Arithmetical Congruence Preservation: from Finite to Infinite

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To Yuri, on his 75th birthday, with thanks for many stimulating discussions on Logic and Computation

Abstract. Various problems on integers lead to the class of functions defined on a ring of numbers (or a subset of such a rings) METTRE RING AU SINGULIER and verifying $a - b$ divides $f(a) - f(b)$ for all a, b. We say that such functions are "congruence preserving". In previous works, we characterized these classes of functions for the cases $\mathbb{N} \to \mathbb{Z}$, $\mathbb{Z} \to \mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ in terms of sums series of rational polynomials (taking only integral values) and the function giving the least common multiple of $1, 2, \ldots, k$. In this paper we relate the finite and infinite cases via a notion of "lifting": if $\pi \colon X \to Y$ is a surjective morphism and f is a function $Y \to Y$ a lifting of f is a function $F : X \to X$ such that $\pi \circ F = f \circ \pi$. We prove that the finite case $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ can be so lifted to the infinite cases $\mathbb{N} \to \mathbb{N}$ and $\mathbb{Z} \to \mathbb{Z}$. We also use such liftings to extend the characterization to the rings of p-adic and profinite integers, using Mahler representation of continuous functions on these rings.

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1 Introduction

A function f (on $\mathbb{N}$ or $\mathbb{Z}$) is said to be congruence preserving if $a - b$ divides $f(a) - f(b)$. Polynomial functions are obvious examples of congruence preserving

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functions. In [3,4] we characterized such functions $\mathbb{N} \to \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z}$ (which we named "functions having the integral difference ratio property"). In [5] we extended the characterization to functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ with $n, m \geq 1$ (for the suitable notion of congruence preservation).

In the present paper, we prove in §2 that every congruence preserving function $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ (with m dividing n) can be lifted to congruence preserving functions $\mathbb{N} \to \mathbb{N}$ and $\mathbb{Z} \to \mathbb{Z}$ (i.e. it is the modular projection of such a function). As a corollary (i) we show that such a lift also works replacing $\mathbb{N}$ with $\mathbb{Z}/qn\mathbb{Z}$ and (ii) we give an alternative proof of a representation (obtained in [5]) of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ as linear sums of "rational" polynomials.

In §3 we consider the rings of p-adic integers (resp. profinite integers) and prove that congruence preserving functions on these rings are inverse limits of congruence preserving functions on the $\mathbb{Z}/p^k\mathbb{Z}$ (resp. on the $\mathbb{Z}/n\mathbb{Z}$). Considering the Mahler representation of continuous functions by series, we prove that congruence preserving functions correspond to those series for which the linear coefficient with rank k is divisible by the least common multiple of $1, \ldots, k$.

2 Switching between finite and infinite

In order to characterize congruence preserving functions on $\mathbb{Z}/n\mathbb{Z}$, we first lift each such function into a congruence preserving function $\mathbb{N} \to \mathbb{N}$. In a second step, we use our characterization of congruence preserving functions $\mathbb{N} \to \mathbb{Z}$ to characterize the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

2.1 Lifting functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ to $\mathbb{N} \to \mathbb{N}$ and $\mathbb{Z} \to \mathbb{Z}$

Definition 1. *Let X be a subset of a commutative ring $(R, +, \times)$. A function $f\colon X \to R$ is said to be congruence preserving if*

$$\forall x, y \in X \quad \exists d \in R \quad f(x) - f(y) = d(x - y)\,, \quad \text{i.e. } x - y \textit{ divides } f(x) - f(y)\,.$$

Definition 2 (Lifting). *Let $\sigma\colon X \to N$ and $\rho\colon Y \to M$ be surjective maps. A function $F\colon X \to Y$ is said to be a (σ, ρ)-*lifting *of a function $f\colon N \to M$ (or simply* lifting *if σ, ρ are clear from the context) if the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \sigma\big\downarrow & & \big\downarrow\rho \\ N & \xrightarrow{f} & M \end{array} \qquad \textit{i.e.} \quad \rho \circ F = f \circ \sigma\,.$$

We will consider elements of $\mathbb{Z}/k\mathbb{Z}$ as integers and vice versa via the following modular projection maps.

Notation 3 *1. Let $\pi_k\colon \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ be the canonical surjective homomorphism associating to an integer its class in $\mathbb{Z}/k\mathbb{Z}$.*

2. Let $\iota_k \colon \mathbb{Z}/k\mathbb{Z} \to \mathbb{N}$ be the injective map associating to an element $x \in \mathbb{Z}/kZ$ its representative in $\{0, \ldots, k-1\}$.
3. Let $\pi_{n,m} \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the map $\pi_{n,m} = \pi_m \circ \iota_n$.
If $m \leq n$ let $\iota_{m,n} \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the injective map $\iota_{m,n} = \pi_n \circ \iota_m$.

Lemma 4. *If m divides n then $\pi_m = \pi_{n,m} \circ \pi_n$ and $\pi_{n,m}$ is a surjective homomorphism.*

The next theorem insures that congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ can be lifted to congruence preserving functions $\mathbb{N} \to \mathbb{N}$ and $\mathbb{N} \to \mathbb{Z}$.

Theorem 5 (Lifting functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to $\mathbb{N} \to \mathbb{N}$). *Let $f \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ with $m \geq 2$. The following conditions are equivalent:*

(1) f is congruence preserving.
(2) f can be (π_n, π_n)-lifted to a congruence preserving function $F : \mathbb{N} \to \mathbb{N}$.
(3) f can be (π_n, π_n)-lifted to a congruence preserving function $F : \mathbb{N} \to \mathbb{Z}$.

In view of applications in the context of p-adic and profinite integers, we state and prove a slightly more general version. As $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ are different rings we use an extension of the notion of congruence preservation introduced in Chen [6] and studied in Bhargava [1]) which we recall below.

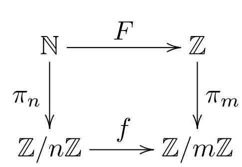
Definition 6. *A function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if*

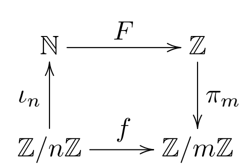
$$\text{for all } x, y \in \mathbb{Z}/n\mathbb{Z}, \quad \pi_{n,m}(x-y) \text{ divides } f(x) - f(y) \text{ in } \mathbb{Z}/m\mathbb{Z}. \tag{1}$$

Theorem 7 (Lifting functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ to $\mathbb{N} \to \mathbb{N}$). *Let $f \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ with m divides n and $m \geq 2$. The following conditions are equivalent:*

(1) f is congruence preserving.
(2) f can be (π_n, π_m)-lifted to a congruence preserving function $F : \mathbb{N} \to \mathbb{N}$.
(3) f can be (π_n, π_m)-lifted to a congruence preserving function $F : \mathbb{N} \to \mathbb{Z}$.

