Coherence of Gray categories via rewriting

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Abstract

Over the recent years, the theory of rewriting has been extended in order to provide systematic techniques to show coherence results for strict higher categories. Here, we investigate a further generalization to low-dimensional weak categories, and consider in details the first non-trivial case: presentations of tricategories. By a general result, those are equivalent to the stricter Gray categories, for which we introduce a notion of rewriting system, as well as associated tools: critical pairs, termination orders, etc. We show that a finite rewriting system admits a finite number of critical pairs and, as a variant of Newman's lemma in our context, that a convergent rewriting system is coherent, meaning that two parallel 3-cells are necessarily equal. This is illustrated on rewriting systems corresponding to various well-known structures in the context of Gray categories (monoids, adjunctions, Frobenius monoids). Finally, we discuss generalizations in arbitrary dimension.

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The rewriting systems which are convergent have a fundamental property, which is a consequence of Newman’s and other classical lemmas in rewriting theory: the space between any two rewriting zigzags with the same source and the same target can be filled with tiles witnessing the confluence of critical branchings. Otherwise said, every diagram commutes modulo the commutation of diagrams induced by critical branchings, which thus axiomatize the coherence of the structure.

Over the recent years, there have been many efforts to generalize the techniques of rewriting from words and terms to morphisms in strict n-categories, starting from the pioneering work of Burroni and Lafont [3, 15, 16]. Those widen the range of applicability of rewriting, and also allow a precise formulation of the above remark initially formulated by Squier, and generalized by Guiraud and Malbos by considering coherent presentations [17, 8, 9]. As a typical example, starting from the 2-category of planar binary forests, which is generated by a binary (\(\mu\)) and a nullary corolla (\(\eta\)), one can consider rewriting rules expressing the fact that \(\mu\) is associative and \(\eta\) is both a left and right unit for \(\mu\). The resulting rewriting system is convergent, and the technique described above allows to prove a coherence theorem for pseudomonoids, of which MacLane’s coherence result is a particular case (a monoidal category is a pseudomonoid in the cartesian 2-category \(\text{Cat}\)).

It is of course of interest to generalize the coherence theorems for classical algebraic structures from strict to weak n-categories. For instance, coherence for pseudomonoids in tricategories is shown in [14]. A rewriting approach in this domain is desirable, but the way one could handle all the coherence morphisms present in weak categories was not
clear. Recently, the use of semistrict weak \(n\)-categories was advocated by the creators of the graphical proof-assistant Globular \([1, 2]\) as a formalism adapted to computer manipulations: without loss of generality most of the coherence morphisms can be considered to be identities, excepting the interchangers and their coherences.

In this article, we develop the theory of rewriting for semistrict 3-categories, also called Gray categories, which is the first dimension in which the strict and weak definitions are not equivalent \([6, 11]\). We illustrate that this provides systematic principles for proving coherence of algebraic structures in Gray categories, allowing us to recover results in this direction recently proved \([14, 4, 20]\) as well as new ones. Moreover, it turns out that this weak framework is better behaved than the strict one in some respects: it was observed that a finite rewriting system on strict 2-morphisms can give rise to an infinite number of critical branchings \([15, 8]\), which is the source of many difficulties \([16]\), both of theoretical and practical nature, whereas in the present setting we show that only a finite number of critical pairs can be generated. Finally, we also hint at generalizations in arbitrary dimension.

1 Coherence of Gray categories via rewriting

1.1 Sesquicategories

We begin by recalling the notion of 2-category as a variant of sesquicategories, details can be found in \([18]\). A sesquicategory, or 2- precategory, \(C\) consists of

- a set \(C_0\) of 0-cells,
- a set \(C_1(x, y)\) of 1-cells \(u : x \to y\) for every 0-cells \(x\) and \(y\),
- a set \(C_2(u, v)\) of 2-cells \(\alpha : u \Rightarrow v : x \to y\) for every parallel 1-cells \(u, v : x \to y\),
- an identity 1-cell \(1_x : x \to x\) for every 0-cell \(x\),
- a composition function which to every 1-cells \(u : x \to y\) and \(v : y \to z\) associates a 1-cell \(u \ast v : x \to z\),
- an identity 2-cell \(1_u : u \Rightarrow u\) for every 1-cell \(u\),
- a vertical composition function which to every 2-cell \(\alpha : v \Rightarrow v'\) and \(\beta : v' \Rightarrow v''\) associates a 2-cell \(\alpha \ast \beta : v \Rightarrow v''\) (middle of (1)),
- a left whiskering composition function which to every 1-cell \(u : x' \to x\) and 2-cell \(\alpha : v \Rightarrow v'\) associates a 2-cell \(u \ast \alpha : u \ast v \Rightarrow u \ast v' : x \to y'\) (left of (1)),
- a right whiskering composition function which to every 2-cell \(\alpha : v \Rightarrow v'\) and 1-cell \(w : y \Rightarrow y'\) associates a 2-cell \(\alpha \ast w : v \ast w \Rightarrow v' \ast w : x \to y'\) (right of (1)).

\[
\begin{align*}
\begin{array}{ccc}
x' & \overset{u}{\longrightarrow} & x \\
\downarrow & \downarrow & \downarrow \\
v \circ \alpha & \Rightarrow & y \\
\end{array} & \quad \begin{array}{ccc}
x & \overset{v}{\longrightarrow} & y \\
\downarrow & \downarrow & \downarrow \\
\psi \circ \beta & \Rightarrow & \psi' \\
\end{array} & \quad \begin{array}{ccc}
x & \overset{v}{\longrightarrow} & y \\
\downarrow & \downarrow & \downarrow \\
v \circ \alpha & \Rightarrow & v' \\
\end{array}
\end{align*}
\]

such that compositions are associative and admit identities as neutral elements: for suitably typed 0-cells \(x, y\), 1-cells \(u, v, w\) and 2-cells \(\alpha, \beta, \gamma\),

\[
\begin{align*}
(u \ast v) \ast w &= u \ast (v \ast w) & 1_x \ast u &= u & u \ast 1_y &= u \\
(\alpha \ast \beta) \ast \gamma &= \alpha \ast (\beta \ast \gamma) & 1_u \ast \alpha &= \alpha & \alpha \ast 1_v &= \alpha \\
1_u \ast (u \ast \alpha) &= (u \ast \alpha) \ast (u \ast \beta) & 1_v \ast 1_v &= 1_{u+v} & (\alpha \ast \beta) \ast w &= (\alpha \ast w) \ast (\beta \ast w) & 1_v \ast w &= 1_{v+w} \\
1_x \ast (u \ast v) &= (u \ast v) \ast \alpha & 1_x \ast \alpha &= \alpha & \alpha \ast (v \ast w) &= (\alpha \ast v) \ast w & \alpha \ast 1_w &= \alpha \\
(u \ast \alpha) \ast w &= u \ast (\alpha \ast w) &
\end{align*}
\]

In a composition, the dimension of the involved cells determines which composition is used, which allows us to unambiguously denote them by the same symbol. In a more terse way,
the category of sesquicategories can be defined as the category of categories enriched over \( \text{Cat} \) equipped with the “funny tensor product” \([5]\).

A 2-category \( C \) is a sesquicategory such that the interchange law holds: this means that for every 2-cells \( \alpha : u \Rightarrow u' : x \rightarrow y \) and \( \beta : v \Rightarrow v' : y \rightarrow z \), we have

\[
(\alpha * v) * (u' * \beta) = (u * \beta) * (\alpha * v')
\]

Since both of the above compositions are equal, we can define the 0-composition of \( \alpha \) and \( \beta \) to be either of them. By contrast, in a sesquicategory, the 0-composition of 2-cells does not make sense: we can only compose 2-cells in codimension 1.

### 1.2 Signatures

In the following, we will be interested in rewriting morphisms in freely generated sesquicategories. Recall that a graph consists of

- a set \( P_0 \) of vertices,
- a set \( P_1 \) of edges,
- functions \( s_0, t_0 : P_1 \rightarrow P_0 \) associating to each edge its source and target vertex.

We write \( P_1^* \) for the set of paths in the graph, \( s_0^*, t_0^* : P_1^* \rightarrow P_0 \) the source and target functions on paths, and \( uv \) for the concatenation of composable paths \( u \) and \( v \).

A signature \( P \) consists of

- a graph \((P_0, s_0, t_0, P_1)\) whose vertices and edges are called 0- and 1-generators,
- a set \( P_2 \) of 2-generators together with functions \( s_1, t_1 : P_2 \rightarrow P_1 \) such that \( s_0^* \circ s_1 = s_0^* \circ t_1 \) and \( t_0^* \circ s_1 = t_0^* \circ t_1 \).

We write \( a : x \rightarrow y \) to indicate that \( a \) is a 1-generator with \( s_0(a) = x \) and \( t_0(a) = y \), and similarly for 2-generators \( \alpha : u \Rightarrow v \) with \( s_1(\alpha) = u \) and \( t_1(\alpha) = v \).

**Example 1 (Monoids).** The signature for monoids is

\[
P_0 = \{ \ast \} \quad P_1 = \{ 1 : \ast \rightarrow \ast \} \quad P_2 = \{ \mu : 2 \Rightarrow 1, \eta : 0 \Rightarrow 1 \}
\]

Note that the set \( P_1^* \) is isomorphic to \( \mathbb{N} \), thus the notation for its elements. The 2-generators of this signature should respectively be understood as a formal multiplication \( (\mu) \) and unit \( (\eta) \), which we will use below to express the structure of a monoid.

A signature \( P \) freely generates a sesquicategory with \( P_0 \) as 0-cells, \( P_1^* \) as 1-cells (composition being concatenation and identities empty paths), and whose 2-cells are generated by \( P_2 \). We write \( P_2^* \) for its set of 2-cells, whose elements can be described explicitly as follows.

**Proposition 2.** The 2-cells in \( P_2^* \) can be described as the sequences of the form

\[
(u_1 * \alpha_1 * w_1) * (u_2 * \alpha_2 * w_2) * \ldots * (u_n * \alpha_n * w_n)
\]

with \( u_i : x \rightarrow x_i \) in \( P_1^* \), \( \alpha_i : v_i \Rightarrow v_i' : x_i \rightarrow y_i \) in \( P_2 \), \( w_i : y_i \rightarrow y \) in \( P_1^* \) (the compositions above are formal ones). The canonical inclusion \( P_2 \rightarrow P_2^* \) sends a 2-generator \( \alpha : u \Rightarrow v : x \rightarrow y \) to \((1_x * \alpha * 1_y)\), vertical composition is given by concatenation, left whiskering the above morphism by \( u \) amounts to replace each \( u_i \) by \( uu_i \), and similarly for right whiskering.

**Proof.** The above sequences are the normal forms for a suitable orientation of the relations (2) as a convergent rewriting system on formal expressions.
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As customary, such morphisms can be pictured using string diagrams. For instance, in the signature of monoids (Ex. 1), if we draw \( \mu \) by \( \Uparrow \), we can picture the following morphisms:

\[
(0 * \mu * 2) * (1 * \mu * 0) * \mu = \begin{array}{c}
\mu \\
\downarrow \\
(2 * \mu * 0) * (0 * \mu * 1) * \mu = \begin{array}{c}
\mu \\
\downarrow \\
\end{array}
\end{array}
\]

Note that in these pictures, there can be only one generator at a given height, and the relative heights matter, so that the two 2-cells are not considered to be equal (contrarily to 2-categories).

