On Lattices of Regular Sets of Natural Integers Closed under Decrementation

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Abstract

We consider lattices of regular sets of non negative integers, i.e. of sets definable in Presburger arithmetic. We prove that if such a lattice is closed under decrement then it is also closed under many other functions: quotients by an integer, roots, etc.

Keywords. Lattices, lattices of subsets of $\mathbb{N}$, regular subsets of $\mathbb{N}$, closure properties.

1 Introduction

1.1 Roadmap

We follow the terminology according to which a function $f : \mathbb{N} \to \mathbb{N}$ is non decreasing if $a \leq b \Rightarrow f(a) \leq f(b)$ for all $a, b \in \mathbb{N}$.

We prove in this paper the following result:

Theorem 1.1. *Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a non decreasing function. The following conditions are equivalent:*

(1) *Every lattice $\mathcal{L}$ of regular subsets of $\mathbb{N}$ which is closed under decrement (i.e. $L \cap L'$, $L \cup L'$ and $L - 1$ are in $\mathcal{L}$ whenever $L, L' \in \mathcal{L}$) is also closed under f^{-1} (i.e. $L \in \mathcal{L}$ implies $f^{-1}(L) \in \mathcal{L}$).*

(2) *The function f satisfies the following properties:*
(i) $f(a) \geq a$ for all $a \in \mathbb{N}$,
(ii) $f(a) - f(b) \equiv 0 \pmod{(a - b)}$ for all $a, b \in \mathbb{N}$.

Particular exemples of such functions f are division by n and n-root for any $n \geq 1$.

This problem, for finite sets and division by n, was submitted to us by Jean-Éric Pin & Zoltán Ésik, [2]. Jean-Éric Pin & Pedro Silva announce, in the framework of profinite topologies and uniformly continuous fonctions, a result related to our theorem 1.1 (see [4, 5]).

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Any regular subset L of $\mathbb{N}$ is ultimately periodic (cf. Lemma 1.9). For an arithmetic progression L, the fact that $f^{-1}(L)$ is a union of decrements of L is an easy result (cf. Proposition 4.1). Difficulties arise with:

(1) the finite set coming from the grouping of arithmetic progressions which constitutes the periodic part of L,

(2) the other finite set before periodicity (these two finite sets are the sets B and A of Proposition 1.8).

Prior to the general result (cf. Theorems 3.2 & 5.1), we prove particular instances, namely division by n and nth root, which give a clearer insight to the proof (cf. Theorems 2.1 and 2.2).

1.2 Lattices closed under decrementation

We recall some definitions and fix some notation.

Definition 1.2. *A lattice $\mathcal{L}$ over a set X is any non empty family of subsets of X such that $L \cup M$ and $L \cap M$ are in $\mathcal{L}$ whenever L, M are in $\mathcal{L}$.*

Definition 1.3. *Let L be a subset of $\mathbb{N}$, $i \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$. The sets*

$$L-i = \{x \in \mathbb{N} \mid x+i \in L\} \ , \ L \div k = \{x \in \mathbb{N} \mid kx \in L\} \ , \ \sqrt[k]{L} = \{x \in \mathbb{N} \mid x^k \in L\}$$

*are respectively called the i-*decrement*, k-*quotient *and k-*root *of L. Observe that the i-decrement is defined as a subset of $\mathbb{N}$, excluding negative integers.*

Let $\mathcal{D}(L)$ denote the family $\{L - i \mid i \in \mathbb{N}\}$ of decrements of L.

Example 1.4. 1) Let $L = \{5, 6\} + 4\mathbb{N} = \{5, 6, 9, 10, 13, 14, \ldots\}$, then $L \div 2 = 3 + 2\mathbb{N} = \{3, 5\} + 4\mathbb{N}$. Moreover, for any integer x, $x^2 \equiv 0 \pmod 4$ or $x^2 \equiv 1 \pmod 4$, hence

$$\begin{aligned} x^2 \in \{5,6\} + 4\mathbb{N} \quad &\Longleftrightarrow \quad x^2 \geq 5 \wedge x^2 \equiv 1 \pmod 4 \\ &\Longleftrightarrow \quad x \geq 3 \wedge x \equiv 1, 3 \pmod 4 \\ &\Longleftrightarrow \quad x \in \{3,5\} + 4\mathbb{N} \end{aligned}$$

Hence also $\sqrt{L} = \{3, 5\} + 4\mathbb{N} = L \div 2$.

