Describing free ω-categories

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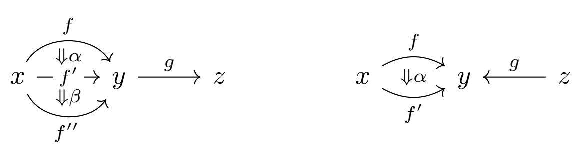
***Abstract*—The notion of pasting diagram is central in the study of strict ω-categories: it encodes a collection of morphisms for which the composition is defined unambiguously. As such, we expect that a pasting diagram itself describes an ω-category which is freely generated by the cells constituting it. In practice, it seems very difficult to characterize this notion in full generality and various definitions have been proposed with the aim of being reasonably easy to compute with, and including common examples (e.g. cubes or orientals). One of the most tractable such structure is *parity complexes*, which uses sets of cells in order to represent the boundaries of a cell. In this work, we first show that parity complexes do not satisfy the aforementioned freeness property by providing a mechanized proof in Agda. Then, we propose a new formalism that satisfies the freeness property and which can be seen as a corrected version of parity complexes.**

I Introduction

Pasting diagrams. Informally, a *pasting diagram* in a strict ω-category is a finite collection of cells, called *generators*, which can be composed unambiguously. This means that one can write a formal expression consisting of well-typed compositions of all the generators and, moreover, any two such expressions should be equal modulo the axioms of ω-categories. For instance, we expect that the cells on the left form a pasting diagram



(1)

since they can be composed as $(\alpha *_1 \beta) *_0 g$ and any other composition "involving all the generators", such as $(\alpha *_0 g) *_1 (\beta *_0 g)$, will give rise to the same result because of the axioms of ω-categories. However, we do not expect that the collection of cells on the right admits a composition, because the 1-cell g is "oriented in the wrong direction", preventing us from writing a well-typed composition expression involving α and g.

As one can expect, this notion of pasting diagram has many applications since it allows for concrete and mechanized computations on ω-categories [9]. For instance, is has been used by Street in order to define and study the combinatorics of a higher-categorical analogue of simplices, called orientals, from which one can define a notion of nerve for strict ω-categories [13], [14]; and similar computations can be performed with other shapes such as cubes [14], [7] or opetopes [12]. It has also been used by Kapranov and Voevodsky to study the relationship between strict ω-groupoids and homotopy types [8], or by Steiner in order to provide a combinatorially pleasant definition of the Crans-Gray tensor product on ω-categories [11].

Formalisms for pasting diagrams. In dimension 1, a pasting diagram is precisely a composable sequence of 1-cells:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n$$

In higher dimensions, the situation is much more complicated and it turns out that characterizing all pasting diagrams is out of reach for now. However, it is sufficient in practice to have access to a "nice" subclass of pasting diagrams. This class should be large enough, so that it includes the examples one wants to consider (e.g. orientals). It should also allow for easy explicit computations, both on paper and with a computer: this means that the data structures used to describe pasting diagrams should be relatively common and easy to manipulate (e.g. sets, multisets, free groups, etc.) and the coherence conditions imposed on those should be relatively easy to check.

Three main formalisms have been proposed for pasting diagrams: Street's *parity complexes* [14], [15] (which we mainly focus on in this article), Johnson's *pasting schemes* [6], and Steiner's *augmented directed complexes* [11] (we should also mention here the work of Power [10]). They mainly differ by the way they encode the source and target of generator (respectively, for an $(n{+}1)$-generator, as the sets of n-generators occurring in the source and target, by relations encoding all the generators inside the boundary, or by linear maps associating to a generator the sum of generators occurring in its boundary), but also by the subtle conditions imposed on those in order to ensure that they form reasonable classes of pasting diagrams, as explained above. In this article, we introduce a new formalism for pasting diagrams, called *torsion-free complexes*, adding a new item to the list. To the best of our knowledge, these various notions have never been formally related: this is the object of a forthcoming article [3] showing that all the previous notions of pasting diagrams can be embedded in the one introduced here,

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which can thus serve as a unifying framework for comparison.

The freeness property. At the beginning of this introduction, we have mentioned the main property that a notion of pasting diagram should satisfy: it should give rise to a unique composition. This can more abstractly be reformulated by saying that a pasting diagram should itself induce an ω-category which is freely generated by the generators of the pasting diagram, what we call here the *freeness property*. In this article, we show that this property does not hold for parity complexes (nor for Johnson's pasting schemes), although it was claimed so. Our counter-example consists in a parity complex that gives rise to two composition expressions which are not equal modulo the axioms of ω-categories. In order to have full confidence in our counter-example, we have fully implemented the free ω-category generated by the parity complex in Agda and formally shown that the two composites are not equal. Apart from the result itself, this provides a good test case for the use of formal methods for higher categories. In the same vein, most of the other properties of parity complexes had been checked in Coq excepting – of course – the freeness property [1] (see Remark 6.2 there).

A corrected formalism. This shortcoming has motivated our introduction of a new formalism for pasting diagrams, for which the freeness property does hold. We describe it here along with its main properties. A much more detailed exposition, along with fully detailed proofs can be found in [3]. In particular, we mention here that it can be used in order to compare the various preexisting notions of pasting diagrams. We believe that this detailed study of the axiomatics is relevant, because it has helped unravel subtle flaws such as the one presented here on structures which were introduced almost 30 years ago, which have remained unnoticed, at least publicly.

Plan. We recall basic definitions for ω-categories (section II), describe a free 3-category in which two composites are shown to be distinct (section III), recall parity complexes and provide a counter-example to the freeness property (section IV), propose a formalism for which this property holds (section V) and conclude (section VII).

II Strict ω-categories

In this section, we begin by recalling elementary definitions about ω-categories.

Graded set. A graded set C is a set together with a partition

$$C = \bigsqcup_{n\in\mathbb{N}} C_n\,,$$

the elements of C_n being *of dimension* n. Any subset D of C is canonically graded by setting $D_n = D \cap C_n$ for every $n \in \mathbb{N}$. Such a subset is *homogeneous* of dimension $k \in \mathbb{N}$ when $D_n = \emptyset$ for $n \neq k$. For $n \in \mathbb{N}$ and $S \subseteq C$, denote $S_{\leq n} = \bigcup_{i\leq n} S_i$.

Globular set. A *globular set* C is a graded set, the elements of dimension n being called n*-cells*, together with functions $\partial^-_n, \partial^+_n : C_{n+1} \to C_n$, respectively associating to an $(n{+}1)$-cell its *source* and *target*, in such a way that the globular identities are satisfied for every $n \in \mathbb{N}$:

$$\partial^-_n \circ \partial^-_{n+1} = \partial^-_n \circ \partial^+_{n+1} \qquad \partial^+_n \circ \partial^-_{n+1} = \partial^+_n \circ \partial^+_{n+1} \qquad (2)$$

Given $m, n \in \mathbb{N}$ with $n \leq m$, we write $\partial^-_n : C_m \to C_n$ for the function $\partial^-_n = \partial^-_n \circ \partial^-_{n+1} \circ \ldots \circ \partial^-_{m-1}$ and similarly for ∂^+_n, and $C_m \times_i C_n$ for the pullback

$$\begin{array}{ccc} C_m \times_i C_n & \dashrightarrow & C_n \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \partial^-_i} \\ C_m & \xrightarrow[\partial^+_i]{} & C_i\,. \end{array}$$

Strict n-category. Given $n \in \mathbb{N} \cup \{\omega\}$, an n*-category* C is a globular set such that $C_i = \emptyset$ for $i > n$ and which is equipped with composition and identity operations

$$*_j : C_i \times_j C_i \to C_i \qquad \text{and} \qquad \mathrm{id}_{i+1} : C_i \to C_{i+1}$$

for $i < n + 1$ and $j \leq i$, which moreover satisfy several axioms. For $j \leq i$ and $x \in C_j$, we denote $\mathrm{id}_i(x)$ for $\mathrm{id}_i \circ \ldots \circ \mathrm{id}_{j+1}(x)$. The required axioms are the following:

(i) if $x *_i y$ is defined, then $\partial^\epsilon_j(\alpha *_i \beta)$ is defined by

$$\begin{cases} \partial^\epsilon_j(\alpha) = \partial^\epsilon_j(\beta) & \text{if } j < i, \\ \partial^-_i(\alpha) & \text{if } j = i \text{ and } \epsilon = -, \\ \partial^+_i(\beta) & \text{if } j = i \text{ and } \epsilon = +, \\ \partial^\epsilon_j(\alpha) *_i \partial^\epsilon_j(\beta) & \text{if } j > i, \end{cases}$$

(ii) for $x \in C_i$,

$$\partial^-_i(\mathrm{id}_{i+1}(x)) = \partial^+_i(\mathrm{id}_{i+1}(x)) = x$$

(iii) for any i,

$$(x *_i y) *_i z = x *_i (y *_i z)$$

(iv) for $x \in C_i$ and $j \leq i$,

$$\mathrm{id}_i(\partial^-_j x) *_j x = x = x *_j \mathrm{id}_i(\partial^+_j x)$$

(v) if $j < i$ and $(x *_i y) *_j (x' *_i y')$ is defined, then

$$(x *_i y) *_j (x' *_i y') = (x *_j x') *_i (y *_j y')$$

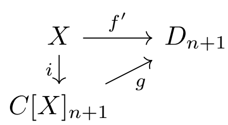
(vi) if $x *_j y$ is defined, then

$$\mathrm{id}_{i+1}(x *_j y) = \mathrm{id}_{i+1}(x) *_j \mathrm{id}_{i+1}(y)$$

Given an $(n{+}k)$-category C, we write $C_{\leq n}$ for the underlying n-category. In the following, we allow ourselves to implicitly consider an n-category as an ω-category with only identities in dimension $k > n$. Moreover, for $m, n \in \mathbb{N}$, $i \leq \min(m, n)$, $k = \max(m, n)$, and $(x, y) \in C_m \times_i C_n$, we often write $x *_i y$ for $\mathrm{id}_k(x) *_i \mathrm{id}_k(y)$.

