Games over finite graphs

Exercise 1

For the whole exercise we assume that we are given a finite graph $G = (V, E)$, with V a finite set of vertices and $E \subseteq V \times V$ a finite subset of edges. Moreover, we assume that there is no dead-end (that is for all $v_1 \in V$ there exists $v_2 \in V$ such that $(v_1, v_2) \in E$). We additionally fix a partition $V_E \cup V_A$ of the vertices V (that is $V_E \cap V_A = \emptyset$ and $V_E \cup V_A = V$). This partition is then use to define an arena $\mathcal{G} = (G, V_E, V_A)$ as explained during the lectures.

Question 1: Let $F_1, \cdots, F_n$ be a collection of subsets of V that are pairwise disjoint (for all $i \neq j$, $F_i \cap F_j = \emptyset$). We consider the following winning condition for Eve :

$$\Omega = V^* F_1 V^* F_2 \cdots V^* F_n V^\omega$$

and we denote by $\mathbb{G} = (\mathcal{G}, \Omega)$ the associated game. Hence in this game, a play $\lambda = v_0 v_1 v_2 \cdots$ is won by Eve if and only if the sets $F_1, \cdots, F_n$ are visited, in this order, at least one, *i.e.* there are $i_1 < i_2 < \cdots i_n$ such that for all $1 \leq j \leq n$, $v_{i_j} \in F_j$.

(a) Give an algorithm that decide for a given vertex whether it is winning for Eve.
(b) What is the complexity of your algorithm ?
(c) Does Eve always has a winning positional strategy from any winning vertex in $\mathbb{G}$?
(d) Can you build a winning strategy for Eve from any winning vertex for her in $\mathbb{G}$?

Answer. We define the following sets : $W_{n+1} = V$ and $W_i = \mathit{Attr}(W_{i+1} \cap F_i)$ for all $1 \leq i \leq n$, where Attr is the usual attractor for Eve as seen in the course. We have the following property : for all $1 \leq i \leq n+1$, W_i consists of those vertices from which Eve has a winning strategy in the game whose winning condition is $\Omega_i = V^* F_i V^* F_{i+2} \cdots V^* F_n V^\omega$ (with the convention that $\Omega_{n+1} = V^\omega$). Then the property is proved by (decreasing) induction on i. For $i = n+1$ the property is obvious. Then assume the property holds for some $i+1$. By definition of the attractor, Eve has from $W_i = \mathit{Attr}(W_{i+1} \cap F_i)$ a positional strategy φ to reach the set $W_{i+1} \cap F_i$ and by induction hypothesis, she has a strategy φ to ensure, from any vertex in W_{i+1}, that the play belongs to Ω_{i+1}. Now consider the strategy φ_{i+1} of Eve that is defined as follows : play accordingly to φ as long as $W_{i+1} \cap F_i$ is not reached and then play accordingly to φ_{i+1} forever. Consider a play λ that starts in W_i and where Eve respects φ_{i+1} : first, it eventually visits $W_{i+1} \cap F_i$ (as Eve first respects φ), and is of the form $\lambda_1 v \lambda_2$ with $v \in W_{i+1} \cap F_i$ and λ_1 not visiting $W_{i+1} \cap F_i$. Then $v\lambda_2$ is a play where Eve respects φ_{i+1}, hence $\lambda_2 \in \Omega_{i+1}$, hence $\lambda \in \Omega_i$. This proves that from any vertex in W_i Eve has a strategy to ensure that the resulting play is in Ω_i. Conversely, consider now a vertex $v \notin W_i$. By definition, Adam has a strategy ψ to prevent reaching $W_{i+1} \cap F_i$, and (by induction hypothesis), for any element not in W_{i+1} Adam has a strategy ψ_i to prevent to produce a play in Ω_{i+1}. Now consider the following strategy for Adam : play according to ψ as long as F_i is not visited, and if this eventually happens, then play according to ψ_i. Consider a play λ starting from a vertex not in W_i and where Adam respects ψ : then either λ never reach F_i (hence $\lambda \notin \Omega_i$) or $\lambda = \lambda_1 v \lambda_2$ with $v \in F_i$ (hence $v \notin W_{i+1}$) and λ_1 not visiting F_i and $v\lambda_2$ is a play where Adam respects ψ_i. Hence $\lambda_2 \notin \Omega_{i+1}$ and therefore $\lambda \notin \Omega_i$. This concludes the induction. Thus the winning vertices in the original game are those in W_1.

Computing W_1 requires n computation of the attractor set, which can be achieved in $\mathcal{O}(n \times |E|)$

Eve may not have a positional strategy in the game. For instance consider a three vertex game where Eve is playing alone and such that the vertices are u,v and w and the edges $\{(u,v)(v,u)(u,w)(w,u)\}$ and $F_1 = \{v\}$ and $F_2 = \{w\}$. Then any positional strategy will produce a play that visits only $\{u,v\}$ or only $\{u,w\}$, hence loosing. Of course Eve has winning strategy in this game.

From the proof above, one can construct a winning strategy (as φ_i is built from φ_{i+1} and φ a positional strategy in a reachability game). In particular one can note that the resulting strategy only uses finite memory (of size n).

Question 2: We again consider a collection $F_1, \cdots, F_n$ of subsets of V that are pairwise disjoint (for all $i \neq j$, $F_i \cap F_j = \emptyset$). We consider the following winning condition for Eve :

$$\Omega = \bigcap_{i=1}^{n} V^* F_i V^\omega$$

and we denote by $\mathbb{G} = (\mathcal{G}, \Omega)$ the associated game. Hence in this game, a play $\lambda = v_0 v_1 v_2 \cdots$ is won by Eve if and only if the sets $F_1, \cdots, F_n$ are visited, in any order, at least one, *i.e.* there are $i_1, i_2, \cdots, i_n$ such that for all $1 \leq j \leq n$, $v_{i_j} \in F_j$.