Proof. (2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1). Assume f lifts to the congruence preserving function $F : \mathbb{N} \to \mathbb{Z}$, i.e. $f \circ \pi_n = \pi_m \circ F$. Since $\pi_n \circ \iota_n$ is the identity we get $f = i_m \circ F \circ \iota_n$. The following diagrams are thus commutative:





Let $x, y \in \mathbb{Z}/n\mathbb{Z}$. As F is congruence preserving, $\iota_n(x) - \iota_n(y)$ divides $F(\iota_n(x)) - F(\iota_n(y))$, hence $F(\iota_n(x)) - F(\iota_n(y)) = (\iota_n(x) - \iota_n(y))\,\delta$. Since π_m is a morphism and $\pi_m \circ \iota_n = \pi_{n,m}$, we get $\pi_m(F(\iota_n(x))) - \pi_m(F(\iota_n(x))) = \pi_{n,m}(x-y)\,\pi_m(\delta)$. As F lifts f we have $\pi_m(F(\iota_n(x))) - \pi_m(F(\iota_n(y))) = f(x) - f(y)$ whence (1).

(1) $\Rightarrow$ (2). By induction on $t \in \mathbb{N}$ we define a sequence of functions $\varphi_t \colon \{0, \ldots, t\} \to \mathbb{N}$ for $t \in \mathbb{N}$ such that φ_{t+1} extends φ_t and (*) and (**) below hold.

$$\left\{ \begin{array}{rl} (*) & \varphi_t \text{ is congruence preserving,} \\ (**) & \pi_m(\varphi_t(u)) = f(\pi_n(u)), \text{ for all } u \in \{0, \ldots, t\}, \\ & \text{i.e. the following diagram commutes:} \quad \begin{array}{ccc} \{0, \ldots, t\} & \xrightarrow{\varphi_t} & \mathbb{Z} \\ \pi_n \downarrow & & \downarrow \pi_m \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/m\mathbb{Z} \end{array} \end{array} \right.$$

Basis. We choose $\varphi_0(0) \in \mathbb{N}$ such that $\pi_m(\varphi_0(0)) = f(\pi_n(0))$. Properties (*) and (**) clearly hold for φ_0.

Induction: from φ_t to φ_{t+1}. Since the wanted φ_{t+1} has to extend φ_t to the domain $\{0, \ldots, t, t+1\}$, we only have to find a convenient value for $\varphi_{t+1}(t+1)$. By the induction hypothesis, (*) and (**) hold for φ_t; in order for φ_{t+1} to satisfy (*) and (**), we have to find $\varphi_{t+1}(t+1)$ such that $t+1-i$ divides $\varphi_{t+1}(t+1) - \varphi_t(i)$, for $i = 0, \ldots, t$, and $\pi_m(\varphi_{t+1}(t+1)) = f(\pi_n(t+1))$. Rewritten in terms of congruences, these conditions amount to say that $\varphi_{t+1}(t+1)$ is a solution of the following system of congruence equations:

$$\left. \begin{array}{r|ll} \star(0) & \varphi_{t+1}(t+1) \equiv \varphi_t(0) & \pmod{t+1} \\ & \vdots & \\ \star(\mathrm{i}) & \varphi_{t+1}(t+1) \equiv \varphi_t(i) & \pmod{t+1-i} \\ & \vdots & \\ \star(\mathrm{t\text{-}1}) & \varphi_{t+1}(t+1) \equiv \varphi_t(t-1) & \pmod{2} \\ \star\star & \varphi_{t+1}(t+1) \equiv \iota_m(f(\pi_n(t+1))) & \pmod{m} \end{array} \right\} \qquad (2)$$

Recall the Generalized Chinese Remainder Theorem (cf. §3.3, exercice 9 p. 114, in Rosen's textbook [13]): a system of congruence equations

$$\bigwedge_{i=0,\ldots,t} x \equiv a_i \pmod{n_i}$$

has a solution if and only if $a_i \equiv a_j \mod \gcd(n_i, n_j)$ for all $0 \leq i < j \leq t$.

Let us show that the conditions of application of the Generalized Chinese Remainder Theorem are satisfied for system (2).

- Lines $\star$(i) and $\star$(j) of system (2) (with $0 \leq i < j \leq t-1$).
  Every common divisor to $t+1-i$ and $t+1-j$ divides their difference $j-i$ hence $\gcd(t+1-i, t+1-j)$ divides $j-i$. Since φ_t satisfies (*), $j-i$ divides $\varphi_t(j) - \varphi_t(i)$ and a fortiori $\gcd(t+1-i, t+1-j)$ divides $\varphi_t(j) - \varphi_t(i)$.
- Lines $\star$(i) and $\star\star$ of system (2) (with $0 \leq i \leq t-1$).
  Let $d = \gcd(t+1-i, m)$. We have to show that d divides $\iota_m(f(\pi_n(t+1))) - \varphi_t(i)$. Since f is congruence preserving, $\pi_{n,m}(\pi_n(t+1) - \pi_n(i))$ divides $f(\pi_n(t+1)) - f(\pi_n(i))$. As m divides n, by Lemma 4, $\pi_{n,m}(\pi_n(t+1) - \pi_n(i)) =$

$\pi_m(t+1)-\pi_m(i)=\pi_m(t+1-i)$ and $f(\pi_n(t+1))-f(\pi_n(i))=k\pi_m(t+1-i)$ for some $k\in\mathbb{Z}/m\mathbb{Z}$. Applying ι_m, there exists $\lambda\in\mathbb{Z}$ such that

$$\iota_m(f(\pi_n(t+1)))-\iota_m(f(\pi_n(i)))=\iota_m(k)\iota_m(\pi_m(t+1-i))+\lambda m$$

as $\iota_m(\pi_m(u))\equiv u \pmod{m}$ for every $u\in\mathbb{Z}$, there exists $\mu\in\mathbb{Z}$ such that

$$\iota_m(f(\pi_n(t+1)))-\iota_m(f(\pi_n(i)))=\iota_m(k)(t+1-i)+\mu m+\lambda m\,. \qquad (3)$$

Since φ_t satisfies (**), we have $\pi_m(\varphi_t(i))=f(\pi_n(i))$ hence $\varphi_t(i)\equiv\iota_m(f(\pi_n(i))) \pmod{m}$. Thus equation (3) can be rewritten

$$\iota_m(f(\pi_n(t+1)))-\varphi_t(i)=(t+1-i)\iota_m(k)+\nu m \quad \text{for some } \nu\,. \qquad (4)$$

As $d=\gcd(t+1-i,m)$ divides m and $t+1-i$, (4) shows that d divides $\iota_n(f(\pi_n(t+1)))-\varphi_t(i)$ as wanted.

Thus, we can apply the Generalized Chinese Theorem and get the wanted value of $\varphi_{t+1}(t+1)$, concluding the induction step.

Finally, taking the union of the φ_t's, $t\in\mathbb{N}$, we get a function $F:\mathbb{N}\to\mathbb{N}$ which is congruence preserving and lifts f. □

Example 8 (counterexample to Theorem 7). Lemma 4 and Theorem 7 do not hold if m does not divide n. Consider $f\colon\mathbb{Z}/6\mathbb{Z}\to\mathbb{Z}/8\mathbb{Z}$ defined by $f(0)=0$, $f(1)=3$, $f(2)=4$, $f(3)=1$, $f(4)=4$, $f(5)=7$. Note first that, in $\mathbb{Z}/8\mathbb{Z}$, 1, 3 and 5 are invertible, hence f is congruence preserving iff for $k\in\{2,\ 4\}$, for all $x\in\mathbb{Z}/6\mathbb{Z}$, k divides $f(x+k)-f(x)$ which is easily checked; nevertheless, f has no congruence preserving lift $F\colon\mathbb{Z}\to\mathbb{Z}$. If such a lift F existed, we should have

(1) because F lifts f, $\pi_8(F(0))=f(\pi_6(0))=0$ and $\pi_8(F(8))=f(\pi_6(8))=f(2)=4$;
(2) as F is congruence preserving, 8 must divide $F(8)-F(0)$; we already noted that 8 divides $F(0)$, hence 8 divides $F(8)$ and $\pi_8(F(8))=0$, contradicting $\pi_8(F(8))=4$.