Free ω-category. An *n-cellular extension* (C, X) of an n-category C is a collection X of formal $n{+}1$-cells for C: formally, it consists in a set X together with functions $s, t : X \to C_n$ such that, when $n > 0$, $\partial^- \circ s = \partial^- \circ t$ and $\partial^+ \circ s = \partial^+ \circ t$. A *morphism* $f : (C, X) \to (D, Y)$ of cellular extensions is an n-functor $f : C \to D$ together with a function $f' : X \to Y$ such that $s \circ f' = f \circ s$ and $t \circ f' = f \circ t$. We denote $\mathbf{Cat}_n^+$ for the category of n-cellular extensions.

There is a forgetful functor $U : \mathbf{Cat}_{n+1} \to \mathbf{Cat}_n^+$ sending an $(n{+}1)$-category C to the cellular extension $(C_{\leq n}, C_{n+1})$, which admits a left adjoint sending an n-cellular extension (C, X) to the $(n{+}1)$-category $C[X]$, called the *free extension* of C by X: the set X^* of $(n{+}1)$-cells of this category consists in formal composites of elements of X, considered modulo the axioms of categories. It satisfies the following universal property: for every $(n{+}1)$-category D, functor $f : C \to D_{\leq n}$ and function $f' : X \to D_{n+1}$ such that $\partial^- \circ f' = f \circ s$ and $\partial^+ \circ f' = f \circ t$, there exists a unique $(n{+}1)$-functor g whose underlying n-functor is f and such that



(3)

where i is the canonical inclusion.

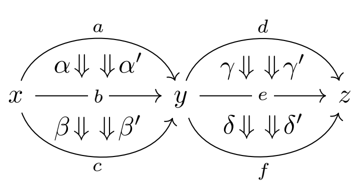
An n-category C is *free* when it is *freely generated* by a set X of cells (of any dimension), i.e., we have $C_{\leq k+1} = C_{\leq k}[X_{k+1}]$ with $X_{k+1} = X \cap C_{k+1}$ for $0 \leq k < n$. In this case, the category C is entirely described by the sets X_k, for $0 \leq k \leq n$, together with the associated maps $s_k, t_k : X_{k+1} \to X_k^*$, for $0 \leq k < n$, since

$$C = X_0[X_1] \ldots [X_n] .$$

This data is sometimes called a *polygraph* or a *computad*.

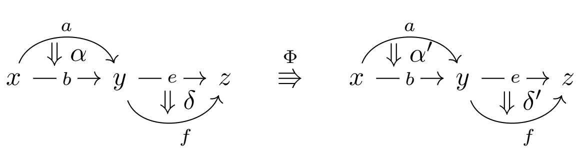
III An ambiguous composition

In order to understand our counter-example to the freeness property of parity complexes, consider the 3-category C freely generated by the 2-cells

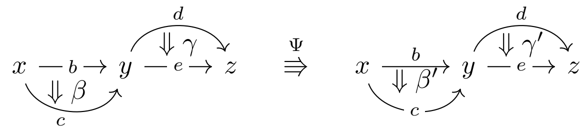


(4)

together with the two 3-cells



and



(formally, this is the 3-category associated to the polygraph corresponding to the above diagrams). In this 3-category, one can consider two composites involving Φ and Ψ, namely:

$$\Gamma = ((a *_0 \gamma) *_1 \Phi *_1 (\beta *_0 f)) *_2 ((\alpha' *_0 d) *_1 \Psi *_1 (c *_0 \delta')) \quad (5)$$

$$\Delta = ((\alpha *_0 d) *_1 \Psi *_1 (c *_0 \delta)) *_2 ((a *_0 \gamma') *_1 \Phi *_1 (\beta' *_0 f)) \quad (6)$$

whose types are both

$$(\alpha *_1 \beta) *_0 (\gamma *_1 \delta) \Rrightarrow (\alpha' *_1 \beta') *_0 (\gamma' *_1 \delta') .$$

We claim here that, in the category C, these composites are different:

$$\Gamma \neq \Delta \quad (7)$$

meaning that the compositions defining Γ and Δ are not the same modulo the axioms of 3-categories, which seems difficult to show directly since the standard tools to handle such axiomatic theories (e.g. convergent rewriting) do not seem to apply here. The technique we employed to show this result consisted in fully describing the category C and showing by computation that the resulting 3-cells are not the same. Due to the high combinatorial complexity of this construction (see the figures below), we have employed the proof assistant Agda which ensures that no corner case has been overlooked.

The full development can be found in [4]; due to an unexpected change in the latest versions of Agda, which is currently being investigated [2], version 2.5.3 must be used. The careful reader will notice below that we did not prove the freeness of our construction, so that a priori we have only described a quotient of the above category C, which is still enough to conclude that (7) holds in C.

3-categories in Agda. We begin by describing our formalization of 3-categories in Agda. A direct description as a 3-globular set would be extremely inconvenient since it would require proving and propagate the globular identities (2) as equality proofs. A much more practical definition of 3-categories consists in encoding the source and target of cells in dependent types, and rely on the very good support of Agda for dependent pattern matching in order to implicitly handle the globular identities. Moreover, we can make most of these variables implicit, leaving the task of inferring them to the proof assistant.

We thus begin by defining *3-precategories* as follows. Those consist in all the structure present in a 3-category (cells, compositions, identities) without requiring the axioms to be satisfied (those will be enforced afterward). The definition is

```
record PCat₃ (C : Set)
  (_→₁_ : (x y : C) → Set)
  (_→₂_ : {x y : C} (f g : x →₁ y) → Set)
  (_→₃_ : {x y : C} {f g : x →₁ y} (F G : f →₂ g) → Set)
: Set where field
id₀ : (x : C) → x →₁ x
id₁ : {x y : C} (f : x →₁ y) → f →₂ f
...
comp₁₀ : {x y z : C} (f : x →₁ y) (g : y →₁ z) → x →₁ z
comp₂₀ : {x y z : C} {f f′ : x →₁ y} {g g′ : y →₁ z}
  (F : f →₂ f′) (G : g →₂ g′) →
  (comp₁₀ f g) →₂ (comp₁₀ f′ g′)
...
```

Such a structure thus consists in a set C (the 0-cells) together with, for every elements $x, y \in C$ of a set $x \to_1 y$ (the 1-cells from x to y) and, for every $x, y \in C$ and $f, g \in (x \to_1 y)$, a set $f \to_2 g$ (the 2-cells from f to g) and so on. Note that for 2-cells, the typing system ensures that the 1-cells f and g are parallel and moreover the 0-cells x and y are implicit (they are declared in curly braces) since they can be inferred from the type of f and g. We also require the definition of identities and compositions, e.g. there is a function which, to every 0-cell x, associates a 1-cell $\mathsf{id}_0\, x \in (x \to_1 x)$.

In order to define 3-categories, we need to put axioms on the operations of a 3-precategory. In order to do so, we define types corresponding to the expected axioms presented in section II:

- is-assoc$_{ij}$: for $0 \leq j < i \leq 3$, represent the associativity of i-cells regarding j-composition (axiom (iii)),
- is-unit$_{ij}$-l and is-unit$_{ij}$-r: for $0 \leq j < i \leq 3$, represent the unitality of j-identities for i-cells (axiom (iv)),
- is-comp$_{ij}$-id: for $0 \leq j < i < 3$, represent the compatibility between j-composition and identities for i-cells (axiom (vi)),
- is-ich$_{ijk}$: for $0 \leq k < j < i \leq 3$, represent the exchange law for i-cells regarding j- and k-compositions (axiom (v)).

Note that the other axioms of categories are enforced by typing. The statement of the axioms is straightforward in low dimensions (they are all of type Set):

```
is-unit₁₀-l =
  {x y : C} {f : x →₁ y} → comp₁₀ (id₀ x) f ≅ f
is-assoc₁₀ =
  {x y z w : C} (f : x →₁ y) (g : y →₁ z) (h : z →₁ w) →
  comp₁₀ (comp₁₀ f g) h ≅ comp₁₀ f (comp₁₀ g h)
is-comp₁₀-id =
  {x y z : C} (f : x →₁ y) (g : y →₁ z) →
  id₁ (comp₁₀ f g) ≅ comp₂₀ (id₁ f) (id₁ g)
is-ich₂₁₀ =
  {x y z : C} {f f′ f″ : x →₁ y} {g g′ g″ : y →₁ z}
  (F : f →₂ f′) (F′ : f′ →₂ f″) (G : g →₂ g′) (G′ : g′ →₂ g″) →
  comp₂₀ (comp₂₁ F F′) (comp₂₁ G G′) ≅
  comp₂₁ (comp₂₀ F G) (comp₂₀ F′ G′)
```

Note that we use here the *heterogeneous* identity type $\cong$ to state the axioms, which allows comparing terms of a different type, thus removing the need of painful type coercions. Indeed, take for instance

```
is-assoc₂₀ : Set
is-assoc₂₀ assoc₁₀ =
  {x y z w : C} {f f′ : x →₁ y} {g g′ : y →₁ z} {h h′ : z →₁ w}
  (F : f →₂ f′) (G : g →₂ g′) (H : h →₂ h′) →
  comp₂₀ (comp₂₀ F G) H ≅ comp₂₀ F (comp₂₀ G H)
```

The two sides of the equation do not have the same type:

$$
\begin{aligned}
(F *_0 G) *_0 H &: ((f *_0 g) *_0 h \to_2 (f' *_0 g') *_0 h') \\
F *_0 (G *_0 H) &: (f *_0 (g *_0 h) \to_2 f' *_0 (g' *_0 h'))
\end{aligned}
$$

making the following *homogeneous* identity type ill-typed:

$$
(F *_0 G) *_0 H \equiv F *_0 (G *_0 H)
$$

Although we could coerce the left-hand side using a proof of is-assoc$_{10}$ in order to type-check, this would make the statements of the axioms and their proofs unnecessarily complicated.