1.3 Rewriting systems

A rewriting system consists of a signature \( \mathcal{P} \) together with a set \( \mathcal{P}_3 \) of 3-generators, or rewriting rules, equipped with source and target functions \( s_2, t_2 : \mathcal{P}_3 \to \mathcal{P}_2 \). A rewriting step

\[
\chi_\ast (u * A * v) \ast \chi_\ast : \chi_\ast (u * \phi * v) \ast \chi_\ast \Rightarrow \chi_\ast (u * \psi * v) \ast \chi_\ast
\]

consists of a rewriting rule \( A : \phi \Rightarrow \psi : v_- \Rightarrow v_+ : x \to y \) together with 1-cells \( u : x' \to x, v : y \to y' \) and 2-cells \( \chi_- : v_- \Rightarrow v_-, \chi_+ : v_+ \Rightarrow v_+' \) as on the right above. A rewriting path is a finite sequence of composable rewriting steps \( \phi_i \Rightarrow \psi_i \), with \( \phi_{i+1} = \psi_i \).

Example 3. The rewriting system for monoids has, on the signature of Ex. 1, the rules

\[
A : (\mu * 1) * \mu \Rightarrow (1 * \mu) * \mu \quad L : (\eta * 1) * \mu \Rightarrow \mu \quad R : (1 * \eta) * \mu \Rightarrow \mu
\]

There is, for instance, a rewriting step

\[
(3 * \mu * 1) * (1 * A * 1) * ((\mu * 1) * \mu) \Rightarrow \begin{array}{c}
\end{array}
\]

We write \( \mathcal{P}^\ast_i \) for the set of rewriting paths, and \( s_2^\ast, t_2^\ast : \mathcal{P}^\ast_i \to \mathcal{P}^\ast_2 \) for the associated source and target functions. We can form a 3-precategory, noted \( \mathcal{P}^\ast \), with \( \mathcal{P}^\ast_i \) as \( i \)-cells for \( i = 0, 1, 2, 3 \) (by convention \( \mathcal{P}^\ast_0 = \mathcal{P}_0 \)) and expected compositions. The notion of 3-precategory will be detailed in Sec. 4.1, but we can already say that it is the expected generalization of 2-precategories (see Sec. 1.1) in dimension 3: a 3-precategory consists of a set \( \mathcal{C}_i \) of \( i \)-cells for \( i = 0, 1, 2, 3 \) together with their source and target in lower dimension (except for 0-cells), identities for 0, 1, 2-cells, and compositions between composable \( i \) and \( j \)-cells, with \( i, j = 1, 2, 3 \), so that compositions are associative and unital in a suitable way. Note that, contrarily to 3-categories, there is only one kind of composition between \( i \) and \( j \)-cells: those can only be composed in codimension \( i \wedge j - 1 \) (we write \( i \wedge j \) for the minimum of \( i \) and \( j \)), which again allows to unambiguously use the same symbol for all compositions. A morphism of 3-precategories is called a 3-functor. By generalizing the argument of Prop. 2, one can show that \( \mathcal{P}^\ast \) enjoys the following universal property, see Sec. 4.2 for details:

Proposition 4. The 3-precategory \( \mathcal{P}^\ast \) is the free 3-precategory whose underlying 2-precateg- 

ory is the one generated by the underlying signature of \( \mathcal{P} \) and containing the rewriting rules as 3-cells.
Following the terminology of [9], we say that two rewriting paths \( P \) and \( Q \) are Peiffer-equivalent when they differ only by successively permuting adjacent rewriting steps at disjoint positions, what we write \( P \equiv Q \) below. For instance, with the notations of Ex. 3, the two following paths are Peiffer-equivalent:

More generally, we can define the Peiffer-equivalence in a 3-precategory as the smallest congruence (w.r.t. compositions) such that, with cells as on the left, we have the relation on the right:

\[
\begin{align*}
\quad & (u_1 \ast A_1 \ast w_1) \ast x \ast (u_2 \ast A_2 \ast w_2) \\
\equiv & ((u_1 \ast \psi_1 \ast w_1) \ast x \ast (u_2 \ast A_2 \ast w_2)) \\
\equiv & ((u_1 \ast \phi_1 \ast w_1) \ast x \ast (u_2 \ast A_2 \ast w_2)) \\
\equiv & ((u_1 \ast \phi_1 \ast w_1) \ast x \ast (u_2 \ast \psi_2 \ast w_2))
\end{align*}
\]

\[\text{(4)}\]

1.4 \((3,2)\)-precategories

A \((3,2)\)-precategory is a 3-precategory in which every 3-cell \( P : \phi \Rightarrow \psi \) is invertible: there exists a 3-cell \( Q : \psi \Rightarrow \phi \) such that \( P \ast Q = 1_\phi \) and \( Q \ast P = 1_\psi \).

Given a rewriting system \( P \), consider the rewriting system \( Q \) with \( Q_i = P_i \) for \( i = 0, 1, 2 \) and \( Q_3 = P_3 \sqcup P_3^\perp \) where \( P_3^\perp = \{ A^- : \psi \Rightarrow \phi \mid A : \phi \Rightarrow \psi \in P_3 \} \) is the set of formally reverted rules in \( P_3 \). We write \( P_3^\perp \) for the set of 3-cells in \( Q_3 \) quotiented by the smallest congruence such that \( A \ast A^- = 1_\phi \) and \( A^- \ast A = 1_\psi \) for every generator \( A \) in \( P_3 \), and call its elements rewriting zigzags. We can form a 3-precategory, noted \( P^\perp \), with \( P^\perp_i \) as \( i \)-cells for \( i = 0, 1, 2 \), \( P^\perp_3 \) as 3-cells, and expected compositions: it is defined as \( P^\ast \) excepting that 3-cells are rewriting zigzags instead of rewriting paths.

\[\text{Proposition 5. The 3-precategory } P^\perp \text{ is the free } (3,2)\text{-precategory on the 3-precategory } P^\ast.\]

According to the above proposition, we generalize the above notation and write \( P^\perp \) for the inverse of an arbitrary 3-cell \( P \). Note that any 3-cell decomposes as \( P_1^- \ast Q_1 \ast \ldots \ast P_n^- \ast Q_n \) where \( P_i \) and \( Q_i \) are rewriting paths (thus the terminology of rewriting zigzag). In the following, we will be mostly interested in \((3,2)\)-precategories (as opposed to 3-precategories).

1.5 Presentations

In order to describe interesting 3-precategories using rewriting systems, we need to be able to quotient the 3-cells in the freely generated 3-precategory. A presentation consists of a rewriting system \( P \) equipped with a set \( P_4 \) of relations together with functions \( s_3, t_3 : P_4 \rightarrow P_3^\perp \) indicating the source and the target rewriting path of a relation, such that \( s_3^2 \circ s_3 = s_3^2 \circ t_3 \) and \( t_3^2 \circ s_3 = t_3^2 \circ t_3 \) (relations are between rewriting paths with same source and same target). We often write \( P \not\Rightarrow Q \) to indicate that \( \Gamma \) is a relation with \( s_3(\Gamma) = P \) and \( t_3(\Gamma) = Q \). We denote by \( \equiv_{P_4} \) (or sometimes even \( \equiv \)), the smallest congruence on 3-cells in \( P_3^\perp \) such that \( P \equiv_{P_4} Q \) for every relation \( \Gamma : P \not\Rightarrow Q \) in \( P_4 \).

The \((3,2)\)-precategory \( P \) presented by a presentation \( P \) is the 3-category obtained from \( P^\perp \) by quotienting 3-cells under \( \equiv_{P_4} \).
Remark. A typical relation that one would like to express in the rewriting system of
monoids (Ex. 3) is the fact that the two ways of multiplying the unit by itself are the same,
as pictured below. However, in order to do so, we need to be able to “exchange” the two
units (the first 3-cell on the right), which there is no way to achieve for now. This motivates
looking at rewriting systems with more structure in next section.

1.6 Presentations of Gray categories

We have seen above that a rewriting system freely generates a 3-precategory. In practice, we
will be interested in describing 3-precategories having some additional structure and axioms.

A Gray category $C$ is a 3-precategory equipped, for every pair of 2-cells $\phi$ and $\psi$ as on
the left, of an invertible 3-cell $X_{\phi, \psi}$ as on the right, called interchanger:

$$x \xrightarrow{u} y \xleftarrow{v} z \xrightarrow{\psi} \psi'$$

$$\xrightarrow{\phi \ast (u \ast \psi) \ast (\phi \ast v')}$$

such that

1. Peiffer-equivalences are identities,
2. interchangers are compatible with compositions and identities in all sensible ways: for
   example,

   $$X_{\phi_1 \ast \phi_2, \psi} = ((\phi_1 \ast v) \ast X_{\phi_2, \psi} \ast (X_{\phi_1, \psi} \ast (\phi_2 \ast v'))$$

   and

   $$X_{1_{u, \psi}} = 1_{u \ast \psi}$$

3. interchangers are natural: in the situation (5), given a 3-cell $P : \phi \Rightarrow \phi'$

   $$((P \ast v) \ast (u' \ast \psi)) \ast X_{\phi', \psi} = X_{\phi, \psi} \ast ((u \ast \psi) \ast (P \ast v'))$$

   and symmetrically.

Alternatively, a Gray category can be defined to be category enriched over the category $\text{Cat}_2$
of 2-categories equipped with a suitable tensor product, called the Gray tensor product [7].

A Gray $(3, 2)$-category is a Gray category in which every 3-cell is invertible. A Gray functor $f : C \to D$ between Gray categories is a 3-prefunctor preserving interchangers (we only
consider the strict flavor of such functors here).

The notion of Gray category generalizes 3-categories by asking for explicit interchange
cells: a 3-category is precisely a Gray category where all interchange 3-cells are identities.
The relevance of Gray categories is that, although they are quite strict (compositions are
strictly associative), they capture the full generality of weak 3-categories, as shown by the
coherence theorem of Gordon, Power and Street [6, 11]:

Theorem 6. Every tricategory is (suitably) equivalent to a Gray category.

In order to present Gray categories, we should ensure that our presentations generate
interchangers and satisfy the required axioms. A Gray presentation $P$ is a presentation such
that

1. for every pair of 3-generators $A_1$ and $A_2$, as well as morphisms as on the left of (4), there
   is a relation as on the right of (4) called a Peiffer generator,
2. for every 2-generators $\alpha$ and $\beta$ and 1-cell $v$ as below:

\[
x \xrightarrow{\alpha} x' \xrightarrow{v} y' \xrightarrow{\beta} z
\]

left, there is a 3-generator $X_{\alpha,v,\beta}$, called interchange generators, as below:

\[
X_{\alpha,v,\beta} : (\alpha * v * w) * (u' * v * \beta) \Rightarrow (u * v * \beta) * (\alpha * v * w')
\]

\[
x \xrightarrow{\alpha} x' \xrightarrow{v} y' \xrightarrow{\beta} z \Rightarrow x \xrightarrow{\alpha} x' \xrightarrow{v} y' \xrightarrow{\beta} z
\]

and we write $P_X \subseteq P_3$ for the set of interchange generators,

3. for every 3-generator $A$, 1-cell $v$ and 2-generator $\alpha$ as on the left or on the right below

\[
x \xrightarrow{\alpha} x' \xrightarrow{v} y' \xrightarrow{\psi} y
\]

\[
x \xrightarrow{\alpha} x' \xrightarrow{v} y' \xrightarrow{\phi} y
\]

there is respectively a relation, called interchange naturality generator,

\[
((A * v * w) * (u' * v * \alpha)) * X_{\phi,v,*} \Rightarrow X_{\psi,v,*} * ((u * w * \alpha) * (A * v * w'))
\]

\[
((\alpha * v * w) * (u' * v * A)) * X_{\alpha,v,\psi} \Rightarrow X_{\alpha,v,*} * ((u * v * A) * (\alpha * v * w'))
\]

where the interchangers $X_{\alpha,v,\psi}$ are suitable composite of interchange generators (see proposition below).

