$F \subseteq V$ final vertex

$\Omega = V^* F V^\omega$

$\text{Attr}_o(F) = F$

$\text{Attr}_i(F) = \text{Attr}_i(F)$

$\text{Attr}_u(F) = \{ v \in V \mid \exists v' \in \text{Attr}_i(F) \text{ s.t. } v \rightarrow v' \}$

$\Omega \{ v \in V \mid \forall v' \in \text{Attr}_i(F) \text{ then } v' \in \text{Attr}_i(F) \}$

$(\text{Attr}_i)_7$

$\subseteq \text{Attr}_i(F)$ Permit

$\text{Th}: \text{Attr}_i(F) \subseteq WE$ + positional start
$\text{Att}_\omega^u (F)$

$\text{Att}_\omega^{u+1} (F)$
Buchi condition:

$F \subseteq V$ final vertex.

Eve wins a play if it visits $F$ often.

$\Omega = \bigcap \left\{ V' : V^* F V' \supseteq \{w_0 v_1 v_2 \ldots | \exists^n : v_i \in F \} \right\}$

\[
\begin{align*}
\text{Attr}_0^+(S) &= \emptyset \\
\text{Attr}_{i+1}^+(S) &= \text{Attr}_i^+(S) \cup \{v \in V \mid \exists v' \text{ s.t. } v \rightarrow v' \text{ and } v' \in \text{Attr}_i^+(S) \cup S\} \\
&\quad \cup \{v \in V \mid \forall v' \text{ s.t. } v \rightarrow v' \text{ then } v' \in \text{Attr}_i^+(S) \cup S\}.
\end{align*}
\]

$\text{Attr}^+(S) = \text{Rim}(\text{Attr}_i^+(S)) ; i \geq 0$

$\geq$ vertices from which Eve can force to revisit $S$ in at least one step.

Proof: Like reachability games.
\( \text{Attr}^+(F) \)

\[
\begin{align*}
(zi) & \rightarrow 1 \\
Z_0 & = F \\
Z_{i+1} & = \text{Attr}^+(Z_i) \cap F \\
Z_\infty & = \text{lim}_{i \to \infty} Z_i \\
L_\ast & \text{ greatest fixed point}
\end{align*}
\]

\[
Z_\infty = \text{Attr}^+(Z_\infty) \cap F
\]

Ls Eve has a strat to go back to Z∞ from Z∞ in at least one step.

\[
\text{Th: } W_E = \text{Attr}^+(Z_\infty)
\]
Proof: $\text{Att}_n (Z^\omega) \subseteq \text{We} \subseteq \text{Att}_n (Z^\omega)$

Define $\varphi$ positional on $\text{Att}_n (Z^\omega)$ by:

$$\varphi(v) = \varphi_i(v) \text{ if } v \in Z^n$$

$$= \varphi_0(v) \text{ if } v \notin Z^n$$

Let us prove that $\varphi$ play starting in $\text{Att}_n (Z^\omega)$ where $\text{Eve}$ respects $\varphi$ is winning for $\text{Eve}$.

$$l = v_0,v_1,v_2,\ldots$$

$\forall i: v_i \in \text{Att}_n (Z^\omega)$ because $v_0$ does

$\varphi_0$ if $v_i$ does then $v_{i+1}$ does too

$$l = v_0 \overline{\varphi_0} \overline{\varphi_i} \overline{\varphi_0} \ldots$$

$$= v_0 \overline{\varphi_0} \overline{\varphi_{k_0}} \overline{\varphi_{k_0+1}} \overline{\varphi_{k_1}} \ldots \overline{\varphi_{k_i}} \overline{\varphi_{k_i+1}} \ldots \overline{\varphi_{k_i}} \Rightarrow \text{winning}$$

$\exists \in Z^\omega \overline{\varphi_{k_i}} \leq f$
\( V \setminus \text{Att}(Z_0) \subset W_A \)
\[ v \notin \text{Att}(Z_0) \implies \exists i : 1 \leq i \leq \text{Att}(Z_i) \]
Call \( \kappa(v) = \text{smallest } i \text{ s.t. } v \in \text{Att}(Z_i) \setminus \text{Att}(Z_{i-1}) \)

**Property:** \( \forall v \notin \text{Att}(Z_0) \text{ then we have:} \)
- \( v \in \text{Att}(Z_0) \), \( v \) has a successor \( v' \) s.t. \( \kappa(v') < \kappa(v) \)
  and if \( v \in F \) then the inequality is strict
- \( \forall v \in F \), \( \forall v' \text{ s.t. } v \rightarrow v' \) one has \( \kappa(v') \leq \kappa(v) \)
  and if \( v \in F \) the inequality is strict.