Once these axioms are defined, we can finally state the definition of a 3-category: it is a 3-precategory satisfying the axioms of 3-categories, i.e., with an inhabitant for each of the type associated to each axiom.

Formalizing the example. We can finally formalize the free 3-category (4) in Agda. This example is a sweet spot since it is too big to be handled by hand, but still small enough to be attacked with formal methods. To give an idea there are 3 0-cells, 18 1-cells, 146 2-cells and 166 3-cells. Of course, those do not have to be listed one by one and can be written in a somewhat generic way. For instance, the set of 0- and 1-cells of the category are inductively defined by

```
data C₀ : Set        data C₁ : C₀ → C₀ → Set where
where                C₁-id : (x : C₀) → C₁ x x
C₀-x : C₀            C₁-xy : C₁-abc → C₁ C₀-x C₀-y
C₀-y : C₀            C₁-yz : C₁-def → C₁ C₀-y C₀-z
C₀-z : C₀            C₁-xz : C₁-abc → C₁-def → C₁ C₀-x C₀-z
```

where C$_1$-abc and C$_1$-def are respectively defined by

```
data C₁-abc : Set where     data C₁-def : Set where
C₁-a : C₁-abc               C₁-d : C₁-def
C₁-b : C₁-abc               C₁-e : C₁-def
C₁-c : C₁-abc               C₁-f : C₁-def
```

Above, there are three constructors C$_0$-x, C$_0$-y, C$_0$-z corresponding to the 0-cells x, y and z, and a 1-cell is either

- an identity on a 0-cell x,
- one of the generators a, b, c,
- one of the generators d, e, f,
- a formal composite of (a, b or c) and (c, d or f).

The sets C_2 and C_3 of 2- and 3-cells can be defined in a similar way. Finally, identities and compositions are defined in the expected way, e.g.

```
id₀ x = C₁-id x
comp₁₀ (C₁-id x) g = g
comp₁₀ (C₁-xy f) (C₁-id .C₀-y) = C₁-xy f
comp₁₀ (C₁-xy f) (C₁-yz g) = C₁-xz f g
comp₁₀ (C₁-yz f) (C₁-id .C₀-z) = C₁-yz f
comp₁₀ (C₁-xz f g) (C₁-id .C₀-z) = C₁-xz f g
```

This ends the definition of the underlying 3-precategory of the category (4).

The remaining part is now to check that the axioms of categories are satisfied on our precategory C. In low dimensions, this is easily done by hand, e.g.

```
assoc₁₀ : is-assoc₁₀ C
assoc₁₀ (C₁-id x) g h = refl
assoc₁₀ (C₁-xy f) (C₁-id .C₀-y) g = refl
assoc₁₀ (C₁-xy f) (C₁-yz g) (C₁-id .C₀-z) = refl
assoc₁₀ (C₁-yz f) (C₁-id .C₀-z) g = refl
assoc₁₀ (C₁-xz f g) (C₁-id .C₀-z) h = refl
```

where refl, standing for "reflexivity", is the only constructor of the identity type $\cong$. Note that the case splitting of Agda takes dependent types in account and thus only produce typable cases, i.e., we only have to handle composable sequences of morphisms (this is one of the main reasons of our choice of Agda over Coq). Some other matches are too long to be done by hand and a generator of Agda code was written (in OCaml). The worse case is the one of the proof of the exchange law $\mathsf{is\text{-}ich}_{210}$ for 2-cells with respect to compositions $*_0$ and $*_1$: our program generates a proof involving 306386 cases (resulting in a file of more than 55MB). As is, this proof cannot be verified by Agda: the typechecking crashes after 7 minutes, topping at 12GB of memory consumption. Fortunately, this can be proven differently. First, we show that the model satisfies a distributivity of the right 0-composition over the 1-composition: given cells as in the left of (1), we have

$$(\alpha *_1 \beta) *_0 g = (\alpha *_0 g) *_1 (\beta *_0 g)\,.$$

(and dually, we have a distributivity on the left). Both are proved using trivial reflexivity on a convenient splitting of the parameters. Next we can prove that, given 0-composable 2-cells $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$, we have

$$\alpha *_0 \beta = (\alpha *_0 \mathrm{id}_g) *_1 (\mathrm{id}_{f'} *_0 \beta) = (\mathrm{id}_f *_0 \beta) *_1 (\alpha *_0 \mathrm{id}_{g'})$$

The exchange law of type $\mathsf{is\text{-}ich}_{210}$ is easily deduced from the above lemmas.

Finally, the inequality (7) has an immediate proof: we define the cells cell-Γ and cell-Δ as the respective composites (5) and (6) and conclude

```
main-lemma : ¬(cell-Γ ≅ cell-Δ)
main-lemma ()
```

The full formalization of our example category has 6990 lines in total and can be typechecked in almost 45 minutes, using 3.5GB. We conclude:

Theorem 1. *In the above free 3-category C, $\Gamma \neq \Delta$.*

Another proof can be given by exhibiting a suitable functor $F : C \to \mathbf{Cat}_2$ such that $F(\Gamma) \neq F(\Delta)$, entailing theorem 1, see [3]. Here, $\mathbf{Cat}_2$ is the 3-category of 2-categories, 2-functors, natural transformations and modifications. The above proof has the advantage of being conceptually more clear (we directly formalize the free category instead of finding an appropriate interpretation) and is a good test-case of the applicability of formal methods in the context of strict higher-categories.

Finally, we should note that related counter-examples have independently been found by Henry, studying subcategories of polygraphs which form presheaf categories [5].

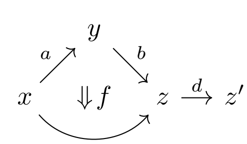
IV Parity complexes

In this section, we recall the definition of parity complexes as given in [14] with the corrections given in [15]. A few minor changes of notations were made to the original presentation. In a free ω-category, the source and the target of an $(n+1)$-generator are free n-cells, i.e., formal composites of n-generators modulo the axioms of categories. In practice, such a direct representation is difficult to manipulate and it turns out that, in many situations, the source and target n-cells are in fact entirely characterized by the set of n-generators they are constituted of. The notion of parity complex provides a formal framework for this intuition, the difficult part being to provide suitable restrictions so that it holds.

ω-hypergraph. An *ω-hypergraph* P is a graded set, the elements of dimension n being called *n-generators*, together with, for each generator $x \in P_{n+1}$, two finite subsets $x^-, x^+ \subseteq P_n$ called the *source* and *target* of x.

Given a subset $X \subseteq P$, we extend the above notation by setting $X^\epsilon = \bigcup_{x \in X} x^\epsilon$ for $\epsilon \in \{-,+\}$. In the following, we write x^{--} for $(x^-)^-$, and similarly for other signs. Given a subset $X \subseteq P$, we define $X^\mp$ as $X^- \setminus X^+$ and $X^\pm$ as $X^+ \setminus X^-$. $X^\mp$ and $X^\pm$ should be understood respectively as the negative and positive "borders" of X.

Example 2. The diagram



(8)

can be encoded as the ω-hypergraph P with

$$P_0 = \{x, y, z, z'\} \qquad P_1 = \{a, b, c, d\} \qquad P_2 = \{f\}$$

and $P_n = \emptyset$ for $n \geq 3$, source and target being

$$a^- = \{x\} \qquad a^+ = \{y\} \qquad f^- = \{a, b\} \qquad f^+ = \{c\}$$

and so on. Moreover, we have

$$f^{--} = \{x, y\} \quad f^{-+} = \{y, z\} \quad f^{-\mp} = \{x\} \quad f^{-\pm} = \{z\}\,.$$

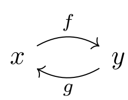
Fork-freeness. Given $n \in \mathbb{N}$, a subset $X \subseteq P_n$ is *fork-free* (also called *well-formed* in [14]) when:

- if $n = 0$ then $|X| = 1$,
- if $n > 0$ then for all $x, y \in X$ and $\epsilon \in \{-,+\}$, if $x^\epsilon \cap y^\epsilon \neq \emptyset$ then $x = y$.

Example 3. The subset $\{a, c\}$ of (8) is not fork-free since $a^- \cap c^- = \{x\}$.

Dependency order. For $n > 0$ and $X \subseteq P_n$, the relation $\triangleleft_X$ on X is the smallest transitive relation such that, for $x, y \in X_n$, $x \triangleleft_X y$ when $x^+ \cap y^- \neq \emptyset$. We extend this relation on subsets $Y, Z \subseteq P$ by writing $Y \triangleleft_X Z$ when there exist $y \in Y$ and $z \in Z$ such that $y \triangleleft_X z$. We will see (lemma 29) that the relation $\triangleleft_X$ can be seen as a *dependency order* corresponding to the order in which the generators of X might be composed. We also define the relation $\triangleleft$ on P as $\triangleleft = \bigcup_{i>0} \triangleleft_{P_i}$. The ω-hypergraph P is *acyclic* when $\triangleleft$ is irreflexive. For $Y \subseteq X$, we say that Y *is a segment for* $\triangleleft_X$ when for all $x, y, z \in X$ with $x \triangleleft_X y \triangleleft_X z$ and $x, z \in Y$, it holds that $y \in Y$.

