Unifying notions of pasting diagrams

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Abstract
In this work, we relate the three main formalisms for the notion of pasting diagram in strict $\omega$-categories: Street’s parity complexes, Johnson’s pasting schemes and Steiner’s augmented directed complexes. We first show that parity complexes and pasting schemes do not induce free $\omega$-categories in general, contrarily to the claims made in their respective papers, by providing a counter-example. Then, we introduce a new formalism that is a strict generalization of augmented directed complexes, and corrected versions of parity complexes and pasting schemes, which moreover satisfies the aforementioned freeness property. Finally, we show that there are no other embeddings between these four formalisms.

Introduction

From an original idea of S. Mimram.

Pasting diagrams. Central to the theory of strict $\omega$-categories is the notion of pasting diagram, which describes collections of morphisms for which a composite is expected to be defined and unambiguous. Reasonable definitions are easy to achieve in low dimensions, but the notion is far from being straightforward in general. The three main proposals are Johnson’s pasting schemes [8], Street’s parity complexes [19, 20] and Steiner’s augmented directed complexes [16, 17]. Even though the ideas underlying the definitions of those formalisms are quite similar, they differ on many points and comparing them precisely is uneasy, and actually, to the best of our knowledge, no formal account of the differences was ever made. In this article, we achieve the task of formally relating them. It turns out that the three notions are incomparable in terms of expressive power (each of the three allows a pasting diagram which is not allowed by others), and the way the comparison is performed here is by embedding them into a generalization of parity complexes which is able to encompass all the various flavors of pasting diagrams.

Originally, the motivation behind pasting diagrams was to give a simpler representation of formal composition of cells in (free) $\omega$-categories. More precisely, given the data, for $i \geq 0$, of generating $i$-cells with their source and target boundaries (under the form of a polygraph [2], also called computad [18]), the cells of the associated free $\omega$-category can be described as the formal composites of generators quotiented by the axioms of $\omega$-categories. This representation
is difficult to handle in practice, because the equivalence relation induced by
the axioms is hard to describe. Instead, a graphical representation of the cells
involved in the composite appeared to be sufficient to designate a cell. For
instance, consider the two formal composites
\[
\alpha \ast_0 (\alpha \ast_1 \beta) \ast_0 ((\gamma \ast_0 h) \ast_1 (\delta \ast_0 h))
\]
and
\[
(a \ast_0 \alpha \ast_0 \varepsilon \ast_0 h) \ast_1 (a \ast_0 \varepsilon \ast_0 \gamma \ast_0 h) \ast_1 (a \ast_0 \beta \ast_0 \delta \ast_0 h).
\]
Under the axioms of $\omega$-categories, it can be checked, though it is not immediate,
that both represent the same cell. However, both are formal composites of the
elements of the following diagram

```
\begin{aligned}
\text{(1)} \quad u & \xrightarrow{a} v & \xrightarrow{b} w & \xrightarrow{c} x & \xrightarrow{h} y \\
\downarrow{\alpha} & \downarrow{\beta} & \downarrow{\gamma} & \downarrow{\delta} & \downarrow{\varepsilon} & \downarrow{\delta} & \downarrow{\gamma} & \downarrow{\varepsilon} & \downarrow{\alpha} & \downarrow{\beta} & \downarrow{\gamma} & \downarrow{\delta} & \downarrow{\varepsilon} & \downarrow{\alpha} & \downarrow{\beta} & \downarrow{\gamma} & \downarrow{\delta} & \downarrow{\varepsilon}
\end{aligned}
```

In fact, all formal composites involving all the generators of this diagram are
equal and the data of the diagram enables to refer to the cell obtained by
composing $u$, $v$, ..., $y$, $a$, $b$, ..., $h$, $\alpha$, $\beta$, $\gamma$, $\delta$ together unambiguously without
giving an explicit composite for them. We call \textit{pasting diagrams} the diagrams
satisfying this property. It can be observed that this pasting diagram is made
of smaller pasting diagrams like

```
\begin{aligned}
\text{v} & \xrightarrow{\beta} w & \xrightarrow{\alpha} u & \xrightarrow{\beta} w & \xrightarrow{\alpha} u \\
\downarrow{\beta} & \downarrow{\alpha} & \downarrow{\beta} & \downarrow{\alpha} & \downarrow{\beta} & \downarrow{\alpha}
\end{aligned}
```

Moreover, the two can be composed along $w$ by taking the union of the pasting
diagrams. More generally, given a set of generators and their source-target
borders satisfying sufficient properties, one can obtain a category of pasting
diagrams on such a set, which is isomorphic to the free category mentioned
earlier, justifying the use of pasting diagrams instead of formal composites to
designate particular cells.

Hence, pasting diagram formalisms give effective descriptions of free $\omega$-cate-
gories. In particular, they give a precise definition of the notion of commutative
diagrams and model generic compositions. Moreover, they make it possible to
study higher categories by probing them through pasting diagrams. For example,
augmented directed complexes were used to give an effective description of the Gray tensor product in [17]. In a related manner, Kapranov and Voevodsky
studied topological properties of pasting schemes in [10] and used them in an
attempt to give a description of $\omega$-groupoids in [11], but their results were shown
paradoxical [15].

Several other works studied pasting diagrams. In [1], Buckley gives a mechan-
ized Coq proof of the results of [19] but stops at the excision theorem [19,
Theorem 4.1]. In particular, the proof of the freeness claim [19, Theorem 4.2] was
not formally verified, and could not be, since this claim does not hold in general,
as is shown in the present paper. In [3], Campbell isolates a common structure behind parity complexes and pasting schemes, called parity structure, and gives stronger axioms than the ones of parity complexes and pasting schemes, taking an opposite path from this work which seeks a more general formalism. In [13], Nguyen studies pre-polytopes with labeled structures and shows that they give a parity structure that satisfies a variant of Campbell’s axioms that are enough to obtain another correct notion of pasting diagrams. In [6], Henry defines a theoretical notion of pasting diagrams, called polyplexes, to show that certain classes of polygraphs are presheaf categories, and uses them to prove a variant of the Simpson’s conjecture in [7]. However, his pasting diagrams can involve some looping behaviors, and are then out of the scope of the formalisms studied in the present work. Using similar ideas, Hadzihasanovic [5] defines a class of pasting diagrams, called regular polygraphs, that is “big enough” to study semi-strict categories and which is well-behaved for several constructions (notably, their realizations as topological spaces are CW complexes).

**Pasting diagrams in 1-categories.** The most simple instance of a pasting diagram is in a 1-category: in this case, those are of the form

\[ x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \rightarrow \cdots \rightarrow x_n \]  

(2)

and admit \( a_n \circ \cdots \circ a_1 \) as composite. On the contrary, diagrams such as

\[ y \xleftarrow{a} x \xrightarrow{b} z \]  

or \( x \xrightleftharpoons{a} \) are not expected to be pasting diagrams: in the first one, the two arrows are not even composable, and the second one is ambiguous in the sense that it might denote \( a \), or \( a \circ a \), etc. Also note that the diagram (2) can be freely obtained as the composite of generating diagrams of the form

\[ x_1 \xrightarrow{a} x_{i+1} \]

(composition amounts here to identify the target object of a diagram with the source of the second), whereas this is not the case for the diagrams of (3). Note that the pasting diagrams of the form (2) can be characterized as the graphs which are acyclic, connected and non-branching (in the sense that no two arrows have the same source or the same target).

**Pasting diagrams in higher categories.** In order to extend this construction to higher dimensions, we first need to generalize the notion of graph: an \( \omega \)-hypergraph is a sequence of sets \( (P_i)_{i \geq 0} \) together with, for \( i \geq 0 \) and \( x \in P_{i+1} \), two subsets \( x^- , x^+ \subseteq P_i \) representing the source and target elements of \( x \). Then, pasting diagrams can be generalized to higher dimensions inductively: an \( n \)-pasting diagram is given by source and target \( (n-1) \)-pasting diagrams and a (compatible) set of \( n \)-generators. Conditions need to be put on the \( \omega \)-hypergraphs in order for the pasting diagrams to have the structure of an \( \omega \)-category. But, contrarily to dimension one, giving such conditions is hard in higher dimensions, because guessing which formal composites are going to be identified by the axioms of \( \omega \)-categories can be tricky. For example, the order in which we are supposed to compose the elements of (1) is ambiguous. Considering only the
2-generators, the orders of composition $\alpha, \beta, \gamma, \delta$ and $\alpha, \gamma, \delta, \beta$ are both possible. However, it can be proved that all possible orders of composition are equivalent by the axioms of strict $\omega$-categories, so this ambiguity is not important. On the contrary, given the 2-cells $\alpha$ and $\beta$ described by the diagrams

\[ \begin{array}{c}
    & a & x \\
    w & \downarrow \alpha & b \\
    & a' & y \\
\end{array} \quad \text{and} \quad \begin{array}{c}
    & b & y \\
    b & \downarrow \beta & c \\
    & b & y \\
\end{array} \]

$\alpha$ and $\beta$ can be composed together in two possible orders: $\alpha$ then $\beta$ or $\beta$ then $\alpha$, which can be represented as

\[ \begin{array}{c}
    & a & x \\
    w & \downarrow \alpha & y \\
    & a' & b \\
\end{array} \quad \text{and} \quad \begin{array}{c}
    & b & y \\
    b & \downarrow \beta & c \\
    & b & y \\
\end{array} \]

But here, these two composites are different. Even more subtle problems arise starting from dimension three, justifying the somewhat sophisticated axioms given for parity complexes and pasting schemes.

**Pasting diagrams as cells.** Given an $\omega$-hypergraph $P$, the pasting diagrams on $P$ can be described as cells on $P$, that is, as organized collections of generators of $P$. For good enough axioms on $P$, we expect these cells to be $\omega$-categorical cells. There are different flavours for these cells, which reflects as different formalisms for pasting diagrams.

A first notion of cell is given by tuples of elements of $P$ that are kept organized by dimension and by source/target status. This is the solution adopted by parity complexes. For example, the pasting diagram (1) is represented by five sets

\[ X_2 = \{\alpha, \beta, \gamma, \delta\}, \]
\[ X_{1,-} = \{a, b, e, h\}, \quad X_{1,+} = \{a, d, g, h\}, \]
\[ X_{0,-} = \{a\}, \quad X_{0,+} = \{y\} \]

where $X_{i,-}$ represent the $i$-source, $X_{i,+}$ the $i$-target, and $X_2$ the 2-dimensional part of the diagram.

Another notion of cell is given by sets that gather all the elements appearing in the pasting diagram, regardless of their dimension or source/target status. This is the solution adopted by pasting schemes. For example, (1) will be represented by the set

\[ X = \{u, v, w, x, y, a, b, c, d, e, f, g, h, \alpha, \beta, \gamma, \delta\}. \]
This notion of cell seems the most natural for our goals since it enables one to refer to the “cell obtained by composing together the generators of a set $S$” directly as the set $S$. However, it is arguably harder to work with.

A last notion of cells can be obtained by interpreting $\omega$-hypergraph as directed complexes of abelian groups. Similarly to the first notion of cell, cells are then given by group elements for each dimension and source/target status. This is the notion adopted by augmented directed complexes. For example, (1) will be represented by the 5 group elements

\[
\begin{align*}
X_2 &= \alpha + \beta + \gamma + \delta, \\
X_{1,-} &= a + b + e + h, \\
X_{1,+} &= a + d + g + h, \\
X_{0,-} &= u, \\
X_{0,+} &= y.
\end{align*}
\]

This has the advantage of allowing tools from group theory or linear algebra, and is currently the most widely used formalism.

**Outline and results.** In Section 1, we recall the definitions of each formalism: parity complexes (Subsection ??), pasting schemes (Subsection ??) and augmented directed complexes (Subsection ??). Then, we introduce generalized parity complexes (Subsection ??, with axioms given in Paragraph ??). We relate each definition to the unifying notion of $\omega$-hypergraph (Paragraph ??): a formalism is then a class of $\omega$-hypergraphs (given by axioms) together with a notion of cell and operations on these cells. In Paragraph ??, we discuss the counter-example to the freeness property of parity complexes, which involves the diagram made of

\[
\begin{tikzcd}
& y \\
& z \\
& \delta \\
\beta & \gamma \arrow{r}{c} & \delta' \arrow{r}{f} & \gamma' \\
\alpha & \beta' \arrow{r}{b} & \gamma' \arrow{r}{d} & \gamma' \arrow{r}{d} & \gamma' \\
\end{tikzcd}
\]

(together with two 3-generators

\[
\begin{tikzcd}
x & \alpha \arrow{r}{b} & y \arrow{r}{\delta} & z \arrow{r}{\delta'} & y \arrow{r}{\delta'} & z, \\
\end{tikzcd}
\]

\[
\begin{tikzcd}
x & \beta \arrow{r}{c} & y \arrow{r}{\gamma} & z \arrow{r}{\gamma'} & \gamma' \arrow{r}{\gamma'} & \gamma' \arrow{r}{\gamma'} & \gamma', \\
\end{tikzcd}
\]

In Paragraph ??, we explain that the counter-example above also contradicts the freeness property claimed for pasting schemes. In Paragraph ??, we give alternative axioms for generalized parity complexes that are simpler to check in practice.
In Section 2, we show that, given a generalized parity complex $P$, the set of cells $\text{Cell}(P)$ on $P$ has the structure of an $\omega$-category. In Subsection 2.2, we prove an adapted version of [19, Lemma 3.2] which is the main tool to build new cells from known cells (Theorem 2.2.3). In Subsection 2.3, we use this property to show that cells on a generalized parity complex have the structure of an $\omega$-category (Theorem 2.3.3).