**Proof:** \( v \) with \( \kappa(v) = i \) and \( v \notin \text{Att}(Z_0) \)
\[ \Rightarrow v \in \text{Att}(Z_{i+1}) \Rightarrow \text{Adam can fire to stay outside of } \text{Att}(Z_{i+1}) \]
\[ \Rightarrow \text{to stay in vertices of } \kappa \leq i \text{, } v \in \text{Att}^+(Z_i) \]
\( v \in F \), if \( \forall v' \text{ s.t. } v \rightarrow v' \text{ one has } \kappa(v') = i \Rightarrow \forall F \Rightarrow \kappa(v) = z_{i+1} \Rightarrow \kappa(v) = z_{i+1} \)
Define $\Psi$ positional strict for Adam by letting

$$\Psi(v) = v'$$

for some $v'$ s.t. $\Delta v(u') \leq \Delta v(u)$

strict if $v \in \mathcal{F}$. \\

$2 = \ldots v_0, v_1, v_2 \ldots$ where Adam respects $\Psi$ and starting outside of $\mathcal{F}$. \\
Then: $\Delta v(u) < \Delta v(u')$ and decreases strictly when $v \in \mathcal{F}$.

$\Rightarrow$ this can only happen finitely often.

$\Rightarrow \sum_{i=1}^{\infty} 1 \leq \mathcal{F}$.

$\Rightarrow$ Adam wins in $2$.

$\text{Adam}(200) \leq \text{WA}$
\[ G = (V, E) \]

\[ C = \{0, \ldots, d \} \subseteq \mathbb{N} \text{ colours} \]

\[ E : V \rightarrow C \text{ : coloring set.} \]

\[ \Omega = \{ \omega = v_0 v_1 \ldots \mid \lim \sup \{(v_i)_{i>0} \} > 0 \text{ even} \} \]

**Th:** One can compute winning region + positional strat that traps in the winning region

- Subarena: \( U \) is a subarena if \( G \cup U \) has no dead-end
- \( G(U, E \cap U \times U) \)

**Trap:** \( U \subseteq V \) is a trap for player \( \sigma \) if \( \bar{\sigma} \) has a strat to trap the play in \( U \) if it starts from \( \bar{\sigma} \)

If \( U \) is a trap then \( U \) is a subarena

The complement of an attractor for \( \bar{\sigma} \) is a trap for \( \sigma \)
Proof is by induction on the number $\sigma$ of colours.

**Base case:** $\sigma = 1$ → trivial.

**Induction:** $\sigma = \sigma' + 1$.

Start at a position $d$ in $C$.

Call $\sigma$ the play that wins if $d$ is as often repeated.

We will construct an infinite sequence $(W^k, Y^k)$ such that:

1. $W^k$ is a trap for $\sigma$ and $Y^k$ is winning on it and traps the play in $W^k$.
2. $(W^k)_{k=1}^\infty$ is an inhabitable sequence.
3. $d^k$ geometrically dominates $d$.

For each $k$, $W^k$ is an inhabitable sequence.

$W_0^0 = \varnothing$ and $Y_0^0$ is never.

Vertices with colour $d$: $W^k_{\sigma} = W^k_{\sigma} \cup U \cup Z_{\sigma}$

Subsequence with $\sigma-1$ colours.
\[ w^k_l \mid \psi^k_l \]

\[ X_k = \text{Add}^\sigma (w^k_l) \mid \psi \text{ path} \]

\[ T = V \setminus X_k \]

\[ Z_k = T_k \setminus \text{Add}^\sigma (N_k) \text{ when } N_k = \{ v \in T_k \mid c(v) = 0 \} \]

L subarena with \( n-1 \) colours

\[ \Rightarrow Z^k_l \mid Z^k_\sigma \]

\[ \psi^k_l \mid \psi^k_\sigma \}

by induction hypothesis on the \# colours

\[ w_{k+1}^{l+1} = X_k \cup Z_k^l \]

\[ \psi_{k+1}^l (v) = \begin{cases} \psi_\sigma^l (v) & \text{if } v \in \overline{Z^l_\sigma} \\ \psi^k_\sigma (v) & \text{if } v \in W^k_\sigma \end{cases} \]

W = Point (w^k_\sigma) \text{ winning by IH on } \psi^k_\sigma
Consider $V \setminus W^{\omega}$

situation when
six point is reached

Define $\Psi$ strat in $\sigma$

$$
\Psi(v) = \begin{cases} 
\Psi_0(v) & \text{if } v \in N \\
\Psi_{\text{oth}}(v) & \text{if } v \in \text{Adj}^\circ(N) \setminus N \\
\Psi_0(v) & \text{if } v \in Z_0
\end{cases}
$$

1. play starting in $V \setminus W^{\omega}$ when $\sigma$ respects $\Psi$

   1) either \( \Delta \) stays in $Z_0$ eventually forever $\Rightarrow$ winning by $5+$

   2) \( \epsilon \) often gets in $\text{Adj}^\circ(N)$ $\Rightarrow \epsilon$ often visits $N \Rightarrow$ man \( \epsilon \) often visited
color is \( c \) $\Rightarrow \sigma$ wins.
Corollary: Solving parity game is in \( \text{NP} \cap \text{co-NP} \).

**Proof**

**NP:**

Eve wins \( \iff \) she has a positional winning strategy.

**Algorithm:**

1. Guess a positional strategy for Eve
2. Check that it is winning

\( \subseteq \text{PTIME} \)

\( \text{co-NP:} \) duality problem

**Big question:** Is it in \( P \)?