countable spaces

These theories can be unified

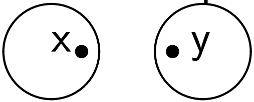
keeping **rich Descriptive Set Theory**

Breakthrough done by Matthew de Brecht, 2011

Life in a non Hausdorff world

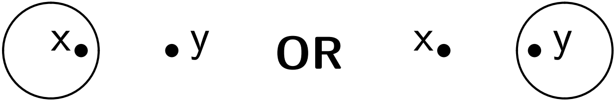
Some separation axioms

(T_2) Haussdorf   Open sets separate points



(T_1) Fréchet   Singleton sets are closed

(T_0) Kolmogorov Open sets distinguish points



$T_0 \rightsquigarrow$ Specialization order

$x \leq y$ iff $\forall \mathit{open}\ U\ (x \in U \Rightarrow y \in U)$ iff $x \in \overline{\{y\}}$

Open sets $\Longrightarrow$ up-sets

Borel hierarchy in a T_0 space E

DISTORSION at LEVEL 2

$$\begin{aligned}
\boldsymbol{\Sigma}^0_1(E) &= \textit{open subsets of } E \\
\boldsymbol{\Sigma}^0_2(E) &= \textit{countable unions of} \\
&\quad \textit{DIFFERENCES of open sets} \\
\boldsymbol{\Sigma}^0_\alpha(E) &= \textit{countable unions of sets in} \\
&\quad \bigcup_{\beta<\alpha} \boldsymbol{\Sigma}^0_\beta(E) \qquad \textit{in case } \alpha \geq 3 \\
\boldsymbol{\Pi}^0_\alpha(E) &= \{E \setminus X \mid X \in \boldsymbol{\Sigma}^0_\alpha(E)\} \\
\boldsymbol{\Delta}^0_\alpha(E) &= \boldsymbol{\Sigma}^0_\alpha(E) \cap \boldsymbol{\Pi}^0_\alpha(E)
\end{aligned}$$

CARE $\begin{cases} \mathbf{F}_\sigma(E) \subseteq \boldsymbol{\Sigma}^0_2(E) \\ \mathbf{G}_\delta(E) \subseteq \boldsymbol{\Pi}^0_2(E) \end{cases}$ may be strict $\subsetneq$

Scott domains

ω-algebraic domains: paradigmatic example

Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq)$

Countable Basis $= \mathcal{P}_{<\omega}(\mathbb{N})$

Algebraic: $\begin{cases} \text{Every } X = \text{union of directed set } \mathcal{P}_{<\omega}(X) \\ X \textit{ finite} \subseteq \bigcup_i Z_i \Longrightarrow \exists i \; X \subseteq Z_i \end{cases}$

Scott Topology of positive information

Basis: $\mathcal{O}_A = \{X \subseteq \mathbb{N} \mid A \subseteq X\}$, A finite

$\mathcal{O}_A$ open trivially quasi-compact NOT closed

$\subseteq$ = specialization order

Comparing with Cantor

(= topology of positive and negative information)

$$\mathbf{\Sigma}^0_n(\mathcal{P}(\mathbb{N})) \subsetneq \mathbf{\Sigma}^0_n(2^{\mathbb{N}}) \subsetneq \mathbf{\Sigma}^0_{n+1}(\mathcal{P}(\mathbb{N}))$$

$$\mathbf{\Sigma}^0_{\omega+\alpha}(\mathcal{P}(\mathbb{N})) = \mathbf{\Sigma}^0_{\omega+\alpha}(2^{\mathbb{N}})$$

Another example of ω-algebraic domain

$\widetilde{[0,1]} = [0,1] \cup (D_2 \times \{+\})$ $\qquad q \rightsquigarrow$ pair $q < (q,+)$

Duplicate $D_2 =$ dyadic rationals

$q < r < x \Longrightarrow (q,+) < (r,+) < x$

Countable Basis $= D_2 \times \{+\}$

Algebraic: $\begin{cases} \text{Every } x = \sup\{(q,+) \mid (q,+) \le x\} \\ (q,+) \le \sup_i x_i \Longrightarrow \exists i \ (q,+) \le x_i \end{cases}$

Poset $\widetilde{[0,1]} \approx (2^{\mathbb{N}}, \textit{lexico})$ $\qquad$ Gives sense to

$0.\varepsilon_1 \ldots \varepsilon_k 0111 \ldots < 0.\varepsilon_1 \ldots \varepsilon_k 1000 \ldots$

Example of algebraic domain

$(\omega_1 + 1, \leq)$ successor of first uncountable ordinal

Uncountable Basis = all successor ordinals

Algebraic: $\begin{cases} \text{Every ordinal is sup of successors} \\ \alpha + 1 \leq \sup_i \alpha_i \implies \exists i \ \alpha + 1 \leq \alpha_i \end{cases}$

Order topology: intervals $]\alpha, \omega_1]$

The Borel hierarchy collapses:

$$\begin{aligned} \text{Borel} &= \boldsymbol{\Sigma}^0_2 \cup \boldsymbol{\Pi}^0_2 \\ &= \text{countable or co-countable subsets} \end{aligned}$$