In Section 3, we show the freeness result for generalized parity complexes. In Subsection 3.2, we prove that the atomic cells (i.e. cells induced by one generator) are generating $\text{Cell}(P)$ (Theorem 3.2.2). In Subsection 3.4, we introduce contexts that are used to obtain canonical form for the cell of an $\omega$-category (as given by Lemma 3.4.4). In Subsection 3.5, we formally define the notion of freeness we are using for $\omega$-categories. In Subsection 3.6, we prove that $\text{Cell}(P)$ is free (Theorem 3.6.18).

In Section 4, we define other notions of cells for generalized parity complexes, namely maximal-well-formed and closed-well-formed sets. Closed-well-formed sets should be understood as the equivalent of the notion of cell for pasting scheme in generalized parity complexes. Maximal-well-formed sets are then a convenient intermediate for proofs between the original notion of cell for parity complexes (as defined in Paragraph ??) and closed-well-formed sets. We show that the both new notions induce $\omega$-categories of cells isomorphic to $\text{Cell}(P)$ (Theorem 4.5.5 and Theorem 4.5.7).

In Section 5, we relate generalized parity complexes to the three other formalisms. In Subsection 5.1, we show that parity complexes are generalized parity complexes (Theorem 5.1.3). In Subsection 5.2, we show that loop-free pasting schemes are generalized parity complexes (Theorem 5.2.9) and that both formalisms induce isomorphic $\omega$-categories (Theorem 5.2.10). In Subsection 5.3, we show that loop-free unital augmented directed complexes are generalized parity complexes (Theorem 5.3.23) and that both formalisms induce isomorphic $\omega$-categories (Theorem 5.3.24). In Subsection 5.4, we give counter-examples to other embeddings between the formalisms.

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1 Definitions

In this section, we recall the definition of strict $\omega$-categories and then we present the three main formalisms for pasting diagrams studied in this article: parity complexes [19], pasting schemes [8] and augmented directed complexes [17]. They all roughly follow the same pattern: starting from what we call an $\omega$-hypergraph, encoding the generating elements of the considered $\omega$-category, they define a notion of cell, consisting of sub-hypergraphs satisfying some conditions. Then, they give conditions on those $\omega$-hypergraph such that these cells can be composed and form an $\omega$-category.

2 The category of cells

In this section, we show that $\text{Cell}(P)$ has a structure of an $\omega$-category. For this purpose, we adapt the proofs of [19] and take the opportunity to simplify them.

2.1 Movement properties

Here, we state several useful properties of movement (as defined in Paragraph ??), some of which coming from [19].

In the following, we suppose given an $\omega$-hypergraph $P$. The first property gives another criterion for movement.

Lemma 2.1.1 ([19, Proposition 2.1]). For $n \in \mathbb{N}$, finite subsets $U \subseteq P_n$ and $S \subseteq P_{n+1}$, there exists $V \subseteq P_n$ such that $S$ moves $U$ to $V$ if and only if $S^\pm \subseteq U$ and $U \setminus S^+ = \emptyset$.

Proof. If $S$ moves $U$ to $V$, then, by definition,

$$S^\mp \subseteq (V \cup S^-) \setminus S^+ = U$$

and

$$U \cap S^+ = ((V \cup S^-) \setminus S^+) \cap S^+ = \emptyset.$$  

Conversely, if $S^\mp \subseteq U$ and $U \cap S^+ = \emptyset$, let $V = (U \cup S^+) \setminus S^-$. Then

$$(V \cup S^-) \setminus S^+ = (U \cup S^+ \cup S^-) \setminus S^+$$

$$= (U \setminus S^+) \cup (S^- \setminus S^+)$$

$$= U \cup S^\mp$$  

(since $U \cap S^+ = \emptyset$)

$$= U$$  

(since $S^\mp \subseteq U$)

and $S$ moves $U$ to $V$.

The next property states that it is possible to modify a movement by adding or removing “independent” elements.

Lemma 2.1.2 ([19, Proposition 2.2]). Let $n \in \mathbb{N}$, $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$ be finite subsets such that $S$ moves $U$ to $V$. Then, for all $X, Y \subseteq P_n$ with $X \subseteq U$, $X \cap S^\pm = \emptyset$ and $Y \cap (S^- \cup S^+) = \emptyset$, $S$ moves $(U \cup Y) \setminus X$ to $(V \cup Y) \setminus X$.  

Similarly, then $S$. Proof. We compute $S,T$ (Lemma 2.1.4) Lemma 2.1.3. The next property gives sufficient conditions for decomposing movements.

Proof. By Lemma 2.1.1, $S^+ \subseteq U$ and $U \cap S^+ = \emptyset$. Using the hypothesis, we can refine both equalities to $S^+ \subseteq (U \cup Y) \setminus X$ and $((U \cup Y) \setminus X) \cap S^+ = \emptyset$. Using Lemma 2.1.1 again, $S$ moves $(U \cup Y) \setminus X$ to $W$ where

\[
W = (((U \cup Y) \setminus X) \cup S^+) \setminus S^-
\]

\[
= (((U \cup S^+ \cup Y) \setminus X) \setminus S^- \quad \text{(since } X \cap S^+ \subseteq U \cap S^+ = \emptyset)\]

\[
= (((U \cup S^+) \setminus S^-) \cup Y) \setminus X \quad \text{(since } Y \cap S^- = \emptyset)\]

\[
= (V \cup Y) \setminus X.
\]

The following property gives sufficient conditions for composing movements.

Lemma 2.1.3 ([19, Proposition 2.3]). For $n \in \mathbb{N}$, finite subsets $U,V,W \subseteq P_n$ and $S,T \subseteq P_{n+1}$ such that $S$ moves $U$ to $V$ and $T$ moves $V$ to $W$, if $S^- \cap T^+ = \emptyset$ then $S \cup T$ moves $U$ to $W$.

Proof. We compute $(U \cup (S \cup T)^+) \setminus (S \cup T)^-$:

\[
(U \cup S^+ \cup T^+) \setminus (S^- \cup T^-) = (((U \cup S^+) \setminus S^-) \cup T^+) \setminus T^-
\]

\[
= (V \cup T^+) \setminus T^-
\]

\[
= W.
\]

Similarly, $(W \cup (S \cup T)^-) \setminus (S \cup T)^+ = U$ and $S \cup T$ moves $U$ to $W$. $\square$

The next property gives sufficient conditions for decomposing movements.

Lemma 2.1.4 ([19, Proposition 2.4]). For $n \in \mathbb{N}$, finite subsets $U,W \subseteq P_n$, $S,T \subseteq P_{n+1}$ such that $S \cup T$ moves $U$ to $W$ and $S^+ \subseteq U$, if $S \perp T$ then there exists $V$ such that $S$ moves $U$ to $V$ and $T$ moves $V$ to $W$.

Proof. Let $R = S \cup T$. By Lemma 2.1.1, $R^+ \subseteq U$ and $U \cap S^+ \subseteq U \cap R^+ = \emptyset$. By Lemma 2.1.1 again, $S$ moves $U$ to $V = (U \cup S^+) \setminus S^-$. Moreover,

\[
S^- \cap T^+ = S^+ \cap T^+ \quad \text{(since } S^+ \cap T^+ = \emptyset, \text{ by } S \perp T)\]

\[
\subseteq U \cap T^+ \quad \text{(since } S^+ \subseteq U, \text{ by hypothesis)}\]

\[
\subseteq U \cap (S \cup T)^+ \quad \text{(by Lemma 2.1.1)}.
\]

Therefore,

\[
R^+ \subseteq U
\]

\[
\iff ((S^- \cup T^-) \setminus T^+) \setminus S^+ \subseteq U
\]

\[
\iff ((T^- \setminus T^+) \setminus S^-) \setminus S^+ \subseteq U \quad \text{(because } S^- \cap T^+ = \emptyset)\]

\[
\iff T^+ \cup S^- \subseteq U \cup S^+ \quad \text{(since } T^+ \cap S^- = \emptyset, \text{ by } S \perp T).\]

Hence, $T^+ \subseteq (U \cup S^+) \setminus S^- = V$ and

\[
V \cap T^+ \subseteq (U \cup S^+) \cap T^+ \subseteq (U \cap R^+) \cup (S^+ \cap T^+) = \emptyset.
\]

\[8\]
By Lemma 2.1.1, $T$ moves $V$ to $(V \cup T^+) \setminus T^-$. Moreover,
\[
S^- \cap T^+ = S^- \cap T^+ \quad \text{(since } S \perp T) \\
\subseteq U \cap R^+ \quad \text{(since } S^\perp \subseteq U \text{ by hypothesis)} \\
= \emptyset.
\]
Therefore,
\[
(V \cup T^+) \setminus T^- = ((V \cup S^+) \setminus S^-) \cup T^+ \setminus T^- \\
= (U \cup S^+ \cup T^+) \setminus (S^- \cup T^-) \quad \text{(since } S^- \cap T^+ = \emptyset) \\
= W.
\]
Hence, $T$ moves $V$ to $W$. \qed

The last properties (not in [19]) describe which elements are touched or left untouched by movement.

**Lemma 2.1.5.** For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if $S$ moves $U$ to $V$, then $S^\perp = U \setminus V$ and $S^\pm = V \setminus U$. In particular, if $T$ moves $U$ to $V$, then $S^\perp = T^\perp$ and $S^\pm = T^\pm$.

**Proof.** By the definition of movement, we have

$V = (U \cup S^+) \setminus S^-$ and $U = (V \cup S^-) \setminus S^+$

and therefore

$U \cap V = U \cap ((U \setminus S^-) \cup S^\pm) \\
= U \setminus S^\perp \quad \text{(since } U \cap S^+ = \emptyset).$

Similarly, $U \cap V = V \setminus S^\pm$. Hence, $S^\perp = U \setminus V$ and $S^\pm = V \setminus U$. \qed

**Lemma 2.1.6.** For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if $S$ moves $U$ to $V$, then

\[
U \setminus S^- = U \setminus S^\perp = U \cap V = V \setminus S^\pm = V \setminus S^+. 
\]

**Proof.**

\[
U \setminus S^- = U \setminus S^\perp \quad \text{(since } U \cap S^+ = \emptyset, \text{ by definition of movement)} \\
= U \cap V \quad \text{(by Lemma 2.1.5)} \\
= V \setminus S^\pm \\
= V \setminus S^+ \quad \text{(since } V \cap S^- = \emptyset, \text{ by definition of movement)} \quad \square
\]

**Lemma 2.1.7.** For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if $S$ moves $U$ to $V$, then

$U = (U \cap V) \cup S^\perp$ and $V = (U \cap V) \cup S^\pm$. 

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Proof. We have
\[ U = (V \cup S^-) \setminus S^+ \]
\[ = (V \setminus S^+) \cup (S^- \setminus S^+) \]
\[ = (U \cap V) \cup S^\top \] (by Lemma 2.1.6)
and
\[ (U \cap V) \cap S^\top \subseteq V \cap S^- \]
\[ = ((U \cup S^-) \setminus S^-) \cap S^- \]
\[ = \emptyset. \]
Hence, \( U = (U \cap V) \cup S^\top \). Similarly \( V = (U \cap V) \cup S^\pm \).

2.2 An adapted proof of Street’s Lemma 3.2

Here, we state and prove a property similar to [19, Lemma 3.2] which enables to build new cells from other cells. We adapt the proof to the new set of axioms and simplify it (notably, we remove the need for the notion of receptivity and the apparent circularity of the proof). In the following, we suppose given an \( \omega \)-hypergraph \( P \).

2.2.1 Gluings and activations. Let \( n \in \mathbb{N} \), \( X \) be an \( n \)-pre-cell of \( P \) and \( G \subseteq P_{n+1} \) be a finite subset. We say that \( G \) is glueable on \( X \) if \( G^\pm \subseteq X_n \). If so, we call gluing of \( X \) on \( G \) the \((n+1)\)-pre-cell \( Y \) of \( P \) such that
\[ Y_{n+1} = G, \]
\[ Y_{n,-} = X_n, \]
\[ Y_{n,+} = (X_n \cup G^+) \setminus G^-, \]
\[ Y_{i,\epsilon} = X_{i,\epsilon}. \]

We denote \( Y \) by \( \text{Glue}(X,G) \). Moreover, we call activation of \( G \) on \( X \) the \( n \)-pre-cell \( \partial_{n}^\epsilon(\text{Glue}(X,G)) \) and we denote it by \( \text{Act}(X,G) \). We say that \( G \) is dually glueable on \( X \) when \( G^\pm \subseteq X_n \) and we define the dual gluing \( \text{Glue}(X,G) \) and the dual activation \( \text{Act}(X,G) \) similarly.

For example, take the \( \omega \)-hypergraph \((??)\). Then \( \{A\} \) is glueable of \( X \) and \( \text{Glue}(X,\{A\}) = Y \) and \( \text{Act}(X,\{A\}) \) is the 2-pre-cell \( X' \) with
\[ X'_2 = \{\alpha_1, \alpha'_2, \alpha'_3, \alpha_4\}, \]
\[ X_{1,-} = \{a, b\}, \quad X_{1,+} = \{c, d', e\}, \]
\[ X_{0,-} = \{z\}, \quad X_{0,+} = \{z\}. \]

2.2.2 The gluing theorem. We prove the following theorem which is an adaptation of [19, Lemma 3.2].

Theorem 2.2.3. Suppose that \( P \) satisfies axioms (G0), (G1), (G2) and (G3). Given \( n \in \mathbb{N} \), an \( n \)-cell \( X \) of \( P \) and a finite fork-free set \( G \subseteq P_{n+1} \) such that \( G \) is glueable on \( X \), we have that
Figure 1: Cells involved and their movements in Theorem 2.2.3

(a) $\operatorname{Act}(X,G)$ is a cell, and $G^+ \cap X_n = \emptyset$,

(b) $\operatorname{Glue}(X,G)$ is a cell,

(c) for $G' \subseteq P_{n+1}$ finite, fork-free and dually glueable on $X$, $G'^- \cap G^+ = \emptyset$.