(a) Give an algorithm that decide for a given vertex whether it is winning for Eve. One might design a new *equivalent* game (with a simpler winning condition studied in the course).

(b) What is the complexity of your algorithm ?

(c) Does Eve always has a winning positional strategy from any winning vertex in $\mathbb{G}$?

(d) Can you build a winning strategy for Eve from any winning vertex for her in $\mathbb{G}$?

Answer. We define a new graph $G' = (V', E')$ where $V' = V \times 2^{\{1,\cdots,n\}}$ and E' consists of the pairs $((v_1, S_1), (v_2, S_2))$ such that $(v_1, v_2) \in V$ and $S_2 = S_1 \cup \{i \mid v_2 \in F_i\}$. We partition V' by letting $V'_E = V_E \times 2^{\{1,\cdots,n\}}$, which leads to define an arena $\mathcal{G}'$. Then one define a set of final vertices $F' = V \times \{1, \ldots, n\}$ and let $\mathbb{G}'$ be the reachability game induced by F' on $\mathcal{G}'$. We claim that Eve has a winning strategy in $\mathbb{G}$ from a vertex v iff she has a winning strategy in $\mathbb{G}'$ from $(v, \{i \mid v \in F_i\})$. One can note that in $\mathcal{G}'$, from some vertex (v, S) if there is an edge to some (v', S') then S' is uniquely determined from v' and S. In particular, it means that any strategy φ in $\mathbb{G}$ can be lifted to a strategy in $\mathbb{G}'$ by letting $\varphi'((v_1, S_1) \cdots (v_k, S_k)) = (v, S)$ where $v = \varphi(v_1 \cdots v_k)$ and S is the unique possible set. Assume that Eve wins in $\mathbb{G}$ from v and call φ a winning strategy. Consider a (finite) play $\lambda = (v_1, S_1)(v_2, S_2) \cdots$ in $\mathbb{G}'$ from $(v, \{i \mid v \in F_i\})$ where Eve respects φ'. It is straightforward to check that $S_k = \{i \mid \exists j \leq k \text{ s.t. } v_k \in F_i\}$ (this is independent of φ'). Now by definition of φ', one has that $v_1 v_2 \cdots$ is a play in $\mathbb{G}$ where Eve respects φ, hence it is winning, hence there is some k such that $S_k = \{1, \cdots, n\}$, meaning that φ' is winning.

Now assume that Eve wins in $\mathbb{G}'$ from $(v, \{i \mid v \in F_i\})$ and call φ' a winning strategy. One can assume that φ' is positional as $\mathbb{G}'$ is a reachability game. Define a strategy φ in $\mathbb{G}$ by letting $\varphi(v_0 \cdots v_k) = v_{k+1}$ where $(v_{k+1}, S_{k+1}) = \varphi'(v_k, \{i \mid \exists j \leq k \text{ s.t. } v_k \in F_i\})$. Consider a play $v_0 v_1 \cdots$ where Eve respects φ. Then the play $(v_0, S_0)(v_1, S_1) \cdots$ where we let $S_k = \{i \mid \exists j \leq$

k s.t. $v_k \in F_i\}$ is a play in $\mathbb{G}$ where Eve respects φ'. Therefore it is a winning play, meaning that $v_0 v_1 \cdots$ is winning for Eve in $\mathbb{G}$. This concludes the proof.

Complexity is exponential as the game $\mathbb{G}'$ is (and solving a reachability game is linear).

There is no memoryless strategy in general for the same reason as previously. From the proof it follows that one can always construct a winning strategy in $\mathbb{G}$ (as one can built one in $\mathbb{G}'$ and lift it back to $\mathbb{G}$.

Question 3: We again consider a collection $F_1, \cdots, F_n$ of subsets of V that are pairwise disjoint (for all $i \neq j$, $F_i \cap F_j = \emptyset$). We consider the following winning condition for Eve : a play $\lambda = v_0 v_1 v_2 \cdots$ is won by Eve if and only if the each of the set $F_1, \cdots, F_n$ is visited infinitely often. Equivalently, for all $1 \leq j \leq n$ there exists $i_1 < i_2 < i_3 \cdots$ such that for all $1 \leq k$, $v_{i_k} \in F_j$.

(a) Give an algorithm that decide for a given vertex whether it is winning for Eve. One might design a new *equivalent* game (with a simpler winning condition studied in the course).

(b) What is the complexity of your algorithm ?

Answer. It suffices to remark that the winning condition is unchanged is one force the order in which the sets F_i are visited, *i.e.* she has to infinitely visit F_1 and then F_2 and then $F_3 \ldots F_n$. Hence it suffices to build an new game with an extra component recalling which F_i should be visited next, and to define as a winning condition to infinitely often switch from the component being n to the component being 1.

Question 4: **[Difficult]** We go back to the setting of the second question, but we additionally assume that all the F_j are singleton. Give a *polynomial time* algorithm to decide whether a vertex is winning for Eve.

Answer. For all pair (i, j) of vertices one check whether i belongs to the attractor of j. Then one builds a graph whose vertices are $1, \ldots, n$ and where there is an edge from i to j if and only if i belongs to the attractor of j. Then in this graph there is a path that visits all vertices if and only if Eve has a winning strategy (this is easy to check). Looking for such a path is checked in polynomial time (one essentially builds the strongly connected components of the graph).