The above families of 3- and 4-cells are called the structural generators of the presentation. We will not insist much about it in the following, but the choice of structural cells is implicitly supposed to be part of a Gray presentation.

\begin{itemize}
  \item \textbf{Proposition 7.} Given a Gray presentation $P$, the presented $(3,2)$-precategory $P$ is canonically a Gray $(3,2)$-category.
  \item \textbf{Example 8.} The Gray presentation of monoids consists of the rewriting system of Ex. 3, as well as additional interchange generators
\end{itemize}
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as well as Peiffer generators, e.g.

for an arbitrary 2-cell $\chi : n + 1 \Rightarrow n + 3$, and interchange naturality generators, e.g.

In the following, when describing a Gray presentation, we will not mention the structural cells which are always implicitly supposed to be present.

A model of a presentation $P$ in a Gray category $C$ is a Gray functor $P \to C$ from the presented Gray $(3,2)$-category to $C$.

▶ Example 9. A model of the presentation $P$ of monoids (Ex. 8) in a Gray category $C$ consists in a 1-cell $a : x \to x$ together with 2-cells $\mu : a \ast a \Rightarrow a$ and $\eta : 1_x \Rightarrow a$ and invertible 3-cells $A, L, R$ (as in Ex. 3) satisfying suitable relations (as in Ex. 8). This is precisely what is usually called a pseudomonoid in $C$.

▶ Remark. A notion of presented Gray category (as opposed to $(3,2)$-category) can also be defined: it is slightly more involved since we still need to formally invert (by a localization) some morphisms, at least the interchangers. Similarly, we could consider their models which are functors to Gray categories. However, in practice people consider algebraic structures with invertible 3-cells (e.g. pseudomonoids), which explains why we are mostly interested in Gray $(3,2)$-categories here for simplicity.

Our goal is to show that some presentations are coherent, meaning that all the diagrams made of structural morphisms commute in the models. Formally, a Gray category is coherent when between any pair of parallel 2-cells there is at most one 3-cell and a Gray presentation is coherent when the associated Gray $(3,2)$-category is.

2 Rewriting

2.1 Confluence

Every rewriting system induces an abstract rewriting system (i.e., a graph) with 2-cells in $P^*_3$ as vertices and rewriting steps as edges (the set of paths thus being $P^*_4$), from which we can use the classical notions and properties of rewriting theory, detailed below. We slightly depart from the tradition by, for confluence properties, asking that diagrams should be closed and commute modulo the relations in $P_4$.

Given a rewriting path $P : \phi \Rightarrow \psi$, we say that $\phi$ rewrites to $\psi$. A normal form is a 2-cell $\phi$ such that the only rewriting path with source $\phi$ is the empty one. A branching is
a pair of coinitial rewriting paths $P_1 : \phi \Rightarrow \phi_1$ and $P_2 : \phi \Rightarrow \phi_2$; it is local when both $P_1$ and $P_2$ are rewriting steps, it is joinable when there exists a pair of cofinal rewriting paths $Q_1 : \phi_1 \Rightarrow \psi$ and $Q_2 : \phi_2 \Rightarrow \psi$, it is confluent when there exists a pair of cofinal rewriting paths $Q_1 : \phi_1 \Rightarrow \psi$ and $Q_2 : \phi_2 \Rightarrow \psi$ such that $P_1 * Q_1 \equiv P_2 * Q_2$, see left of (9). Similarly, a rewriting zigzag $P : \phi_1 \Rightarrow \phi_2$ in $P_3^T$ is confluent when there exists a pair of cofinal rewriting paths $Q_1 : \phi_1 \Rightarrow \psi$ and $Q_2 : \phi_2 \Rightarrow \psi$ such that $P * Q_2 \equiv Q_1$, see right of (9)

\[
\begin{array}{ccc}
P_1 & \Rightarrow & P_2 \\
\phi_1 & \equiv & \phi_2 \\
\quad & \Downarrow & \quad \\
Q_1 & \Rightarrow & Q_2
\end{array}
\]  

(9)

A rewriting system is
- terminating when every sequence of composable rewriting steps is finite,
- (locally) confluent when every (local) branching is confluent,
- Church-Rosser when every rewriting zigzag is confluent,
- convergent if both terminating and locally confluent.

In a terminating rewriting system, every 2-cell $\phi$ rewrites to a normal form $\hat{\phi}$. The classical proof by well-founded induction of Newman’s lemma [19], can be directly adapted (as in [8, Thm. 3.1.6]) in order to show

\textbf{Theorem 10.} A convergent rewriting system is confluent.

Finally, for abstract rewriting systems it is well known that confluence implies the Church-Rosser property. In this setting, this translates as the following theorem, which adapts in our setting, the proof of Squier’s theorem for coherent presentations of categories, see [17, Thm. 5.2] and [8, Thm. 4.3.2]:

\textbf{Theorem 11.} A convergent presentation $P$ is Church-Rosser and coherent.

\textbf{Proof.} Suppose given a rewriting path $P : \phi \Rightarrow \psi$. Since $P$ is terminating, there is a rewriting path $P_\phi : \phi \Rightarrow \hat{\phi}$ (resp. $P_\psi : \psi \Rightarrow \hat{\psi}$) from $\phi$ (resp. $\psi$) to a normal form $\hat{\phi}$ (resp. $\hat{\psi}$). Moreover, by confluence, we have $\hat{\phi} = \hat{\psi}$ and $P_\phi \equiv P * P_\phi$, see the left of (10). Therefore, we have equivalences $P \equiv P_\phi * P_\phi$ and $P^- \equiv P_\phi * P_\phi^-$, as in the middle and right of (10):

\[
\begin{array}{ccc}
\phi & \Rightarrow & \psi \\
P_\phi & \equiv & P_\phi \\
\quad & \Downarrow & \quad \\
\phi & \approx & P_\phi
\end{array}
\]  

(10)

Finally, as explained above, a 3-cell of $\overline{P}$ is a zigzag of rewriting paths $P_1^- * Q_1 * \ldots * P_n^- * Q_n$ which is equivalent (modulo relations and axioms for inverses) to $P_\phi * P_\phi^- :$

\[
\begin{array}{cccccccc}
\phi & \Rightarrow & P^- & \Rightarrow & Q_1 & \Rightarrow & \ldots & \Rightarrow & P^- & \Rightarrow & \psi \\
\quad & \Downarrow & \quad & \Downarrow & \quad & \Downarrow & \quad & \Downarrow & \quad & \Downarrow & \quad & \Downarrow \\
\phi & \equiv & P_\phi & \equiv & P_\phi^- & \equiv & \ldots & \equiv & P_\phi & \equiv & P_\phi^- & \equiv
\end{array}
\]

Note that the 3-cell $P_\phi * P_\phi^-$ only depends on the source $\phi$ and the target $\psi$. We immediately deduce that two parallel 3-cells in $\overline{P}$ are equal. \hfill \blacksquare
2.2 Termination

Termination of a presentation is usually proved by checking that rules are decreasing according to some suitable order. A termination order is a well-founded partial order < on parallel 2-cells of a presentation $P$ such that
- for every rewriting rule $A : \phi \Rightarrow \psi$ we have $\phi > \psi$,
- given composable 2-cells $\phi, \psi_1$ and $\phi'$ (resp. $\phi, \psi_2$ and $\phi'$) such that $\psi_1 > \psi_2$, we have $\phi * \psi_1 * \phi' > \phi * \psi_2 * \phi'$
- given 2-cells $\phi > \psi$ and composable 1-cells $u$ and $w$, we have $u * \phi * w > u * \psi * w$.

**Proposition 12.** A rewriting system equipped with a termination order is terminating.

**Example 13.** A termination order for the rewriting system of monoids (Ex. 3) can be constructed as follows. Firstly, the three non-structural rewriting rules can be shown to be terminating exactly as for 3-polygraph of monoids [15, Sect. A.2] (roughly $L$ and $R$ decrease the number of generators and $A$ puts $\mu$ generators on the right), by a termination order for which the interchangers are left invariant. Secondly, the interchangers make 2-cells decrease in the following sense. A 2-cell corresponds to a forest of leveled planar binary trees (where nodes correspond to 2-generators), i.e., trees equipped with a total “vertical” order refining the depth order. The interchanger rules decrease the sum, for each generators, of the number of generators above (w.r.t. to the vertical order) and on the left (which is easily defined for such forests).

2.3 Critical branchings

Given a local branching $(P_1, P_2)$, the following situations can occur. The branching is
- trivial when $P_1 = P_2$,
- non-minimal when there is another branching $(Q_1, Q_2)$ such that $P_i = \phi * (u_i * Q_i * v_i) * \psi_i$ for $i = 0, 1$ for some 1-cells $u, v$ and 2-cells $\phi, \psi$, not all identities,
- independent, or Peiffer, when there are morphisms of the form (4) such that
  
  $P_1 = ((u_i * A_i * f_1) * \chi * (u_2 * \phi_2 * w_2))$ \quad $P_2 = ((u_1 * \phi_1 * w_1) * \chi * (u_2 * A_2 * w_2))$

- natural when there are morphisms as on the left of (6) such that
  
  $P_1 = ((A * v * w) * (u' * v * A))$ \quad $P_2$ is the first rewriting step of $X_{\phi,v+w}$, and similarly for the situation on the right of (6),
- critical when it is of none of the above forms.

Since, by definition of Gray presentations, non-critical branchings are necessarily confluent, we have:

**Theorem 14.** A presentation is locally confluent if and only if every critical branching is confluent.

As usual, critical branchings can be computed by considering the ways two left members $\phi_1$ and $\phi_2$ of rules can overlap non-trivially (sharing at least one 2-generator). Graphically, the following generic situations can happen, where the two regions respectively represent $\phi_1$ and $\phi_2$, the square $\psi$ in the middle being the intersection (overlap) of both, which is supposed not to be an identity 2-cell:

\[
\begin{array}{c}
\phi_1 \left\{ \begin{array}{c}
\psi_1 \\
\phi_2
\end{array} \right. \\
\phi_1 = \phi_1' * (u * \psi) \\
\phi_2 = (v * \psi) * \phi_2' \\
\phi_2 = (u * \psi) * \phi_2'
\end{array}
\quad
\begin{array}{c}
\phi_1 \left\{ \begin{array}{c}
\psi_1 \\
\phi_2
\end{array} \right. \\
\phi_1 = \phi_1' * (u * \psi) \\
\phi_2 = (v * \psi) * \phi_2' \\
\phi_2 = (u * \psi) * \phi_2'
\end{array}
\quad
\begin{array}{c}
\phi_1 \left\{ \begin{array}{c}
\psi_1 \\
\phi_2
\end{array} \right. \\
\phi_1 = \phi_1' * \psi \\
\phi_2 = (u * \psi) * \phi_2' \\
\phi_2 = \psi * \phi_2'
\end{array}
\quad
\begin{array}{c}
\phi_1 \left\{ \begin{array}{c}
\psi_1 \\
\phi_2
\end{array} \right. \\
\phi_1 = \phi_1' * (u * \psi) * \phi_2' \\
\phi_2 = \psi * \phi_2'
\end{array}
\quad
\begin{array}{c}
\phi_1 \left\{ \begin{array}{c}
\psi_1 \\
\phi_2
\end{array} \right. \\
\phi_1 = \phi_1' * (u * \psi) * \phi_2' \\
\phi_2 = \psi
\end{array}
\end{array}
\]
(and also the situations obtained by swapping \( \phi_1 \) and \( \phi_2 \)). From this, one deduces that any pair of rules can give rise to a finite number of critical branchings which can effectively be computed (the algorithmic aspects will be detailed in future works). Moreover, note that a non-structural rewriting rule \( R : \phi \Rightarrow \psi \) can only give rise to a finite number of critical branchings with interchangers: if the two \( 2 \)-generators involved in an interchanger \( X_{\alpha,\nu,\beta} \) are too far apart horizontally (i.e., \( \nu \) is a composite of too many \( 1 \)-cells), the branching is necessarily an exchange branching, e.g. left of (8). Similarly, that two interchangers never make a critical branching (all such branchings are natural), e.g. right of (8). From the above considerations, we deduce:

\[ \textbf{Theorem 15.} \] A presentation with a finite number of \( 2 \)-generators and of non-structural \( 3 \)-generators, with non-identity \( 2 \)-cells as sources, has a finite number of critical branchings.