Note that $\pi_{6,8}$ is neither a homomorphism nor surjective and $0=\pi_8(8)\neq\pi_{6,8}\circ\pi_6(8)=2$. □

We can also lift congruence preserving functions from $\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}\to\mathbb{Z}$ instead of $\mathbb{N}\to\mathbb{N}$.

Theorem 9 (Lifting functions $\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}\to\mathbb{Z}$). *Let $f\colon\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}$ with m divides n and $m\geq 2$. The following conditions are equivalent:*

(1) f is congruence preserving.
(2) f can be (π_n,π_m)-lifted to a congruence preserving function $F:\mathbb{Z}\to\mathbb{Z}$.

Proof. (2) ⇒ (1). The proof is the same as that of (3) ⇒ (1) in Theorem 7.
(1) ⇒ (2). The argument is a slight modification of that for the same implication in Theorem 7. We define the lift $F:\mathbb{Z}\to\mathbb{Z}$ of $f:\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}$ as the union of a series of functions φ_t, $t\in\mathbb{N}$ such that
- φ_{2t} has domain $\{-t,\dots,t\}$ and φ_{2t+1} has domain $\{-t,\dots,t+1\}$,
- φ_{t+1} extends φ_t,
- φ_t is congruence preserving. The induction step is done exactly as in Theorem 7 via a system of congruence equations and an application of the Generalized Chinese Remainder Theorem.

2.2 Representation of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$

As a first corollary of Theorem 7 we get a new proof of the representations of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ as finite linear sums of polynomials with rational coefficients (cf. [5]). Let us recall the so-called binomial polynomials.

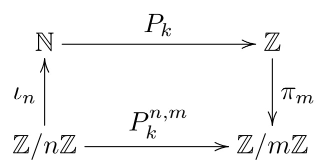
Definition 10. *For $k \in \mathbb{N}$, let $P_k(x) = \binom{x}{k} = \frac{1}{k!} \prod_{\ell=0}^{\ell=k-1} (x-\ell)$.*

Though P_k has rational coefficients, it maps $\mathbb{N}$ into $\mathbb{Z}$. Also, observe that $P_k(x)$ takes value 0 for all $k > x$. This implies that for any sequence of integers $(a_k)_{k\in\mathbb{N}}$, the infinite sum $\sum_{k\in\mathbb{N}} a_k\, P_k(x)$ reduces to a finite sum for any $x \in \mathbb{N}$ hence defines a function $\mathbb{N} \to \mathbb{Z}$.

Definition 11. *We denote by $lcm(k)$ the least common multiple of integers $1, \ldots, k$ (with the convention $lcm(0) = 1$).*

Definition 12. *To each binomial polynomial P_k, $k \in \mathbb{N}$, we associate a function $P_k^{n,m} \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ which sends an element $x \in \mathbb{Z}/n\mathbb{Z}$ to $(\pi_m \circ P_k \circ \iota_n)(x) \in \mathbb{Z}/m\mathbb{Z}$.*

In other words, consider the representative t of x lying in $\{0, \ldots, n-1\}$, evaluate $P_k(t)$ in $\mathbb{N}$ and then take the class of the result in $\mathbb{Z}/m\mathbb{Z}$. Hence, the following diagram commutes:



Lemma 13. *If $lcm(k)$ divides a_k in $\mathbb{Z}$, then the function $\pi_m(a_k)P_k^{n,m} \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ (represented by $a_k P_k$) is congruence preserving.*

Proof. In [3] we proved that if $lcm(k)$ divides a_k then $a_k P_k$ is a congruence preserving function on $\mathbb{N}$. Let us now show that $\pi_m(a_k)P_k^{n,m} \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is also congruence preserving. Let $x, y \in \mathbb{Z}/n\mathbb{Z}$: as $a_k P_k$ is congruence preserving, $\iota_n(x) - \iota_n(y)$ divides $a_k P_k(\iota_n(x)) - a_k P_k(\iota_n(y))$. As m divides n, π_m is a morphism (cf. Lemma 4) hence $\pi_m(\iota_n(x)) - \pi_m(\iota_n(y))$ divides $\pi_m(a_k)\pi_m(P_k(\iota_n(x))) - \pi_m(a_k)\pi_m(P_k(\iota_n(y))) = \pi_m(a_k)P_k^{n,m}(x) - \pi_m(a_k)P_k^{n,m}(x)$. As $\pi_m \circ \iota_n = \pi_{n,m}$ we have $\pi_m(\iota_n(x)) - \pi_m(\iota_n(y)) = \pi_{n,m}(x)) - \pi_{n,m}(y)$ and we conclude that $\pi_m(a_k)P_k^{n,m}$ is congruence preserving. □

Corollary 14 ([5]). *Let $1 \leq m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$, p_i prime. Suppose m divides n and let $\nu(m) = \max_{i=1,\ldots,\ell}\ p_i^{\alpha_i}$. A function $f \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if it is represented by a finite $\mathbb{Z}$-linear sum $f = \sum_{k=0}^{\nu(m)-1} \pi_m(a_k)P_k^{n,m}$ such that $lcm(k)$ divides a_k (in $\mathbb{Z}$) for all $k < \nu(m)$. Moreover, such a representation is unique.*

Proof. Assume $f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is congruence preserving. Applying Theorem 7, lift f to $F\colon \mathbb{N} \to \mathbb{N}$ which is congruence preserving.

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\;F = \sum_{k=0}^{\nu(m)-1} a_k\, P_k\;} & \mathbb{Z} \\ \pi_n \downarrow & & \downarrow \pi_m \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\quad f \quad} & \mathbb{Z}/m\mathbb{Z} \end{array} \qquad f \circ \pi_n = \pi_m \circ F$$

We proved in [5] that every congruence preserving function $F\colon \mathbb{N} \to \mathbb{N}$ is of the form $F = \sum_{k=0}^{\infty} a_k P_k$ where $lcm(k)$ divides a_k for all k. As π_m is a morphism (because m divides n) and F lifts f, we have, for $u \in \mathbb{Z}$

$$\begin{aligned} f(\pi_n(u)) \;&=\; \pi_m(F(u)) \;=\; \pi_m(\sum_{k=0}^{\infty} a_k\, P_k(u)) \\ &= \sum_{k=0}^{\infty} \pi_m(a_k)\, \pi_m(P_k(u)) \;=\; \sum_{k=0}^{k=\nu(m)-1} \pi_m(a_k)\, \pi_m(P_k(u)) \end{aligned} \tag{5}$$

The last equality is obtained by noting that for $k \geq \nu(m)$, m divides $lcm(k)$ hence as a_k is a multiple of $lcm(k)$, $\pi_m(a_k) = 0$. From (5) we get $f(\pi_n(u)) = \sum_{k=0}^{k=\nu(m)-1} \pi_m(a_k)\, \pi_m(P_k(u)) = \pi_m(\sum_{k=0}^{k=\nu(m)-1} a_k\, P_k(u))$. This proves that f is lifted to the rational polynomial function $\sum_{k=0}^{k=\nu(m)-1} a_k\, P_k$. Since $P_k(k) = 1$ for all $k \in \mathbb{N}$, and $P_k(i) = 0$ for $k > i$, we obtain the unicity of the representation.