2) Let $L = \{1, 2\} + 4\mathbb{N}$, then $L \div 3 = \{2, 3\} + 4\mathbb{N}$ and $\sqrt{L} = \{1, 3\} + 4\mathbb{N}$.

The following results are straightforward.

Proposition 1.5 (Composing decrements)**.** *$(L - i) - j = L - (i + j)$.*

Proposition 1.6. *For $L \subseteq \mathbb{N}$ let $\mathcal{L}(L)$ be the family of sets of the form $\bigcup_{j \in J} \bigcap_{i \in I_j} (L - i)$ where J and the I_j's are finite non empty subsets of $\mathbb{N}$. Then the family $\mathcal{L}(L)$ is the smallest sublattice of $\mathcal{P}(\mathbb{N})$ containing L and closed under decrement.*

Proof. Observe that $\left(\bigcup_{j \in J} \bigcap_{i \in I_j} (L - i)\right) - k = \bigcup_{j \in J} \bigcap_{i \in I_j} \left(L - (i + k)\right)$. $\square$

1.3 Regular sets of natural integers

Definition 1.7. *1. A set $L \subseteq \mathbb{N}$ is periodic with period r if, for every x, $x \in L \Longrightarrow x + r \in L$.*
2. A set $L \subseteq \mathbb{N}$ is ultimately periodic with period r if there exists $q \in \mathbb{N}$ such that $L \cap \{x \mid x \geq q\}$ is periodic with period r, i.e. for every $x \geq q$, $x \in L \Longrightarrow x + r \in L$.

As we here we work with a semigroup and not a group, namely $(\mathbb{N}, +)$, the definition of periodicity is not given by an equivalence $x \in L \Longleftrightarrow x + r \in L$ but by an implication $x \in L \Longrightarrow x + r \in L$.

Regular subsets of $\mathbb{N}$ are subsets which are recognized by finite automata in unary notation (cf. [1], pages 100–103). Here, we will only use the following classical characterization of regular subsets of $\mathbb{N}$ which goes back to Myhill, 1957 [3]. Recall that an arithmetic progression is a subset of $\mathbb{N}$ of the form $a + r\mathbb{N}$.

Proposition 1.8. *Let $L \subseteq N$. The following conditions are equivalent:*

(i) *L is regular,*

(ii) *L is the union of a finite set with finitely many arithmetic progressions,*

(iii) *$L = A \cup (q + B + r\mathbb{N})$, where $q \in \mathbb{N}$, $r \in \mathbb{N} \setminus \{0\}$, $A \subseteq \{0, 1, \ldots, \max(0, q-1)\}$ and $B \subseteq \{0, 1, \ldots, r-1\}$.*

Observe that in case $B = \emptyset$, the set $A \cup (q + B + r\mathbb{N})$ reduces to the finite set A. The following lemmas will be useful.

Lemma 1.9. *Any regular set is ultimately periodic and its family of decrements is finite.*

More precisely, suppose $L = A \cup (q + B + r\mathbb{N}) \subseteq \mathbb{N}$ where $q \in \mathbb{N}$, $r \in \mathbb{N} \setminus \{0\}$, $A \subseteq \{0, 1, \ldots, q-1\} \cap \mathbb{N}$, and $B \subseteq \{0, 1, \ldots, r-1\}$. Then

(1) $\forall x \geq q \ (x \in L \Longleftrightarrow x + r \in L)$

(2) *The family $\mathcal{D}(L)$ of decrements of L is equal to $\{L - i \mid 0 \leq i < q + r\}$.*

Proof. (1) Let $x \geq q$, so that $x = q + i + kr$ for some $0 \leq i < r$, $k \geq 0$. Then $x \in L = A \cup (q + B + r\mathbb{N}) \iff q + i + kr \in q + B + r\mathbb{N} \iff i \in B$. Similarly, $x + r \in L \iff q + i + (k+1)r \in q + B + r\mathbb{N} \iff i \in B$. Thus, $x \in L \iff x + r \in L$.
(2) Let $j \geq q$. Then $j = q + i + kr$ for some $0 \leq i < r$, $k \geq 0$. For any $x \in \mathbb{N}$, we have $x \in L - j \Leftrightarrow x + j \in L \Leftrightarrow x + q + i + kr \in L \Leftrightarrow x + q + i \in L \Leftrightarrow x \in L - (q + i)$, the third in place of equivalence being obtained by applying k times point (1). □

Example 1.10. (Example 1.4 continued) 1) For $L = \{5, 6\} + 4\mathbb{N}$, the set $\mathcal{D}(L)$ consists of 7 sets $L, L - 1 = \{4, 5\} + 4\mathbb{N}, \ldots, L - 5 = \{0, 1\} + 4\mathbb{N}, L - 6 = \{0, 3\} + 4\mathbb{N}, L - 7 = \{2, 3\} + 4\mathbb{N} = L - 3$.