It should be noted that this theorem contrasts with the situation for presentations of \((3,2)\)-categories (where interchangers are identities), where a finite presentation can give rise to an infinite number of critical branchings \cite{15,8}. Our formalization of rewriting systems avoids this problem, at the cost of having to explicitly handle interchangers.

\[ \textbf{Example 16.} \] The presentation for monoids (Ex. 8) has five critical branchings:

\[ \begin{array}{cccccccc}
\varphi & \Rightarrow & \psi & \Rightarrow & \varphi & \Rightarrow & \psi & \Rightarrow \\
\varphi & \Rightarrow & \psi & \Rightarrow & \varphi & \Rightarrow & \psi & \Rightarrow \\
\varphi & \Rightarrow & \psi & \Rightarrow & \varphi & \Rightarrow & \psi & \Rightarrow \\
\end{array} \]

2.4 A coherent completion procedure

The general methodology for constructing confluent presentations is the following one. Suppose given a presentation \( P \) (usually containing no relation in \( P \) excepting structural ones).

1. Find a termination order for the rules of \( P \): if none can be found try to reorient some rules. Conclude that \( P \) is terminating by Prop. 12.
2. Compute the critical branchings and check that they are joinable: if a critical branching is not joinable, add a new rule to make it confluent (this is the Knuth-Bendix completion procedure \cite{13}).
3. For every critical branching, choose a way to join it and add a corresponding relation in \( P \) (if not already present).
4. Conclude that \( P \) is locally confluent by Thm. 14, thus confluent by Thm. 10 and thus coherent by Thm. 11.
5. Optionally, remove some redundant rules and relations in order to achieve a smaller presentation.

This methodology is illustrated in next section. Note that steps 2 and 3 can be combined, giving rise to a “homotopical completion procedure” and 5 can be partly automated: this is detailed in the case of coherent presentations of monoids in \cite{10} and left for future work for Gray presentations.

3 Applications

3.1 Pseudomonoids

Consider the presentation \( P \) for monoids given in Ex. 8 whose termination was shown in Ex. 13. There are five critical branchings, given in Ex. 16, which are all joinable. If we add five corresponding relations in \( P \) we obtain a convergent, and thus coherent, presentation. Note however that the presentation \( P \) given in Ex. 8 has only two relations: in fact, three of the five relations are derivable from the other and can thus be removed (the argument
given in [8] for pseudomonoids in 3-categories can directly be adapted to our setting). This allows us to recover the coherence theorem of [14].

3.2 Adjunctions

The presentation for adjunctions is given by $P_0 = \{x, y\}$, $P_1 = \{a : x \to y, b : y \to x\}$ and $P_2 = \{\eta : 1_x \Rightarrow ab, \varepsilon : ba \Rightarrow 1_y\}$ where $\eta$ and $\varepsilon$ are respectively pictured as $\cap$ and $\cup$. The two rules are shown on the left below and the relations corresponding to the two critical branchings are on the right:

They are sometimes called the swallowtail relations. A model for this presentation in the 2-category $\text{Cat}$ (seen as a (3,2)-precategory with only identity 3-cells) is precisely an adjunction. Termination can be shown by observing that the two non-structural rules decrease the number of generators and the structural rules decrease the number of generators which are “on the left and above”, as in the previous case. We deduce that this presentation is coherent, thus recovering a variant of the coherence theorem shown in [4] (see below).

3.3 Self-dualities

The theory for self-dualities is the following variant of the previous one. We have $P_0 = \{\star\}$, $P_1 = \{a : \star \to \star\}$, $P_2 = \{\eta : 1_\star \Rightarrow aa, \varepsilon : aa \Rightarrow 1_\star\}$} where $\eta$ and $\varepsilon$ are respectively pictured as $\cap$ and $\cup$. The two rules are those on the left of (11). Note that because of the difference in “typing” of 0- and 1-cells, the rewriting system is not anymore terminating, since we have the reduction

Moreover, this endomorphism 3-cell is not an identity, preventing any hope for the presentation to be coherent. Following [4], we can still aim at showing a partial coherence result by restricting to 2-cells which are connected, i.e., whose graphical representation is connected (we do not give the formal definition here). In this case, termination can actually be shown by using the same arguments as in Sec. 3.2. However, the critical pairs are not joinable either since, for instance, we have

(for which there is little hope that a Knuth-Bendix completion will provide a reasonably small presentation). However, one can obtain a rewriting system which is terminating on connected 2-cells and confluent by orienting the interchangers as follows

The relations generated by critical branchings can be pictured as on the right of (11).
3.4 Frobenius monoids

The presentation for (non-unital) Frobenius monoids is given by \( P_0 = \{ \star \} \), \( P_1 = \{ 1 : \star \to \star \} \) and \( P_2 = \{ \mu : 2 \Rightarrow 1, \delta : 1 \Rightarrow 2 \} \). If we respectively picture \( \mu \) and \( \delta \) by \( \nabla \) and \( \Lambda \), we have the four rewriting rules on the left below:

\[
\begin{align*}
\nabla & \Rightarrow \Lambda \\
\Lambda & \Rightarrow \nabla \\
\nabla \cup \Lambda & \Rightarrow \Lambda \\
\nabla \cup \Lambda & \Rightarrow \nabla
\end{align*}
\]

(and interchangers are oriented as usual). By Knuth-Bendix completion, we add the two rules on the right. The resulting rewriting system has 19 joinable critical pairs, to each of which corresponds a relation. We conjecture that the rewriting system is terminating, which would give rise to a coherence theorem for Frobenius monoids. A coherence theorem using a different set of generators and relations is shown in [4].

4 Rewriting systems in higher dimension

4.1 Precategories

Given \( n \in \mathbb{N} \), an \( n \)-globular set \( C \) is a diagram of sets

\[
C_0 \xrightarrow{t_0} C_1 \xleftarrow{s_1} C_2 \xrightarrow{\cdots} C_n
\]

such that \( s_i \circ s_{i+1} = s_i \circ t_{i+1} \) and \( t_i \circ s_{i+1} = t_i \circ t_{i+1} \) for \( 0 \leq i < n-1 \). A morphism \( f : C \to D \) between \( n \)-globular sets is a family of morphisms \( f_i : C_i \to D_i \), with \( 0 \leq i \leq n \), such that \( s_i \circ f_{i+1} = f_i \circ s_i \). The resulting category is denoted by \( \text{Glob}_n \). Given \( i, j, k \in \mathbb{N} \) with \( k < i \) and \( k < j \), we write \( G_i \times_k G_j \) for the pullback of the diagram \( C_i \xrightarrow{t_k \circ \cdots \circ t_{i-1}} C_k \xleftarrow{s_k \circ \cdots \circ s_{j-1}} C_j \).

An \( n \)-precategory \( C \), see [12], is an \( n \)-globular set equipped with

- identity functions \( 1_i : G_i \to G_{i+1} \) for \( 0 \leq i < n \),
- composition functions \( s_{i,j} : G_i \times_{i \wedge j-1} G_j \to G_{i\vee j} \) for \( 0 < i, j \leq n \).

As previously, since the dimension of cells determines the functions to be used, we omit the indices from \( s, t, 1 \) and \( \star \). For composition, it is sometimes useful to write \( u \star_k v \) to indicate that \( k = i \wedge j - 1 \), where \( i \) is the dimension of \( u \) and \( j \) is the dimension of \( v \). We require the following axioms:

- for \( (u, v) \in C_i \times_{i \wedge j-1} C_j \) with \( 0 < i, j \leq n \),
  \[
  s(u \star v) = \begin{cases} 
  u \star s(v) & \text{if } i < j \\
  s(u) & \text{if } i = j \\
  s(u) \star v & \text{if } i > j
  \end{cases}
  \quad t(u \star v) = \begin{cases} 
  u \star t(v) & \text{if } i < j \\
  t(u) & \text{if } i = j \\
  t(u) \star v & \text{if } i > j
  \end{cases}
  \]

- for every \( u \in C_i \) with \( 0 \leq i < n \), \( s(1_u) = u \star t(1_u) \)
- for every \( (u, v) \in C_i \times_{i \wedge j-1} C_j \) with \( 0 < i, j \leq n \),
  \[
  1_u \star v = \begin{cases} 
  v & \text{if } i \leq j \\
  1 \star v & \text{if } i > j
  \end{cases}
  \quad u \star 1_v = \begin{cases} 
  u & \text{if } i \geq j \\
  1 & \text{if } i < j
  \end{cases}
  \]

such that, for composable cells \( u, v, w \), with \( k < l \),

\[
(u \star_k v) \star_k w = u \star_k (v \star_k w) \quad u \star_k (v \star_l w) = (u \star_k v) \star_l (u \star_k w)
\]

A morphism of \( n \)-precategories, called an \( n \)-prefunctor, is a morphism between the underlying globular sets which preserves identities and compositions as expected. We write \( \text{PCat}_n \) for the category of \( n \)-precategories. This category is locally presentable and thus complete and cocomplete. Given an \( n \)-precategory \( C \), we write \( C_0 \) for its set of 0-cells seen as an \( n \)-category.
with empty sets of \( i \)-cells for \( 0 < i \leq n \). The “funny tensor product” \( C \boxtimes D \) of two \( n \)-precategories \( C \) and \( D \) is defined as the pushout on the right where the arrows are the obvious inclusions. This makes \( \text{PCat}_n \) into a monoidal category and we have:

\[ \text{Proposition 17.} \] An \( n+1 \)-precategory is the same as a category enriched in \( \text{PCat}_n \) equipped with the funny tensor product.

### 4.2 Prepolygraphs

We now briefly introduce the notion of prepolygraph which generalizes in arbitrary dimension the notion of rewriting system, by a direct adaptation the definition invented by Burroni for \( n \)-categories [3]. We write \( \text{PCat}_n^+ \) for the pullback on the right where the arrow on the top is the forgetful functor and the one on the left is the truncation functor (forgetting the set of \( n+1 \)-cells in an \( n+1 \)-globular set). An object in this category consists of an \( n \)-precategory equipped with a set of \( n+1 \)-cells (for which there is no notion of composition). There is a forgetful functor \( \text{PCat}_{n+1} \to \text{PCat}_n^+ \) which amounts to forget about compositions involving \( n+1 \)-cells, which admits a right adjoint \( L_n : \text{PCat}_n^+ \to \text{PCat}_{n+1} \), generating all the formal compositions of \( n+1 \)-cells.