The converse follows from Lemma 13 and the fact that any finite sum of congruence preserving functions is congruence preserving. □

2.3 Lifting functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/s\mathbb{Z}$

As a second corollary of Theorem 7 we can lift congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to congruence preserving functions $\mathbb{Z}/qn\mathbb{Z} \to \mathbb{Z}/qn\mathbb{Z}$.

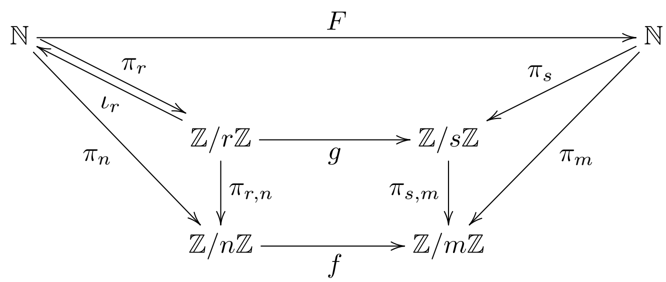
We state a slightly more general result.

Corollary 15. *Assume $m, n, s, r \geq 1$, m divides both n and s, and n, s both divide r. If $f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is congruence preserving then it can be $(\pi_{r,n}, \pi_{s,m})$-lifted to $g\colon \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/s\mathbb{Z}$ which is also congruence preserving.*

n DIVIDES r IS A CONSEQUENCE
NON je ne pense pas

Proof. As m divides n, using Theorem 7, we lift f to a congruence preserving $F : \mathbb{N} \to \mathbb{N}$ and set $g = \pi_s \circ F \circ \iota_r$.

We first show that the rectangular subdiagram around f, g commutes:



$$
\begin{aligned}
\pi_{s,m} \circ g &= \pi_{s,m} \circ (\pi_s \circ F \circ \iota_r) \\
&= (\pi_m \circ F) \circ \iota_r && m \text{ divides } s \text{ yields } \pi_m = \pi_{s,m} \circ \pi_s \text{ (Lemma 4)} \\
&= (f \circ \pi_n) \circ \iota_r && \text{since } F \text{ lifts } f \\
&= f \circ \pi_{r,n} && \text{since } \pi_n \circ \iota_r = \pi_{r,n}
\end{aligned}
$$

Thus, $\pi_{s,m} \circ g = f \circ \pi_{r,n}$, i.e. g lifts f.
Finally, if $x, y \in \mathbb{Z}/r\mathbb{Z}$ then $\iota_r(x) - \iota_r(y)$ divides $F(\iota_r(x)) - F(\iota_r(y))$ (by congruence preservation of F). As π_s is a morphism, and $\pi_s = \pi_{r,s} \circ \pi_r$ (because s divides r), and $\pi_r \circ \iota_r$ is the identity on $\mathbb{Z}/r\mathbb{Z}$, we deduce that $\pi_s(\iota_r(x)) - \pi_s(\iota_r(y)) = (\pi_{r,s} \circ \pi_r \circ \iota_r)(x) - (\pi_{r,s} \circ \pi_r \circ \iota_r)(y) = \pi_{r,s}(x - y)$ divides $\pi_s(F(\iota_r(x))) - \pi_s(F(\iota_r(y)) = g(x) - g(y)$ (by definition of g). We thus conclude that g is congruence preserving. □

Remark 16. Let us check that the previous diagram is completely commutative. The large trapezoid around F, f commutes because F lifts f. The upper trapezoid F, g, ι_r, π_s commutes by definition of g. The upper trapezoid F, g, π_r, π_s commutes since $g \circ \pi_r = (\pi_s \circ F \circ \iota_r) \circ \pi_r = \pi_s \circ F$ (as $\iota_r \circ \pi_r$ is the identity). The left and right triangles $\pi_n, \pi_r, \pi_{r,n}$ and $\pi_m, \pi_s, \pi_{s,m}$ commute by Lemma 4 as n divides r and m divides s. Finally, the triangle $\pi_n, \iota_r, \pi_{r,n}$ commutes by definition of $\pi_{r,n}$ (cf. Notation 3).

3 Congruence preservation on p-adic/profinite integers

All along this section, p is a prime number; we study congruence preserving functions on the rings $\mathbb{Z}_p$ of p-adic integers and $\widehat{\mathbb{Z}}$ of profinite integers. $\mathbb{Z}_p$ is the projective limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ relative to the projections π_{p^n,p^m}. Usually, $\widehat{\mathbb{Z}}$ is defined as the projective limit $\varprojlim \mathbb{Z}/n\mathbb{Z}$ of the finite rings $\mathbb{Z}/n\mathbb{Z}$ relative to the projections $\pi_{n,m}$, for m dividing n. We here use the following equivalent definition which allows to get completely similar proofs for $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$.

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n!\mathbb{Z} = \{\hat{x} = (x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}/n!\mathbb{Z} \mid \forall m < n,\ x_m \equiv x_n \pmod{m!}\}$$

Recall that $\mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}}$) contains the ring $\mathbb{Z}$ and is a compact topological ring for the topology given by the ultrametric d such that $d(x, y) = 2^{-n}$ where n is largest such that p^n (resp. $n!$) divides $x - y$, i.e. x and y have the same first n digits in their base p (resp. base factorial) representation. We refer to the Appendix for some basic definitions, representations and facts that we use about the compact topological rings $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$.

We first prove that on $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$ every congruence preserving function is continuous (Proposition 18).

3.1 Congruence preserving functions are continuous

Definition 17. *1. Let $\mu : \mathbb{N} \to \mathbb{N}$ be increasing. A function $\Psi : \mathbb{Z}_p \to \mathbb{Z}_p$ admits μ as modulus of uniform continuity if and only if $d(x, y) \leq 2^{-\mu(n)}$ implies $d(\Psi(x), \Psi(y)) \leq 2^{-n}$.*
2. Φ is 1-Lipschitz if it admits the identity as modulus of uniform continuity.

Since the rings $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$ are compact, every continuous function admits a modulus of uniform continuity. For congruence preserving function, we get a tight bound on the modulus.

Proposition 18. *Every congruence preserving function $\Psi : \mathbb{Z}_p \to \mathbb{Z}_p$ is 1-Lipschitz (hence continuous). Idem with $\widehat{\mathbb{Z}}$ in place of $\mathbb{Z}_p$.*

Proof. If $d(x, y) \leq 2^{-n}$ then p^n divides $x - y$ hence (by congruence preservation) p^n also divides $\Psi(x) - \Psi(y)$ which yields $d(\Psi(x), \Psi(y)) \leq 2^{-n}$. □

The converse of Proposition 18 is false: a 1-Lipschitz function is not necessarily congruence preserving as will be seen in Example 31.