Example 4. The ω-hypergraph



is not acyclic since we have $f \triangleleft g \triangleleft f$.

Cell. For $n \in \mathbb{N}$, an *n-pre-cell* is a tuple

$$X = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n)$$

of finite subsets of P, such that $X_{i,\epsilon} \subseteq P_i$ for $0 \leq i < n$ and $\epsilon \in \{-,+\}$, and $X_n \subseteq P_n$. By convention, we sometimes write $X_{n,-}$ or $X_{n,+}$ for X_n. The collection of pre-cells of P is canonically equipped with a structure of globular set: given $n \geq 0$, $\epsilon \in \{-,+\}$ and an $(n{+}1)$-pre-cell X, define the n-pre-cell $\partial^\epsilon X$ as

$$\partial^\epsilon X = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_{n,\epsilon})$$

The globular conditions $\partial^\epsilon \circ \partial^- = \partial^\epsilon \circ \partial^+$ are then trivially satisfied.

Given sets $X, Y \in P_n$ and $F \in P_{n+1}$, for some $n \in \mathbb{N}$, we say that F *moves* X to Y when

$$X = (Y \cup F^-) \setminus F^+ \qquad \text{and} \qquad Y = (X \cup F^+) \setminus F^-\,.$$

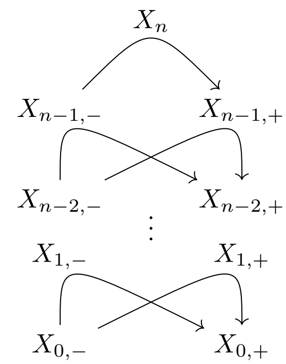
The idea here is that Y is the subset obtained from X by replacing the "negative border" of F by its "positive border".

Example 5. In (8), $\{f\}$ moves $\{a, b, d\}$ to $\{c, d\}$.

An *n-cell* is a pre-cell as above, such that,

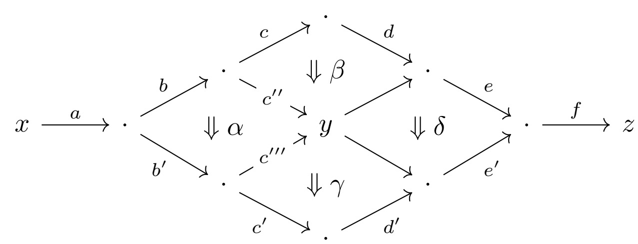
- $X_{i,\epsilon}$ is fork-free for $0 \leq i \leq n$ and $\epsilon \in \{-,+\}$,
- $X_{i+1,\epsilon}$ moves $X_{i,-}$ to $X_{i,+}$ for $0 \leq i < n$, $\epsilon \in \{-,+\}$.

We denote $\mathrm{Cell}(P)$ the graded set of cells, which inherits a structure of globular set from pre-cells. The n-cell X can then be represented the following way:



where the arrows $X \overset{F}{\curvearrowright} Y$ mean that F moves X to Y.

Example 6. Consider the pasting diagram



It contains (among other) the cells

$$(\{x\})$$
$$(\{x\}, \{y\}, \{a, b, c''\}, \{a, b', c'''\}, \{\alpha\})$$
$$(\{x\}, \{z\}, \{a, b, c, d, e, f\}, \{a, b', c', d', e', f\}, \{\alpha, \beta, \gamma, \delta\})\,.$$

Composition of cells and identity. For X a cell of dimension n, define the *i-source* $\partial_i^- X$ by induction on i by $\partial_n^- X = X$ and $\partial_i^- X = \partial^-(\partial_{i+1}^- X)$, and the the *i-target* $\partial_i^+ X$ is defined similarly. This equips the collection $\mathrm{Cell}(P)$ of all cells of P with the structure of a globular set. We say that two n-cells X and Y are *i-composable* when $\partial_i^+ X = \partial_i^- Y$, for some $0 \leq i < n$. In this case, we define their i-composite $X *_i Y$ as the pre-cell

$$Z = (Z_{0,-}, Z_{0,+}, \ldots, Z_{n-1,-}, Z_{n-1,+}, Z_n)$$

such that

$$Z_{j,\epsilon} \quad = \quad \begin{cases} X_{j,\epsilon} & \text{if } j < i, \\ X_{i,-} & \text{if } j = i \text{ and } \epsilon = -, \\ Y_{i,+} & \text{if } j = i \text{ and } \epsilon = +, \\ X_{j,\epsilon} \cup Y_{j,\epsilon} & \text{if } j > i. \end{cases}$$

The *identity of an n-cell* X is the $(n{+}1)$-cell id_X such that:

$$\mathrm{id}_X = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n, X_n, \emptyset)$$

Atom. The *atom* associated to a generator $x \in P_n$ is the pre-cell $\langle x \rangle$ corresponding to this generator. Formally, it is defined as

$$\langle x \rangle = (\langle x \rangle_{0,-}, \langle x \rangle_{0,+}, \ldots, \langle x \rangle_{n-1,-}, \langle x \rangle_{n-1,+}, \langle x \rangle_n)$$

with $\langle x \rangle_n = \{x\}$ and, for $0 \leq i < n$,

$$\langle x \rangle_{i,-} = \langle x \rangle^{\mp}_{i+1,-} \qquad \text{and} \qquad \langle x \rangle_{i,+} = \langle x \rangle^{\pm}_{i+1,+} .$$

A generator x is *relevant* when the atom $\langle x \rangle$ is a cell.

Example 7. The atom associated to f in the example (8) is $\langle f \rangle$ with

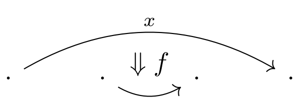
$$\langle f \rangle_{0,-} = \{x\} \qquad \langle f \rangle_{0,+} = \{z\} \qquad \langle f \rangle_2 = \{f\}$$
$$\langle f \rangle_{1,-} = \{a, b\} \qquad \langle f \rangle_{1,+} = \{c\}$$

Tightness. A subset $X \subseteq P_n$ is *tight* when, for all $x, y \in P_n$ such that $x \triangleleft y$ and $y \in X$, we have $x^- \cap X^\pm = \emptyset$. For example, in example 6, $X = \{\beta, \gamma\}$ is not tight since $\alpha \triangleleft \gamma$ and $c'' \in \alpha^- \cap X^\pm$. This is a correction appearing in [15] but which will not be used for defining torsion-free complexes.

Parity complex. A *parity complex* P is an ω-hypergraph satisfying the following axioms:

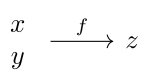
- (C0) for $n \geq 1$ and $x \in P_n$, $x^- \neq \emptyset$ and $x^+ \neq \emptyset$,
- (C1) for $n \geq 2$ and $x \in P_n$, $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$,
- (C2) for $n \geq 1$ and $x \in P_n$, x^- and x^+ are fork-free,
- (C3) P is acyclic,
- (C4) for $n \geq 1$, $x, y \in P_n$, $z \in P_{n+1}$, if $x \triangleleft y$, $x \in z^\epsilon$ and $y \in z^\eta$ for some $\epsilon, \eta \in \{-, +\}$, then $\epsilon = \eta$,
- (C5) for $i < n$ and $x \in P_n$, $\langle x \rangle_{i,-}$ is tight.

Axiom (C1) ensures that generators have globular shapes, so that the following ω-hypergraph is disallowed:



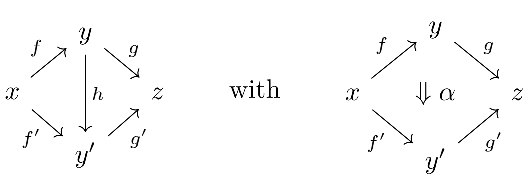
(9)

Axiom (C2) forbids generators to have parallel generators in their source or their target. For example, in the ω-hypergraph



(10)

does not satisfy (C2) since f^- is not fork-free. Axiom (C3) ensures that the hypergraph is acyclic, forbidding ω-hypergraphs like example 4. Axiom (C4) roughly says that we cannot have "a bridge between the source and the target of a generator": for instance, the ω-hypergraph



(11)

is disallowed because we have $\alpha^- \ni f \triangleleft h \triangleleft g' \in \alpha^+$, i.e., h is a "bridge" between the source and the target of α.

The axioms (C0) to (C4) above were originally the only ones [14], but they appeared not to be sufficient and (C5) was added afterward [15]. This axiom relates to a property that we call the *segment axiom* that will be discussed in next section.

Based on these axioms, Street states two claims about parity complexes [14]. First, that the above operations equip it with a structure of an ω-category:

Theorem 8 ([14, Theorem 3.6])**.** *Given a parity complex P,* Cell(P) *is an ω-category.*

Second, that this ω-category is free:

Claim 9 ([14, Theorem 4.2])**.** *Given a parity complex P, the ω-category* Cell(P) *is freely generated by the set $\{\langle x \rangle \mid x \in P,\ x$ is relevant$\}$.*

However, we will see now that the last claim does not hold as is, by providing a counter-example based on the situation described in section III, and propose a corrected axiomatic for parity complexes in section V.