We now define by induction on \( n \in \mathbb{N} \), the category \( \text{Pol}_n \) of \( n \)-prepolygraphs together with a functor \( F_n : \text{Pol}_n \to \text{PCat}_n \) associating to each \( n \)-prepolygraph the associated freely generated \( n \)-precategory. For \( n = 0 \), we set \( \text{Pol}_0 = \text{Set} \) and \( F_0 \) is the identity functor (\( \text{PCat}_0 \) is isomorphic to \( \text{Set} \)). The category of \( n+1 \)-prepolygraphs is defined by the pullback on the right where the vertical arrow is the expected forgetful functor, and we define the functor \( F_{n+1} = L_{n+1} \circ F_n^+ \). More explicitly, an \( n \)-prepolygraph consists in a diagram of sets

\[
\begin{array}{cccccc}
& & & & & \\
& P_0 & & P_1 & & P_2 & \ldots \\
& s_0 & & s_1 & & s_2 & \ldots \\
& t_0 & & t_1 & & t_2 & \ldots \\
& P_0^* & & P_1^* & & P_2^* & \ldots \\
& s_0^{-1} & & s_1^{-1} & & s_2^{-1} & \ldots \\
& t_0^{-1} & & t_1^{-1} & & t_2^{-1} & \ldots \\
& P_0'^* & & P_1'^* & & P_2'^* & \ldots \\
& s_{n-2}^{-1} & & s_{n-1}^{-1} & \ldots \\
& t_{n-2}^{-1} & & t_{n-1}^{-1} & \ldots \\
& P_{n-1} & & P_n & \\

\end{array}
\]

such that \( s_i^* \circ s_{i+1} = s_i^* \circ t_{i+1} \) and \( t_i^* \circ s_{i+1} = t_i^* \circ t_{i+1} \), together with a structure of \( n \)-precategory on the globular set on the bottom row: \( P_i \) is the set of \( i \)-generators, \( s_i, t_i : P_{i+1} \to P_i^* \) respectively associate to each \( i+1 \)-generator its source and target, and \( P_i^* \) is the set of \( i \)-cells, i.e., formal compositions of \( i \)-generators.

The cells in such prepolygraphs are particularly easy to manipulate because of the following normal form, generalizing Prop. 2 and its proof. We plan to investigate algorithmic aspects (for computing critical pairs, etc.) based on this representation in future works.

\[ \text{Theorem 18.} \] A non-identity \( k \)-cell \( P \) in an \( n \)-prepolygraph decomposes uniquely as \( P = R^1 \ast R^2 \ast \ldots \ast R^p \) with each \( R^i \) being a \( k \)-rewriting step, i.e., a composite of the form \( R^i = w_{k-1}^i \ast (\ldots \ast (w_2^i \ast (w_1^i \ast A^i \ast w_1^i) \ast w_2^i) \ast \ldots ) \ast w_{k-1}^i \) where \( A^i \) is a \( k \)-generator and \( w_j^i \) and \( v_j^i \) are \( j \)-cells.

Interestingly, this formalization based on prepolygraphs corresponds precisely to the one proposed by Bar and Vicary [2]: their representation is more economical thanks to the use of integers in order to encode cells, but somewhat obscures the universal properties it satisfies.

\[ \text{Proposition 19.} \] The \( n \)-signatures of [2] correspond to the \( n \)-prepolygraphs defined above.
Their work gives hints at a way to generalize Gray presentations in order to present semistrict tetracategories, by providing the adapted collections of structural cells. We plan to investigate this, as well as an adaptation of our techniques in order to provide automation to their tool Globular [1] in future work.

5 Proofs

▷ Proposition 20. The 3-precategory $P^\top$ is the free $(3,2)$-precategory on the 3-precategory $P^*$.

Proof. Define the canonical functor $\theta : P^* \to P^\top$ by the universal property satisfied by $P^*$: the generators of $P$ are sent to their associated classes in $P^\top$.

We now show that this functor is universal as a functor from $P^*$ to a $(3,2)$-precategories. So let $Q$ be a $(3,2)$-precategory and $F : P^* \to Q$ be a functor.

Let $P_{inv}^i$ be the rewriting system such that $P_{inv}^i = P_i$ for $i \leq 2$ and $P_{inv}^3 = P_3^1 \sqcup P_3^{-1}$ where $P_1^i = \{\alpha \mid \alpha \in P_i\}$ and $P_3^{-1} = \{\alpha^{-1} \mid \alpha \in P_3\}$. Note that $P^\top$ is obtained by quotienting $(P_{inv}^i)^*$ by the congruence generated by $\alpha^1 \ast \alpha^{-1} \equiv 1_{\alpha^1}$ and $\alpha^{-1} \ast \alpha^1 \equiv 1_{\alpha^1}$.

So we define the factoring functor $G : P^\top \to Q$ first by defining $G' : (P_{inv}^i)^* \to Q$, which amounts to defining images for the generators of $P_{inv}^i$ using the universal property of $(P_{inv}^i)^*$: set $G'(x) = F(x)$ for $x \in P_0 \sqcup P_1 \sqcup P_2$, and $G'(\alpha^1) = F(\alpha)$, $G'((\alpha^{-1}) = F(\alpha)^{-1}$ for $\alpha \in P_3$.

Note that $G'(\beta) = G'(\gamma)$ for $\beta \equiv \gamma$. So $G'$ can be factored as $G : P^\top \to Q$. One readily checks that $G \circ \theta(x) = F(x)$ for $x$ a generator of $P$. By universality, it entails $F = G \circ \theta$.

We show the unicity of $G$. Let $H$ a functor such that $F = H \circ \theta$. Then, for $x \in P_0 \sqcup P_1 \sqcup P_2$, $H(x) = F(x) = G(x)$ and similarly $H(\alpha^1) = G(\alpha^1)$ for $\alpha^1 \in P_3^1$. For $\alpha^{-1} \in P_3^{-1}$,

$$1_{H(x')} = H(1_{x'}) = H(\alpha^1 \ast \alpha^{-1}) = H(\alpha^1) \ast H(\alpha^{-1})$$

and similarly for $G$.

So $H(\alpha^{-1}) = H(\alpha)^{-1} = G(\alpha)^{-1} = G(\alpha^{-1})$. By universality, $H = G$.

▷ Proposition 21. Given a Gray presentation $P$, the presented $(3,2)$-precategory $\overline{P}$ is canonically a Gray groupoid.

Proof. Let $P$ a Gray presentation. We show that $\overline{P}$ has a structure of Gray groupoid.

Let $x, y$ two 0-cells. We show that $\text{Hom}(x, y)$ is a 2-category. It is already known that it is a 2-precategory. So the only property to show is the exchange law for 2-cells, which is sufficient to prove for a situation involving only 2-cell generators of $\text{Hom}(x, y)$, that is, 3-cell generators of $P$.

So consider the following situation where $\theta, \theta'$ are 3-generators of $P$ seen as 2-generators of $\text{Hom}(x, y)$:

$$f \xrightarrow{\alpha \theta \gamma} f' \xrightarrow{\beta \theta' \gamma'} g \xrightarrow{\gamma \theta \gamma'} g'$$

(12)

We need to show the following in $\text{Hom}(x, y)$:

$$(\theta \ast_0 \beta \ast_0 \gamma) \ast_1 (\alpha' \ast_0 \beta \ast_0 \theta') = (\alpha \ast_0 \beta \ast_0 \theta') \ast_1 (\theta \ast_0 \beta \ast_0 \gamma')$$

which is equivalent to the following equation in $C$:

$$(\theta \ast_1 \beta \ast_1 \gamma) \ast_2 (\alpha' \ast_1 \beta \ast_1 \theta') = (\alpha \ast_1 \beta \ast_1 \theta') \ast_2 (\theta \ast_1 \beta \ast_1 \gamma')$$
Coherence of Gray categories via rewriting

But this is precisely an equation entailed by a Peiffer generator.
So \( \text{Hom}(x, y) \) is a 2-category for all \( x, y \).

Now, we show that a composition operation \( \text{Hom}(x, y) \otimes \text{Hom}(y, z) \xrightarrow{\text{comp}_{x,y,z}} \text{Hom}(x, z) \) can be defined, where \( \otimes \) is the Gray tensor product. Following ??, recall that if \( C, C' \) are two 2-categories, \( C \otimes C' \) is the 2-category such that

- the 0-cells are the pairs \( (f, g) \in C_0 \times C'_0 \)
- the 1-cells are generated by two types of elements:
  - for \( \alpha : f \to f' \in C_1 \) and \( g \in C'_0 \), an element \((\alpha, g) : (f, g) \to (f', g)\)
  - for \( f \in C_0 \) and \( \beta : g \to g' \in C'_1 \), an element \((f, \beta) : (f, g) \to (f, g')\)
- the 2-cells are generated by three types of elements:
  - for \( A : \alpha \to \alpha' \in C_0 \) and \( g \in C'_0 \), an element \((A, g) : (\alpha, g) \to (\alpha', g)\)
  - for \( f \in C_0 \) and \( B : \beta \to \beta' \in C'_1 \), an element \((f, B) : (f, \beta) \to (f, \beta')\)
  - for \( \alpha : f \to f' \in C_1 \) and \( \beta : g \to g' \in C'_1 \), an element \((\alpha, \beta) : (f, g) \to (f', \beta) \to (f, g')\)

and such that several relations are satisfied.

Let \( Q \) the polygraphs whose \( i \)-generators are the \( i \)-generators described just before. By the universal property of \( Q^* \), there is a canonical functor \( F : Q^* \to \text{Hom}(x, y) \otimes \text{Hom}(y, z) \).

In order to define \( \text{comp}_{x,y,z}^l : Q^* \to P^* \) by defining \( \text{comp}_{x,y,z}^l \) on \( Q \):

- Images of 0-generators : for \( (f, g) \in C_0 \times C'_0 \), \( \text{comp}_{x,y,z}^l(f, g) = f *_0 g \)
- Images of 1-generators:
  - for \( \alpha : f \to f' \in C_1 \) and \( g \in C'_0 \), \( \text{comp}_{x,y,z}^l(\alpha, g) = \alpha *_0 g \)
  - for \( f \in C_0 \) and \( \beta : g \to g' \in C'_1 \), \( \text{comp}_{x,y,z}^l(f, \beta) = f *_0 \beta \)
- Images of 2-generators:
  - for \( A : \alpha \to \alpha' \in C_0 \) and \( g \in C'_0 \), \( \text{comp}_{x,y,z}^l(A, g) = A *_0 g \)
  - for \( f \in C_0 \) and \( B : \beta \to \beta' \in C'_1 \), \( \text{comp}_{x,y,z}^l(f, B) = f *_0 B \)
  - for \( \alpha : f \to f' \in C_1 \) and \( \beta : g \to g' \in C'_1 \), \( \text{comp}_{x,y,z}^l(\alpha, \beta) = X_{\alpha,\beta} \)

where \( X_{\alpha,\beta} \) is defined in the following way : if \( \alpha \) (resp. \( \beta \)) is an identity, then \( X_{\alpha,\beta} = 1_{\alpha \times \beta} \) (resp. \( 1_{\alpha \times \beta} \)). Otherwise, if \( \alpha = \prod_{i=1}^n u_i *_{0,0} v_i \) and \( \beta = \prod_{j=1}^m u'_j *_{0,0} v'_j \) where \( \alpha_i \) and \( \beta_j \) are generators in \( P \), then

\[
X_{\alpha,\beta} = \prod_{j=1}^m \prod_{i=1}^n (u_1 *_{0,0} s_{0,0} v_1 *_{0,0} \prod_{l=1}^{j-1} u'_l *_{0,0} \beta_l *_{0,0} v'_l )
\]

\[
\ast_1 (\prod_{k=1}^{i-1} u_k *_{0,0} \alpha_k *_{0,0} v_k ) *_{0,0} u'_i *_{0,0} s_{0,0} \beta_j *_{0,0} v'_j
\]

\[
\ast_1 (u_i *_{0,0} X_{\alpha_i, v_i *_{0,0} u'_i, \beta_j} *_{0,0} v'_j)
\]

\[
\ast_1 (\prod_{k=1}^{i+1} u_k *_{0,0} \alpha_k *_{0,0} v_k ) *_{0,0} u'_i *_{0,0} t_{0,0} \beta_l *_{0,0} v'_l
\]

\[
\ast_1 (u_n *_{0,0} t_{0,0} \alpha_n *_{0,0} v_n *_{0,0} \prod_{l=j+1}^m u'_l *_{0,0} \beta_l *_{0,0} v'_l )
\]

This 3-cell is the operation that moves sequentially each \( \beta_j \) above the sequence of \( \alpha_i \), using the elementary exchange generators provided by the presentation.
One readily checks that this definition is compatible with the source and target conditions of the polygraph $\mathcal{P}$, which entails a functor $\text{comp}^\prime_{x,y,z} : \mathcal{Q}^\star \to \mathcal{P}^\top$.