This theorem naturally admits a dual statement, when $G$ is dually glueable on $X$.

Proof. See Figure 1 for a representation of the cells in the statement of the theorem. The proof of this theorem (and its dual) is made with an induction on $n$. For a given $n$, there are four steps. Firstly, we show that (a) holds when $|G| = 1$. Secondly, we use the first step to show that (a) holds for all possible $G$. Thirdly, we prove that (b) holds. And fourthly, we prove that (c) holds. In the following, let $S$ be $\operatorname{Act}(X,G)_n = (X_n \cup G^+) \setminus G^-$.

Step 1: (a) holds when $|G| = 1$. Let $x \in P_{n+1}$ be such that $\{x\} = G$. If $n = 0$, then there exists $y \in P_0$ such that $X_0 = \{y\}$. By axioms (G1) and (G2), there exists $z \in P_0$ with $y \neq z$ such that $x^- = \{y\}$ and $x^+ = \{z\}$. So $\operatorname{Act}(X,G) = \{z\}$ is a cell. Otherwise, $n > 0$. Then, $S = (X_n \cup x^+) \setminus x^-$ and in order to prove that $\operatorname{Act}(X,G)$ is a cell, we need to show that

- $S$ moves $X_{n-1,-}$ to $X_{n-1,+}$;
- $S$ is fork-free.

Using the segment Axiom (G3), we get that $x^-$ is a segment in $X_n$ for $\preceq_{X_n}$. By elementary properties of partial orders, we can decompose $X_n$ as the partition

$$X_n = U \cup x^- \cup V$$

with $U$ initial and $V$ final for $\preceq_{X_n}$, which implies that $U^\uparrow \subseteq X_{n-1,-}$ and $V^\downarrow \subseteq X_{n-1,+}$. As subsets of the fork-free set $X_n$, $U$, $x^-$ and $V$ are fork-free and $U \perp x^-$, $x^- \perp V$, $U \perp V$. Using Lemma 2.1.4, we get $A, B \subseteq P_{n-1}$ such that
Figure 2: The decomposition of $X_n$

- $U$ moves $X_{n-1,-}$ to $A$,
- $x^-$ moves $A$ to $B$,
- $V$ moves $B$ to $X_{n-1,+}$

as pictured on Figure 2. In the following, for $Z \subseteq P_{n-1}$, we write $D(Z)$ for the $(n-1)$-pre-cell of $P$ defined by

$$D(Z)_{n-1} = Z,$$
$$D(Z)_{i,\epsilon} = X_{i,\epsilon} \quad \text{for} \quad i < n - 1 \quad \text{and} \quad \epsilon \in \{-, +\}.$$

Since $D(A) = \text{Act}(D(X_{n-1,-}), U)$, $D(B) = \text{Act}(D(A), x^-)$, $D(X_{n-1,-}) = \partial^- X$ is an $(n-1)$-cell and both $U$ and $x^-$ are fork-free, by using two times the induction hypothesis of Theorem 2.2.3, first on $D(X_{n-1,-})$, then on $D(A)$, we get that

$$D(A) \text{ and } D(B) \text{ are cells.} \quad (5)$$

By (G2), we have that

$$x^+ \text{ is fork-free.} \quad (6)$$

Since $x^-$ moves $A$ to $B$, by Lemma 2.1.1,

$$A \cap x^- = \emptyset. \quad (7)$$

By (G2), it holds that $x^+ \subseteq A$. By (5) and (6), using the induction hypothesis of Theorem 2.2.3 on $D(A)$, we get

$$A \cap x^+ = \emptyset. \quad (8)$$
By Lemma 2.1.1, there exists $B'$ such that $x^+$ moves $A$ to $B'$, and
\begin{align*}
B' &= (A \cup x^+) \setminus x^+ \\
&= (A \setminus x^+) \cup (x^+ \setminus x^+) \\
&= (A \setminus x^+) \cup x^+ \pm 
\quad \text{(by (8))}
&= (A \setminus x^+) \cup x^- \pm 
\quad \text{(since $x^+ \pm = x^- \mp$, by (G2))}
&= (A \cup x^-) \setminus x^-
\end{align*}

Hence,
\[x^+ \text{ moves } A \text{ to } B. \quad (9)\]

Since $x^+ \pm \subseteq D(A)_{n-1}$ and $U^\pm \subseteq D(A)_{n-1}$, using the induction hypothesis of Theorem 2.2.3, by (c) we get that
\[U^- \cap x^+ = \emptyset. \quad (10)\]

Similarly with $D(B)$, we get that
\[x^- \cap V^+ = \emptyset. \quad (11)\]

By definition, $U$ moves $X_{n-1,-}$ to $A$ and $x^+$ moves $A$ to $B$. Moreover, by (10), $U^- \cap x^+ = \emptyset$. Using Lemma 2.1.3, we deduce that
\[U \cup x^+ \text{ moves } X_{n-1,-} \text{ to } B. \quad (12)\]

Since $U$ and $V$ are disjoint and respectively initial and terminal for $\prec_{X_n}$, we have that $U^- \cap V^+ = \emptyset$. Also, by (11), we have $(x^- \cap V^+) = \emptyset$, therefore
\[(U \cup x^+)^- \cap V^+ \subseteq (U^- \cap V^+) \cup (x^- \cap V^+) = \emptyset.\]

Using (12) and Lemma 2.1.3, knowing that $S = U \cup x^+ \cup V$, we deduce that
\[S \text{ moves } X_{n-1,-} \text{ to } X_{n-1,+.} \quad (13)\]

The set $U \cup V$ is fork-free as a subset of the fork-free $X_n$, and $x^+$ is fork-free since $x$ is relevant by (G2). Moreover,
\begin{align*}
U^- \cap x^+ &= U^- \cap x^+ (\text{by (10)}) \\
&\subseteq U^- \cap A \quad \text{(by (9) and Lemma 2.1.1)} \\
&= \emptyset \quad \text{(since $U$ moves $X_{n-1,-}$ to $A$),}
\end{align*}

\begin{align*}
U^+ \cap x^+ &= U^+ \cap x^+ (\text{by (10)}) \\
&\subseteq A \cap x^+ \quad \text{(by Lemma 2.1.1 since $U$ moves $X_{n-1,-}$ to $A$)} \\
&= \emptyset \quad \text{(by (9) and Lemma 2.1.1).}
\end{align*}

So $U \perp x^+$. Similarly, $x^+ \perp V$. Hence, since $S = U \cup x^+ \cup V$,
\[S \text{ is fork-free.} \quad (14)\]
Then, by (13) and (14),

\[ \text{Act}(X, G) \] is a cell.

Finally, we prove the second part of (a). By Axiom (G1), \( x^- \cap x^+ = \emptyset \). Since \( U \perp x^+ \) and \( x^+ \perp V \) (by (14)), using (G0), we deduce that

\[ U \cap x^+ = x^+ \cap V = \emptyset \]

Hence,

\[ X_n \cap x^+ = (U \cup x^- \cup V) \cap x^+ = \emptyset \]

and it concludes the proof of the Step 1.

Step 2: (a) holds. We prove this by induction on \(|G|\). If \(|G| = 0\), then the result is trivial and the case \(|G| = 1\) was proved in Step 1. So suppose that \(|G| \geq 2\).

Since the relation \( \leq \) is acyclic by (G1), we can consider a minimal \( x \in G \) for \( \leq \).

Let \( \tilde{G} \) be \( G \setminus \{x\} \) and recall that we defined \( S \) as \( (X_n \cup G^+) \setminus G^- \). In order to show that \( \text{Act}(X, G) \) is a cell, we need to prove the following:

- \( S \) moves \( X_{n-1,-} \) to \( X_{n-1,+} \);
- \( S \) is fork-free.

For this purpose, we will first move \( X_n \) with \( \{x\} \) to \( U := (X_n \cup x^+) \setminus x^- \) and use Step 1, then move \( U \) by \( \tilde{G} \) to \( V := (U \cup \tilde{G}^+) \setminus \tilde{G}^- \) and use the induction of Step 2. Finally, we will prove that \( V = S \). So, using Step 1 with \( X \) and \( \{x\} \), we get that

- \( \text{Act}(X, \{x\}) \) is a cell;
- in particular, \( U \) is fork-free and, if \( n > 0 \), \( U \) moves \( X_{n-1,-} \) to \( X_{n-1,+} \);
- \( X_n \cap x^+ = \emptyset \).

By Lemma 2.1.1, we deduce that \( \{x\} \) moves \( X_n \) to \( U \). Moreover,

\[
\tilde{G}^- = \tilde{G}^+ \setminus \tilde{G}^+
= (G^- \setminus x^-) \setminus (G^+ \setminus x^+) \quad (\text{since fork-freeness implies that } G^\leq = \cup_{u \in G} u^e)
\subseteq ((G^- \setminus x^-) \setminus G^+) \cup x^+
= ((G^- \setminus G^+) \setminus x^-) \cup x^+
\subseteq (X_n \setminus x^-) \cup x^+ \quad (\text{since } G^\leq \subseteq X_n \text{ by Lemma 2.1.1})
\subseteq (X_n \cup x^+) \setminus x^+ \quad (\text{since } x^- \cap x^+ = \emptyset \text{ by (G1)})
= U.
\]

Also, \( \tilde{G} \) is fork-free as a subset of the fork-free set \( G \). Using the induction hypothesis of Step 2 for \( \tilde{G} \), we get that

- \( \text{Act}(\text{Act}(X, \{x\}), \tilde{G}) \) is a cell;
- In particular, \( V := (U \cup \tilde{G}^+) \setminus \tilde{G}^- \) is fork-free, and, if \( n > 0 \), \( V \) moves \( X_{n-1,-} \) to \( X_{n-1,+} \);
- \( U \cap \tilde{G}^+ = \emptyset \).
By Lemma 2.1.1, we deduce that $\bar{G}$ moves $U$ to $V$. Also note that $x^− \cap \bar{G}^+ = \emptyset$ since $x$ was taken minimal in $G$. Using Lemma 2.1.3, we deduce that $G = \{x\} \cup \bar{G}$ moves $X_n$ to $V$. But $S = (X_n \cup G^+) \setminus G^−$. So $S = V$.

One still needs to show the second part of (a), that is, $X_n \cap G^+ = \emptyset$:

$$X_n \cap G^+ = (U \cup x^- \setminus x^+) \cap G^+$$

(by Lemma 2.1.1, since $\{x\}$ moves $X_n$ to $U$)

$$= ((U \cup x^-) \cap G^+) \setminus x^+$$

(since $x^- \cap G = x^- \cap (x^+ \cup \bar{G}) = \emptyset$)

$$= (U \cap G^+) \setminus x^+$$

$$= U \cap G^+$$

which ends the proof of Step 2.

**Step 3**: (b) holds. By (a), $\text{Act}(X,G)$ is a cell. To conclude, we need to show that $G$ moves $X_n$ to $S$. By definition of $S$, we have that $S = (X_n \cup G^+) \setminus G^-$. Also:

$$(S \cup G^-) \setminus G^+ = (((X_n \cup G^+) \setminus G^-) \cup G^-) \setminus G^+$$

$$= (X_n \cup G^±) \setminus G^+$$

$$= X_n \cup G^±$$

(since $X_n \cap G^+ = \emptyset$ by (a))

$$= X_n$$

(since $G$ is glueable on $X$).

Hence, $\text{Glue}(X,G)$ is a cell.

**Step 4**: (c) holds. By contradiction, suppose that $G'^- \cap G^+ \neq \emptyset$. By definition, there are $x \in G'$, $y \in G$ and $z \in x^- \cap y^+$. Consider

$$U = \{x' \in G' \mid x \not\in x'\} \cup \{x\},$$

$$V = \{y' \in G \mid y' \not\in y\} \cup \{y\}.$$ 

By the acyclicity Axiom (G1), we have

$$U^+ \cap V^- = \emptyset.$$ 

Since $U$ is a terminal set for $\text{Act}(G')$, we have in particular $U^+ \cap G'^- \subseteq U^-$. So,

$$U^+ = (U^+ \setminus G'^-) \cup (U^+ \cap G'^-) \subseteq G'^+ \cup U^-.$$ 

Hence, $U^\pm \subseteq G'^\pm \subseteq X_n$ (since $G'$ is dually glueable on $X$). Similarly, $V^\mp \subseteq X_n$. Using the dual version of (a), $Y := \overline{\text{Act}}(X,U)$ is an $n$-cell with $Y_n = (X_n \cup U^-) \setminus U^+$ (see Figure 3) and we have

$$V^\mp = V^\mp \setminus U^+$$

(since $V^- \cap U^+ = \emptyset$)

$$\subseteq X_n \setminus U^+$$

(since $V^\mp \subseteq X_n$)

$$\subseteq (X_n \cup U^-) \setminus U^+$$

$$= Y_n.$$
Using Theorem 2.2.3(a) with $Y$ and $V$, we get
\[ Y_n \cap V^+ = \emptyset. \]
But, since $z \in U^\perp \subseteq Y_n$ (by (G1)) and $U^\perp \subseteq Y_n$, $z \in Y_n \cap V^+$, which is a contradiction. Hence,
\[ G'^- \cap G^+ = \emptyset, \]
which ends the proof of (c). $\square$

2.3 Cell($P$) is an $\omega$-category

Here, we finally prove that Cell($P$) has a structure of an $\omega$-category. In the following, we suppose given an $\omega$-hypergraph $P$ which satisfies (G0), (G1), (G2) and (G3).