Freenesslessness. The diagram (4) defining the 3-category of section III can be interpreted as an ω-hypergraph P with

$$P_0 = \{x, y, z\} \qquad P_1 = \{a, b, c, d, e, f\}$$
$$P_2 = \{\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'\} \qquad P_3 = \{\Phi, \Psi\}$$

and figured sources and targets, e.g.

$$\Phi^- = \{\alpha, \delta\} \quad \Phi^+ = \{\alpha', \delta'\} \quad \Psi^- = \{\beta, \gamma\} \quad \Psi^+ = \{\beta', \gamma'\}$$

which is easily checked to satisfy the axioms (C0) to (C5) of parity complexes. Moreover, all the generators can be shown relevant. In the ω-category Cell(P), consider the two compositions (5) and (6) (respectively corresponding to Γ and Δ), interpreting the generators as the corresponding atom, i.e.,

$$((\langle a \rangle *_0 \langle \gamma \rangle) *_1 \langle \Phi \rangle *_1 (\langle \beta \rangle *_0 \langle f \rangle)) *_2 ((\langle \alpha' \rangle *_0 \langle d \rangle) *_1 \langle \Psi \rangle *_1 (\langle c \rangle *_0 \langle \delta' \rangle))$$
$$((\langle \alpha \rangle *_0 \langle d \rangle) *_1 \langle \Psi \rangle *_1 (\langle c \rangle *_0 \langle \delta \rangle)) *_2 ((\langle a \rangle *_0 \langle \gamma' \rangle) *_1 \langle \Phi \rangle *_1 (\langle \beta' \rangle *_0 \langle f \rangle))$$

These two composite induce the same 3-cell X with

$$X_3 = \{\Phi, \Psi\}$$
$$X_{2,-} = \{\alpha, \beta, \gamma, \delta\} \qquad X_{2,+} = \{\alpha', \beta', \gamma', \delta'\}$$
$$X_{1,-} = \{a, d\} \qquad X_{1,+} = \{c, f\}$$
$$X_{0,-} = \{x\} \qquad X_{0,-} = \{z\}$$

As a consequence, we can conclude that claimed theorem 9 does not hold with the above parity complex P:

Theorem 10. *The ω-category* Cell(P) *is not freely generated by its atoms.*

Proof. Suppose that Cell(P) is freely generated by the atoms of P. Then, by the universal property of free extensions, there is a functor $I : \text{Cell}(P) \to C$, where C is the free ω-category described in section III, sending each atom to the corresponding cell (e.g. $I(\langle \Phi \rangle) = \Phi$).

By functoriality, the two above composites are respectively sent to Γ and Δ, i.e., $\Gamma = F(X) = \Delta$, but we have shown $\Gamma \neq \Delta$ in theorem 1, thus reaching a contradiction. $\square$

V Torsion-free complexes

In this section, we propose a new set of axioms on ω-hypergraphs which entails theorem 9, thus fixing the definition of parity complexes. We begin by introducing two conditions which will be explained below.

Segment condition. Given an m-generator $x \in P_m$ of an hypergraph P, we say that x *satisfies the segment condition* when, for all $i < m$ and X i-cell such that $\langle x\rangle_{i,-} \subseteq X_i$, it holds that $\langle x\rangle_{i,-}$ is a segment for $\triangleleft_{X_i}$, and dually with $\langle x\rangle_{i,+}$.

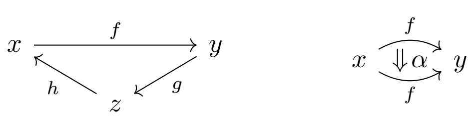
Torsion. Given an m-cell X and two generators x and y of respective dimensions i and j with $i, j > m > 0$, x and y are *in torsion with respect to* X, when $\langle x\rangle_{m,+} \subseteq X_m$, $\langle y\rangle_{m,-} \subseteq X_m$, $\langle x\rangle_{m,+} \cap \langle y\rangle_{m,-} = \emptyset$, and $\langle x\rangle_{m,+} \triangleleft_{X_m} \langle y\rangle_{m,-} \triangleleft_{X_m} \langle x\rangle_{m,+}$.

Torsion-free complexes. An ω-hypergraph P is a *torsion-free complex* when it satisfies the following axioms:

- (T0) for $n \geq 1$ and $x \in P_n$, $x^- \neq \emptyset$ and $x^+ \neq \emptyset$,
- (T1) P is acyclic,
- (T2) for all $x \in P$, x is relevant,
- (T3) for $m \geq 0$ and $x \in P_m$, x satisfies the segment condition,
- (T4) for all $m > 0$, $i, j > m$, $x \in P_i$, $y \in P_j$ and X m-cell, x and y are not in torsion with respect to X.

Axiom (T0), also called *non-emptiness axiom*, corresponds to (C0) which enforces that the source and the target of each generator of the ω-hypergraph are not empty, so that the following ω-hypergraphs are forbidden:

Axiom (T1), called *acyclicity axiom*, corresponds to (C3) and forbids ω-hypergraphs involving some kind of loop as in the following

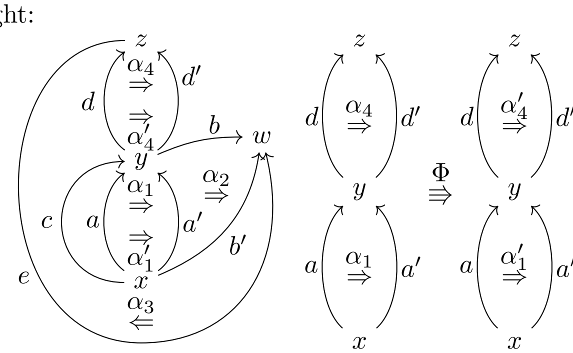


Axiom (T2), called *relevance axiom*, asks that generators of the ω-hypergraph induce cells. For example, the ω-hypergraphs 9 and 10 are forbidden by this axiom.

Lemma 11. *An ω-hypergraph satisfying (T2) also satisfies axioms (C1) and (C2) of parity complexes.*

The axioms (T3), called *segment axiom*, and (T4), called *torsion-freeness axioms*, are more complicated and are detailed below. They roughly respectively ensure that the atoms are generating and that compositions are free in the ω-category Cell(P).

Segment axiom. Recall that our goal is to obtain a category of cells which is freely generated by the atoms. A necessary condition for this is that all cells should be *decomposable*, that is, obtainable by compositions of atoms. But the definition of cells does not require this property and in fact there are cells that are not decomposable under axioms (T0) to (T2). Consider the ω-hypergraph on the left below, with an additional 3-generator Φ as on the right:



(12)

where $\alpha_3^- = \{b'\}$ and $\alpha_3^+ = \{c, d, e\}$. In this example, there is a maximal cell Y given by:

$$
\begin{aligned}
&Y_3 = \{\Phi\} \\
&Y_{2,-} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} && Y_{2,+} = \{\alpha'_1, \alpha_2, \alpha_3, \alpha'_4\} \\
&Y_{1,-} = \{a, b\} && Y_{1,+} = \{c, d', e\} \\
&Y_{0,-} = \{x\} && Y_{0,+} = \{z\}
\end{aligned}
\tag{13}
$$

and Y is not decomposable (explanations follow).

Axiom (T3), called *segment axiom*, prevents this kind of problem. In particular, the ω-hypergraph (12) is forbidden by this axiom. Indeed, the 2-cell X with:

$$
\begin{aligned}
&X_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \\
&X_{1,-} = \{a, b\} && X_{1,+} = \{c, d', e\} \\
&X_{0,-} = \{x\} && X_{0,+} = \{z\}
\end{aligned}
\tag{14}
$$

satisfies $\langle\Phi\rangle_{2,-} = \{\alpha_1, \alpha_4\} \subseteq X_2$ but $\alpha_1 \triangleleft_{X_2} \alpha_2 \triangleleft_{X_2} \alpha_3 \triangleleft_{X_2} \alpha_4$ and $\alpha_2, \alpha_3 \notin \langle\Phi\rangle_{2,-}$. So $\langle\Phi\rangle_{2,-}$ is not a segment for $\triangleleft_{X_2}$.

In order to understand how the decomposability of cells relates to (T3), we make the following observations. Firstly, $\triangleleft$ restricts in which order generators can be composed. In (12), if there exist $x_1, x_2, x_3, x_4 \in P_2$ and X^1, X^2, X^3, X^4 2-cells with $X_2^i = \{x_i\}$ such that X (14) satisfies $X = X^1 *_1 X^2 *_1 X^3 *_1 X^4$, then, since $\alpha_1 \triangleleft \alpha_2 \triangleleft \alpha_3 \triangleleft \alpha_4$, the only possible case is $x_i = \alpha_i$. Secondly, the decomposition property that we want asks for some orders of compositions to be allowed. In (12), note that $\partial_2^- Y = X$. Then, a necessary condition for Y to be decomposable into atoms is that there should exist an order of composition of X in which α_1, α_4 are consecutive. This does not hold since $\triangleleft_{X_n}$ forbids it, so Y is not decomposable and then this ω-hypergraph should be forbidden. Hence, the segment axiom is a necessary requirement to conciliate the restrictions imposed by $\triangleleft$ on compositions and the properties needed to have decomposable cells.

Lemma 12. *The segment axiom is a consequence of the axioms of parity complexes.*

Torsion-freeness axiom. A situation with torsion can be exhibited in the ω-hypergraph (4). Indeed, consider the 2-cell X given by

$$X_2 = \{\alpha', \beta, \gamma, \delta'\}$$
$$X_{1,-} = \{a, d\} \qquad X_{1,+} = \{c, f\}$$
$$X_{0,-} = \{x\} \qquad X_{0,+} = \{z\}$$

Then, $\langle\Phi\rangle_{2,+} \subseteq X_2$ and $\langle\Psi\rangle_{2,-} \subseteq X_2$ and Φ and Ψ are in torsion with respect to X since $\alpha' \triangleleft_{X_2} \beta$ and $\gamma \triangleleft_{X_2} \delta'$, and we have seen that this is an example of a polygraph P such that the category $\mathrm{Cell}(P)$ is not free. The torsion-freeness axiom excludes such parity complexes in which there are cells which are ambiguous, in the sense that they can be obtained as two different and non-equivalent composites. Indeed, given x, y and X as in the statement of (T4), one can exhibit a cell Y that is obtained by composing "x then y" or alternatively "y then x", but the "torsion" between x and y prevents from exchanging them, in a way that the two previous compositions can not be related by the axioms of ω-categories.