Example of ω-continuous domain

$([0,1], \leq)$
Continuous basis $= \{0\} \cup$ any dense set D
NOT algebraic:
Every $x \neq 0$ is non trivial sup of elements of D

$[0,1]$ is retract of $\widetilde{[0,1]} = [0,1] \cup D_2 \times \{+\}$

$$[0,1] \overset{\iota}{\hookrightarrow} \widetilde{[0,1]} \qquad \widetilde{[0,1]} \overset{p}{\twoheadrightarrow} [0,1] \qquad p \circ \iota = Id_{[0,1]}$$

identity $\qquad (q,+) \mapsto q$

Towards formal definitions of continuous/algebraic domains

INTUITION FROM COMPUTATIONS

- Put together possibly "infinitary" objects

& "finitary" approximations (= informations)

- Informations go increasing & are compatible

  $\Longrightarrow$ directed set and its sup

- Approximations may miss "negative" info.

  This is why $(\mathcal{P}(\mathbb{N}), \text{Scott}) \neq$ Cantor

- may never know if a computation is infinite
- recursively enumerable set = only positive info

Dcpo's and the way-below relation

- DCPO (directed complete poset)
  Every directed set has a supremum
- Relation "way-below" (or approximation):
  $x \ll y \iff \forall Z \; \textit{directed}$
  $(y \leq \sup Z \Rightarrow \exists z \in Z \; x \leq z)$

- x unavoidable piece of information for y
- x appears in any system of approximations of an element $\geq y$

In $\mathcal{P}(\mathbb{N})$ $\quad X \ll Y \iff (X \; \textit{finite} \wedge X \subseteq Y)$

Continuous/algebraic domains

$x \ll y \iff \forall Z$ *directed*

$$(y \leq \sup Z \Rightarrow \exists z \in Z\ x \leq z)$$

Continuous domain = dcpo + basis B s.t.

$\forall x \quad B \cap {\downarrow\!\!\!\downarrow} x$ *is directed* $\wedge \ x = \sup(B \cap {\downarrow\!\!\!\downarrow} x)$

every element is the directed sup of its unavoidable minorants

x compact if $x \ll x$ (any inequality $\sup Z \geq x$ is trivial: $\exists z \in Z\ z \geq x$)

Algebraic domain = dcpo + compact elements form a basis

ω-continuous/ω-algebraic = countable basis

Scott topology on a dcpo $(D, \leq)$

X Scott closed ≡ X down-set closed under directed sup

X Scott open ≡ X up-set only trivially accessible by directed sup

$\{x \mid x \not\leq a\}$ is Scott open

T_0 topology specialization order $= \leq$

Some properties of continuous domains

Continuous base B :

$B \cap \Downarrow x$ directed and $x = \sup(B \cap \Downarrow x)$

Interpolation in continuous domains.

"Density" $m \ll x \Rightarrow \exists y \;\; m \ll y \ll x$

Care! $u \ll v$ does not exclude $u = v$

Interpolation if M finite

$(\forall m \in M \;\; m \ll x) \Rightarrow \exists y \; \forall m \in M \;\; m \ll y \ll x$

Open sets in continuous domains.

$\{\Uparrow x \mid x \in B\}$ topological basis

U open iff $U = \bigcup_{x \in U} \Uparrow x$ iff $U = \bigcup_{x \in U \cap B} \Uparrow x$

Quasi-Polish spaces

Quasi-metric

Give up the symmetry axiom of metrics

Quasi-metric on E

map $d : E \times E \to [0, +\infty[$ such that

$$x = y \iff d(x,y) = d(y,x) = 0$$

$$d(x,z) \leq d(x,y) + d(y,z)$$

Topology generated by open balls

$$B_d(a,r) = \{x \in E \mid d(a,x) < r\}$$

Fundamental example: $\mathcal{P}(\mathbb{N})$ is quasi-metric

$$d(X,Y) = \sup\{2^{-n} \mid n \in X \smallsetminus Y\}$$

$d(A,Y) < 2^{-n} \iff A \cap \{p \mid p \leq n\} \subseteq Y$

$\{Y \mid A \subseteq Y\} = \bigcap_{a \in A} B_d(\{a\}, 2^{-a})$

Quasi-metric versus metric

$$d^{-1}(x,y) = d(y,x)$$
$$\widehat{d}(x,y) = \max(d(x,y), d(y,x))$$

$$(E,d) \text{ quasi-metric} \Rightarrow \begin{cases} (E, d^{-1}) & \textit{quasi-metric} \\ (E, \widehat{d}) & \textit{metric} \end{cases}$$

(Kunzi) (E,d) has countable base
iff $(E,\widehat{d})$ has countable dense set

Quasi-Polish spaces

Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ if $\lim_{n\to+\infty} \sup_{p\geq n} d(x_n, x_p) = 0$

Complete quasi-metric Every Cauchy sequence converges wrt the metric $\widehat{d}(x,y) = \max(d(x,y), d(y,x))$

Quasi-Polish space (Hans Peter Kunzi) Topology associated to a complete quasi-metric with countable topological basis