**Lemma 2.3.1.** Let $n > 0$ and $X,Y$ be two $n$-cells of $P$ that are $(n-1)$-composable. Then,

(a) $X_n^- \cap Y_n^+ = \emptyset$,

(b) $X_n \cap Y_n = \emptyset$,

(c) $X *_{n-1} Y$ is an $n$-cell of $P$.

**Proof.** Using Theorem 2.2.3(c) with $\partial^+ X$, $X_n$ and $Y_n$, we get
\[ X_n^- \cap Y_n^+ = \emptyset. \]
Moreover,
\[ X_n^+ \cap Y_n^+ = X_n^+ \cap Y_n^+ \quad \text{(since $X_n^- \cap Y_n^+ = \emptyset$)} \]
\[ \subseteq X_{n-1,+} \cap Y_n^+ \]
\[ = Y_{n-1,-} \cap Y_n^+ \]
\[ = \emptyset \quad \text{(by Theorem 2.2.3(a))}. \]
By (G0), it implies that $X_n \cap Y_n = \emptyset$. Similarly,
\[ X_n^- \cap Y_n^- = \emptyset. \]
So $X_n \cup Y_n$ is fork-free. For $X \ast_{n-1} Y$ to be a cell, $X_n \cup Y_n$ must move $X_{n-1} \ast_{n-1}$ to $Y_{n-1} \cup_{n-1}$. But, since $X$ and $Y$ are cells and are $(n-1)$-composable, we know that $X_n$ moves $X_{n-1} \ast_{n-1}$ to $X_{n-1} \ast_{n-1}$, $Y_n$ moves $Y_{n-1} \ast_{n-1}$ to $Y_{n-1} \cup_{n-1}$, and $X_{n-1} \ast_{n-1} = Y_{n-1} \cup_{n-1}$. Since $X_n \cup Y_n$, using Lemma 2.1.3, we get that $X_n \cup Y_n$ moves $X_{n-1} \ast_{n-1}$ to $Y_{n-1} \cup_{n-1}$. Hence, $X \ast_{n-1} Y$ is a cell. 

Lemma 2.3.2. Let $0 \leq i < n$ and $X, Y$ be two $n$-cells of $P$ that are i-composable. Then,

(a) for $i < j \leq n$, $(X_j^- \cup X_{j+1}^+) \cap (Y_j^+ \cup Y_{j+1}^+) = \emptyset$,

(b) $X \ast_i Y$ is a cell.

Proof. By induction on $n - i$. If $n - i = 1$, the properties follow from Lemma 2.3.1. So suppose that $n - i > 1$. For $\epsilon, \eta \in \{-, +\}$, by induction with $\partial^\epsilon X$ and $\partial^\eta Y$, we get that

$$X_{n-1, \epsilon}^- \cap Y_{n-1, \eta}^+ = \emptyset.$$

Therefore,

$$(X_{n-1, -}^- \cup X_{n-1, +}^-) \cap (Y_{n-1, -}^+ \cup Y_{n-1, +}^+) = \emptyset.$$

We also get that

$$(X_j^- \cup X_{j+1}^-) \cap (Y_j^+ \cup Y_{j+1}^+) = \emptyset \quad \text{for } i < j < n - 1.$$

Let $Z = \partial^+ X \ast_i \partial^- Y$. Then, by induction, $Z$ is a $(n-1)$-cell and $Z_{n-1} = X_{n-1, +} \cup Y_{n-1, -}$. Using Theorem 2.2.3(c), we get

$$X_n^- \cap Y_n^+ = \emptyset$$

which concludes the proof of (a).

For (b), we already know that $\partial^- X \ast_i \partial^- Y$ and $\partial^+ X \ast_i \partial^+ Y$ are cells by induction. So, in order to prove that $X \ast_i Y$ is a cell, we just need to show that $X_n \cup Y_n$ is fork-free and moves $X_{n-1} \cup Y_{n-1}$ to $X_{n-1, +} \cup Y_{n-1, +}$. But

$$X_n^+ \cap Y_n^+ = X_n^+ \cap Y_n^+$$

(by (a))

$$\subseteq Z_{n-1} \cap Y_n^+$$

$$= \emptyset$$

(by Theorem 2.2.3(a)).

Similarly,

$$X_n^- \cap Y_n^- = \emptyset$$

so $X_n \cup Y_n$ is fork-free. Using the dual of Theorem 2.2.3(a) with $Z$ and $X_n$, we get that

$$X_n^- \cap (X_{n-1, +} \cup Y_{n-1, -}) = X_n^- \cap Y_{n-1, -} = \emptyset.$$

Similarly, if $Z' = \partial^+ X \ast_i \partial^- Y$ then $Z'_{n-1} = X_{n-1, -} \cup Y_{n-1, -}$. Using Theorem 2.2.3(a) with $Z'$ and $X_n$, we get that

$$X_n^+ \cap (X_{n-1, -} \cup Y_{n-1, -}) = X_n^+ \cap Y_{n-1, -} = \emptyset.$$
Hence, \( X \) that set generated by \( \text{Cell}(Z) \) so suppose given \( n \) manipulate the cells of an \( \omega \)-category. We first define the freeness notion we are using and give some tools to prove completeness for \( \text{Cell}(P) \). In this section, we give a complete proof of freeness for \( \text{Cell}(P) \).

**Theorem 2.3.3.** \((\text{Cell}(P), \partial^-, \partial^+, \ast, \text{id})\) is an \( \omega \)-category.

**Proof.** We already know that \((\text{Cell}(P), \partial^-, \partial^+)\) is a globular set. By Lemma 2.3.2, the composition operation \( \ast \) is well-defined on composable cells. Moreover, all the axioms of \( \omega \)-categories (given in Paragraph ??), follow readily from the definitions of \( \partial^-, \partial^+, \ast, \text{id} \). For example, consider the exchange law ??.

Given \( j < i \leq n \), \( X, X', Y, Y' \in \text{Cell}(P)_n \) such that \( X, Y \) are \( i \)-composable, \( X', Y' \) are \( j \)-composable, let

\[
Z = (X \ast_i Y') \ast_j (Y \ast_i Y') \quad \text{and} \quad Z' = (X \ast_j Y) \ast_i (X' \ast_j Y').
\]

For \( k \leq n \) and \( \epsilon \in \{-, +\} \), we have

\[
Z_{k,\epsilon} = Z'_{k,\epsilon} = \begin{cases} 
X_{k,\epsilon} \cup Y_{k,\epsilon} \cup X'_{k,\epsilon} \cup Y'_{k,\epsilon} & \text{when } k > i, \\
X_{i,-} \cup X'_{i,-} & \text{when } k = i \text{ and } \epsilon = -, \\
Y_{i,+} \cup Y'_{i,+} & \text{when } k = i \text{ and } \epsilon = +, \\
X_{k,\epsilon} \cup X'_{k,\epsilon} & \text{when } j < k < i, \\
X_{j,-} & \text{when } k = j \text{ and } \epsilon = -, \\
X'_{j,+} & \text{when } k = j \text{ and } \epsilon = +, \\
X_{k,\epsilon} & \text{when } k < j,
\end{cases}
\]

so \( Z = Z' \). Thus, \( \text{Cell}(P) \) satisfies axiom ?? and the others as well. Hence, 
\((\text{Cell}(P), \partial^-, \partial^+, \ast, \text{id})\) is an \( \omega \)-category. \( \square \)

### 3 The freeness property

In this section, we give a complete proof of freeness for \( \omega \)-categories. We first define the freeness notion we are using and give some tools to manipulate the cells of an \( \omega \)-category.

#### 3.1 Generating sets

Suppose given \( n \in \mathbb{N} \cup \{\omega\} \), \( \mathcal{C} \) an \( n \)-category and \( S \) a subset of \( \bigsqcup_{0 \leq i < n+1} \mathcal{C}_i \). The *set generated by \( S \) in \( \mathcal{C} \)*, denoted by \( S^* \), is the smallest subset \( T \subseteq \mathcal{C} \) such that

- \( S \subseteq T \),
- if \( X, Y \in T \cap \mathcal{C}_i \) and \( \partial_j^+ X = \partial_j^- Y \) for some \( j < i \), then \( X \ast_j Y \in T \).
Conversely, suppose that for all \( n \) the \( \text{id}_{i+1}(X) \in T \), and in this case, we say that \( S \) \emph{generates} \( T \) in \( C \). Note that if \( S \) generates \( C \), every cell of \( C \) can be written by an expression involving only cells in \( S \), compositions and identities.

For example, consider the following \( \omega \)-hypergraph:

\[
\begin{array}{c}
\circ \quad f \\
\downarrow \quad A \\
\circ \quad B \\
\downarrow \quad g \\
\circ \quad y \\
\downarrow \quad h \\
\end{array}
\]

Then, the set \( S_1 = \{ A, B \} \) is \emph{not} generating, whereas the set \( S_2 = \{ x, y, f, g, h, A, B \} \) is generating.

The notion of generating set for an \( \omega \)-category can be reduced to the notion of generating set for an \( n \)-category, with \( n \in \mathbb{N} \):

**Lemma 3.1.1.** Let \( C \) be an \( \omega \)-category and \( S \subseteq \bigcup_{n \geq 0} C_n \). Then \( S \) generates \( C \) in \( C \) if and only if for all \( n \geq 0 \), \( S_{\leq 0} \) generates \( C_{\leq n} \) in \( C_{\leq n} \).

**Proof.** Let \( n \geq 0 \). Since \( S_{\leq n} \subseteq (S^*)_{\leq n} \), we have \( (S_{\leq n})^* \subseteq (S^*)_{\leq n} \). For the other side, since \( S \subseteq (S_{\leq 0})^* \cup \bigcup_{n \geq 1} C_n \), we have \( S^* \subseteq (S_{\leq n})^* \cup \bigcup_{n \geq 1} C_n \), so \( (S^*)_{\leq n} \subseteq (S_{\leq n})^* \). Hence, \( (S_{\leq n})^* = (S^*)_{\leq n} \).

Suppose that \( S \) is generating \( C \). Then, for all \( n \geq 0 \), \( (S_{\leq n})^* = (S^*)_{\leq n} = C_{\leq n} \). Conversely, suppose that for all \( n \geq 0 \), \( (S_{\leq n})^* = C_{\leq n} \). For all \( n \), we have \( C_{\leq n} = (S_{\leq n})^* = (S^*)_{\leq n} \subseteq S^* \). Hence, \( C = \bigcup_{n \geq 0} C_{\leq n} = S^* \).

### 3.2 Atoms are generating

Here, we show that the atoms are generating by adapting the results and proofs of [19].

In this subsection, we suppose given an \( \omega \)-hypergraph \( P \). We define the rank of an \( n \)-cell \( X \) of \( P \) as the \( n \)-tuple

\[
\text{Rank}(X) = ([X_{1,-} \cap X_{1,+}], \ldots, [X_{n-1,-} \cap X_{n-1,+}], [X_n]).
\]

We order the ranks using a lexicographic ordering \( <_{\text{lex}} \): if \( (a_1, \ldots, a_n) \) and \( (b_1, \ldots, b_n) \) are two \( n \)-tuples and there exists \( 1 \leq k \leq n \) such that \( a_i = b_i \) for \( i > k \) and \( a_k < b_k \), then \( (a_1, \ldots, a_n) <_{\text{lex}} (b_1, \ldots, b_n) \). Note \( <_{\text{lex}} \) is well-founded.

For example, in the \( \omega \)-hypergraph (??), the 2-cell

\[
X = (\{ t \}, \{ z \}, \{ a, b, c, d, e, f \}, \{ a, b', c', d', e', f \}, \{ a, \beta, \gamma, \delta \})
\]

has rank

\[
\text{Rank}(X) = ([\{ a, f \}], [\{ a, \beta, \gamma, \delta \}]) = (2, 4).
\]

**Theorem 3.2.1** (Excision of extremals). Suppose that \( P \) satisfies \((G0)\), \((G1)\), \((G2)\) and \((G3)\). Let \( n \in \mathbb{N} \), \( X \) be an \( n \)-cell of \( P \) and \( u \in X_n \) such that \( X \not= \langle u \rangle \). Then there exist \( i < n \) and \( n \)-cells \( Y, Z \) such that \( \text{Rank}(Y) <_{\text{lex}} \text{Rank}(X) \), \( \text{Rank}(Z) <_{\text{lex}} \text{Rank}(X) \) and \( X = Y \ast_* Z \).
Proof. Since $X \neq (u)$, there is a least $i \geq -1$ such that $i < n$ and

$$(X_{j,-}, X_{j,+}) = ((u)_{j,-}, (u)_{j,+}) \quad \text{for } i + 1 < j \leq n.$$ 

In fact, $i \geq 0$. If $i = -1$, then $X_{1,-} = (u)_{1,-}$ and, since $X$ and $(u)$ are cells, $X_{0,-} = X_{0,-} = (u)_{0,-}$ and similarly $X_{0,+} = (u)_{0,+}$, contradicting $X \neq (u)$. If $i < n - 1$, since $X$ is a cell, $X_{i+2,\epsilon} = (u)_{i+2,\epsilon}$ moves $X_{i+1,-}$ to $X_{i+1,+}$, and, by Lemma 2.1.7, we have

$$X_{i+1,\epsilon} = (u)_{i+1,\epsilon} \cup (X_{i+1,-} \cap X_{i+1,+}) \quad \text{for } \epsilon \in \{-, +\}.$$ 