2) If $L = \{1, 2\} + 4\mathbb{N}$, then $\mathcal{D}(L) = \{L, \{0, 1\} + 4\mathbb{N}, \{0, 3\} + 4\mathbb{N}, \{2, 3\} + 4\mathbb{N}\}$.

In case of an arithmetic progression, Proposition 1.6 can be simplified.

Lemma 1.11. *Let $L = q + r\mathbb{N}$ be the range of an arithmetic sequence, $r > 0$.*
1. The family $\mathcal{D}(L)$ of decrements of L is equal to

$$\mathcal{D}(L) = \{s + r\mathbb{N} \mid 0 \leq s \leq \max(r-1, q)\} = \{L - j \mid 0 \leq j \leq \max(r-1, q)\} .$$

2. The smallest lattice $\mathcal{L}(L)$ containing L and closed under decrement is equal to the family of sets *(i)* $\{A + r\mathbb{N} \mid A \subseteq \{0, \ldots, \max(r-1, q)\}$ *if* $r \geq 2$,

(ii) $\{s + \mathbb{N} \mid 0 \leq s \leq q\}$ *if* $r = 1$.

In particular, every nonempty set of $\mathcal{L}(L)$ *is a finite union of decrements of* L*, and the empty set is in* $\mathcal{L}(L)$ *just in case* $r \geq 2$ *(obtained with* $A = \emptyset$*).*

Proof. 1. In case $j \leq q$ then $L - j = s + r\mathbb{N}$ with $s = q - j \leq q$. If $j \geq q$, i.e. $j = q + i + kr$ with $0 \leq i < r$ and $k \in \mathbb{N}$, then $L - j = \{x \in \mathbb{N} \mid x + (q + i + kr) \in q + r\mathbb{N}\} = \{x \in \mathbb{N} \mid x + i \in r\mathbb{N}\} = r\mathbb{N} - i$. If $i = 0$ then $L - j = r\mathbb{N} = L - q$. If $0 < i < r$ then $L - j = r\mathbb{N} - i = (r - i) + r\mathbb{N} = L - (q - (r - i))$.
2. Observe that the intersection of two sets in the family $\mathcal{D}(L)$ is either empty (possible in case $r \geq 2$ only) or equal to the smallest one. Then apply Proposition 1.6, noting that for $r = 1$, $A + r\mathbb{N} = \min(A) + \mathbb{N}$. □

2 Closure under quotient and root

The following result was suggested for lattices of finite sets by Ésik & Pin [2].

Theorem 2.1. *Any lattice of regular subsets of* $\mathbb{N}$ *which is closed under decrement is also closed under* k*-quotient, for* $k \in \mathbb{N} \setminus \{0\}$*.*

Proof. The case $k = 1$ is trivial. We prove the theorem by induction on $k \geq 1$. For pedagogical reasons, we explicit the case $k = 2$.

Case $k = 2$. Consider some $L \in \mathcal{L}$ and let $J_a = (L - a) \cap \bigcap_{i \in L - a} (L - i)$ for any $a \in \mathbb{N}$. By Lemma 1.9), there are finitely many distinct sets $(L - i)$'s, so that J_a is a finite intersection of decrements of L. The assumed closure properties of $\mathcal{L}$ insure that $J_a \in \mathcal{L}$.

In case $a \in L \div 2$, i.e. $2a \in L$, the following properties are true.

(1) $a \in J_a$. In fact, $a \in L - i$ for any $i \in L - a$ and $a \in L - a$ because $2a \in L$.

(2) $J_a \subseteq L \div 2$. Indeed, if $b \in J_a$ then $b \in L - a$ and b is in all the $(L - i)$'s, for $i \in L - a$. Letting $i = b$, we get $b \in L - b$, i.e $2b \in L$ and $b \in L \div 2$.

Since there are finitely many $L - a$'s, there are finitely many J_a's. Using closure under finite union, we see that $K = \bigcup_{a \in L \div 2} J_a$ is in $\mathcal{L}$. Clearly, $K = L \div 2$ because each element $a \in L \div 2$ is in J_a and each J_a is included in $L \div 2$.

