Quantitative Languages of Infinite Words

M2 training period proposition

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Definition 1. *Let A be a finite alphabet. A* **quantitative language** *of infinite words over A is a mapping from the set of infinite words A^ω to the set of reals $\mathbb{R}$.*

Definition 2. *A (non-deterministic) weighted* **automaton** *is a tuple $\mathcal{A} = (Q, A, q_0, \Delta)$ where Q is a finite set of control states, $q_0 \in Q$ is the initial state, A is the input alphabet and $\Delta \subseteq Q \times A \times \mathbb{R} \times Q$.*

A run of $\mathcal{A}$ on $u = a_0a_1a_2\cdots$ is a sequence $q_0, r_0, q_1, r_1, q_2, \cdots$ such that q_0 is the initial state of $\mathcal{A}$ and for every $i \geqslant 0$ one has $(q_i, a_i, r_i, q_{i+1}) \in \Delta$.

From now on fix a weighted automaton $\mathcal{A} = (Q, A, q_0, \Delta)$. For simplicity we assume that we have for any $(q, a) \in Q \times A$ that $|\{(r, q') \in \mathbb{R} \times Q \mid (q, a, r, q') \in \Delta\}| = 2$.

With any word $u \in A^\omega$ we can consider all runs of $\mathcal{A}$ on u and associate a complete binary tree t_u whose nodes are labelled by elements of Q and arcs are labelled by reals: a branch in this tree corresponds to a run of $\mathcal{A}$ on u.

Definition 3. *A* **payoff function** *is a map $C : \mathbb{R}^\omega \to \mathbb{R}$.*

Typical payoff functions can be the lim-inf-average function

$$\mathrm{LimInfAv} : r_0r_1r_2\cdots \mapsto \liminf(\frac{1}{n}\sum_{j=0}^{n-1} r_j)_{n\geqslant 0}$$

or the discounted payoff function (for some $0 < \lambda < 1$)

$$\mathrm{D}_\lambda : r_0r_1r_2\cdots \mapsto \lim_{n\to\infty}\sum_{j=0}^{n} r_j\lambda^j$$

Payoff functions can naturally be used to associate with any run a payoff (simply take the value of the payoff function on the sequence of weights along the run).

Hence, for a fixed payoff function, for a word $u \in A^\omega$, we associate with each branch in t_u a real as the payoff corresponding to the run associated with the branch. Our goal during this internship is to find a robust way to associate with t_u (equivalently with u) a unique payoff. Trivial ways to do so is to take *e.g.* the infimum over all possible payoff but we would rather have a notion corresponding to the expected payoff one could obtain by randomly choosing a run (and getting the corresponding payoff). Of course such a definition may need extra requirement on the payoff function.

Once good notions are found, we would like to study closure properties of the quantitative languages that we obtain and also investigate other characterisations of those languages.

Depending on whether we get results on the infinite words setting, we may later investigate extensions to the infinite trees setting.