This equality is still valid when $i = n - 1$ since $(u)_{i+1,\epsilon} = \{x\} \subseteq X_n$. By definition of $i$, there exists $w \in (X_{i+1,-} \cap X_{i+1,+}) \setminus (u)_{i+1,\epsilon}$. Let $x$ be minimal in $X_{i+1,-}$ such that $x < X_{i+1,-} w$ or $x = w$ and let $y$ be maximal for $x < X_{i+1,-}$ such that $w < X_{i+1,-} y$ or $w = y$. By Axiom (G3), either $x \notin (u)_{i+1,-}$ or $y \notin (u)_{i+1,-}$. By symmetry, we can suppose that $x \notin (u)_{i+1,-}$. By minimality, we have $x^\perp \subseteq X_{i,-}$. Since $\partial^-_i X$ is a cell, by Theorem 2.2.3, $Y := \text{id}_n(\text{Glue}(\partial^-_i X, \{x\}))$ is an $n$-cell with

$$Y_{j,\epsilon} = \emptyset \quad \text{for } i + 1 < j \leq n \text{ and } \epsilon \in \{-, +\}$$

$$Y_{i+1,\epsilon} = \{x\} \quad \text{for } \epsilon \in \{-, +\}$$

$$Y_{i,-} = X_{i,-}$$

$$Y_{i,+} = (X_{i,-} \cup x^+) \setminus x^-$$

$$Y_{j,\epsilon} = X_{j,\epsilon} \quad \text{for } j < i \text{ and } \epsilon \in \{-, +\}.$$ 

For $\epsilon \in \{-, +\}$, since $X_{i+1,-} = \{x\} \cup (X_{i+1,\epsilon} \setminus \{x\})$ is fork-free and moves $X_{i,-}$ to $X_{i,+}$, by Lemma 2.1.4, we have that

$$\{x\} \text{ moves } X_{i,-} \text{ to } (X_{i,-} \cup x^+) \setminus x^-$$

and

$$X_{i+1,\epsilon} \setminus \{x\} \text{ moves } (X_{i,-} \cup x^+) \setminus x^- \text{ to } X_{i,+}.$$ 

If $i + 2 \leq n$, for $\epsilon \in \{-, +\}$, since $X_{i+2,\epsilon} = (u)_{i+2,\epsilon}$ moves $X_{i+1,-}$ to $X_{i+1,+}$ and $X_{i+2,\epsilon} \cap \{x\} = \emptyset$, using Lemma 2.1.2, we have that

$$X_{i+2,\epsilon} \text{ moves } X_{i+1,-} \setminus \{x\} \text{ to } X_{i+1,+} \setminus \{x\}.$$ 

So the following $n$-pre-cell $Z$ is a cell:

$$Z_{j,\epsilon} = X_{j,\epsilon} \quad \text{for } i + 1 < j \leq n \text{ and } \epsilon \in \{-, +\}$$

$$Z_{i+1,\epsilon} = X_{i+1,\epsilon} \setminus \{x\} \quad \text{for } \epsilon \in \{-, +\}$$

$$Z_{i,-} = (X_{i,-} \cup x^+) \setminus x^-$$

$$Z_{i,+} = X_{i,+}$$

$$Z_{j,\epsilon} = X_{j,\epsilon} \quad \text{for } j < i \text{ and } \epsilon \in \{-, +\}.$$ 

One readily checks that that $\text{Rank}(Y) <_{\text{lex}} \text{Rank}(X)$, $\text{Rank}(Z) <_{\text{lex}} \text{Rank}(X)$ and $X = Y \ast_i Z$. 

\[ \Box \]

**Theorem 3.2.2.** Suppose that $P$ satisfies (G0), (G1), (G2) and (G3). Let $S = \{\langle x \rangle \mid x \in P\}$. Then $S$ is generating $\text{Cell}(P)$. 

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Proof. Let \( T \) be the generated set by \( S \) in \( \text{Cell}(P) \). We show that \( T = \text{Cell}(P) \) by induction on the dimension of the cells. By the constraints on cells, the 0-cells of \( P \) are necessarily atoms. For \( n > 0 \), we do a second induction on the rank of the \( n \)-cell. So let \( X \) be an \( n \)-cell of \( P \) such that all \( n \)-cells with lower ranks are in \( T \). If \( X = \emptyset \), then \( X = \text{id}_n(X') \) for some \( X' \). By induction, \( X' \in T \), so \( X \in T \). Otherwise, if \( X \neq \emptyset \), then either \( X \) is an atom, in which case \( X \in T \), or by the excision Theorem 3.2.1, there are \( i < n \) and \( Y, Z \) \( n \)-cells of \( P \) with lower ranks than \( X \) such that \( X = Y * Z \). By induction, \( Y, Z \in T \) so \( X \in T \). So \( \text{Cell}(P)_n \subseteq T \) for all \( n \). Hence, \( T = \text{Cell}(P) \).

### 3.3 Properties of the exchange law

Here, we show some useful properties related to the exchange law.

In this subsection, we suppose given an \( m \)-category \( C \) with \( m \in \mathbb{N} \cup \{ \omega \} \). The first lemma states that low level compositions commute with sequences of high level compositions.

**Lemma 3.3.1.** Let \( j < i \leq n < m + 1 \), \( p \geq 0 \) and \( x_1, \ldots, x_p, y_1, \ldots, y_p \) \( n \)-cells of \( C \) such that \((x_1, \ldots, x_p)\) and \((y_1, \ldots, y_p)\) are \( i \)-composable, and \( \partial_j^+ x_k = \partial_j^- y_k \) for \( 1 \leq k \leq p \). Then

\[
(x_1 \ast_i \cdots \ast_i x_p) \ast_j (y_1 \ast_i \cdots \ast_i y_p) = (x_1 \ast_j y_1) \ast_i \cdots \ast_i (x_p \ast_j y_p).
\]

**Proof.** We do an induction on \( p \). If \( p = 1 \), the result is trivial. So suppose \( p > 1 \). Then

\[
(x_1 \ast_i \cdots \ast_i x_p) \ast_j (y_1 \ast_i \cdots \ast_i y_p)
= ((x_1 \ast_i \cdots \ast_i x_{p-1}) \ast_i x_p) \ast_j ((y_1 \ast_i \cdots \ast_i y_{p-1}) \ast_i y_p)
= ((x_1 \ast_i \cdots \ast_i x_{p-1}) \ast_j (y_1 \ast_i \cdots \ast_i y_{p-1})) \ast_i (x_p \ast_j y_p)
= ((x_1 \ast_j y_1) \ast_i \cdots \ast_i (x_{p-1} \ast_j y_{p-1})) \ast_i (x_p \ast_j y_p)
\]

(by induction)

The next lemma states that compositions with identities distribute over sequences of compositions.

**Lemma 3.3.2.** Let \( j < i \leq n < m + 1 \), \( p \geq 0 \) and \((x_1, \ldots, x_p)\) \( n \)-cells of \( C \) that are \( i \)-composable and \( y \) an \( i \)-cell of \( C \).

- If \( \partial_j^+ x_k = \partial_j^- y \) for \( 1 \leq k \leq p \), then
  \[
  (x_1 \ast_i \cdots \ast_i x_p) \ast_j \text{id}_n(y) = (x_1 \ast_j \text{id}_n(y)) \ast_i \cdots \ast_i (x_p \ast_j \text{id}_n(y)).
  \]

- If \( \partial_j^- x_k = \partial_j^+ y \) for \( 1 \leq k \leq p \), then
  \[
  \text{id}_n(y) \ast_j (x_1 \ast_i \cdots \ast_i x_p) = (\text{id}_n(y) \ast_j x_1) \ast_i \cdots \ast_i (\text{id}_n(y) \ast_j x_p).
  \]

**Proof.** Note that \( \text{id}_n(y) = \underbrace{\text{id}_n(y) \ast_i \cdots \ast_i \text{id}_n(y)}_p \). Using Lemma 3.3.1, we conclude the proof.

The next lemma states that composing two sequences of cells in low dimension is equivalent to composing the first one and then the second one in high dimension.
Lemma 3.3.3. Let \( j < i \leq n < m + 1 \), \( p, q \geq 0 \) and \( n \)-cells \( x_1, \ldots, x_p \) and \( y_1, \ldots, y_q \) such that \( (x_1, \ldots, x_p) \) and \( (y_1, \ldots, y_q) \) are \( i \)-composable and \( \partial^+_j x_k = \partial^-_j y_{k'} \) for \( 1 \leq k \leq p \) and \( 1 \leq k' \leq q \). Then,

\[
(x_1 * \cdots * x_p) * j (y_1 * \cdots * y_q) \\
= (x_1 * j \operatorname{id}_n(\partial^-_i y_1)) * \cdots * (x_p * j \operatorname{id}_n(\partial^-_i y_1)) \\
\times (\operatorname{id}_n(\partial^+_i x_p) * j y_1) * \cdots * (\operatorname{id}_n(\partial^+_i x_p) * j y_q) \\
= (\operatorname{id}_n(\partial^-_i x_1) * j y_1) * \cdots * (\operatorname{id}_n(\partial^-_i x_1) * j y_q) \\
\times (x_1 * j \operatorname{id}_n(\partial^+_i y_q)) * \cdots * (x_p * j \operatorname{id}_n(\partial^+_i y_q)).
\]

Proof. We have that

\[
(x_1 * \cdots * x_p) = x_1 * \cdots * x_p \underbrace{\operatorname{id}_n(\partial^+_i x_p)}_{q} * \cdots * \underbrace{\operatorname{id}_n(\partial^+_i x_p)}_{q}
\]

and

\[
(y_1 * \cdots * y_q) = \underbrace{\operatorname{id}_n(\partial^-_i y_1)}_{p} * \cdots * \operatorname{id}_n(\partial^-_i y_1) * y_1 * \cdots * y_p.
\]

So, using Lemma 3.3.1, we get that

\[
(x_1 * \cdots * x_p) * j (y_1 * \cdots * y_q) \\
= (x_1 * j \operatorname{id}_n(\partial^-_i y_1)) * \cdots * (x_p * j \operatorname{id}_n(\partial^-_i y_1)) \\
\times (\operatorname{id}_n(\partial^+_i x_p) * j y_1) * \cdots * (\operatorname{id}_n(\partial^+_i x_p) * j y_q).
\]

Similarly,

\[
(x_1 * \cdots * x_p) * j (y_1 * \cdots * y_q) \\
= (\operatorname{id}_n(\partial^-_i x_1) * j y_1) * \cdots * (\operatorname{id}_n(\partial^-_i x_1) * j y_q) \\
\times (x_1 * j \operatorname{id}_n(\partial^+_i y_q)) * \cdots * (x_p * j \operatorname{id}_n(\partial^+_i y_q)).
\]

The next lemma is a special case of Lemma 3.3.3.

Lemma 3.3.4. Let \( j < i \leq n < m + 1 \) and \( n \)-cells \( x, y \) such that \( \partial^+_j x = \partial^-_j y \). Then,

\[
x * j y = (x * j \operatorname{id}_n(\partial^-_i y)) * i (\operatorname{id}_n(\partial^+_i x) * j y) \\
= (\operatorname{id}_n(\partial^-_i x) * j y) * i (x * j \operatorname{id}_n(\partial^+_i y)).
\]

Proof. By Lemma 3.3.3.

3.4 Contexts

Here, we define contexts, a tool to manipulate the cells of an \( \omega \)-category. They can be thought as a composite of cells with one “hole” that can be filled by another cell. Using contexts, we are able to define a canonical form for cells.

In the following, we suppose given an \( m \)-category \( C \) with \( m \in \mathbb{N} \cup \{\omega\} \).
3.4.1 Definition. Given \( n < m + 1 \), two \( n \)-cells \( y, z \in C \) are said parallel when \( n = 0 \) or \( \partial^y \equiv \partial^z \) and \( \partial^+ y = \partial^+ z \). Given \( p \) with \( n \leq p < m + 1 \) and a \( p \)-cell \( x \), we say that \( x \) is adapted to \((y, z)\) when \( \partial^y x = y \) and \( \partial^+_n x = z \). Given \( x, y, z \) as above, we define an \( n \)-context \( E \) of type \((y, z)\) and the evaluation \( E[x] \) of \( E \) on \( x \) by induction on the dimension \( n \) of \( y \) and \( z \) as follows.

- For \( 0 \)-cells \( y \) and \( z \), there is a unique context of type \((y, z)\), noted \([-\).
  
- Given a \( p \)-cell \( x \) as above, the induced cell is \([x] = x\).

Given parallel \( n \)-cells \( x \) and \( y \), a context of type \((y, z)\) is given by a context \( E \) of type \((\partial^- y, \partial^+ z)\), together with a pair of \((n+1)\)-cells \( x_{n+1} \) and \( x'_{n+1} \) such that

\[
\partial^+(x_{n+1}) = \tilde{E}[\partial^- y] \quad \text{and} \quad \partial^-(x'_{n+1}) = \tilde{E}[\partial^+ z].
\]

Given a \( p \)-cell \( x \) as above, the evaluation of \( E \) on \( x \) is

\[
E[x] = x_{n+1} *_n \tilde{E}[x] *_p x'_{n+1}
\]

so that we sometimes write

\[
E = x_{n+1} *_n \tilde{E} *_n x'_{n+1}
\]

for the context \( E \) itself.

Given parallel \( n \)-cells \( y \) and \( z \), a context of type \((y, z)\) thus consists of pairs of suitably typed \( k \)-cells \((x_k, x'_k)\), for \( 0 \leq k \leq n \), and is of the form

\[
E = x_n *_{n-1} (\cdots *_1 (x_1 *_0 [-] *_0 x'_1) \cdots) *_{n-1} x'_p
\]

We write \( \pi^- k E = x_k \) and \( \pi^+ k E = x'_k \) for the \( k \)-cells of the context \( E \). Moreover, an \( m \)-context \( E \) will be said adapted to \( x \) when its type is \((\partial^-_m x, \partial^+_m x)\).

3.4.2 Canonical forms. Here, we show that contexts can be used to give a canonical form to the cells of an \( \omega \)-category. First, we prove that the composition of an evaluated context with a cell results in an evaluated context.