Inductive case. Assuming $\mathcal{L}$ is closed under k-quotient, we prove that it is closed under $(k+1)$-quotient. For $L \in \mathcal{L}$, set $J_a = ((L - a) \div k) \cap \bigcap_{i \in (L - a) \div k} (L - ki)$ for any $a \in \mathbb{N}$. By Lemma 1.9, there are finitely many distinct $(L - i)$'s, so that J_a is a finite intersection of decrements of L and of a k-quotient of L. The assumed closure properties of $\mathcal{L}$ and induction hypothesis insure that $J_a \in \mathcal{L}$.

In case $a \in L \div (k+1)$, i.e. $(k+1)a \in L$, the following properties are true.

(1) $a \in J_a$. In fact, $a \in L - ki$ for any $i \in (L - a) \div k$. Also, since $(k+1)a \in L$, we have $ka \in L - a$ hence $a \in (L - a) \div k$.

(2) $J_a \subseteq L \div (k+1)$. If $b \in J_a$ then $b \in (L - a) \div k$ and b is in all the $L - ki$'s, for $i \in (L - a) \div k$. Letting $i = b$, we get $b \in L - kb$, i.e $(k+1)b \in L$ and $b \in L \div (k+1)$.

Since there are finitely many $(L - a)$'s, there are finitely many $(L - a) \div k$'s hence finitely many J_a's. Using closure under finite union, we see that the set $K = \bigcup_{a \in L \div (k+1)} J_a$ is in $\mathcal{L}$. Clearly, $K = L \div (k+1)$ because each element $a \in L \div (k+1)$ is in J_a and each J_a is included in $L \div (k+1)$. □

Theorem 2.2. *Any lattice of regular subsets of* $\mathbb{N}$ *which is closed under decrement is also closed under* k*-root, for* $k \in \mathbb{N} \setminus \{0\}$.

Proof. Adapt the above proof: substitute $\times$ and division for $+$ and subtraction, so that $L - i$ becomes $L \div i$. In the argument, finiteness of the family $\{L - i \mid i \in \mathbb{N}\}$ is replaced by that of $\{L \div k \mid k \in \mathbb{N} \setminus \{0\}\}$ which holds since, by Lemma 1.9 and Proposition 1.6, $\mathcal{L}(L)$ is always finite when L is regular. $\square$

Example 2.3. (Examples 1.4 and 1.10 continued) If $L = \{1, 2\} + 4\mathbb{N}$, then $L \div 3 = \{2 + 4\mathbb{N}\} \cup \{3 + 4\mathbb{N}\} = L - 3$ and $\sqrt{L} = \{1, 3\} + 4\mathbb{N} = (L - 5) \cup (L - 3)$.

For $L = \{5, 6\} + 4\mathbb{N}$, we have $L \div 2 = \sqrt{L} = \{3, 5\} + 4\mathbb{N} = \big((L - 2) \cap (L - 3)\big) \cup \big(L \cap (L - 1)\ \big)$.

3 More induced closures

We extend closure under quotient (cf. Theorem 2.1) and under n-root (cf. Theorem 2.2) to a more general class of functions $f : \mathbb{N} \to \mathbb{N}$. Given a regular set $L \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, the set $L - n = \{x \in \mathbb{N} \mid x + n \in L\}$ is regular. Also, by Lemma 1.9, the family $\{L - n | n \in \mathbb{N}\}$ is finite.

Lemma 3.1. *For any set* $L \subseteq \mathbb{N}$ *and for any function* $f : \mathbb{N} \to \mathbb{N}$ *such that* $f(x) - f(y) \in (x - y)\mathbb{N}$ *for every* $x, y \in \mathbb{N}$*, and such that* $f(x) \geq x$ *for every* $x \in \mathbb{N}$*, we have:*

$$f^{-1}(L) \;=\; \bigcup_{a \in f^{-1}(L)} \left(\bigcap_{n \in L - a} L - n \right) \tag{1}$$

Proof. Let us first consider $a \in f^{-1}(L)$. Notice that for every $n \in L - a$, we have $a + n \in L$ and thus $a \in L - n$. We deduce that a is in $\bigcap_{n \in L - a} L - n$ and the inclusion $\subseteq$ is proved.

For the other inclusion, let $a \in f^{-1}(L)$ and $b \in \bigcap_{n \in L - a} L - n$. By the assumption on f, there exists $k \in \mathbb{N}$ such that $f(a) - f(b) = k(a - b)$. Assume by contradiction that $f(b) \notin L$. Since $f(a) \in L$ we get $f(a) \neq f(b)$, and in particular $a \neq b$.