Comparing with parity complexes. We have the following comparison theorem between parity complexes and torsion-free complexes:

Theorem 13. *A parity complex satisfying (T2) and (T4) is a torsion-free complex.*

At first sight, the formulation of this theorem seems to imply that there are parity complexes which are not captured by the notion of torsion-free complex, the former notion thus being more general. However, although parity complexes can have generators that are not relevant, i.e., do not necessarily satisfy condition (T2), the non-relevant generators play no role in the cells and can be removed without changing the associated ω-category of cells. Moreover, there is no known example of a parity complex not satisfying (T4) whose associated ω-category is free, and we conjecture that there exists none. Under such an assumption, (T4) would be the weakest axiom to add to obtain a corrected version of parity complexes, meaning that torsion-free complexes are not missing any free ω-categories from parity complexes.

Computable axioms. The axioms (T3) and (T4) stated above are computationally expensive and inconvenient for day-to-day use. In this section, we give stronger axioms that are easier to compute and imply the previous ones. In the following, suppose given an ω-hypergraph P satisfying the axioms (T0), (T1) and (T2).

Given $m \geq 0$, $x, y \in P_m$, write $x \curvearrowright y$ when there exists $z \in P_{m+1}$ such that $x \in z^-$ and $y \in z^+$. Denote $\curvearrowright^*$ the reflexive transitive closure of $\curvearrowright$. When $S, T \subseteq P_m$, we write $S \curvearrowright^* T$ when there exist $s \in S$ and $t \in T$ such that $s \curvearrowright^* t$. We define the following alternate version of axioms (T3) and (T4):

- (T3') for $m > 0$, $x \in P_m$, $k < m$, $\neg(\langle x\rangle_{k,+} \curvearrowright^* \langle x\rangle_{k,-})$,
- (T4') for $m > 0$, $i, j > m$, $x \in P_i, y \in P_j$, if $\langle x\rangle_{m,+} \cap \langle y\rangle_{m,-} = \emptyset$, then at most one of the following is true:

$$\langle x\rangle_{m-1,+} \curvearrowright^* \langle y\rangle_{m-1,-} \quad \text{or} \quad \langle y\rangle_{m-1,+} \curvearrowright^* \langle x\rangle_{m-1,-}\,.$$

Then, (T3) and (T4) can be independently replaced by their computable counterparts in the axiomatic of generalized parity complexes.

Lemma 14. *If P satisfies (T3'), then it satisfies (T3).*

Lemma 15. *If P satisfies (T4'), then it satisfies (T4).*

VI The ω-category of cells

Finally, in this section, we provide the main steps of the proof showing that given a torsion-free complex P, the set $\mathrm{Cell}(P)$ of cells of P can canonically be equipped with a structure of ω-category and moreover, this ω-category is freely generated by the atoms. We only give here sketches and a detailed account can be found in [3]. The structure of the proof follows the original one of Street [14] with two major improvements: firstly, it avoids introducing the notion of *receptive* cell which is quite odd and turns out to be satisfied by every cell and, secondly, the proof of freeness is detailed and corrected.

Cells form an ω-category. We begin by showing that $\mathrm{Cell}(P)$ is an ω-category. The first (and main) step is to prove an analogue of [14, Lemma 3.2], see theorem 17 below. Let X an m-pre-cell and $U \subseteq P_{m+1}$. We say that U is *glueable on X* if $U^\mp \subseteq X_m$. If so, we call *gluing of U on X* the $m{+}1$-pre-cell Y such that $Y_{m+1} = U$ and

$$Y_{m,-} = X_m \quad Y_{m,+} = (X_m \cup U^+) \setminus U^- \quad Y_{i,\epsilon} = X_{i,\epsilon}\,.$$

We denote Y as $\mathrm{Glue}(X, U)$. Moreover, we call *activation of U on X* the pre-cell $\partial^+_m(\mathrm{Glue}(X, U))$ and we denote it $\mathrm{Act}(X, U)$ (see figure 1).

Example 16. Consider the ω-hypergraph from (12) and the cells X of (14) and Y of (13). Then $\{\Phi\}$ is glueable on X and $\mathrm{Glue}(X, \{\Phi\}) = Y$ and $\mathrm{Act}(X, \{\Phi\})$ is the 2-cell X' with:

$$X'_2 = \{\alpha'_1, \alpha_2, \alpha_3, \alpha'_4\}$$
$$X_{1,-} = \{a, b\} \qquad X_{1,+} = \{c, d', e\}$$
$$X_{0,-} = \{x\} \qquad X_{0,+} = \{z\}$$

Theorem 17. *Given a torsion-free complex, an m-cell X, and $U \subseteq P_{m+1}$ a finite fork-free set such that U is glueable on X. Then*

1) $\mathrm{Act}(X, U)$ *is a cell, and* $U^+ \cap X_m = \emptyset$,
2) $\mathrm{Glue}(X, U)$ *is a cell,*
3) *if* $V \subseteq P_{m+1}$ *such that* $V^\pm \subseteq X_m$ *then* $V^- \cap U^+ = \emptyset$.

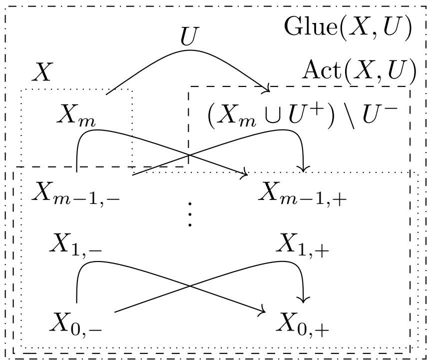


Figure 1. Cells involved and their movements in theorem 17.

The main use of Theorem 17 is that it enables to construct cells from other cells. In particular, it entails that cell composition is well-defined.

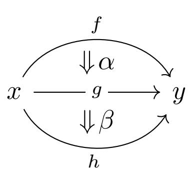
Lemma 18. *Let $m > n \geq 0$ and X,Y two m-cells that are n-composable. Then, $X *_n Y$ is a cell. Moreover, for $n < k \leq m$ and $\epsilon \in \{-,+\}$, $(X^\epsilon_{k,-} \cup X^\epsilon_{k,+}) \cap (Y^\epsilon_{k,-} \cup Y^\epsilon_{k,+}) = \emptyset$.*

The structure of ω-category on $\mathrm{Cell}(P)$ follows readily, since checking the axioms of ω-categories involve only trivial properties on sets.

Theorem 19. $(\mathrm{Cell}(P), \partial^-, \partial^+, *, \mathrm{id})$ *is an ω-category.*

Atoms are generating. Suppose given an ω-category C and a set G of cells of C. We say that C is *generated* by G when every cell of C can be obtained as a composite of elements of G and identities.

Example 20. Consider the ω-category C freely generated by the polygraph:



The set $G' = \{\alpha, \beta\}$ is not generating whereas $G = \{x, y, f, g, h, \alpha, \beta\}$ is.

Lemma 21. *A set G of cells in an ω-category C generates it if and only if $G_{\leq m}$ generates $C_{\leq m}$ in $C_{\leq m}$ for all $m \in \mathbb{N}$.*

Define the *rank* of an m-cell X to be the m-tuple of cardinals:

$$\mathrm{Rank}(X) = (|X_{1,-} \cap X_{1,+}|, ..., |X_{m-1,-} \cap X_{m-1,+}|, |X_m|)$$

On those, the *lexicographic ordering* $<_{\mathrm{lex}}$ is well-founded: it is defined by $(p_1, \ldots, p_m) <_{\mathrm{lex}} (q_1, \ldots, q_m)$ if there exists $1 \leq k \leq m$ such that $p_i = q_i$ for $i > k$ and $p_k < q_k$. Then, a cell which is not an atom can be written as a composite of smaller cells.

Lemma 22 (Excision of extremals)**.** *Let X an m-cell such that $x \in X_m$ and $X \neq \langle x \rangle$. Then there exist $n < m$ and m-cells Y, Z such that $\mathrm{Rank}(Y) <_{\mathrm{lex}} \mathrm{Rank}(X)$, $\mathrm{Rank}(Z) <_{\mathrm{lex}} \mathrm{Rank}(X)$ and $X = Y *_n Z$.*

Then, we can deduce the generating property of the atoms.

Theorem 23. *Given a torsion-free complex P, the set of atoms is generating* $\mathrm{Cell}(P)$.

Proof. Since $<_{\mathrm{lex}}$ is well-founded, for every cell X, we conclude by lemma 22. ☐

Contexts. Our last task consists in showing that the ω-category of cells is freely generated by the atoms. In order to do so, we first need to study the notion of context in an ω-category C.

Given $m \in \mathbb{N}$, suppose fixed two m-cells $y, z \in C$ which are *parallel*, i.e., have the same source and the same target (by convention any two 0-cells are parallel). We define below the notion of context E of type (y, z): it can be thought of as an m-cell composed of k-cells, of dimension at most $m-1$, together with a "hole" cell from y to z, which can be substituted by an actual cell. Given an n-cell x, with $n > m$, such that $\partial^-_m x = y$ and $\partial^+_m x = z$, we say that E is *adapted* to x: in this situation, we will have an induced n-cell, noted $E[x]$, which will be defined along with the notion of k-context. The notion of m-*context* (or simply *context*) E of type (y, z) is defined by induction on the dimension m of y and z as follows.