Note the following quite expectable result.

Corollary 19. *There are functions $\mathbb{Z}_p \to \mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$) which are not continuous hence not congruence preserving.*

Proof. As $\mathbb{Z}_p$ has cardinality $2^{\aleph_0}$ there are $2^{2^{\aleph_0}}$ functions $\mathbb{Z}_p \to \mathbb{Z}_p$. Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, $\mathbb{Z}_p$ is a separable space, hence there are at most $2^{\aleph_0}$ continuous functions. □

3.2 Congruence preserving functions and inverse limits

In general an arbitrary continuous function on $\mathbb{Z}_p$ is not the inverse limit of a sequence of functions $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$'s. However, this is true for congruence preserving functions. We first recall how any continuous function $\Psi : \mathbb{Z}_p \to \mathbb{Z}_p$ is the inverse limit of an inverse system of continuous functions $\psi_n : \mathbb{Z}/p^{\mu(n)}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$, $n \in \mathbb{N}$, i.e. the diagram of Figure 1 commutes for any $m \leq n$. For legibility, we use notations adapted to $\mathbb{Z}_p$.

Notation 20 *We write $\widehat{\pi_n}$ for $\pi_{p^n} : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ and $\widehat{\iota_n}$ for $\iota_{p^n} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p$.*

Lemma 4 has an avatar in the profinite framework.

Lemma 21. *$\widehat{\pi_n} \circ \widehat{\iota_n}$ is the identity on $\mathbb{Z}/p^n\mathbb{Z}$. If $m \leq n$ then $\widehat{\pi_m} = \pi_{p^n,p^m} \circ \widehat{\pi_n}$.*

Proposition 22. *Consider $\Psi : \mathbb{Z}_p \to \mathbb{Z}_p$ and a strictly increasing $\mu : \mathbb{N} \to \mathbb{N}$. Define $\psi_n : \mathbb{Z}/p^{\mu(n)}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ as $\psi_n = \widehat{\pi_n} \circ \Psi \circ \widehat{\iota_{\mu(n)}}$ for all $n \in \mathbb{N}$. Then the following conditions are equivalent :*

(1) Ψ is uniformly continuous and admits μ as a modulus of uniform continuity.
(2) The sequence $(\psi_n)_{n\in\mathbb{N}}$ is an inverse system with Ψ as inverse limit (in other words, for all $1 \le m \le n$, the diagrams of Figure 1 commute)
(3) For all $n \ge 1$, the upper half (dealing with Ψ and ψ_n) of the diagram of Figure 1 commutes.

Idem with $\widehat{\mathbb{Z}}$ in place of $\mathbb{Z}_p$.

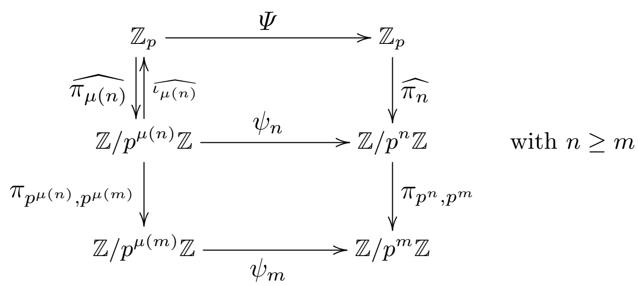


Fig. 1. The inverse system $(\psi_n)_{n\in\mathbb{N}}$ and its inverse limit Ψ.

Proof. (1) $\Rightarrow$ (2). We first show $\widehat{\pi_n} \circ \Psi = \psi_n \circ \widehat{\pi_{\mu(n)}}$. Let $u \in \mathbb{Z}_p$. Since $\widehat{\pi_{\mu(n)}} \circ \widehat{\iota_{\mu(n)}}$ is the identity on $\mathbb{Z}/p^{\mu(n)}\mathbb{Z}$, we have $\widehat{\pi_{\mu(n)}}(u) = \widehat{\pi_{\mu(n)}}(\widehat{\iota_{\mu(n)}}(\widehat{\pi_{\mu(n)}}(u)))$ hence $p^{\mu(n)}$ (considered as an element of $\mathbb{Z}_p$) divides the difference $u - \widehat{\iota_{\mu(n)}}(\widehat{\pi_{\mu(n)}}(u))$, i.e. the distance between these two elements is at most $2^{-\mu(n)}$. As μ is a modulus of uniform continuity for Ψ, the distance between their images under Ψ is at most 2^{-n}, i.e. p^n divides their difference, hence $\widehat{\pi_n}(\Psi(u)) = \widehat{\pi_n}(\Psi(\widehat{\iota_{\mu(n)}}(\widehat{\pi_{\mu(n)}}(u))))$. By definition, $\psi_n = \widehat{\pi_n} \circ \Psi \circ \widehat{\iota_{\mu(n)}}$. Thus, $\widehat{\pi_n}(\Psi(u)) = \psi_n(\widehat{\pi_{\mu(n)}}(u))$, which proves that Ψ lifts ψ_n.
We now show $\pi_{p^n,p^m} \circ \psi_n = \psi_m \circ \pi_{p^{\mu(n)},p^{\mu(m)}}$. Observe that, since $n \ge m$ and μ is increasing, p^m divides p^n and $p^{\mu(m)}$ divides $p^{\mu(n)}$. We just proved above equality $\widehat{\pi_m} \circ \Psi = \psi_m \circ \widehat{\pi_{\mu(m)}}$. Applying three times Lemma 21, we get

$$\begin{aligned} \widehat{\pi_m} \circ \Psi \circ \widehat{\iota_{\mu(n)}} &= \psi_m \circ \widehat{\pi_{\mu(m)}} \circ \widehat{\iota_{\mu(n)}} \\ (\pi_{p^n,p^m} \circ \widehat{\pi_n}) \circ \Psi \circ \widehat{\iota_{\mu(n)}} &= \psi_m \circ (\pi_{p^{\mu(n)},p^{\mu(m)}} \circ \widehat{\pi_{\mu(n)}}) \circ \widehat{\iota_{\mu(n)}} \\ \pi_{p^n,p^m} \circ \psi_n &= \psi_m \circ \pi_{p^{\mu(n)},p^{\mu(m)}} \qquad \text{as } \widehat{\pi_{\mu(n)}} \circ \widehat{\iota_{\mu(n)}} \text{ is the identity.} \end{aligned}$$

The last equality means that ψ_n lifts ψ_m.
(2) $\Rightarrow$ (3). Trivial
(3) $\Rightarrow$ (1). The fact that Ψ lifts ψ_n shows that two elements of $\mathbb{Z}_p$ with the same first $\mu(n)$ digits (in the p-adic representation) have images with the same first n digits. This proves that μ is a modulus of uniform continuity for Ψ. ☐

For congruence preserving functions $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$, the representation of Proposition 22 as an inverse limit gets smoother since then $\mu(n) = n$.

Theorem 23. *For a function $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$, letting $\varphi_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ be defined as $\varphi_n = \widehat{\pi_n} \circ \Phi \circ \widehat{\iota_n}$, the following conditions are equivalent.*

(1) Φ is congruence preserving.
(2) All φ_n's are congruence preserving function and the sequence $(\varphi_n)_{n\geq 1}$ is an inverse system with Φ as inverse limit (in other words, for all $1 \leq m \leq n$, the diagrams of Figure 2 commute).