Now we check that the axioms on the generators given by the Gray tensor product are satisfied:

\[
\text{comp}^\prime_{x,y,z}((\alpha_1 * \alpha_2, g)) = \text{comp}^\prime_{x,y,z}(\alpha_1, g) *_1 \text{comp}^\prime_{x,y,z}(\alpha_2, g)
\]

\[
\text{comp}^\prime_{x,y,z}(f, \beta_1 * \beta_2) = \text{comp}^\prime_{x,y,z}(f, \beta_1) *_1 \text{comp}^\prime_{x,y,z}(f, \beta_2)
\]  

(13)

\[
\text{comp}^\prime_{x,y,z}(\theta_1 * \theta_2, g) = \text{comp}^\prime_{x,y,z}(\theta_1, g) *_2 \text{comp}^\prime_{x,y,z}(\theta_2, g)
\]

\[
\text{comp}^\prime_{x,y,z}(f, \theta'_1 * \theta'_2) = \text{comp}^\prime_{x,y,z}(f, \theta'_1) *_2 \text{comp}^\prime_{x,y,z}(f, \theta'_2)
\]  

(14)

\[
\text{comp}^\prime_{x,y,z}(\alpha, 1, g) = 1_{\alpha *_0 g}
\]

\[
\text{comp}^\prime_{x,y,z}(1_f, \beta) = 1_{f *_0 \beta}
\]  

(15)

\[
\text{comp}^\prime_{x,y,z}(\alpha_1 * \alpha_2, \beta) = (\text{comp}^\prime_{x,y,z}(\alpha_1, g) *_1 \text{comp}^\prime_{x,y,z}(\alpha_2, \beta)) *_2 (\text{comp}^\prime_{x,y,z}(\alpha_1, \beta) *_1 \text{comp}^\prime_{x,y,z}(\alpha_2, g'))
\]

\[
\text{comp}^\prime_{x,y,z}(\alpha, \beta_1 * \beta_2) = (\text{comp}^\prime_{x,y,z}(\alpha, \beta_1) *_1 \text{comp}^\prime_{x,y,z}(f', \beta_2)) *_2 (\text{comp}^\prime_{x,y,z}(f', \beta_1) *_1 \text{comp}^\prime_{x,y,z}(\alpha, \beta_2))
\]  

(16)

\[
\text{comp}^\prime_{x,y,z}(\alpha, \beta) = (\text{comp}^\prime_{x,y,z}(\alpha, g) *_1 (f, \theta)) *_2 \text{comp}^\prime_{x,y,z}(\alpha, g')
\]

\[
\text{comp}^\prime_{x,y,z}(\alpha, \beta) = (\text{comp}^\prime_{x,y,z}(f, \beta) *_1 \text{comp}^\prime_{x,y,z}(\theta, g'))
\]

\[
\text{comp}^\prime_{x,y,z}(\theta, g) *_1 \text{comp}^\prime_{x,y,z}(f, \beta) *_2 \text{comp}^\prime_{x,y,z}(\alpha, \beta')
\]  

(17)

For 13, we only prove the first equation since the other is similar. So,

\[
\text{comp}^\prime_{x,y,z}(\alpha_1 * \alpha_2, g)
\]

\[
=(\alpha_1 * \alpha_2) *_0 g
\]

\[
=(\alpha_1 *_0 g) *_1 (\alpha_2 *_0 g)
\]

\[
=\text{comp}^\prime_{x,y,z}(\alpha_1, g) *_1 \text{comp}^\prime_{x,y,z}(\alpha_2, g)
\]

For 14, we only prove the first equation since the other is similar. So,

\[
\text{comp}^\prime_{x,y,z}(\theta_1 * \theta_2, g)
\]

\[
=(\theta_1 * \theta_2) *_0 g
\]

\[
=(\theta_1 *_0 g) *_2 (\theta_2 *_0 g)
\]

\[
=\text{comp}^\prime_{x,y,z}(\theta_1, g) *_2 \text{comp}^\prime_{x,y,z}(\theta_2, g)
\]

For 15, we only prove the first equation since the other is similar. So,
\[ comp'_{x,y,z}(\alpha,1_g) = X_{\alpha,1_g} = 1_{\alpha*0_g} \]

For 16, the second equation is obtained by unrolling the definition. The first one needs more details.

Let \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_m\} \) two sets of letters. We consider the category \( W \) whose objects are shuffles of \( a_1a_2\ldots a_n \) and \( b_1b_2\ldots b_m \) and whose morphisms are freely generated by the arrows \( t_{r,s} : w_1 \to w_2 \) where \( w_1 = ra_ib_js \) and \( w_2 = rb_ia Js \) for words \( r \) and \( s \) that make \( w_1 \) and \( w_2 \) objects of \( W \) (when either \( w_1 \) or \( w_2 \) is known, we will refer to \( t_{r,s} \) by \( t_{i,j} \)).

Note that for \( w = ra_ia_jma_ia_jbsr \), there is the branching \( (t_{i_1,j_1}, t_{i_2,j_2}) \) that is confluent:

Define a relation \( \approx \) on the parallel arrows of \( W \) to be the smallest equivalence relation satisfying the following axioms:
- if \( w = ra_ia_jma_ia_jbsr \), then
  \[
  \begin{array}{c}
  ra_ia_jma_ia_jbsr \\
  \approx
  \\
  rb_ia_ia_jma_ia_jbsr
  \end{array}
  \]
- if \( g_1 \approx g_2 \) then \( f *_0 g_1 *_0 h \approx f *_0 g_2 *_0 h \)

Denote \( \text{size}(f) \) to be equal to \( n \) when \( f = f_1 * \ldots * f_n \) where \( f_i \)'s are generators.

Remark the following property:

\[ \text{Lemma 22. If there is a morphism } f : w_1 \to w_2 \text{ where } a_i \text{ appears just before } b_j \text{ in } w_1, \text{ but appears after in } w_2, \text{ then there is a morphism } f' \text{ such that } w_1 \overset{t_{i,j}}{\to} w \overset{f'}{\to} w_2 \text{ with } \text{size}(f') = \text{size}(f) - 1. \]

\[ \text{Proof. We prove this by induction on } \text{size}(f) : \]
- if \( \text{size}(f) = 0 \), then \( w_1 = w_2 \), which contradicts the fact that \( a_i \) and \( b_j \) appear in different orders. So this case is impossible
if \( f = f_1 \ast \ldots \ast f_n \) with \( n > 0 \), then if \( f_1 = t_{i,j} \), \( f' = f_2 \ast \ldots \ast f_n \) satisfies the wanted property. Otherwise, if \( f_1 : w_1 \rightarrow w \), then \( a_i \) is still just before \( b_j \) in \( w \), so by induction, we get \( f'' \) such that \( w \xrightarrow{t_{i,j}} w' \xrightarrow{f''} w_2 \) with \( \text{size}(f'') = n - 2 \). Also, it is clear that we can find \( f'_1 \) such that \( w \xrightarrow{t_{i,j}} w' \xrightarrow{f'_1} w_2 \).

(Note that this is not a commuting diagram)

So \( f' = f'_1 \ast f'' \) satisfies the wanted property

Now, we show that \( \approx \) is actually full:

\[\begin{align*}
\text{Lemma 23.} & \quad \text{For all } f, f' : w_1 \rightarrow w_2 \text{ in } \mathcal{W}, \; f \approx f' \\
\text{Proof.} & \quad \text{Note that one can readily show by induction that parallel arrows have the same size. We prove this lemma by induction on } \text{size}(f).
\end{align*}\]

- if \( \text{size}(f) = 0 \), then \( f = 1_{sf} = f' \) so \( f \approx f' \)
- if \( \text{size}(f) > 0 \), then let \( f = f_1 \ast \ldots \ast f_n \) and \( f' = f'_1 \ast \ldots \ast f'_n \). If \( f_1 = f'_1 \), then by induction, we have \( f_2 \ast \ldots \ast f_n \approx f'_2 \ast \ldots \ast f'_n \), so \( f \approx f' \) using the stability of \( \approx \). Otherwise, if \( f_1 : w_1 \rightarrow w \) and \( f'_1 = t_{i,j} \), then \( a_i \) appears just before \( b_j \) in \( w \) and appear after in \( w_2 \), so we can find \( f'' \) such that \( w \xrightarrow{t_{i,j}} w' \xrightarrow{f''} w_2 \).

By induction, we show that \( f_2 \ast \ldots \ast f_n \approx t_{i,j} \ast f'' \) and \( f'_1 \ast f'' \approx t_{i,j} \ast f'_1 \). Plus, by definition of \( \approx \), \( f_1 \ast t_{i,j} \approx t_{i,j} \ast f'_1 \). So, using stability and transitivity of \( \approx \), we prove that \( f \approx f' \).
We now define a functor $F : W \to \text{Hom}(u_1 * s\alpha_1 * v_1 * u'_1, u_n * t\alpha_n * v_n * u'_n * \beta_n * v'_n)$. Let $m_i, m'_j$ such that $m_0 = u_1 * s\alpha_1, m'_0 = u'_1 * s\beta_1$ and for $i, j > 0$, $m_i = u_i * t\alpha_i * v_i$ and $m'_j = u'_j * t\beta_j * v'_j$. If $w = x_1...x_{n+m}$ is a shuffle of $a_1...a_n$ and $b_1...b_m$, denote $n_{a,k} = \{ l | l < \text{kand} a_i = x_l \}$ and $n_{b,k} = \{ l | l < \text{kand} x_i = b_j \}$. Define $\gamma_{w,k}$ as follows:

$$\gamma_{w,k} = \begin{cases} u_i * 0 \alpha_i * 0 v_i * 0 m_{n_{a,k}} & \text{if } x_k = a_i \\ m_{n_{a,k}} * 0 u'_j * 0 \beta_j * 0 v'_j & \text{if } x_k = b_j \end{cases}$$

Now, we define $F$ on objects by setting $F(w) = \prod_{k=1}^{n+m} \gamma_{w,k}$. To define $F$ on morphisms, we only need to define it on generators. So, if $w = ra_1 b_j s$ and $w' = rb_j a_1 s$, with $|r| = l$, and $t_{i,j} : w \to w'$, then $F(t_{i,j}) = (\prod_{k=1}^{l+m} \gamma_{w,k} \ast (u_i * 0 X_{\alpha_{i},v_{i},u_{i}',\beta_{j},v_{j}'}) \ast 0 v_{j}') \ast 1 (\prod_{k=l+2}^{n+m} \gamma_{w,k})$.