Assume first that $a < b$. We consider the minimal natural number $r \in \mathbb{N}$ such that $f(a) + r(b - a) \notin L$. Note that such a natural number exists since $f(a) + k(b - a) = f(b) \notin L$. Moreover, since $f(a) \in L$ we get $r \geq 1$. By minimality of r, we get $f(a) + (r - 1)(b - a) \in L$. Thus, $n + a \in L$ with $n = f(a) + r(b - a) - b$. Since $f(a) \geq a$, we get $n \geq (r - 1)(b - a) \geq 0$. Now $n + a \in L$ implies $n \in L - a$ and thus $b \in L - n$; hence $n + b = f(a) + r(b - a) \in L$, contradicting the definition of r.

Assume next that $a > b$ and consider the minimal natural number $r \in \mathbb{N}$ such that $f(b) + r(a - b) \in L$. Again, such a natural number exists since $f(b) + k(a - b) = f(a) \in L$. Moreover, since $f(b) \notin L$, we get $r \geq 1$. Let $n = f(b) - b + (r - 1)(a - b)$. Since $f(b) \geq b$ and $a - b \geq 0$ we get $n \geq 0$. Moreover, as $n + a = f(b) + r(a - b) \in L$, we get $n \in L - a$. Thus, $b \in L - n$ and we get $n + b \in L$. That means $n + b = f(b) + (r - 1)(a - b) \in L$ which contradicts the minimality of r.

We have proved by contradiction that $f(b) \in L$. Thus, $b \in f^{-1}(L)$ and we get the other inclusion. $\square$

We can now prove the (2) $\Rightarrow$ (1) implication of our main theorem 1.1.

Theorem 3.2. *Let $f : \mathbb{N} \to \mathbb{N}$ be non decreasing and such that (i) $f(a) \geq a$ and (ii) $f(a) - f(b) \equiv 0 \pmod{(a-b)}$ for all $a, b \in \mathbb{N}$. Every lattice of regular subsets of $\mathbb{N}$ closed under decrement is also closed under f^{-1}.*

Proof. Let $\mathcal{L}$ be a lattice of regular sets closed under decrement and let $L \in \mathcal{L}$. Consider the representation of $f^{-1}(L)$ given by formula (1) of Lemma 3.1. In order to ensure that $f^{-1}(L)$ belongs to the lattice $\mathcal{L}$, we have to show that both the intersection and the union are finite: since L is regular, the family $\{L - n | n \in \mathbb{N}\}$ is finite by Lemma 1.9; this concludes the proof. □

Remark 3.3. Every non decreasing polynomial with integral coefficients mapping $\mathbb{N}$ into $\mathbb{N}$ satisfies the conditions of Theorem 3.2. Thus, Theorems 2.1 and 2.2 are consequences of Theorem 3.2; their proof gives a first idea and a better understanding to prove the more general Theorem 3.2).

4 About arithmetic progressions

For arithmetic progressions we sharpen Theorem 3.2 and give a simpler proof.

Proposition 4.1. *Let $f : \mathbb{N} \to \mathbb{N}$ non decreasing be such that for all $a, b \in \mathbb{N}$ (i) $f(a) \geq a$ and (ii) $f(a) - f(b) \equiv 0 \pmod{(a-b)}$. For every arithmetic progression $L = q + r\mathbb{N}$, with $q, r \in \mathbb{N}$, $r \geq 1$, the following conditions hold:*

(1) *$f^{-1}(L)$ is the union of at most r decrements of L,*

(2) *the smallest lattice $\mathcal{L}(L)$ closed under decrement and such that $L \in \mathcal{L}$ is closed under f^{-1}.*

Proof. (1) If $f(a) \in q + r\mathbb{N}$ then, using monotonicity of f and property (ii), for every $k \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that $f(a + kr) = f(a) + \ell r$ hence $f(a + kr) \in q + r\mathbb{N}$ and $a + r\mathbb{N} \subseteq f^{-1}(L)$. Thus, $f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} a + r\mathbb{N}$. Now, if $a < b$ and $a \equiv b \pmod{r}$ then $b + r\mathbb{N} \subseteq a + r\mathbb{N}$. Hence the last equality can be rewritten $f^{-1}(L) = \bigcup_{a \in M} a + r\mathbb{N}$ where M picks the minimum element of $f^{-1}(L) \cap (i + r\mathbb{N})$ for each i such that $0 \leq i < r$ and $f^{-1}(L) \cap (i + r\mathbb{N})$ is nonempty. In particular, M has at most r elements.