**Lemma 3.4.3.** Let \( n \leq p < m + 1 \), a \( p \)-cell \( x \) and an \( n \)-context \( E \) adapted to \( x \). Given \( i \leq q \leq n \) and a \( q \)-cell \( y \) such that \( \partial^+_i y = \partial^+_i E[x] \), there exists an \( n \)-context \( E' \) adapted to \( x \) such that \( E[x] *_i y = E'[x] \).

**Proof.** Let \( z = E[x] *_i y \). Then

\[
z = (\pi^- n E *_{n-1} (\cdots (\pi^- 1 E *_0 x *_0 \pi^+_1 E) \cdots) *_{n-2} \pi^-_{n-1} E) *_{n-1} \pi^+_n E) *_i y.
\]

We prove this lemma by an induction on \((n, q)\) (ordered lexicographically). If \( i = n \), then, since \( y \) is of dimension \( q \leq n \),

\[
z = E[x].
\]

Otherwise, if \( i = n - 1 \), then

\[
z = (\pi^- n E) *_{n-1} (\cdots (\pi^- 1 E *_0 x *_0 \pi^+_1 E) \cdots) *_{n-2} (\pi^-_{n-1} E) *_{n-1} (\pi^+_n E *_{n-1} y)
\]

which is of the form \( E'[x] \). Otherwise, \( i < n - 1 \). Then,
- if \( q < n \), using Lemma 3.3.2, we get
  \[
  z = (\pi_n^{-} E \ast_1 y) \ast_{n-1} (E[x] \ast_1 y) \ast_{n-1} (\pi_n^{+} E \ast_1 y)
  \]
  By induction hypothesis on the middle part, we get an \((n-1)\)-context \(E'\) such that
  \[
  z = (\pi_n^{-} E \ast_1 y) \ast_{n-1} E'[x] \ast_{n-1} (\pi_n^{+} E \ast_1 y).
  \]
  From this, one easily deduces an \(n\)-context \(E''\) such that \(z = E''[x]\).

- Otherwise, if \( q = n \), using Lemma 3.3.3, we get
  \[
  z = (E[x] \ast_1 \partial_{n-1}^{-} y) \ast_{n-1} (\partial_{n-1}^{+} E[x] \ast_1 y).
  \]
  Using the induction hypothesis on the left-hand side, we get an \(n\)-context \(E'\) such that
  \[
  z = E'[x] \ast_{n-1} (\partial_{n-1}^{+} E[x] \ast_1 y).
  \]
  Thus,
  \[
  z = (\pi_n^{-} E' \ast_{n-1} (\cdots \ast_{n-1} \pi_n^{+} E')) \ast_{n-1} (\partial_{n-1}^{+} (\pi_n^{+} E) \ast_1 y)
  \]
  which is the evaluation of the form on \(x\).

Contexts can be used to obtain the canonical form for the cells of an \(\omega\)-category, as in the following lemma.

**Lemma 3.4.4.** Suppose that \( P \subseteq C \) generates \( C \). Then, for \( n \geq 0 \), every \(n\)-cell \(x\) of \(C\) can be written in one of the following form:

- \( \text{id}_n(x') \) with \( x' \) an \((n-1)\)-cell.

- \( E_1[x_1] \ast_{n-1} E_2[x_2] \ast_{n-1} \cdots \ast_{n-1} E_p[x_p] \) with \( p > 0 \), \( x_1, \ldots, x_p \in P_n \) and \( E_1, \ldots, E_p \) \((n-1)\)-contexts.

**Proof.** Let \( x \) be an \(n\)-cell of \(C\). Since \( P \) is generating, there is a formal expression involving only compositions, identities and elements of \(P\) that is equal to \(x\). We make an induction on the structure of such an expression.

If \( x \in P \), then

\[
  x = \partial_{n-1}^{-}(x) \ast_{n-1} (\cdots (\partial_{0}^{-} x \ast_0 x \ast_0 \partial_{0}^{-} x) \cdots) \ast_{n-1} \partial_{n-1}^{+}(x)
\]

which has the form of an evaluated context. Otherwise, if \( x = \text{id}_n(x') \) with \( x' \) of dimension \((n-1)\), which is one of the desired forms. Otherwise, \( x = y \ast_1 z \).

By induction, we can write \( y \) and \( z \) in one of the desired form. There are four cases:

- \( y = \text{id}_n(y') \) and \( z = \text{id}_n(z') \), with \( y', z' \) \((n-1)\)-cells. Then \( x = \text{id}_n(y' \ast_1 z') \) which is a desired form.

- \( y = E_1[y_1] \ast_{n-1} \cdots \ast_{n-1} E_p[y_p] \) and \( z = \text{id}_n(z') \). If \( i = n-1 \), then \( y \ast_i z = y \) and it is a desired form. Otherwise, \( i < n-1 \) and, using Lemma 3.3.2,

  \[
  y \ast_i z = (E_1[y_1] \ast_i z) \ast_{n-1} \cdots \ast_{n-1} (E_p[y_p] \ast_i z).
  \]

By applying Lemma 3.4.3 to all \( E_k[y_k] \ast_i z \) for \( 1 \leq k \leq p \), we get the desired form.
- \( y = \text{id}_n(y') \) and \( z = E_i[z_1] *_{n-1} \cdots *_{n-1} E_p[z_p] \). This case is similar to the previous one.

- \( y = E_i[y_1] *_{n-1} \cdots *_{n-1} E_p[y_p] \) and \( z = E_i'[z_1] *_{n-1} \cdots *_{n-1} E'_q[z_q] \). If \( i = n - 1 \), we can concatenate the two decompositions directly. Otherwise \( i < n - 1 \) and, by Lemma 3.3.4,

\[
y *_i z = (y *_i \partial^-_{n-1}(z)) *_{n-1} (\partial^+_{n-1}(y) *_{n-1} z).
\]

Using Lemma 3.3.2,

\[
y *_i z = (E_i[y_1] *_i \partial^-_{n-1}(z)) *_{n-1} \cdots *_{n-1} (E_p[y_p] *_i \partial^-_{n-1}(z)) *_{n-1} (\partial^+_{n-1}(y) *_i E'_q[z_q]).
\]

By applying Lemma 3.4.3 to all \( E_k[y_k] *_i \partial^-_{n-1}(z) \) for \( 1 \leq k \leq p \), and to all \( \partial^+_{n-1}(y) *_i E'_q[z_k] \) for \( 1 \leq k \leq q \), we get a desired form. \( \square \)

3.5 Free \( \omega \)-categories

Here, we define briefly the notion of freeness we use for the \( \omega \)-category of cells. We refer to [12, 2] for a more complete presentation.

3.5.1 Cellular extensions. An \( n \)-cellular extension

\[
\mathcal{C} \xleftarrow{s} \xrightarrow{t} S
\]

is given by an \( n \)-category \( \mathcal{C} \), a set \( S \) and functions \( s,t : S \to C_n \) such that, if \( n > 0 \), \( \partial^- \circ s = \partial^- \circ t \) and \( \partial^+ \circ s = \partial^+ \circ t \). When there is no ambiguity, we denote by \( (\mathcal{C}, S) \) such an extension. A morphism \((F,f)\) between two \( n \)-cellular extensions

\[
\mathcal{C} \xleftarrow{s} \xrightarrow{t} S \quad \text{and} \quad \mathcal{D} \xleftarrow{s'} \xrightarrow{t'} T
\]

is given by an \( n \)-functor \( F : \mathcal{C} \to \mathcal{D} \) and a function \( f : S \to T \) such that the following diagrams commute:

\[
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow s & & \downarrow t' \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\quad
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow s' & & \downarrow t' \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\]

Following [2], we denote by \( n\text{-Cat}^+ \) the category of \( n \)-cellular extensions. Note that there is a forgetful functor \( U : (n+1)\text{-Cat} \to n\text{-Cat}^+ \) where

\[ U(\mathcal{C}) = C_{\leq n} \xrightarrow{\partial^-} \xleftarrow{\partial^+} C_{n+1}. \]

By generic categorical arguments, it can be shown that this functor has a left adjoint

\[ [-,-] : n\text{-Cat}^+ \to (n+1)\text{-Cat} \]

sending an \( n \)-cellular extension \((\mathcal{C}, S)\) to an \((n+1)\)-category \( C[S] \). A free extension of \( \mathcal{C} \) by \( S \) is a categorical extension of \( \mathcal{C} \) isomorphic to \( C[S] \). Note
that $\mathcal{C}[S]_{\leq n}$ is isomorphic to $\mathcal{C}$ and $\mathcal{C}[S]_{n+1}$ is the set of all the formal composites made with elements of $S$ (considered as $(n+1)$-cells with sources and targets given by $s$ and $t$) and cells of $\mathcal{C}$.

Given an $n$-cellular extension $(\mathcal{C}, S)$, the unit of the adjunction induces a morphism

$$\eta_{(\mathcal{C}, S)} : (\mathcal{C}, S) \to (\mathcal{C}, \mathcal{C}[S]_{n+1})$$

satisfying the following universal property: for all $(n+1)$-category $\mathcal{D}$ and all morphism of cellular extension $(F, f) : (\mathcal{C}, S) \to (\mathcal{D}_{\leq n}, \mathcal{D}_{n+1})$ there is a unique $(n+1)$-functor $G : \mathcal{C}[S] \to \mathcal{D}$ making the following diagram commute:

$$\begin{array}{ccc}
(\mathcal{C}, S) & \xrightarrow{(F, f)} & (\mathcal{D}_{\leq n}, \mathcal{D}_{n+1}) \\
\downarrow_{\eta_{(\mathcal{C}, S)}} & & \uparrow_{UG} \\
(\mathcal{C}, \mathcal{C}[S]_{n+1}) & & 
\end{array}$$

### 3.5.2 Polygraphs.

For $n \in \mathbb{N}$, we define inductively on $n$ the notion of an $n$-polygraph $P$ together with the $n$-category $P^*$ generated by $P$:

- A 0-polygraph $P$ is a set $P_0$. The generated 0-category $P^*$ by $P$ is $P_0$ (seen as a 0-category).

- An $(n+1)$-polygraph $P$ is given by an $n$-polygraph $P \leq n$ together with an $n$-cellular extension $(P_\leq n)^* \xleftarrow{t_n} P_{n+1}$

and the $(n+1)$-category $P^*$ generated by $P$ is the free extension

$$(P_\leq n)^*[P_{n+1}].$$

An $\omega$-polygraph $P$ is then a sequence $(P^i)_{i \geq 0}$ where $P^i$ is an $i$-polygraph such that $(P^{i+1}) \leq i = P^i$ and the $\omega$-category generated by $P$ is the colimit $\cup_{i \geq 0}(P^i)^*$.

### 3.5.3 Freeness.

For $n \in \mathbb{N} \cup \{\omega\}$, a free $n$-category is an $n$-category $\mathcal{C}$ such that there exists an $n$-polygraph $P$ with $\mathcal{C} \simeq P^*$. By unfolding the definition, it means that, for $k < n$, $\mathcal{C}_{\leq k+1} \simeq \mathcal{C}_{\leq k}[P_{k+1}]$ for some $k$-cellular extension

$$\mathcal{C}_{\leq k} \xleftarrow{t_k} P_{k+1}.$$ 

### 3.6 Cell$(P)$ is free

Here, we show that generalized parity complexes induce $\omega$-categories as defined in Subsection 3.5.

In the following, we suppose given a generalized parity complex $P$. 

3.6.1 A cellular extension for $\text{Cell}(P)$. For $n \in \mathbb{N}$, there is an $n$-cellular extension

$$\text{Cell}(P) \xrightarrow{\partial^\epsilon(-)} P_{n+1}$$

where, for $x \in P_{n+1}$ and $\epsilon \in \{-, +\}$, $\partial^\epsilon(-)(x) = \partial^\epsilon(x)$ (which is an $n$-cell by (G2)). In the following, we denote by $\text{Cell}(P)_{\leq n}[P_{n+1}]$ the $(n+1)$-category $\text{Cell}(P)_{\leq n}[P_{n+1}]$. We have a morphism of cellular extensions

$$\text{Cell}(P) \xrightarrow{\partial^\epsilon(-)} P_{n+1} \xrightarrow{\text{id}_{\text{Cell}(P)_{\leq n}[P_{n+1}]}} \text{Cell}(P)_{\leq n}[P_{n+1}]$$

where, for all $x \in P_{n+1}$, $\langle - \rangle(x) = (x)$. By the universal property discussed in Subsection 3.5, there is an unique $(n+1)$-functor

$$\text{eval}_{n+1} : \text{Cell}(P)_{\leq n}[P_{n+1}] \rightarrow \text{Cell}(P)_{\leq n+1}$$

such that the following diagram commutes:

$$\xymatrix{ P_{n+1} \ar[r]^-{\langle - \rangle} \ar[d]_{\eta_{\text{Cell}(P)_{\leq n}[P_{n+1}]}} & \text{Cell}(P)_{n+1} \ar[d]^{\text{eval}_{n+1}} \\ \text{Cell}(P)_{\leq n}[P_{n+1}] & \text{Cell}(P)_{\leq n+1}}$$

For $x \in P_{n+1}$, we write $\hat{x}$ for $\eta_{\text{Cell}(P)_{\leq n}[P_{n+1}])(x) \in (\text{Cell}(P)_{\leq n}[P_{n+1}])(x)$. For $m \leq n$ and $y \in P_m$, we write $\bar{y}$ for $(y) \in (\text{Cell}(P)_{\leq n}[P_{n+1}])(y) \in \text{Cell}(P)_{\leq n}[P_{n+1}]$. For conciseness, we will sometimes write eval for $\text{eval}_{n+1}$.

3.6.2 The generators are preserved by eval. In order to show that $\text{eval}$ is an isomorphism and in particular a monomorphism, we state lemmas relating the generators involved in the images of $\text{eval}$ with the generators involved in the arguments of $\text{eval}$. The first lemma states that the evaluation of identities gives cells with no top-level generators.