We then have the following property for $F$:

**Lemma 24.** If $f, f' : w \to w'$, then $F(f) = F(f')$.

**Proof.** We just need to show that $f \approx f'$ implies $F(f) = F(f')$. Since $\approx$ and equality are both equivalences relations, we just need to check the implication for axioms of $\approx$.

So, if we have a square like this in $W$:

$$\begin{array}{ccc}
ra_1 b_1 m a_2 b_2 r & \approx & ra_1 b_1 m b_2 a_2 r \\
rb_j a_1 m a_2 b_2 & \quad & rb_j a_1 m b_2 a_2 r \\
\end{array}$$

we need to show that $F(t_{i,j_1}) \ast F(t_{i,j_2}) = F(t_{i,j_2}) \ast F(t_{i,j_1})$. If $l_1 = |r|$ and $l_2 = |m|$, it amounts to showing that

$$(u_i * 0 X_{\alpha_{i,1},v_{i,1},u_{i,1}',\beta_{j,1},v_{j,1}'}) \ast 1 (\prod_{k=1}^{l_1+l_2} \gamma_{w,k} \ast 1 (u_i * 0 s X_{\alpha_{i,2},v_{i,2},u_{i,2}',\beta_{j,2},v_{j,2}'})$$


which is true, since it is a Peiffer equivalence, and Peiffer equivalences are quotiented in $p^7$.

Plus, one readily shows that if $g, g'$ are parallel arrows in $W$ such that $F(g) = F(g')$, then for $f$ and $h$ compatible arrows, it holds that $F(f \ast g \ast h) = F(f \ast g' \ast h)$.

\[\framebox{\checkmark}\]
This lemma basically says that all ways of moving the $\beta_j$’s up the $\alpha_i$’s are equal as 3-cells. It entails that

$$\text{comp}_{x,y,z}^p(\alpha_1 \ast \alpha_2, \beta) = (\text{comp}_{x,y,z}^p(\alpha_1, g) \ast \text{comp}_{x,y,z}^p(\alpha_2, f)) \ast (\text{comp}_{x,y,z}^p(\alpha_1, \beta) \ast \text{comp}_{x,y,z}^p(\alpha_2, g'))$$

since both sides are images of parallel arrows in $W$.

For 17, we only prove the first one since the other is similar.

We first prove this equality when $\alpha = u \ast_0 \alpha_g \ast_0 v$ where $\alpha_g$ is a generator. Let $\beta = \prod_{j=1}^n u_j' \ast_0 \beta_j \ast_0 v_j'$. The left hand side of the equation is then

$$= u \ast_0 (X_{\alpha_g \ast_0 \alpha} \ast_2 (u \ast_0 \alpha_g \ast_0 v \ast_0 g'))$$

Because of the interchange naturality generator, the last line is equal to

$$= (u \ast_0 (X_{\alpha_g \ast_0 \alpha} \ast_2 (u \ast_0 \alpha_g \ast_0 v \ast_0 \theta) \ast_2 (u \ast_0 X_{\alpha_g \ast_0 \alpha}))$$

which is exactly the right hand-side of 17.

We show this equation for more general $\alpha$.

If $\alpha$ is an identity, then both sides are equal to $\ast'(f, \theta)$.

If $\alpha = \prod_{i=1}^n u_i \ast_0 \alpha_i \ast_0 v_i$, then let

$$\text{Ech}_{\delta,i} = \left(\prod_{k=1}^{i-1} (u_k \ast_0 \alpha_k \ast_0 v_k) \ast_0 \delta \ast_0 \text{comp}_{x,y,z}^p(\alpha_i, \delta) \ast_0 \left(\prod_{k=i+1}^n (u_k \ast_0 \alpha_k \ast_0 v_k) \ast_0 \theta\right)\right)$$

for $\delta \in \{\beta, \beta'\}$ and let $f_i$ for $0 \leq i \leq n$ to be such that $f_0 = u_1 \ast_0 (u_0 \ast_0 v_0 \ast_0 \theta)$ and for $i > 0$, $f_i = u_i \ast_0 \theta \ast_0 v_i \ast_0 \theta$ and let $\phi_k$ for $0 \leq k \leq n$ such that:

$$\phi_k = \left(\prod_{i=n}^{k+1} \text{Ech}_{\beta,i}\right) \ast_2 \left(\prod_{i=1}^{k} (u_i \ast_0 \alpha_i \ast_0 v_i \ast_0 g) \ast_1 (f_k \ast_0 \theta) \ast_1 (\prod_{i=k+1}^n (u_i \ast_0 \alpha_i \ast_0 v_i \ast_0 g)) \ast_2 \left(\prod_{i=k}^{n} \text{Ech}_{\beta',i}\right)\right)$$

The left hand-side of 17 is then

$$\left(\prod_{i=n}^{k+1} \text{Ech}_{\beta,i}\right) \ast_2 \left(\text{comp}_{x,y,z}^p(f_0, \theta) \ast_1 \left(\prod_{i=1}^n (u_i \ast_0 \alpha_i \ast_0 v_i \ast_0 g')\right)\right) = \phi_0$$

and the right hand-side is

$$\left(\prod_{i=1}^n (u_i \ast_0 \alpha_i \ast_0 v_i \ast_0 g) \ast_1 (f_n \ast_0 \theta) \ast_2 \left(\prod_{i=n}^{1} \text{Ech}_{\beta',i}\right)\right) = \phi_n$$

Plus, since
Hom$(\alpha_n * 0 v_i * 0 g)$

$Ech_{\beta,k} * 2 \left( \prod_{i=1}^{k-1} u_i * 0 \alpha_i * 0 v_i * 0 g \right) * 1 (f_{k-1} * 0 \theta) * 1 \left( \prod_{i=k}^{n} u_i * 0 \alpha_i * 0 v_i * 0 g \right)$

$=( \prod_{i=1}^{k-1} u_i * 0 \alpha_i * 0 v_i * 0 g ) * 1 (comp'_{x,y,z}(u_k * 0 \alpha_k * 0 v_k, \beta) * 1 \left( \prod_{i=k}^{n} u_i * 0 \alpha_i * 0 v_i * 0 g \right) ) * 1 (comp'_{x,y,z}(f_{k-1}, \theta) * 1 \left( \prod_{i=k}^{n} u_i * 0 \alpha_i * 0 v_i * 0 g \right) ) * 1 (comp'_{x,y,z}(u_k * 0 \alpha_k * 0 v_k, \beta') * 1 \left( \prod_{i=k}^{n} u_i * 0 \alpha_i * 0 v_i * 0 g \right) ) * 1 (f_k * 0 \theta) * 1 (\prod_{i=k+1}^{n} u_i * 0 \alpha_i * 0 v_i * 0 g ) * 1 \left( \prod_{i=k}^{n} u_i * 0 \alpha_i * 0 v_i * 0 g \right) * 2 Ech_{\beta',k}$

we have $\phi_{k-1} = \phi_k$ for all $1 \leq k \leq n$. So $\phi_0 = \phi_n$ and 17 holds.

So $comp'_{x,y,z}$ can be factored by $F$:

$\xymatrix{ Q^* \ar[r]^{comp'_{x,y,z}} & \ar[d]^F \ar[r]_{comp'_{x,y,z}} & \ar[r] \ar[d] & \ar[r] \ar[d] \ar[r] & \ar[r] \ar[d] \ar[r] & \ar[r] \ar[d] \ar[r] }$

$Hom(x, y) \otimes Hom(y, z) \ar[r] & Hom(x, z)$

It is then easy to check that these composition operations satisfies the associativity and unital conditions imposed in the definition of a Gray-category.

So we defined a structure of Gray-groupoid on $P^\beta$.

▶ **Proposition 25.** A rewriting system equipped with a termination order is terminating.

**Proof.** Consider a sequence of elementary reductions $(A_i : \phi_i \Rightarrow \phi_{i+1})_{1 \leq i < n}$ for some $n \in \mathbb{N} \cup \{\infty\}$. We show that $n \in \mathbb{N}$.

Indeed, since $A_1 = \psi_{1,i} * 1 (u_i * 0 P_i * 0 v_i) * 1 \psi_{2,i}$, and $\phi_i = s_2 A_i = \psi_{1,i} * 1 (u_i * 0 s_2 P_i * 0 v_i) * 1 \psi_{2,i}$, and $\phi_{i+1} = t_2 A_i = \psi_{1,i} * 1 (u_i * 0 t_2 P_i * 0 v_i) * 1 \psi_{2,i}$, using that $s_2 P_i > t_2 P_i$ and the other axioms on $>$, one readily shows that $\phi_i > \phi_{i+1}$. So $(\phi_i)_{1 \leq i < n}$ is a decreasing sequence for $>$ which is terminating. So $n < \infty$.

▶ **Theorem 26.** A presentation with a finite number of 2-generators and of non-structural 3-generators, with non-identity 2-cells as sources, has a finite number of critical branchings.

**Proof.** If $\alpha = \prod_{i=1}^{n} r_i * 0 \alpha_i * 0 s_i$ with the $\alpha_i$'s generators, recall that we call $n$ the size of $\alpha$, also denoted $\text{size}(\alpha)$. Plus, if $0 \leq i \leq \text{size}(\alpha)$, we call $i$-whisker of $\alpha$ to be $r_i * 0 \alpha_i * 0 s_i$ denoted
as $W_2(i)$. It is uniquely defined as a consequence of the unicity of the normal form. Note that $W_2$ can be seen as a function of type $L_n \rightarrow P_2$ where $L_n = \{1, ..., \text{size}(\alpha)\}$. If $(\beta_1, ..., \beta_p)$ is a sequence of composable 2-cells, $\beta = \beta_1 * ... * _1 \beta_p$ and $n = \text{size}(\beta_1) + ... + \text{size}(\beta_p)$ there is a canonical partition of $I_3$ for $(\beta_1, ..., \beta_p)$ $S_1, ..., S_p$ where $S_k = [a_k, b_k]$ with $a_k = 1 + \sum_{i=1}^{k-1} \text{size}(\beta_i)$ and $b_k = \sum_{i=1}^{k} \text{size}(\beta_i)$ such that $W_2(i) = W_2(i) - a_k + 1$ for $i \in S_k$.

If we have a 3-generator $\theta$, we define a context for $\theta$ to be a 4-tuple $E = (u, v, \phi, \psi)$ where $u, v$ are 1-cells and $\phi, \psi$ are 2-cells such that $\phi * 1 (u * 0 \theta * 0 v) * 1 \psi$ exists (we will denote this cell as $E[\theta]$). We also define the action interval (also denoted $AI(\theta, E)$) of $(\theta, E)$ on $s_2E[\theta]$ to be the segment of $\{1, ..., \text{size}(s_2E[\theta])\}$ starting at $\text{size}(\phi) + 1$ and ending at $\text{size}(\phi) + \text{size}(s_2 \theta)$. The size of this interval is the value $\text{size}(s_2 \theta)$.

Note that we can represent a branching as a pair $B = ((\theta, E), (\theta', E'))$ where $E$ is a context for $\theta$ and $E'$ is a context for $\theta'$ such that $s_2E[\theta] = s_2E'[\theta']$. We call the source of the branching $B$, denoted $s_2B$ to be $s_2E[\theta]$.