A similar equivalence also holds for functions $\Phi : \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$.

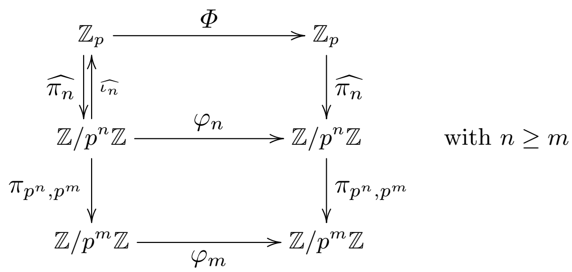


Fig. 2. Φ as the inverse limit of the φ_n's, $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2). Proposition 18 insures that Φ is 1-Lipschitz. The implication (1) $\Rightarrow$ (2) in Proposition 22, applied with the identity as μ, insures that the sequence $(\varphi_n)_{n\geq 1}$ is an inverse system with Φ as inverse limit. It remains to show that φ_n is congruence preserving. Since Φ is congruence preserving, if $x, y \in \mathbb{Z}/p^n\mathbb{Z}$ then $\widehat{\iota_n}(x) - \widehat{\iota_n}(y)$ divides $\Phi(\widehat{\iota_n}(x)) - \Phi(\widehat{\iota_n}(y))$. Now, the canonical projection $\widehat{\pi_n}$ is a morphism hence $\widehat{\pi_n}(\widehat{\iota_n}(x)) - \widehat{\pi_n}(\widehat{\iota_n}(y))$ divides $\widehat{\pi_n}(\Phi(\widehat{\iota_n}(x))) - \widehat{\pi_n}(\Phi(\widehat{\iota_n}(y)))$. As $\widehat{\pi_n} \circ \widehat{\iota_n}$ is the identity on $\mathbb{Z}/p^n\mathbb{Z}$, $x - y$ divides $\widehat{\pi_n}(\Phi(\widehat{\iota_n}(x))) - \widehat{\pi_n}(\Phi(\widehat{\iota_n}(y))) = \varphi_n(x) - \varphi_n(y)$ as wanted.

(2) $\Rightarrow$ (1). Let $x, y \in \mathbb{Z}_p$. Since φ_n is congruence preserving $\widehat{\pi_n}(x) - \widehat{\pi_n}(y)$ divides $\varphi_n(\widehat{\pi_n}(x)) - \varphi_n(\widehat{\pi_n}(y))$. Let

$$U_n^{x,y} = \{u \in \mathbb{Z}/p^n\mathbb{Z} \mid \varphi_n(\widehat{\pi_n}(x)) - \varphi_n(\widehat{\pi_n}(y)) = (\widehat{\pi_n}(x) - \widehat{\pi_n}(y))\, u\}\,.$$

If $m \leq n$ and $u \in U_n^{x,y}$ then, applying π_{p^n,p^m} to the equality defining $U_n^{x,y}$, using the commutative diagrams of Figure 2 and letting $v = \pi_{p^n,p^m}(u)$, we get

$$\begin{aligned}
\varphi_n(\widehat{\pi_n}(x)) - \varphi_n(\widehat{\pi_n}(y)) &= \big(\widehat{\pi_n}(x) - \widehat{\pi_n}(y)\big)\, u\\
\pi_{p^n,p^m}(\varphi_n(\widehat{\pi_n}(x)) - \pi_{p^n,p^m}(\varphi_n(\widehat{\pi_n}(y)) &= \big(\pi_{p^n,p^m}(\widehat{\pi_n}(x)) - \pi_{p^n,p^m}(\widehat{\pi_n}(y)\big)\, v\\
\varphi_m(\pi_{p^n,p^m}(\widehat{\pi_n}(x))) - \varphi_m(\pi_{p^n,p^m}(\widehat{\pi_n}(y))) &= \big(\widehat{\pi_m}(x) - \widehat{\pi_m}(y)\big)\, v\\
\varphi_m(\widehat{\pi_m}(x)) - \varphi_m(\widehat{\pi_m}(y)) &= \big(\widehat{\pi_m}(x) - \widehat{\pi_m}(y)\big)\, v
\end{aligned}$$

Thus, if $u \in U_n^{x,y}$ then $v = \pi_{p^n,p^m}(u) \in U_m^{x,y}$.

Consider the tree $\mathcal{T}$ of finite sequences $(u_0, \ldots, u_n)$ such that $u_i \in U_i^{x,y}$ and $u_i = \pi_{p^n,p^i}(u_n)$ for all $i = 0, \ldots, n$. Since each $U_n^{x,y}$ is nonempty, the tree $\mathcal{T}$ is infinite. Since it is at most p-branching, using König's Lemma, we can pick an infinite branch $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$. This branch defines an element $z \in \mathbb{Z}_p$. The commutative diagrams of Figure 2 show that the sequences $(\widehat{\pi_n}(x) - \widehat{\pi_n}(y))_{n\in\mathbb{N}}$ and $\varphi_n(\widehat{\pi_n}(x)) - \varphi_n(\widehat{\pi_n}(y))$ represent $x - y$ and $\Phi(x) - \Phi(y)$ in Z_p. Equality $\varphi_m(\widehat{\pi_m}(x)) - \varphi_m(\widehat{\pi_m}(y)) = (\widehat{\pi_m}(x) - \widehat{\pi_m}(y))\, \pi_{p^n,p^m}(u)$ shows that (going to the projective limits) $\Phi(x) - \Phi(y) = (x - y)\, z$. This proves that Φ is congruence preserving. □

3.3 Extension of congruence preserving functions $\mathbb{N} \to \mathbb{N}$

Congruence preserving functions $\mathbb{Z}_p \to \mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$) are determined by their restrictions to $\mathbb{N}$ since $\mathbb{N}$ is dense in $\mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}}$). Let us state a (partial) converse result.

Theorem 24. *Every congruence preserving function $F : \mathbb{N} \to \mathbb{Z}$ has a unique extension to a congruence preserving function $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$).*

Proof. Let us denote by $\widetilde{\mathbb{N}}$ and $\widetilde{\mathbb{Z}}$ the canonical copies of $\mathbb{N}$ and $\mathbb{Z}$ in $\mathbb{Z}_p$ and by $\widetilde{F} : \widetilde{\mathbb{N}} \to \widetilde{\mathbb{Z}}$ the copy of F as a partial function on $\mathbb{Z}_p$. As F is congruence preserving so is $\widetilde{F}$, which is thus also uniformly continuous (as a partial function on $\mathbb{Z}_p$). Since $\widetilde{\mathbb{N}}$ is dense in $\mathbb{Z}_p$, $\widetilde{F}$ has a unique uniformly continuous extension $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$. To show that this extension Φ is congruence preserving, observe that Φ, being uniformly continuous, is the inverse limit of the $\varphi_n = \widehat{\pi_n} \circ \Phi \circ \widehat{\iota_n}$. Now, since $\widehat{\iota_n}$ has range exactly $\widetilde{\mathbb{N}}$ we see that $\varphi_n = \widehat{\pi_n} \circ \widetilde{F} \circ \widehat{\iota_n}$; as $\widetilde{F}$ is congruence preserving so is φ_n. Finally, Theorem 23 insures that Φ is also congruence preserving. □

Polynomials in $\mathbb{Z}_p[X]$ obviously define congruence preserving functions $\mathbb{Z}_p \to \mathbb{Z}_p$. But non polynomial functions can also be congruence preserving.