**Lemma 3.6.3.** For $n \in \mathbb{N}$ and $X \in (\text{Cell}(P)_{\leq n}[P_{n+1}])_{n+1}$, $\text{eval}(\text{id}_{n+1}(X))_{n+1} = \emptyset$.

**Proof.** Since eval is an $(n+1)$-functor, we have

$$(\text{id}_{n+1}(X))_{n+1} = (\text{id}_{n+1}(X))_{n+1} = \emptyset.$$  

Hence,

$$\text{eval}(\text{id}_{n+1}(X))_{n+1} = (\text{id}_{n+1}(X))_{n+1} = \emptyset.$$  

The next lemma states that the generators involved in the decomposition of $\text{Cell}(P)_{\leq n}[P_{n+1}]$ are the top-level generators of the evaluation.

**Lemma 3.6.4.** Given $n \in \mathbb{N}$, $k \geq 1$, $x_1, \ldots, x_k \in P_{n+1}$, adapted $n$-contexts $E_1, \ldots, E_k$ of $\text{Cell}(P)_{\leq n}$, and $X = E_1[x_1] \ast \cdots \ast E_k[x_k] \in \text{Cell}(P)_{\leq n}[P_{n+1}]$, we have

(a) $\text{eval}(X)_{n+1}$ is equal to $\{x_1, \ldots, x_k\}$,

(b) for $i \neq j$, $x_i \neq x_j$.  

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In particular, if
\[ E_1[x_1] \ast_n \cdots \ast_n E_k[x_k] = E'_1[\hat{y}_1] \ast_n \cdots \ast_n E'_l[\hat{y}] \]
with \( l \geq 1 \), \( y_1, \ldots, y_l \in P_{n+1} \) and \( E'_1, \ldots, E'_l \) being adapted \( n \)-contexts, then \( k = l \) and
\[ \{x_1, \ldots, x_k\} = \{y_1, \ldots, y_l\}. \]

**Proof.** By definition, for all \( 1 \leq i \leq k \), eval(\( x_i \)) = \( x_i \), thus eval(\( \hat{x}_i \))\( _{n+1} = \{ x_i \} \). With an induction on the structure of \( E_i \), one can show that
\[ \text{eval}(E_i[\hat{x}_i])_{n+1} = \text{eval}(\hat{x}_i) = \{ x_i \}. \]

Moreover,
\[
\text{eval}(X) = \text{eval}(E_1[x_1] \ast_n \cdots \ast_n E_k[x_k])_{n+1} \\
= (\text{eval}(E_1[x_1]) \ast_n \cdots \ast_n \text{eval}(E_k[x_k]))_{n+1} \\
= \text{eval}(E_1[x_1])_{n+1} \cup \cdots \cup \text{eval}(E_k[x_k])_{n+1} \\
= \{x_1, \ldots, x_k\}
\]
so (a) holds. Now let \( 1 \leq i < j \leq k \) and \( Y, Z \) be \((n+1)\)-cells defined by
\[ Y = \text{eval}(E_1[x_1] \ast_n \cdots \ast_n E_i[\hat{x}_i]) \quad \text{and} \quad Z = \text{eval}(E_{i+1}[\hat{x}_{i+1}] \ast_n \cdots \ast_n E_k[\hat{x}_k]). \]

Then \( x_i \in Y_{n+1} \), \( x_j \in Z_{n+1} \) and \( Y, Z \) are \( n \)-composable. Using Lemma 2.3.1, we have that \( Y_{n+1} \cap Z_{n+1} = \emptyset \). Hence, \( x_i \neq x_j \). \( \square \)

**3.6.5 Properties on Cell\((P)\).** We first state a simple criterion for the equality of two cells in Cell\((P)\).

**Lemma 3.6.6.** Given \( n \in \mathbb{N} \), \( \epsilon \in \{-, +\} \) and \( X, Y \in \text{Cell}(P)_n \) such that \( X = Y \), we have \( X = Y \).

**Proof.** If \( n = 0 \), the result is trivial. So suppose \( n > 0 \). By symmetry, we can moreover suppose that \( \epsilon = - \). By the hypothesis, we only need to prove that \( X_{n-1,+} = Y_{n-1,+} \). But
\[ X_{n-1,+} = (X_{n-1,-} \cup X_n^+) \setminus X_n^- = (Y_{n-1,-} \cup Y_n^+) \setminus Y_n^- = Y_{n-1,+}. \]

The next states that, given \( m \in \mathbb{N} \), an evaluated \( m \)-context induces a partition of the \( m \)-generators involved in the associated cell.

**Lemma 3.6.7.** Given \( n \in \mathbb{N}, \ 1 \leq m \leq n, \ x \in P_{n+1}, \ \epsilon \in \{-, +\}, \) an adapted \( m \)-context \( E \) of Cell\((P)\)\( _{\leq n} \), we have that
\[ (\pi_m E)_m, \langle x \rangle_{m, \epsilon}, (\pi_m^+ E)_m \] is a partition of \( (\partial_m^\epsilon E[\hat{x}])_m \).

Moreover, given another \( m \)-context \( E' \) adapted to \( \hat{x} \) such that \( E[\hat{x}] = E'[\hat{x}] \), we have
\[ (\pi_m^- E)_m \cup (\pi_m^+ E)_m = (\pi_m^- E')_m \cup (\pi_m^+ E')_m. \]
Proof. For the first part, note that
\[ \partial_m^\sigma(E[x]) = \pi_m E \ast_{m-1} \partial_m^\tau(x) \ast_{m-1} \pi_m^+ E \]
By the definition of composition,
\[ (\partial_m^\sigma(E[x]))_m = (\pi_m E)_m \cup (\partial_m^\tau(x))_m \cup (\pi_m^+ E)_m \] (15)
and by the definition of atoms, \((\partial_m^\tau(x))_m = (x)_{m,e}\). By Lemma 2.3.1(b), (15) is moreover a partition. The second part is a consequence of the first since we have
\[ (\pi_m^- E)_m \cup (\partial_m^- (x))_m \cup (\pi_m^+ E)_m = (\pi_m^- E')_m \cup (\partial_m^- (x))_m \cup (\pi_m^+ E')_m \]
and both sides are partitions. \qed

3.6.8 Linear extensions. For \( n \in \mathbb{N} \), we write \( \mathbb{N}_n \) for \( \{1, \ldots, n\} \). Given a finite poset \( (S,) \), a linear extension of \((S,\cdot)\) is given by a bijection \( \sigma : \mathbb{N}_{|S|} \to S \) such that, for \( 1 \leq i, j \leq k \), if \( \sigma(i) < \sigma(j) \), then \( i < j \). The linear extensions of \((S,\cdot)\) have a structure of an 1-category \( \text{LinExt}(S) \) where the objects are the linear extensions \( \sigma : \mathbb{N}_{|S|} \to S \) and the morphisms between \( \sigma, \tau : \mathbb{N}_{|S|} \to S \) are the functions \( f : \mathbb{N}_{|S|} \to \mathbb{N}_{|S|} \) such that the triangle
\[
\begin{diagram}
N_{|S|} \arrow{e}{f} \arrow{s}{\sigma} \arrow{sw}{\tau} & N_{|S|} \\
S \arrow{n}{\sigma} \arrow{ne}{\tau}
\end{diagram}
\]
is commutative.

The next lemma states that the morphisms of linear extensions are generated by transpositions.

Lemma 3.6.9. Let \( T \) be the set of consecutive transpositions in \( \text{LinExt}(S) \), that is,
\[ T = \{(i+1) : \sigma \to \tau \in \text{LinExt}(S) \mid \sigma, \tau \in \text{LinExt}(S), i \in \mathbb{N}_{|S|-1} \}. \]
Then, \( \text{LinExt}(S) \) is generated by \( \text{LinExt}(S)_0 \cup T \).

Proof. Let \( \sigma, \tau \in \text{LinExt}(S)_0 \) and \( f : \sigma \to \tau \in \text{LinExt}(S)_1 \). Since \( \tau \circ f = \sigma, f \) is a bijection. We prove the result by induction on the number \( N \) of inversions of \( f \). If \( N = 0 \), then \( f = \text{id}_{\mathbb{N}_{|S|}} = \text{id}_1(\sigma) \). Otherwise, \( N > 0 \). Thus, there exists \( k \in \mathbb{N}_{|S|-1} \) such that \( f(k) > f(k+1) \). Let \( \sigma' = \sigma \circ (k+1) \). \( \sigma' \) is then a linear extension of \((S,\cdot)\) as in
\[
\begin{diagram}
\mathbb{N}_{|S|} \arrow{e}{(k+1)} \arrow{s}{\sigma} \arrow{se}{\sigma'} \arrow{ne}{\tau} & \mathbb{N}_{|S|} \arrow{e}{f(k+1)} & \mathbb{N}_{|S|} \\
S \arrow{n}{\sigma} \arrow{ne}{\tau}
\end{diagram}
\]
Indeed, for \( i \neq j \in \mathbb{N}_{|S|} \) such that \( \sigma'(i) < \sigma'(j) \),
- if \( \{i, j\} \cap \{k, k+1\} = \emptyset \), then \( \sigma(i) < \sigma(j) \) and \( i < j \);
Lemma 3.6.11. Let $i = k$ and $j \neq k + 1$, then $\sigma(i + 1) < \sigma(j)$ and $i + 1 < j$, so $i < j$;
- if $i = k$ and $j = k + 1$, then $i < j$;
- if $i = k + 1$ and $j \neq k$, then $\sigma(k) < \sigma(j)$, so $k < j$, and, since $j \neq i = k + 1$, $i = k + 1 < j$;
- if $i = k + 1$ and $j = k$, then $\sigma(k) < \sigma(k + 1)$, so $\tau(f(k)) < \tau(f(k + 1))$, hence $f(k) < f(k + 1)$, contradicting the hypothesis;
- if $i \notin \{k, k + 1\}$ and $j \in \{k, k + 1\}$, then $i < j$ similarly as when $i \in \{k, k + 1\}$ and $j \notin \{k, k + 1\}$.

Moreover, the number of inversions of $f \circ (k \ k + 1)$ is $N - 1$. By induction, it can be written using elements of $T$. Hence, $f$ can be written as a composite of elements of $T$. 

3.6.10 The technical lemmas. We now state the technical lemmas that we use to prove the freeness property. They give properties satisfied by $\text{Cell}(P)_{\leq n}^+$ and they should be thought as one big lemma since they are mutually dependent. However, we preferred to split them, for clarity.

The first lemma enables to modify a context $E$ adapted a generator $x$ without changing the evaluation.

Lemma 3.6.11. Let $n \in \mathbb{N}$ and $m \leq n$. Let $x \in P_{n+1}$, $E$ be an adapted $m$-context of $\text{Cell}(P)_{\leq n}^+$ with $m \leq n$. Consider the following subsets of $P_m$:

\[
S = (\pi_m^-E)_m \cup (\pi_m^+E)_m, \quad U = \{y \in S \mid y \triangleleft (x)_{m,-}\},
\]

\[
S' = S \cup \langle x \rangle_{m,-}, \quad V = \{y \in S \mid \langle x \rangle_{m,-} \triangleleft y\}.
\]

Then, for every partition $U' \cup V'$ of $S$ such that $U \subseteq U'$, and $V \subseteq V'$, $U'$ initial and $V'$ final for $\triangleleft_S$, there exists an $m$-context $E'$ adapted to $\hat{x}$ such that

\[
(\pi_m^-E')_m = U', \quad (\pi_m^+E')_m = V', \quad E[\hat{x}] = E'[\hat{x}].
\]

Graphically, with $m = 2$, this can be illustrated as

Next, we show that if two $(n+1)$-generators in context do not have common $n$-generators in their source and target then we can apply the exchange rule.

Lemma 3.6.12. Let $n \in \mathbb{N}$ and $m \leq n$. Let $k_1, k_2 \geq 0$ be such that $\max(k_1, k_2) = n + 1$, $x_1 \in P_{k_1}$, $x_2 \in P_{k_2}$, $E_1$, $E_2$ be adapted $m$-contexts
of $\text{Cell}(P)^+_n$, with $0 \leq m < \min(k_1, k_2)$ such that $E_1[\tilde{x}_1]$ and $E_2[\tilde{x}_2]$ are $m$-composable. Then

$$\langle x_1 \rangle_{m,-} \cap \langle x_2 \rangle_{m,+} = \emptyset.$$  

Moreover, if $\langle x_1 \rangle_{m,+} \cap \langle x_2 \rangle_{m,-} = \emptyset$, then there exist $m$-contexts $E'_1, E'_2$ such that

$$E_1[\tilde{x}_1] \ast_mE_2[\tilde{x}_2] = E'_2[\tilde{x}_2] \ast_mE'_1[\tilde{x}_1].$$

An instance of the above lemma with $m = 2$ can be pictured as

We have seen in Lemma 3.6.13 that every $n$-cell can be expressed as a composition of $n$-generators in context. The following lemma states that there is such a decomposition for every linearization of the poset of such $n$-generators under the “dependency order” $\triangleleft$.

**Lemma 3.6.13.** Let $n \in \mathbb{N}$. Let $U = \{x_1, \ldots, x_k\} \subseteq P_{n+1}$ be a set of generators and $E_1, \ldots, E_k$ be adapted $n$-contexts such that the cell

$$X = E_1[\tilde{x}_1] \ast_n \cdots \ast_n E_k[\tilde{x}_k]$$

exists in $\text{Cell}(P)^+_n$. Then

$$x_i \triangleleft x_j \quad \text{implies} \quad i < j$$

for all indices $i, j$ such that $1 \leq i, j \leq k$. Moreover, if $\sigma$ is a linear extension of $(U, \triangleleft_U)$, then there exist $n$-contexts $E'_1, \ldots, E'_k$ such that

$$X = E'_1[\tilde{x}_1[\sigma(1)]] \ast_n \cdots \ast_n E'_k[\tilde{x}_k[\sigma(k)]].$$  

The following lemma states that, in order for two contexts applied to a generator to evaluate to the same cell, it is enough for them to have same source or target.