We say that that the actions of $(\theta, E)$ and $(\theta', E')$ are overlapping at $(i, i')$ for $1 \leq i \leq |AI(\theta, E)|$ and $1 \leq i' \leq |AI(\theta', E')|$ if $\text{size}(\phi) + i = \text{size}(\phi') + i'$. It implies that $\text{size}(\phi) + i \in AI(\theta, E) \cap AI(\theta', E')$.

**Lemma 27.** In a minimal branching $B = ((\theta, E), (\theta', E'))$, $AI(\theta, E) \cup AI(\theta', E') = \{1, ..., \text{size}(s_2B)\}$

**Proof.**

**Lemma 28.** Let $\theta$ and $\theta'$ be two 3-generators and $E = (u, v, \phi, \psi)$, $E' = (u', v', \phi', \psi')$ be contexts for the two such that $B = ((\theta, E), (\theta', E'))$ is a minimal non-independent branching. Then:

- either $\phi$ or $\phi'$ is an identity
- either $\psi$ or $\psi'$ is an identity
- either $u$ or $u'$ is an identity
- either $v$ or $v'$ is an identity

**Proof.** We only prove the first and the third properties, since the others are similar. Let $\beta = s_2B$.

For the first one, suppose neither $\phi$ nor $\phi'$ is an identity. Since $\phi * 1 (u * 0 s_2 \theta * 0 v) * 1 \psi = \phi' * 1 (u' * 0 s_2 \theta' * 0 v') * 1 \psi'$, by the unicity of the decomposition, we can decompose $\phi$ as $(r * 0 \alpha * 0 s) * 1 \phi_{sub}$ and $\phi'$ as $(r * 0 \alpha * 0 s) * 1 \phi'_{sub}$. So we can factor out $(r * 0 \alpha * 0 s)$ and find a smaller branching. So $B$ was not minimal in the first place, which is a contradiction. So either $\phi$ or $\phi'$ is an identity.

For the third one, suppose neither $u$ nor $u'$ is an identity. We can suppose that $\text{size}(u) \leq \text{size}(u')$, so $u$ is a prefix of $u'$. Let $S_1, S_2, S_3$ and $T_1, T_2, T_3$ be the canonical partitions for $(\phi, u * 0 s_2 \theta * 0 v, \psi)$ and $(\phi', u' * 0 s_2 \theta' * 0 v', \psi')$ respectively. So all $W_2(i)$ for $i \in S_2 \cup T_2$ can be factored by $u$. Suppose there is $i \in I_3$ such that $i \notin S_2 \cup T_2$. By symmetry, we can suppose $i \in S_1$. Then $\phi$ is not an identity, then $\phi'$ is and $T_1 = \emptyset$. So $i \in T_3$ and $\psi$ is an identity and $S_3 = \emptyset$. Let $M = I_3 \setminus (S_2 \cup T_2)$. So $E[\theta] = (u * 0 \theta * 0 v) * 1 \prod_{i \in M} W_2(i) * 1 (u' * 0 s_2 \theta' * 0 v) * 1 \psi'$ and $E'[\theta'] = (u * 0 s_2 \theta * 0 v) * 1 \prod_{i \in M} W_2(i) * 1 (u' * 0 s_2 \theta' * 0 v') * 1 \psi$. Then $B$ is an independent branching which is a contradiction. So $I_3 = S_2 \cup T_2$ and $E[\theta] = u * 0 E_{sub}[\theta]$ and $E'[\theta'] = u * 0 E'_{sub}[\theta']$ for convenient $E_{sub}, E'_{sub}$. So $B$ is not minimal, which is a contradiction. So either $u$ or $u'$ is an identity.

**Lemma 29.** Let $\theta$ and $\theta'$ be two 3-generators and $p, p'$ such that $1 \leq p \leq \text{size}(\theta)$ and $1 \leq p' \leq \text{size}(\theta')$. Then there is at most one non-independent minimal branching $B = ((\theta, E), (\theta', E'))$ such that the actions of $(\theta, E)$ and $(\theta', E')$ are overlapping at $(p, p')$. 


Proof. Suppose such a branching $B = ((\theta, E), (\theta', E'))$ exists with $E = (u, v, \phi, \psi)$ and $E' = (u', v', \phi', \psi')$. We show that $E, E'$ are in fact uniquely determined.

Let $\beta = s_2 B, s_2 \theta = \prod_{i=1}^{n} r_i * \alpha_{i} * s_{i}$, and $s_2 \theta' = \prod_{i=1}^{n} r_i' * \alpha_{i}' * s_{i}'$.

According to the hypothesis, we have $u * r_p * s_p * v = w * r_p' * s_p' * v'$. So $u * r_p = w' * r_p'$ and $s_p * v = s_p' * v'$. We can suppose by symmetry that $|r_p| \leq |r_p'|$.

Then, $u \geq u'$. By the previous property, it holds that $u'$ is actually an identity, and then $u \geq r_p = r_p'$. So $u$ is uniquely defined.

In a similar way, $v$ and $v'$ are uniquely defined.

We can suppose by symmetry that $p \leq p'$. Then $|\phi| \geq |\phi'|$, so $\phi'$ is an identity as a consequence of the previous property and $\phi = u * (\prod (p' - r_k) * s_k) * v$. So $\phi$ and $\phi'$ are uniquely determined. Similarly, $\psi$ and $\psi'$ are uniquely determined.

So the branching $B$ is uniquely determined. \hfill $\blacksquare$

We split the branching $B = ((\theta, E), (\theta', E'))$ into three kinds:

- operational-operational: when both $\theta$ and $\theta'$ are operational generators
- operational-structural: when only one of $\theta$ and $\theta'$ is operational
- structure-structural: when both $\theta$ and $\theta'$ are structural

Lemma 30. There is only a finite number of minimal non-independent operational-operational branchings.

Proof. Since there is a finite number of 3-generators that are operational, we just need to show that for given $\theta, \theta'$ operational 3-generators, there is only a finite number of minimal non-independent branching $B = ((\theta, E), (\theta', E'))$.

But, non-independence entails that there is a couple $(p, p')$ such that the action of $E[\theta]$ and $E'[\theta']$ are overlapping at $(p, p')$ and $E, E'$ are unique for such a $(p, p')$ according to 29. And the number of possible couples $(p, p')$ is finite, which concludes the proof. \hfill $\blacksquare$

Lemma 31. There is only a finite number of operational-structural critical branchings.

Proof. Let $B = ((\theta, E), (\theta', E'))$ a minimal non-independent branching where $\theta$ is an operational generator and $\theta'$ is a structural generator. So $\theta'$ is $X_{\gamma_1, w, \gamma_2}$ for some $\gamma_1, \gamma_2, w$.

Note that there is only one possible case among the three following:

- case 1 : the action intervals of $E[\theta]$ and $E'[\theta']$ are overlapping at $(1, 2)$
- case 2 : $AI(\theta', E') \subset AI(\theta, E)$
- case 3 : the action intervals of $E[\theta]$ and $E'[\theta']$ are overlapping at $(size(s_2 \theta), 1)$

Case 1. The overlapping entails $u' * t_1 * t_0 * r_1 * \alpha_1 * s_1 * v = u * r_1 * v$. So $u' * t_1 * t_0 * r_1 * \alpha_1 * s_1 * v$. So $u' * t_1 * t_0 * r_1 * \alpha_1 * s_1 * v = u * r_1 * v$.

There are two possible cases: either $size(w) \geq size(r_1)$ or $size(w) < size(r_1)$.

If $size(w) \geq size(r_1)$, then by minimality, $u'$ and $v$ are identities. Plus, if we write $w = w' \beta, E'[\theta']$ is the first element of the sequence defining $X_{\gamma_1, \gamma_2, s_2 \theta}$, and $E[\theta] = (\gamma_1 * w' * s_1 \theta) * (t_1 * t_0 * w' \theta)$. So $B$ is a natural branching and hence is not a critical branching.

If $size(w) < size(r_1)$, the composition defining $w$ is a suffix of the one defining $r_1$ so there is a finite number of possibilities for $w$. Plus, there is a finite number of possible $\gamma_1$ since there is a finite number of 2-generators. Note that the minimal non-independent branching $B$ is uniquely defined by $\theta$, $\gamma_1$, $w$ and knowing that the action intervals of $E[\theta]$ and $E'[\theta']$ overlap at $(1, 2)$. So there is a finite number of such branchings (and hence critical branchings) for a given $\theta$.

Case 2. Let $\{l, l+1\} = AI(\theta, E) \cap AI(\theta', E')$. Then, $u * r_l * v = u' * t_1 * t_0 * r_1 * \alpha_1 * s_1 * v = u' * t_1 * t_0 * r_1 * \alpha_1 * s_1 * v = u' * t_1 * t_0 * r_1 * \alpha_1 * s_1 * v$. So $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_{l+1}$.
By minimality, \( u \circ_0 r_l = u', s_{t+1} \circ_0 v = v' \). By minimality, \( u \) and \( v \) are identities and \( u' = r_l \) and \( v' = s_{t+1} \).

And since \( w \circ_0 s_{t+1} \circ_0 s_{t+1} = s_t \), \( w \) is uniquely defined for a given \( l \). So the branching \( B \) in the case 2 is completely determined by \( l \). Since \( l \leq \text{size}(s_\theta) \) there is a finite number of such branchings for a given \( \theta \).

**Case 3.** This case is similar to Case 1.

So for a given \( \theta \), there is a finite number of critical branchings \( B = ((\theta, E), (\theta', E')) \) with \( \theta' \) structural. Since there is a finite number of operational generators \( \theta \), there is a finite number of operational-structural critical branchings.

**Lemma 32.** There are no structural-structural critical branchings.

**Proof.** Let \( B = ((\theta, E), (\theta', E')) \) be a minimal non-independent branching where both \( \theta \) and \( \theta' \) are structural 3-generators, so \( \theta = X_{\gamma_1, w_1, \delta_1} \) and \( \theta' = X_{\gamma_2, w_2, \delta_2} \) for some \( \gamma_1, w_1, \delta_1, \gamma_2, w_2, \delta_2 \).

As before, there are still three cases:

- **Case 1:** the action intervals of \( E[\theta] \) and \( E'[\theta'] \) are overlapping at (2, 1)
- **Case 2:** \( AI(\theta', E') = AI(\theta, E) = \{1, 2\} \)
- **Case 3:** the action intervals of \( E[\theta] \) and \( E'[\theta'] \) are overlapping at (1, 2)

**Case 1.** By minimality and identifications, \( u \) and \( v' \) are identities, \( u' = t_1 \gamma_1 \circ_0 w_1 \), \( v = w_2 \circ_0 s_1 \delta_2 \). So \( E[\theta] \) is the first step of \( X_{\gamma_1, w_1, s_1, \delta_1} \) and \( E'[\theta'] = (\gamma_1 \circ_0 w_1 \circ_0 s_1 \theta') \circ_0 (s_1 \gamma_1 \circ_0 w_1 \circ_0 \theta') \).

So \( B \) is a natural branching and hence is not a critical branching.

**Case 2.** By minimality and identifications, \( \gamma_1 = \gamma_2 \), \( w_1 = w_2 \) and \( \delta_1 = \delta_2 \). So \( B \) is a trivial branching.

**Case 3.** This case is similar to case 1.

So there are no structural-structural critical branchings.

From 30, 31 and 32, there are a finite number of critical branchings.

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**References**