Consequence 25 *The extensions to $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$ of the $\mathbb{N} \to \mathbb{Z}$ functions [3,4]*

$$x \mapsto \lfloor e^{1/a}\, a^x\, x! \rfloor \quad (\text{for } a \in \mathbb{Z} \setminus \{0,1\}) \quad , \quad x \mapsto \textbf{\textit{if}}\ x = 0\ \textbf{\textit{then}}\ 1\ \textbf{\textit{else}}\ \lfloor e\, x! \rfloor$$

and the Bessel like function $f(n) = \sqrt{\frac{e}{\pi}} \times \frac{\Gamma(1/2)}{2 \times 4^n \times n!} \int_1^\infty e^{-t/2}(t^2 - 1)^n dt$ *are congruence preserving.*

3.4 Representation of congruence preserving functions $\mathbb{Z}_p \to \mathbb{Z}_p$

We now characterize congruence preserving functions via their representation as infinite linear sums of the P_k's (suitably extended to $\mathbb{Z}_p$). This representation is a refinement of Mahler's characterization of continuous functions (Theorem 28). First recall the notion of valuation.

Definition 26. *The p-valuation (resp. the factorial valuation) $Val(x)$ of $x \in \mathbb{Z}_p$, or $x \in \mathbb{Z}/p^n\mathbb{Z}$ (resp. $x \in \widehat{\mathbb{Z}}$) is the largest s such that p^s (resp. $s!$) divides x or is $+\infty$ in case $x = 0$. It is also the length of the initial block of zeros in the p-adic (resp. factorial) representation of x.*

Note that for any polynomial P_k (or more generally any polynomial), the below diagram commutes for any $m \leq n$ (recall that $P_k^{p^n,p^n} = \pi_{p^n} \circ P_k \circ \iota_{p^n}$, cf. Definition 12):

$$\begin{array}{ccc} \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{P_k^{p^n,p^n}} & \mathbb{Z}/p^n\mathbb{Z} \\ \pi_{p^n,p^m}\downarrow & & \downarrow\pi_{p^n,p^m} \\ \mathbb{Z}/p^m\mathbb{Z} & \xrightarrow{P_k^{p^m,p^m}} & \mathbb{Z}/p^m\mathbb{Z} \end{array} \quad \text{i.e.} \quad \pi_{p^n,p^m} \circ P_k^{p^n,p^n} = P_k^{p^m,p^m} \circ \pi_{p^n,p^m}\,.$$

This allows to define the interpretation $\widehat{P_k}$ of P_k in $\mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}}$) as an inverse limit.

Definition 27. *$\widehat{P_k} \colon \mathbb{Z}_p \to \mathbb{Z}_p$ is the inverse limit of the inverse system $(P_k^{p^n,p^n})_{n\geq 1}$. Otherwise stated, for $x \in \mathbb{Z}_p$ such that $x = \varprojlim_{n\in\mathbb{N}} x_n$, we have*

$$\widehat{P_k}(x) = \varprojlim_{n\in\mathbb{N}} P_k^{p^n,p^n}(x_n) = \varprojlim_{n\in\mathbb{N}} \pi_{p^n}(P_k(\iota_{p^n}(x_n)))$$

Thus, the following diagram commutes for all n :

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\widehat{P_k}} & \mathbb{Z}_p \\ \widehat{\pi_n}\downarrow & & \downarrow\widehat{\pi_n} \\ \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{P_k^{p^n,p^n}} & \mathbb{Z}/p^n\mathbb{Z} \\ \iota_{p^n}\downarrow & & \downarrow\iota_{p^n} \\ \mathbb{N} & \xrightarrow{P_k} & \mathbb{N} \end{array}$$

Recall Mahler's characterization of continuous functions on $\mathbb{Z}_p$ (resp. $\widehat{Z}$).

Theorem 28 (Mahler, 1956 [10]). *1. A series $\sum_{k\in\mathbb{N}} a_k \widehat{P_k}(x)$, $a_k \in \mathbb{Z}_p$, is convergent in $\mathbb{Z}_p$ if and only if $\lim_{k\to\infty} a_k = 0$, i.e. the corresponding sequence of valuations $(Val(a_k))_{k\in\mathbb{N}}$ tends to $+\infty$.*
2. A function $\mathbb{Z}_p \to \mathbb{Z}_p$ is represented by a convergent series if and only if it is continuous. Moreover, such a representation is unique.
Idem with $\widehat{\mathbb{Z}}$.

Theorem 29 refines Mahler's characterization to congruence preserving functions.

Theorem 29. *A function $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$ represented by a series $\Phi = \sum_{k\in\mathbb{N}} a_k \widehat{P_k}$ is congruence preserving if and only if $lcm(k)$ divides a_k for all k.*

Note. The condition "$lcm(k)$ divides a_k for all k" is stronger than $\lim_{k\to\infty} a_k = 0$.

Proof. Suppose Φ is congruence preserving and let $\varphi_n = \widehat{\pi_n} \circ \Phi \circ \widehat{\iota_n}$. Theorem 23 insures that $\Phi = \varprojlim_{n\in\mathbb{N}} \varphi_n$ and the φ_n's are congruence preserving on $\mathbb{Z}/p^n\mathbb{Z}$. Using Corollary 14, we get $\varphi_n = \sum_{k=0}^{\nu(n)-1} b_k^n \, P_k^{p^n,p^n}$ with $lcm(k)$ dividing b_k^n for all $k \le \nu(n)-1$. By Proposition 18, Φ is uniformly continuous hence by Mahler's Theorem 28, $\Phi = \sum_{k\in\mathbb{N}} a_k \widehat{P_k}$ with $a_k \in \mathbb{Z}_p$ such that $\lim_{k\to\infty} a_k = 0$. Equation $\varphi_n = \widehat{\pi_n} \circ \Phi \circ \widehat{\iota_n}$ then yields

$$\varphi_n = \widehat{\pi_n} \circ (\sum_{k\in\mathbb{N}} a_k \widehat{P_k}) \circ \widehat{\iota_n} = \sum_{k\in\mathbb{N}} \widehat{\pi_n}(a_k)\, \widehat{\pi_n} \circ \widehat{P_k} \circ \widehat{\iota_n} = \sum_{k\in\mathbb{N}} \widehat{\pi_n}(a_k)\, P_k^{p^n,p^n} \,.$$

The unicity of the representation of φ_n (cf. Corollary 14) insures that $b_k^n = \widehat{\pi_n}(a_k)$. Similarly, $b_k^m = \widehat{\pi_m}(a_k)$; as for $m \le n$, $\widehat{\pi_m} = \pi_{p^n,p^m} \circ \widehat{\pi_n}$ (Lemma 21), we obtain $b_k^m = \pi_{p^n,p^m}(b_k^n)$. Thus, $(b_k^n)_{n\in\mathbb{N}}$ is an inverse system such that $a_k = \varprojlim_{n\in\mathbb{N}} b_k^n$. Since φ_n is congruence preserving Corollary 14 insures that $lcm(k)$ divides b_k^n; applying Lemma 30, we see that for all n, $\nu_p(k) \le Val(b_k^n)$. Noting that $Val(a_k) = Val(b_k^n)$, we deduce that $\nu_p(k) \le Val(a_k)$, hence $p^{\nu_p(k)}$ and thus also $lcm(k)$ divide a_k. In particular, this implies that $d(a_k, 0) \le 2^{-\nu_p(k)}$ and $\lim_{k\to\infty} a_k = 0$.