It remains to show that, for each $a \in M$, the set $a + r\mathbb{N}$ is a decrement of L. Using Lemma 1.11, this amounts to show that $a \leq \max(r-1, q)$ for each $a \in M$. Let $a \in M$, $a = \min\big(f^{-1}(L) \cap (i + r\mathbb{N})\big)$ with $0 \leq i < r$. By way of contradiction, supposing $a > \max(r-1, q)$, so that $a - r \in \mathbb{N}$, we show that $f(a - r) \in L$.

Case $q < r$. Then $\max(r-1, q) = r-1 < a$. Since $f(a) \in L$ we have $f(a) = q + kr$ for some $k \in \mathbb{N}$. Using property (ii), we get $f(a) \equiv f(a-r) \equiv q \pmod{r}$. Since $q < r$, this yields $f(a-r) = q + \ell r$ for some $\ell \in \mathbb{N}$ and thus $f(a-r) \in L$.

Case $q \geq r$. Then $\max(r-1, q) = q$ and $a > q \geq r$. Let $q = i + kr$ with $0 \leq i < r$ and $k \geq 1$. As above, $f(a-r) \equiv f(a) \equiv i \pmod{r}$ hence $f(a-r) = i + \ell r$ for some $\ell \in \mathbb{N}$. Now, $f(a-r) \geq a-r$ by (i) hence $i + \ell r \geq a - r > q - r = i + (k-1)r$ so that $\ell \geq k$. Thus, $f(a-r) = i + \ell r = q - kr + \ell r = q + (\ell - k)r \in q + r\mathbb{N} = L$.

In both cases, we have $f(a-r) \in L$, contradicting the minimality of a in the intersection of its congruence class modulo r with $f^{-1}(L)$.

(2) By Lemma 1.11 any set K in $\mathcal{L}(L)$ is of the form $K = A + r\mathbb{N}$ with $A \subseteq \{0, \ldots, \max(r-1, q)\}$, hence $\forall a \in A \quad a \leq \max(r-1, q)$. Then $f^{-1}(K) =$

$\cup_{a\in A} f^{-1}(a + r\mathbb{N})$. By (1), each $f^{-1}(a + r\mathbb{N})$ is of the form $A_a + r\mathbb{N}$ with $A_a \subseteq \{0, \ldots, \max(r-1, a)\}$. Hence $f^{-1}(K) = \cup_{a\in A}\big(A_a + r\mathbb{N}\big) = (\cup_{a\in A} A_a) + r\mathbb{N}$, and $\cup_{a\in A} A_a$ is a subset of $\{0, \ldots, \max(r-1, q)\}$. When $r \geq 2$, this concludes the proof that $f^{-1}(K) \in \mathcal{L}(L)$ by Lemma 1.11 2(i). If $r = 1$, we must check also that $f^{-1}(K) \neq \emptyset$: indeed $K = a + \mathbb{N}$ by Lemma 1.11 2(ii), as $f(a) \geq a$ by hypothesis (i), $f(a) \in a + \mathbb{N}$ and $a \in f^{-1}(K)$ which is non empty; this concludes the proof that $f^{-1}(K) \in \mathcal{L}(L)$ for the case $r = 1$. □

Remark 4.2. The statement of Proposition 4.1 is sharper than that of Theorem 3.2 applied to arithmetic progressions. In fact, the proof of Proposition 4.1 also shows that, for an arithmetic progression L, the lattice $\mathcal{L}(L)$ is the smallest join-semilattice containing L and closed under decrement.

Remark 4.3. The proof of Proposition 4.1 cannot be extended to regular sets, not even to periodic sets. Let $f\colon n \mapsto n^2$ and L the periodic set $\{0, 4, 8\} + 3\mathbb{N} = \{0, 3, 4\} \cup (6 + \mathbb{N})$. Then $f^{-1}(L) = \sqrt{L} = \mathbb{N} \setminus \{1\} = L - 4$ is a decrement of L. However, this result cannot be obtained by the proof of Proposition 4.1 because this proof relies on the fact that, whenever $a \in f^{-1}(L)$, then $a + r\mathbb{N}$ is a decrement of L; here however, $2 + 3\mathbb{N}$ is not a decrement of L and does not even belong to $\mathcal{L}(L)$). Indeed, $\mathcal{D}(L)$ consists here of $L, L-1, L-2, L-3, L-4, L-5, L-6 = \mathbb{N}$, all of which are of the form $D_i \cup (6 + \mathbb{N})$ with D_i a finite set. Thus, $2 + 3\mathbb{N}$ cannot be obtained by finite unions, intersections and decrements of such sets, all of which contain *all* the integers larger than 6.