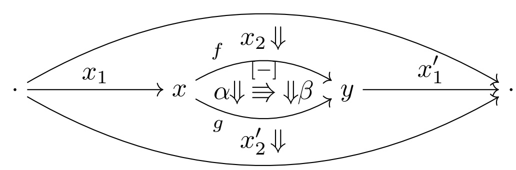
- For 0-cells y and z, there is a unique context of type (y, z), noted $[-]$. Given an n-cell x as above, the induced cell is $[x] = x$.
- For y and z of dimension $m+1$, a context of type (y, z) consists of a context E' of type $(\partial^- y, \partial^+ z)$, together with a pair of $(m+1)$-cells x_{m+1} and x'_{m+1} such that
$$\partial^+(x_{m+1}) = E'[\partial^- y] \quad \text{and} \quad \partial^-(x'_{m+1}) = E'[\partial^+ z]\,.$$
Given an n-cell x as above, the induced cell is
$$E[x] = x_{m+1} *_m E'[x] *_n x'_{m+1}$$
so that we sometimes write
$$E = x_{m+1} *_m E' *_m x'_{m+1}.$$

Given parallel m-cells y and z, a context of type (y, z) thus consists of pairs of suitably typed k-cells (x_k, x'_k), for $0 \leq k \leq m$, and is of the form

$$E = x_m *_{m-1} (\ldots *_1 (x_1 *_0 [-] *_0 x'_1) *_1 \ldots) *_{m-1} x'_n$$

We sometimes write $\pi^-_k E = x_k$ and $\pi^+_k E = x'_k$ for the k-cells of the context E.

Example 24. Given two 2-cells $\alpha, \beta : f \Rightarrow g : x \to y$, a context of type (α, β) can be depicted as



(15)

Lemma 25. *Suppose that C is generated by a set G of cells. Then every m-cell x is either of the form $\mathrm{id}_{x'}$, for some $(m-1)$-cell x', or of the form*

$$E_1[x_1] *_{m-1} E_2[x_2] *_{m-1} \dots *_{m-1} E_k[x_k] \tag{16}$$

for some m-cells $x_1, \dots, x_k \in G$ and adapted $(m-1)$-contexts $E_1, \dots, E_k$.

Atoms are freely generating. Here, we prove that the ω-category $\mathrm{Cell}(P)$ is free by showing, using an induction on m, that the m-category $\mathrm{Cell}(P)_{\leq m}$ is freely generated by its atoms. The base case is immediate and we suppose the property satisfied for $m \in \mathbb{N}$. We have a cellular extension

$$\mathrm{Cell}(P)_{\leq m} \Longleftarrow P_{m+1}$$

where the two functions send a generator $x \in P_{m+1}$ to $\partial^\epsilon\langle x\rangle$, for $\epsilon \in \{-,+\}$. We consider the associated free extension, noted

$$\mathrm{Cell}(P)^+_{\leq m} = \mathrm{Cell}(P)_{\leq m}[P_{m+1}]\,.$$

By its universal property, there is a functor

$$I : \mathrm{Cell}(P)^+_{\leq m} \to \mathrm{Cell}(P)_{\leq m+1}$$

which is the only one, up to isomorphism, whose underlying m-functor is the identity, satisfying (3). For $x \in P_{m+1}$, we write $\widehat{x}$ for its image under the canonical inclusion $P_{m+1} \to \mathrm{Cell}(P)^+_{\leq m}$. By convention, for $x \in P_i$ with $0 \leq i \leq m$, we also write $\widehat{x} = \langle x\rangle$.

The following lemma ensures that there is a well-defined set of $(m+1)$-generators for an $(m+1)$-cell in $\mathrm{Cell}(P)^+_{\leq m}$.

Lemma 26. *Suppose given generators $x_1, \dots, x_k \in P_{m+1}$ and contexts $E_1, \dots, E_k$ in $\mathrm{Cell}(P)_{\leq m}$ such that*

$$X = E_1[\widehat{x_1}] *_m \dots *_m E_k[\widehat{x_k}]$$

exists in $\mathrm{Cell}(P)^+_{\leq m}$. Then

$$I(X)_{m+1} = \{x_1, ..., x_k\}$$

and $x_i \neq x_j$ for $i \neq j$. In the case where $k = 0$, X is of the form $X = \mathrm{id}_Y$ and we have $I(\mathrm{id}_Y)_{m+1} = \emptyset$.

In particular, if

$$E_1[\widehat{x_1}] *_m \dots *_m E_k[\widehat{x_k}] = \tilde{E}_1[\widehat{y_1}] *_m \dots *_m \tilde{E}_{k'}[\widehat{y_{k'}}]$$

for some generators $y_1, \dots, y_{k'} \in P_{m+1}$ and contexts $\tilde{E}_1, \dots, \tilde{E}_{k'}$, then $\{x_1, \dots, x_k\} = \{y_1, \dots, y_{k'}\}$ and $k = k'$.

We now state the main lemmas which are used in order to prove the freeness property (theorem 33 below). Recall that, given a poset $(U, <)$, a subset $V \subseteq U$ is said *initial* (resp. *terminal*) for $<$ when for all $x \in U$, if there exists $y \in V$ such that $x < y$ (resp. $y < x$), then $x \in V$. Fixing $U = \{x_1, \dots, x_k\}$, a *linear extension* is a permutation σ of the indices such that $x_{\sigma(i)} < x_{\sigma(j)}$ implies $i < j$ for $i \neq j$.

The first lemma ensures that in a context of the form (15), we can transfer some generators from x_2 to x'_2 (or the converse) without changing the cells induced by the contexts, as long as the "dependency order" between the generators is preserved.

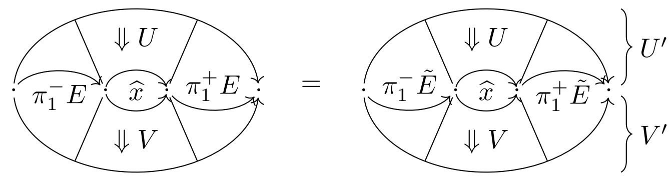
Lemma 27. *Let $x \in P_{m+1}$, E an adapted n-context of $\mathrm{Cell}(P)^+_{\leq m}$ with $n \leq m$ and $X = E[\widehat{x}]$. Let the following subsets of P_n:*

$$S = (\pi^-_n E)_n \cup (\pi^+_n E)_n \qquad U = \{y \in S \mid y \lhd_{S'} \langle x\rangle_{n,-}\}$$
$$S' = S \cup \langle x\rangle_{n,-} = (\partial^-_m X)_n \quad V = \{y \in S \mid \langle x\rangle_{n,-} \lhd_{S'} y\}$$

Then, for every partition $U' \sqcup V'$ of S such that $U \subseteq U'$, and $V \subseteq V'$, U' initial and V' final for $\lhd_S$, there exists an n-context $\tilde{E}$ such that

$$(\pi^-_n \tilde{E})_n = U' \qquad (\pi^+_n \tilde{E})_n = V' \qquad X = \tilde{E}[\widehat{x}]\,.$$

Graphically, with $n = 2$, this can be illustrated as



Proof. By induction, we can reduce to the case where $U' = (\pi^-_n E)_n \setminus \{y\}$ and $V' = (\pi^+_n E)_n \cup \{y\}$. Using lemma 29 inductively, we can suppose that $(\pi^-_n E)_n = E_1[\widehat{y_1}] *_{n-1} \dots *_{n-1} E_k[\widehat{y_k}] *_{n-1} E_y[\widehat{y}]$. By lemma 28 (used inductively with lower n), we have $E_y[\widehat{y}] *_{m-1} E'[\widehat{x}] = \tilde{E}'[\widehat{x}] *_{m-1} \tilde{E}_y[\widehat{y}]$. □

Next, we show that if two $(m+1)$-generators in context do not have common m-generators in their source and target then we can apply the exchange rule.

Lemma 28. *Let $k_1, k_2 \geq 0$ such that $\max(k_1, k_2) = m+1$, $x_1 \in P_{k_1}$, $x_2 \in P_{k_2}$, E_1, E_2 two n-contexts of $\mathrm{Cell}(P)_{\leq m}$ with $0 \leq n < \min(k_1, k_2)$ such that $E_1[\widehat{x_1}] *_n E_2[\widehat{x_2}]$ is an $(m+1)$-cell in $\mathrm{Cell}(P)^+_{\leq m}$. Then*

$$\langle x_1\rangle_{n,-} \cap \langle x_2\rangle_{n,+} = \emptyset$$

Moreover, if $\langle x_1\rangle_{n,+} \cap \langle x_2\rangle_{n,-} = \emptyset$, then there exist n-contexts $\tilde{E}_1, \tilde{E}_2$ such that:

$$E_1[\widehat{x_1}] *_n E_2[\widehat{x_2}] = \tilde{E}_2[\widehat{x_2}] *_n \tilde{E}_1[\widehat{x_1}]$$

Proof. We have $I(E_1[\widehat{x_1}] *_n E_2[\widehat{x_2}]) = E_1[\langle x_1\rangle] *_n E_2[\langle x_2\rangle]$. By lemma 18, $E_1[\langle x_1\rangle]^-_{n+1,-} \cap E_2[\langle x_2\rangle]^+_{n+1,+} = \emptyset$. We deduce the first part since $E_i[\langle x_i\rangle]_{n+1,\epsilon} = \langle x_i\rangle_{n+1,\epsilon}$ and $\langle x_i\rangle_{n,\epsilon} \subseteq \langle x_i\rangle^\epsilon_{n+1,\epsilon}$ for $(i,\epsilon) \in \{(1,-),(2,+)\}$.