Conversely, if $\Phi = \sum_{k\in\mathbb{N}} a_k \widehat{P_k}$ and $lcm(k)$ divides a_k for all k then $lcm(k)$ divides $\widehat{\pi_n}(a_k)$ for all n, k. Thus, the associated φ_n are congruence preserving which implies that so is Φ by Theorem 23. □

Lemma 30. *Let $\nu_p(k)$ be the largest i such that $p^i \le k < p^{i+1}$. In $\mathbb{Z}/p^n\mathbb{Z}$, $lcm(k)$ divides a number x iff $\nu_p(k) \le Val(x)$.*

Proof. In $\mathbb{Z}/p^n\mathbb{Z}$ all numbers are invertible except multiples of p. Hence $lcm(k)$ divides x iff $p^{\nu_p(k)}$ divides x. □

Example 31. Let $\Phi = \sum_{k\in\mathbb{N}} a_k \, P_k$ with $a_k = p^{\nu_p(k)-1}$, with $\nu_p(k)$ as in Lemma 30. Φ is uniformly continuous by Theorem 28. By Lemma 30, $lcm(k)$ does not divide a_k; hence by Theorem 29, Φ is *not* congruence preserving.

4 Conclusion

We here studied functions having congruence preserving properties. These functions appeared as uniformly continuous functions in a variety of finite groups (see [11]).

The contribution of the present paper is to *characterize congruence preserving functions* on various sets derived from $\mathbb{Z}$ such as $\mathbb{Z}/n\mathbb{Z}$, (resp. $\mathbb{Z}_p, \widehat{\mathbb{Z}}$) via polynomials (resp. series) with *rational coefficients* which share the following common property: $lcm(k)$ divides the k-th coefficient. Examples of *non polynomial* (Bessel like) congruence preserving functions can be found in [4].

Acknowledgments

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Appendix

Appendix 1: Basics on p-adic and profinite integers

Recall some classical equivalent approaches to the topological rings of p-adic integers and profinite integers, cf. Lenstra [8,9], Lang [7] and Robert [12].

Proposition 32. *Let p be prime. The three following approaches lead to isomorphic structures, called the topological ring $\mathbb{Z}_p$ of p-adic integers.*

- *The ring $\mathbb{Z}_p$ is the inverse limit of the following inverse system:*
  - *the family of rings $\mathbb{Z}/p^n\mathbb{Z}$ for $n \in \mathbb{N}$, endowed with the discrete topology,*
  - *the family of surjective morphisms $\pi_{p^n,p^m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ for $0 \leq n \geq m$.*
- *The ring $\mathbb{Z}_p$ is the set of infinite sequences $\{0, \ldots, p-1\}^{\mathbb{N}}$ endowed with the Cantor topology and addition and multiplication which extend the usual way to perform addition and multiplication on base p representations of natural integers.*
- *The ring $\mathbb{Z}_p$ is the Cauchy completion of the metric topological ring $(\mathbb{N}, +, \times)$ relative to the following ultrametric: $d(x, x) = 0$ and for $x \neq y$, $d(x, y) = 2^{-n}$ where n is the p-valuation of $|x-y|$, i.e. the maximum k such that p^k divides $x - y$.*

Recall the factorial representation of integers.

Lemma 33. *Every positive integer n has a unique representation as*

$$n = c_k k! + c_{k-1}(k-1)! + ... + c_2 2! + c_1 1!$$

where $c_k \neq 0$ and $0 \leq c_i \leq i$ for all $i = 1, ..., k$.

Proposition 34. *The four following approaches lead to isomorphic structures, called the topological ring $\widehat{\mathbb{Z}}$ of profinite integers.*

- *The ring $\widehat{\mathbb{Z}}$ is the inverse limit of the following inverse system:*
  - *the family of rings $\mathbb{Z}/k\mathbb{Z}$ for $k \geq 1$, endowed with the discrete topology,*
  - *the family of surjective morphisms $\pi_{n,m} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$.*
- *The ring $\widehat{\mathbb{Z}}$ is the inverse limit of the following inverse system:*
  - *the family of rings $\mathbb{Z}/k!\mathbb{Z}$ for $k \geq 1$, endowed with the discrete topology,*
  - *the family of surjective morphisms $\pi_{(n+1)!,n!} : \mathbb{Z}/n!\mathbb{Z} \to \mathbb{Z}/m!\mathbb{Z}$ for $n \geq m$.*
- *The ring $\widehat{\mathbb{Z}}$ is the set of infinite sequences $\prod_{n \geq 1}\{0, \ldots, n\}$ endowed with the product topology and addition and multiplication which extend the obvious way to perform addition and multiplication on factorial representations of natural integers.*
- *The ring $\widehat{\mathbb{Z}}$ is the Cauchy completion of the metric topological ring $(\mathbb{N}, +, \times)$ relative to the following ultrametric: for $x \neq y \in \mathbb{N}$, $d(x, x) = 0$ and $d(x, y) = 2^{-n}$ where n is the maximum k such that $k!$ divides $x - y$.*
- *The ring $\widehat{\mathbb{Z}}$ is the product ring $\prod_{p\ prime} \mathbb{Z}_p$ endowed with the product topology.*

Proposition 35. *The topological rings $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$ are compact and zero dimensional (i.e. they have a basis of closed open sets).*

Appendix 2: $\mathbb{N}$ and $\mathbb{Z}$ in $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$

Proposition 36. *Let $\lambda : \mathbb{N} \to \mathbb{Z}_p$ (resp. $\lambda : \mathbb{N} \to \widehat{\mathbb{Z}}$) be the function which maps $n \in \mathbb{N}$ to the element of $\mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}}$) with base p (resp. factorial) representation obtained by suffixing an infinite tail of zeros to the base p (resp. factorial) representation of n.*
The function λ is an embedding of the semiring $\mathbb{N}$ onto a topologically dense semiring in the ring $\mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}}$).

Remark 37. In the base p representation, the opposite of an element $f \in \mathbb{Z}_p$ is the element $-f$ such that, for all $m \in \mathbb{N}$,

$$(-f)(i) = \begin{cases} 0 & \textit{if } \ \forall s \leq i \ f(s) = 0\,, \\ p - f(i) & \textit{if } \ i \textit{ is least such that } f(i) \neq 0\,, \\ p - 1 - f(i) & \textit{if } \ \exists s < i \ f(s) \neq 0\,. \end{cases}$$

In particular,
– Integers in $\mathbb{N}$ correspond in $\mathbb{Z}_p$ to infinite base p representations with a tail of 0's.
– Integers in $\mathbb{Z} \setminus \mathbb{N}$ correspond in $\mathbb{Z}_p$ to infinite base p representations with a tail of digits $p - 1$.
Similar results hold for the infinite factorial representation of profinite integers.