Remark 4.4. Proposition 4.1 does not hold for finite sets, nor general regular sets, nor periodic sets: unions of decrements are not sufficient to obtain $f^{-1}(L)$, intersections are needed.

Consider $f\colon n \mapsto n^2$. Let $L = \{5, 6\} + 4\mathbb{N}$ (periodic); then (cf. Example 2.3) $L \div 2 = \sqrt{L} = \{3, 5\} + 4\mathbb{N} = \big((L-2) \cap (L-3)\big) \cup \big(L \cap (L-1)\big)$ cannot be obtained as a union of decrements of L: in order to obtain 5, we must include either L, $L-1$, $L-4$ or $L-5$, but each of these decrements contains numbers not in $\sqrt{L}$ (respectively 6, 4, 2 and 0) which must be excluded by a suitable intersection.

Let $L = \{1, 2\}$ then $f^{-1}(L) = \{1\}$; the decrements of L are the sets $\{1, 2\}, \{0, 1\}, \{0\}, \emptyset$, no union of which is $f^{-1}(L)$, intersections are required to get $f^{-1}(L)$.

This is why the proof in both the general and the finite case does exclude the elements which are not in $f^{-1}(L)$ by using carefully chosen intersections.

5 Characterizing induced closures

We characterize the functions $f : \mathbb{N} \to \mathbb{N}$ such that closure under decrement yields closure under f^{-1}.

Theorem 5.1. *Let $f : \mathbb{N} \to \mathbb{N}$. The following conditions are equivalent.*

- (i) *Every lattice of regular subsets of $\mathbb{N}$ closed under decrement is closed under f^{-1}.*
- (ii) *For every finite subset L of $\mathbb{N}$, the lattice $\mathcal{L}(L)$ is closed under f^{-1}.*
- (iii) *For every arithmetic progression $L = q + r\mathbb{N}$, $r > 0$, the lattice $\mathcal{L}(L)$ is closed under f^{-1}.*

(iv) *The map* f *is non decreasing and satisfies* $f(a) \geq a$ *and* $f(a) - f(b) \equiv 0 \pmod{(a-b)}$ *for all* $a, b \in \mathbb{N}$.

Proof. (iv) $\Rightarrow$ (i). This is Theorem 3.2.
(i) $\Rightarrow$ (ii). Finite sets are regular sets.
(i) $\Rightarrow$ (iii). Arithmetic progressions are regular sets.
(ii) $\Rightarrow$ (iv). We first prove that $f(a) \geq a$, for all $a \in \mathbb{N}$. Let $a \in \mathbb{N}$ and $L = \{f(a)\}$. Observe that the smallest lattice containing the set $\{f(a)\}$ and closed under decrementation is the family of subsets of $\{0, 1, \ldots, f(a)-1, f(a)\}$. As a consequence, all elements of $f^{-1}(L)$ must be less than $f(a)$. In particular $a \leq f(a)$, since $a \in f^{-1}(L)$.

We prove now that $f(a) - f(b) \in (a-b)\mathbb{N}$ for all $a, b \in \mathbb{N}$ such that $a > b$. In particular, f is monotone non decreasing and $f(a) - f(b) \equiv 0 \pmod{(a-b)}$. We argue by contradiction. Suppose that $f(a) \notin f(b) + (a-b)\mathbb{N}$. Let

$$\ell = \left\lfloor \frac{f(a) - a}{a-b} \right\rfloor, \quad k = \left\lfloor \frac{f(a)}{a-b} \right\rfloor, \quad L \ = \ \{f(a) - j(a-b) \mid 0 \leq j \leq k\} \ .$$

Since $f(a) \geq a$, we have $\ell \geq 0$; moreover,

$$k = \left\lfloor \frac{f(a)}{a-b} \right\rfloor = \left\lfloor \frac{f(a) - a}{a-b} + 1 + \frac{b}{a-b} \right\rfloor$$

hence $k \geq \ell + 1$.