For the second part, using lemma 29 (with lower m) and axiom (T4), we are able to reduce to the case where $E_1[\widehat{x_1}]$ and $E_2[\widehat{x_2}]$ are respectively of the form

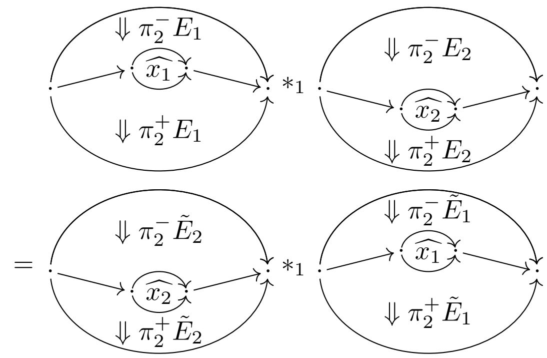
$$\pi^-_n E_1 *_{n-1} E'_1[\widehat{x_1}] *_{n-1} X *_{n-1} E'_2[\partial^-_n \widehat{x_2}] *_{n-1} \pi^+_n E_2$$

and

$$\pi^-_n E_1 *_{n-1} E'_1[\partial^+_n \widehat{x_1}] *_{n-1} X *_{n-1} E'_2[\widehat{x_2}] *_{n-1} \pi^+_n E_2$$

with X an n-cell. For such a form, we can apply an exchange rule and conclude the second part. □

Note that this is the only place where we need (T4). An instance of the above lemma, with $n = 2$ can be pictured as



We have seen in lemma 25 that every m-cell can be expressed as a composition (16) of m-generators in context. We now show that there is such a decomposition for every linearization of the poset of such m-generators under the "dependency order" $\triangleleft$.

Lemma 29. *Consider a set* $U = \{x_1, ..., x_k\} \subseteq P_{m+1}$ *of generators and* m*-contexts* $E_1, \ldots, E_k$ *such that the cell*

$$X = E_1[\widehat{x_1}] *_m \ldots *_m E_k[\widehat{x_k}]$$

exists in $\mathrm{Cell}(P)^+_{\leq m}$. *Then*

$$x_i \triangleleft_D x_j \qquad \textit{implies} \qquad i < j$$

for all indices i, j *such that* $1 \leq i, j \leq k$. *Moreover, if* σ *is a linear extension of* $(U, \triangleleft_U)$, *then there exist* m*-contexts* $\tilde{E}_1, ..., \tilde{E}_k$ *such that*

$$X = \tilde{E}_1[\widehat{x_{\sigma(1)}}] *_m ... *_m \tilde{E}_k[\widehat{x_{\sigma(k)}}] \,.$$

Proof. For the first part, note that $\triangleleft_D$ is the transitive closure of $\triangleleft^1_D$ where for $x, y \in D$, $x \triangleleft^1_D y$ when $x^+ \cap y^- \neq \emptyset$. Then, it is enough to show that $x_i \triangleleft^1_D x_j$ implies $i < j$. By contradiction, suppose that $x_j \triangleleft^1_D x_i$ with $i < j$. By (T1), several swaps of two generators using the second part of lemma 28 can be applied and enable to reduce to the case $j = i + 1$. Hence, $E_i[\widehat{x_i}] *_m E_{i+1}[\widehat{x_{i+1}}]$ exists and $\langle x_i \rangle_{m,-} \cap \langle x_{i+1} \rangle_{m,+} = x_i^- \cap x_{i+1}^+ \neq \emptyset$, contradicting the first part of lemma 28. For the second part, since linear extensions are generated by consecutive transpositions, it is enough to consider the case where $\sigma = (i, i+1)$. Then, the hypothesis enable to apply the second part of lemma 28 to swap $\widehat{x_i}$ and $\widehat{x_{i+1}}$ in the composite X. ☐

We show now that, in order for two contexts applied to a generator to induce the same cell, it is enough for them to have same source.

Lemma 30. *Let* $x \in P_{m+1}$ *and* E_1, E_2 n*-contexts with* $n \leq m$ *such that* $\partial^-_n E_1[\widehat{x}] = \partial^-_n E_2[\widehat{x}]$ *or* $\partial^+_n E_1[\widehat{x}] = \partial^+_n E_2[\widehat{x}]$. *Then* $E_1[\widehat{x}] = E_2[\widehat{x}]$.

Proof. By induction on n, using lemma 27, we change E_1 to match E_2 while preserving the cell obtained by applying E_1 to $\widehat{x}$. ☐

We can deduce that the functor I is injective when restricted to $(m+1)$-cells:

Lemma 31. *Given parallel* $(m+1)$*-cells* X *and* Y *in* $\mathrm{Cell}(P)^+_{\leq m}$, *if* $I(X)_{m+1} = I(Y)_{m+1}$ *then* $X = Y$.

Proof. By lemmas 25 and 26, we can write the cells as

$$X = E_1[\widehat{x_1}] *_m ... *_m E_k[\widehat{x_k}] \quad Y = \tilde{E}_1[\widehat{y_1}] *_m ... *_m \tilde{E}_k[\widehat{y_k}]$$

By permuting the generators using the second part of lemma 29, we can moreover suppose that $x_i = y_i$ for $1 \leq i \leq k$. Then, by applying lemma 30, we get that $E_i[\widehat{x_i}] = \tilde{E}_i[\widehat{y_i}]$. Hence, $X = Y$. ☐

Finally, we can conclude the induction showing that $\mathrm{Cell}(P)_{\leq m+1}$ is a free extension of $\mathrm{Cell}(P)_{\leq m}$ by P_{m+1}:

Lemma 32. $\mathrm{Cell}(P)^+_{\leq m}$ *is isomorphic to* $\mathrm{Cell}(P)_{\leq m+1}$.

Proof. By lemma 31, the functor I is injective. It is moreover surjective: given $X \in \mathrm{Cell}(P)_{m+1}$, by lemma 25 and theorem 23, X can be written

$$X = E_1[\langle x_1 \rangle] *_m ... *_m E_k[\langle x_k \rangle]$$

the cell X is then the image of X' by I where

$$X' = E_1[\widehat{x_1}] *_m ... *_m E_k[\widehat{x_k}]$$

and I is thus an isomorphism of $(m+1)$-categories. ☐

Using those properties, we can show that our new definition of pasting diagrams satisfies the desired freeness property:

Theorem 33. $\mathrm{Cell}(P)$ *is freely generated by the set of atoms* $\{\langle x \rangle \mid x \in P\}$.

Proof. By lemma 32, for $m \in \mathbb{N}$, $\mathrm{Cell}(P)_{\leq m+1}$ is a free extension of $\mathrm{Cell}(P)_{\leq m}$ by $\{\langle x \rangle \mid x \in P_{m+1}\}$. $\mathrm{Cell}(P)$ is thus freely generated by the atoms. ☐

VII Conclusion and related works

In this article, we have shown that the freeness property does not hold for parity complexes and proposed a corrected notion for which it does. Another argument in favor of this new structure is that it also relates to the main other formalisms for pasting diagrams, allowing to study their relationship, which should be detailed elsewhere [3]. In Johnson's *pasting schemes* [6], cells are represented by "closed finite graded sets", which are essentially subhypergraphs of the hypergraph of the pasting scheme. Pasting schemes do not satisfy the freeness property since the ω-hypergraph underlying (4) is also accepted as a pasting scheme, but those satisfying (T4) can be embedded in torsion-free complexes. Steiner's *augmented directed complexes* [11], which are based on complexes of groups (with distinguished submonoids), can also be embedded in torsion-free complexes. Our counter-example (4) to the freeness property does not apply for those since the torsion axiom is entailed by the "loop-freeness" condition imposed on the basis.

References

[1] Mitchell Buckley. A formal verification of the theory of parity complexes. *Journal of Formalized Reasoning*, 8(1):25–48, 2015.

[2] Simon Forest. Agda issue #3466: two definitionally equal terms are not equal, 2018. `https://github.com/agda/agda/issues/3466`.

[3] Simon Forest. Unifying notions of pasting diagrams. Unpublished manuscript, available at `http://www.lix.polytechnique.fr/Labo/Simon.Forest/data/articles/gen-pc.pdf`, 2018.

[4] Simon Forest and Samuel Mimram. Agda formalization of the counter-example for parity complexes. Available at `https://gitlab.inria.fr/sforest/3-pasting-example`, 2018.

[5] Simon Henry. Non-unital polygraphs form a presheaf category. *arXiv preprint arXiv:1711.00744*, 2017.

[6] Michael Johnson. The combinatorics of n-categorical pasting. *Journal of Pure and Applied Algebra*, 62(3):211–225, 1989.

[7] Mikhail Kapranov and Vladimir Voevodsky. Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results). *Cahiers de Topologie et Géométrie différentielle catégoriques*, 32(1):11–27, 1991.

[8] Mikhail Kapranov and Vladimir Voevodsky. ∞-groupoids and homotopy types. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 32(1):29–46, 1991.

[9] Samuel Mimram. An implementation of parity complexes. Available at `https://github.com/smimram/homer/tree/master/src/parity`, 2016.

[10] John Power. An n-categorical pasting theorem. In *Category theory*, pages 326–358. Springer, 1991.

[11] Richard Steiner. Omega-categories and chain complexes. *Homology, Homotopy and Applications*, 6(1):175–200, 2004.

[12] Richard Steiner. Opetopes and chain complexes. *Theory and Applications of Categories*, 26(19):501–519, 2012.

[13] Ross Street. The algebra of oriented simplexes. *Journal of Pure and Applied Algebra*, 49(3):283–335, 1987.

[14] Ross Street. Parity complexes. *Cahiers de topologie et géométrie différentielle catégoriques*, 32(4):315–343, 1991.

[15] Ross Street. Parity complexes: corrigenda. *Cahiers de topologie et géométrie différentielle catégoriques*, 35(4):359–361, 1994.