De Brecht results on quasi-Polish spaces

$B_{d^{-1}}(a,r)$, $B_{\widehat{d}}(a,r)$ are $\boldsymbol{\Sigma}^0_2(E,d)$

(X,d) quasi-Polish $\Longrightarrow$ $(X,\widehat{d})$ Polish

$\mathrm{Borel}(E,d) = \mathrm{Borel}(E,\widehat{d})$

uncountable quasi-Polish $\Longrightarrow$ cardinal $2^{\aleph_0}$

Borel hierarchy does not collapse

Polish spaces ω-continuous domains	} are quasi-Polish

Metrizable + quasi-Polish $\Rightarrow$ Polish

De Brecht results on quasi-Polish spaces

- Baire property for open hence for $\mathbf{G}_\delta$ sets.

  Also true for $\mathbf{\Pi}^0_2$ sets (Becher & SG)

- Hausdorff-Kuratowski property: for $\beta \geq 1$

$$\mathbf{D}^0_{\beta+1} = \bigcup_{\alpha<\omega_1} \mathbf{D}(\mathbf{\Sigma}^0_\alpha)$$

De Brecht results on quasi-Polish spaces

quasi-Polish $\equiv \mathbf{\Pi}^0_2$ in $\mathcal{P}(\mathbb{N})$ (Scott topo.)

$\{X \mid \forall i\ (2i \in X \Leftrightarrow 2i+1 \notin X)\} \approx$ *Cantor* $2^{\mathbb{N}}$

$\mathbf{\Pi}^0_2$ in Scott $\mathcal{P}(\mathbb{N})$

In general,

if $(U_n)_{n\in\mathbb{N}}$ countable open base in X

then $x \mapsto \{n \mid x \in U_n\}$ is an embedding $X \to \mathcal{P}(\mathbb{N})$

Choquet games

Banach-Mazur and Choquet games

X topological space

Banach-Mazur game $BM(X)$

ω rounds, Two players Empty, NonEmpty

alternatively choose non empty open sets

Empty chooses the U_i's, NonEmpty the V_i's

so that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$

Empty wins iff $\bigcap_{i \in \mathbb{N}} U_i = \emptyset$

Choquet game $Ch(X)$ Variant of $BM(X)$

At round i Empty also chooses $x_i \in U_i$ and then NonEmpty picks $V_i \subseteq U_i$ s.t. $x_i \in V_i$.

Special winning strategies

Convergent ws for NonEmpty: The V_i's are a basis of neighborhoods of some $x \in \bigcap_{i \in \mathbb{N}} U_i$

Markov winning strategy: depends only on
- the last move of the opponent
- and the ordinal rank of the move

Stationary winning strategy: depends only on the last move of the opponent

Special ws in the Banach-Mazur game

(Galvin & Telgarsky, 1986)

(1) If NonEmpty has a ws in $BM(X)$ (resp. & convergent) then it has one which depends only on the last two moves (his and that of Empty)

(2) If NonEmpty has a Markov ws in $BM(X)$ (resp. & convergent) then it has one which is stationary

(Debs, 1984)

(1) cannot be improved to stationary

Games and topology

(Oxtoby, 1957) X has the Baire property
iff Empty has no ws in $BM(X)$

(Choquet, 1969) X is Polish iff it is T_1, regular,
and NonEmpty has a ws in $Ch(X)$

(de Brecht, 2011) X is quasi-Polish iff
it is T_0, has a countable basis and
NonEmpty has a convergent ws in $Ch(X)$
(which can also be taken Markov)

(Becher & SG, 2012) Idem as above
with stationary in place of Markov

Approximation spaces

A domain approach to quasi-Polish spaces

(V.Becher & SG, 2012)

Approximation relation $\ll$ on E topological space

$=$ binary relation on a topological base $\mathcal{B}$ s.t.

(1) $U \ll V \Rightarrow V \subseteq U$ *more information in V than U*

(2) $U \subseteq T$ and $U \ll V \Rightarrow T \ll V$

(3) $\forall x \in U \; \exists W \in \mathcal{B} \; (x \in W \wedge U \ll W)$

(4) $U_i \ll U_{i+1}$ for all $i \in \mathbb{N} \Rightarrow \bigcap_{i \in \mathbb{N}} U_i \neq \emptyset$

- $\ll$ convergent approx. relation if

(4bis) $=$ (4) $+$ the V_i's are a neighborhood basis for some $x \in \bigcap_{i \in \mathbb{N}} U_i$.

Flavor of "way-below" relation on continuous dcpo's

$\{(\uparrow x, \uparrow y) \mid x, y \in B,\; x \ll y\}$ is an approx. relation wrt Scott topology if B base of continuous dcpo

Approximation spaces and quasi-Polish spaces

- If there is approximation relation on one base

then there is some in each base

- (V.Becher & SG, 2012) A space is quasi-Polish iff

- it is T_0, has a countable base

- and has a convergent approximation relation

- (V.Becher & SG, 2012) X has an

approximation (resp. convergent) relation

iff NonEmpty has stationary

(resp. & convergent) ws in $Ch(X)$

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Thank you for your attention