For $j \in \{0, \ldots, k\}$, $f(a) \neq f(b) + j(a-b)$ hence $f(b) \neq f(a) - j(a-b)$. Thus, $f(b) \notin L$ and $b \notin f^{-1}(L)$. Of course, $f(a) \in L$ and $a \in f^{-1}(L)$. To get a contradiction, we show that $f^{-1}(L)$ is not in $\mathcal{L}(L)$. Since $f^{-1}(L)$ contains a but not b, it suffices to show that every set $X \in \mathcal{L}(L)$ which contains a also contains b. Since $\mathcal{L}(L)$ is generated by the $L - i$'s, we reduce to show that, for all i, if a is in $L - i$ then so is b. Now, using the definition of ℓ, for all $i \in \mathbb{N}$

$$\begin{aligned} a \in L - i \quad &\iff \quad \exists \alpha \in \{0, \ldots, k\} \ \ a = f(a) - \alpha(a-b) - i \\ &\iff \quad \exists \alpha \in \{0, \ldots, k\} \ \ i = f(a) - a - \alpha(a-b) \\ &\iff \quad \exists \alpha \in \{0, \ldots, \ell\} \ \ i = f(a) - a - \alpha(a-b) \end{aligned}$$

and, for i associated to such an $\alpha \in \{0, \ldots, \ell\}$,

$$L - i \quad = \quad \mathbb{N} \ \cap \ \{a + (\alpha - j)(a-b) \mid j \in \{0, \ldots, k\}\}$$

letting $j = \alpha$ and $j = \alpha + 1$ (which is $\leq k$ since $\alpha \leq \ell < k$), we see that

$$L - i \quad \supseteq \quad \{a, b\} \ .$$

This gives the required contradiction.

(iii) $\Rightarrow$ (iv). Note first that if (iii) holds, f cannot be constant: indeed, for any constant function $f(x) = a$, there exists an arithmetic progression, namely $L = a + 1 + \mathbb{N}$, such that the lattice $\mathcal{L}(L)$ is not closed under f^{-1}. In fact, $f^{-1}(L) = \emptyset \notin \mathcal{L}(L)$ because all sets of $\mathcal{D}(L)$ are of the form $\{\ell + \mathbb{N} \mid 0 \leq \ell \leq a+1\}$, hence all their finite intersections contain $a+1+\mathbb{N}$ and so are not empty.

By Lemma 1.11, if $L = q + r\mathbb{N}$, with $q, r \in \mathbb{N}$, $r \geq 1$, then $\mathcal{L}(L)$ is the family of sets of the form $B + r\mathbb{N}$ with $B \subseteq \{0, \ldots, \max(q, r-1)\}$, and $B \neq \emptyset$ if $r = 1$.

First, we check that f is non decreasing. Let $a < b$ and let $L = f(a) + \mathbb{N}$. Note that, since $r = 1$, a set $B + \mathbb{N}$ is equal to $\min(B) + \mathbb{N}$ and $\emptyset \notin \mathcal{L}(L)$, hence $f^{-1}(L) = s + \mathbb{N}$, with $s \leq f(a)$; as $a \in f^{-1}(L)$, then $a \in s + \mathbb{N}$, i.e. $a \geq s$, and $f(s + \mathbb{N}) \subseteq L$. Since $b > a \geq s$ we get $b \in s + \mathbb{N}$ and $f(b) \in L$ hence $f(b) \geq f(a)$.

Second, we show that $f(b)-f(a) \in (b-a)\mathbb{N}$ whenever $a < b$. Let $L = f(a) + (b-a)\mathbb{N}$. Then, in view of Lemma 1.11, we may write $f^{-1}(L) = A + (b-a)\mathbb{N}$. Since $a \in f^{-1}(L)$, we have $a+(b-a)\mathbb{N} \subseteq f^{-1}(L)$ hence $b = a+(b-a) \in f^{-1}(L)$, i.e. $f(b) \in L$, whence for some k, $f(b) = f(a) + k(b-a)$.

Finally, we show that $f(a) \geq a$ for all $a \in \mathbb{N}$. Suppose that, for some a, $f(a) < a$. Since a divides $f(a) - f(0) \leq f(a) < a$, we have $f(a) = f(0)$ hence f is constant on $\{0, \ldots, a\}$ with value $< a$.

Case 1. There are infinitely many a's such that $f(a) < a$. Then f is constant, contradicting what was proved above.

Case 2. There is a largest a such that $f(a) < a$. Then $f(x) = f(0) < a$ for $x \leq a$ and $f(x) \geq x > a$ for $x > a$. Thus, $\emptyset \neq f^{-1}(a+\mathbb{N}) \subseteq (a+1)+\mathbb{N}$ is not in $\mathcal{L}(a+\mathbb{N})$, contradicting (iii). □

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