**Lemma 3.6.14.** Let $n \in \mathbb{N}$. Let $x \in P_{n+1}$ and $E_1, E_2$ be adapted $m$-contexts with $m \leq n$ such that $\partial^+_m E_1[\tilde{x}] = \partial^+_m E_2[\tilde{x}]$ or $\partial^-_m E_1[\tilde{x}] = \partial^-_m E_2[\tilde{x}]$. Then

$$E_1[\tilde{x}] = E_2[\tilde{x}].$$
The following lemma generalizes Lemma 3.6.14 to other cells.

**Lemma 3.6.15.** Let \( n \in \mathbb{N} \). Let \( X, Y \in \text{Cell}(P)_{\leq n}^+ \) and \( \epsilon \in \{-, +\} \) be such that \( (\text{eval}(X))_{n+1} = (\text{eval}(Y))_{n+1} \) and \( \partial_m^\epsilon X = \partial_m^\epsilon Y \). Then, \( X = Y \).

Finally, we can conclude that \( \text{Cell}(P)_{\leq n+1} \) is a free extension of \( \text{Cell}(P)^{(n)} \) by \( P_{n+1} \).

**Lemma 3.6.16.** \( \text{Cell}(P)_{\leq n}^+ \) is isomorphic to \( \text{Cell}(P)_{\leq n+1} \).

**Proof.** We will prove the lemmas above using an induction on \( n \). For a fixed \( n \) we will prove Lemma 3.6.11 and Lemma 3.6.12 together by induction on \( m \).

**Proof of Lemma 3.6.11.** If \( m = 0 \), the property is trivial. So suppose \( m > 0 \). By Lemma 3.4.4, \( \pi^{-}_m E \) can be written

\[
\pi^{-}_m E = E_1[y_1] \ast_{m-1} \ldots \ast_{m-1} E_p[y_p]
\]

with \( p \in \mathbb{N}, y_1, \ldots, y_p \in P_m \) and \( E_1, \ldots, E_p \) \((m-1)\)-contexts. Let \( Y \) be \( \{y_1, \ldots, y_p\} \). Since \( U' \) is initial for \( \langle s, U' \rangle \), \( U' \cap Y \) is initial for \( \langle s, Y \rangle \), so there exists a linear extension of \( \langle Y, \langle s, Y \rangle \rangle \)
\[
\sigma : N_p \rightarrow Y
\]
such that \( \{i \in N_p \mid \sigma(i) \in U'\} = \{1, \ldots, i_0\} \) for some \( i_0 \in \{0\} \cup N_p \). Since we can use Lemma 3.6.12 to permute the \( y_i \)'s in \( \pi^{-}_m E \) according to \( \sigma \), we can suppose that \( \pi^{-}_m E \) is such that

\[
\{i \in N_p \mid y_i \in U'\} = \{1, \ldots, i_0\}.
\]

Remember that

\[
E[\hat{x}] = \pi^{-}_m E \ast_{m-1} E'[\hat{x}] \ast_{m-1} \pi^{-}_m E
\]

If \( i_0 < p \), we want to swap all the \( E_i[y_i] \) for \( i > i_0 \) and then iterate this procedure for \( i_0 < i < p \). So, suppose that \( i_0 < p \). Let \( T \) be \( E_p[y_p] \ast_{m-1} E'[\hat{x}] \). Since \( y_p \notin U' \), we have \( y_p \notin U \). In particular, \( \langle y_p \rangle_{\{m-1\}+} \cap \langle \hat{x} \rangle_{m-1} \) = \( \emptyset \). So, using Lemma 3.6.12, we get adapted \((m-1)\)-contexts \( F \) and \( F' \) such that

\[
T = F[\hat{x}] \ast_{m-1} F'[\hat{y}_p].
\]

Thus, we obtain an \( m \)-context \( E' \) adapted to \( \hat{x} \) with \( E'[\hat{x}] = E[\hat{x}] \) and

\[
\pi^{-}_m E' = E_1[y_1] \ast_{m-1} \ldots \ast_{m-1} E_{p-1}[y_{p-1}]
\]

\[
\hat{E}' = F[\hat{x}]
\]

\[
\pi^{-}_m E' = F'[\hat{y}_p] \ast_{m-1} \pi^{-}_m E
\]

After iterating this procedure for all \( i > i_0 \), we get an adapted \( m \)-context \( E' \) for \( \hat{x} \) such that \( E'[\hat{x}] = E[\hat{x}] \) and \( (\pi^{-}_m E')_m = (\pi^{-}_m E)_m \cap U' \).

Using a similar procedure to transfer elements from \( \pi^{-}_m E' \) to \( \pi^{-}_m E' \), we get an adapted \( m \)-context \( E'' \) for \( \hat{x} \) such that \( E''[\hat{x}] = E[\hat{x}] \) and
\[(\pi_m^+E''_m) = (\pi_m^+E') \cap V'.\] By Lemma 3.6.7, \((\pi_m^+E')_m \cup (\pi_m^+E')_m \) and \((\pi_m^+E'')_m \cup (\pi_m^+E'')_m \) are partitions of \(S\), as \(U' \cup V'\) (by hypothesis), thus

\[
\begin{align*}
(\pi_m^+E'')_m &= S \setminus (\pi_m^+E'')_m \\
&= S \setminus ((\pi_m^+E')_m \cap V') \\
&= (\pi_m^-E')_m \cup U' \\
&= ((\pi_m^+E)_m \cap U') \cup U' \\
&= U'.
\end{align*}
\]

By partition, we have \((\pi_m^+E'')_m = V'\). Hence, \(E''\) satisfies the wanted properties.

**Proof of Lemma 3.6.12.** We have

\(\text{eval}(E_1[z_1] *_m E_2[z_2]) = \text{eval}(E_1[z_1]) *_m \text{eval}(E_2[z_2]) = E_1[\{z_1\}] *_m E_2[\{z_2\}]\).

By Lemma 2.3.2, \((E_1[\{z_1\}]_{m+1,-}) \cap (E_2[\{z_2\}]_{m+1,+}) = \emptyset\). But

\[
E_1[\{z_1\}]_{m+1,-} = \langle z_1 \rangle_{m+1,-} \quad \text{and} \quad E_2[\{z_2\}]_{m+1,+} = \langle z_2 \rangle_{m+1,+}.
\]

Therefore,

\[
\langle z_1 \rangle_{m,-} \cap \langle z_2 \rangle_{m,+} \subseteq \langle z_1 \rangle_{m+1,-} \cap \langle z_2 \rangle_{m+1,+} = \emptyset.
\]

For the second part, suppose that \(\langle z_1 \rangle_{m,+} \cap \langle z_2 \rangle_{m,-} = \emptyset\). If \(m = 0\), then since \(z_1 \neq z_2\) exists, we have \(\langle z_1 \rangle_{0,+} = \langle z_2 \rangle_{0,-}\) and they are non-empty by Axiom \((G2)\), which contradicts \(\langle z_1 \rangle_{0,+} \cap \langle z_2 \rangle_{0,-} = \emptyset\). Hence, \(m > 0\). Consider the following:

\[
S_i = (\pi_m^-E_i)_m \cup (\pi_m^+E_i)_m \quad \text{for} \ i \in \{1, 2\},
\]

\[
M = \partial_m^+E_1[z_1] \quad \text{(or equivalently} \ \partial_m^+E_2[z_2]),
\]

\[
S' = M_m,
\]

\[
U_i = \{x \in S_i \mid x \not\in \langle z_i \rangle_{m,+}\} \quad \text{for} \ i \in \{1, 2\},
\]

\[
V_i = \{x \in S_i \mid \langle z_i \rangle_{m,+} \subseteq \langle x \rangle\} \quad \text{for} \ i \in \{1, 2\}.
\]

We have \(\langle z_1 \rangle_{m,+} \subseteq S'\) and \(\langle z_2 \rangle_{m,-} \subseteq S'\). By \((G4)\), we do not have both \(\langle z_1 \rangle_{m,+} \not\subseteq S' \langle z_2 \rangle_{m,-}\) and \(\langle z_2 \rangle_{m,-} \not\subseteq S' \langle z_1 \rangle_{m,+}\). By symmetry, we can suppose that \(\not\subseteq \langle z_2 \rangle_{m,-} \not\subseteq S' \langle z_1 \rangle_{m,+}\). Since we can use Lemma 3.6.11 (which is proved for the current values of \(m\) and \(n\)) to change \(E_1\) and \(E_2\), we can suppose that

\[
\langle \pi_m^+E_1 \rangle_m = U_1, \quad \langle \pi_m^+E_2 \rangle_m = S_2 \setminus V_2,
\]

\[
\langle \pi_m^-E_1 \rangle_m = S_1 \setminus U_1, \quad \langle \pi_m^-E_2 \rangle_m = V_2.
\]

Then

\[
(U_1 \cup \langle z_1 \rangle_{m,+}) \cap ((\langle z_2 \rangle_{m,-} \cup V_2) = \emptyset
\]

since, otherwise, it would contradict the condition \(\not\subseteq \langle z_2 \rangle_{m,-} \not\subseteq S' \langle z_1 \rangle_{m,+}\). Consider the following sets:

\[
Q_1 = U_1, \quad Q_2 = \langle z_1 \rangle_{m,+},
\]

\[
Q_3 = S' \setminus (U_1 \cup \langle z_1 \rangle_{m,+} \cup \langle z_2 \rangle_{m,-} \cup V_2),
\]

\[
Q_4 = \langle z_2 \rangle_{m,-}, \quad Q_5 = V_2.
\]

Then \(Q_1, Q_2, Q_3, Q_4, Q_5\) form a partition of \(S'\). Let \(R_1, \ldots, R_5\) be such that \(R_5 = Q_5\) and \(R_i = R_{i+1} \cup Q_i\). Note that

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- $R_5$ is final for $\prec_{S'}$, by segment Axiom (G3) used for $z_2$ and $M$,
- $R_4$ is final for $\prec_{S'}$, by definition of $V_2$,
- $R_3$ is final for $\prec_{S'}$, because $S' \setminus R_3 = Q_1 \cup Q_2$ is initial for $\prec_{S'}$,
- $R_2$ is final for $\prec_{S'}$, by segment Axiom (G3) used for $z_1$ and $M$.

This implies that there exists a linear extension for $(S', \prec_{S'})$

$$\sigma : \mathbb{N}_{S'} \rightarrow S'$$

such that for $i, j \in \mathbb{N}_{S'}$ and $1 \leq k, l \leq 5$, if $\sigma(i) \in Q_k$ and $\sigma(j) \in Q_l$ with $k < l$, then $i < j$. Since $S' = M_m$, using Lemma 3.6.12 inductively, $M$ can be written

$$M = \prod_{i=1}^{\lfloor S' \rfloor} F_i[\widehat{\sigma(i)}]$$

with $F_1, \ldots, F_{\lfloor S' \rfloor}$ adapted $(m-1)$-contexts. By regrouping the terms corresponding to $Q_1, \ldots, Q_5$ respectively, we obtain five $m$-cells $M^1, M^2, M^3, M^4, M^5 \in \text{Cell}(P)_m$ where

$$M^i = \prod_{i \in \sigma^{-1}(Q_i)} F_i[\widehat{\sigma(i)}]$$

and such that

$$M = M^1 \ast_{m-1} M^2 \ast_{m-1} M^3 \ast_{m-1} M^4 \ast_{m-1} M^5.$$ 

Since

$$\partial_{m-1}^{-1}(\pi_m E_1) = \partial_{m-1}^{-1}M = \partial_{m-1}^{-1}M^1$$

and

$$(\pi_m E_1)_m = U_1 = M^1_m,$$

by Lemma 3.6.6, it implies that $\pi_m E_1 = M^1$. Moreover, since

$$\partial_{m-1}^{-1} \hat{E}_1[\hat{\partial}_m E_1^+] = \partial_{m-1}^{-1}(\pi_m E_1) = \partial_{m-1}^{-1}M^1 = \partial_{m-1}^{-1}M^2$$

and

$$(\hat{E}_1[\hat{\partial}_m E_1^+])_m = (z_1)_{m,+} = M^2_m,$$

by Lemma 3.6.6, it implies that $\hat{E}_1[\hat{\partial}_m E_1^+] = M^2$. Similarly, we can show that

$$\hat{E}_2[\hat{\partial}_m E_2^+] = M^4$$

and

$$\pi_m E_2 = M^5.$$ 

Moreover, since

$$(\pi_m E_2)_m = S_2 \setminus V_2$$

$$= S' \setminus ((z_2)_{m,-} \cup V_2)$$

$$= Q_1 \cup Q_2 \cup Q_3$$

and

$$\partial_{m-1}^{-1}(\pi_m E_2) = \partial_{m-1}^{-1}M = \partial_{m-1}^{-1}(M^1 \ast_{m-1} M^2 \ast_{m-1} M^3),$$

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by Lemma 3.6.6, we have that
\[
\pi_m^+ E_2 = M^1 \ast_{m-1} M^2 \ast_{m-1} M^3.
\]
Similarly, we can show that
\[
\pi_m^+ E_1 = M^3 \ast_{m-1} M^4 \ast_{m-1} M^5.
\]
Hence,
\[
E_1[z_1] \ast_n E_2[z_2] = (M^1 \ast_{m-1} \tilde{E}_1[z_1] \ast_{m-1} M^3 \ast_{m-1} \partial_m^+ \tilde{E}_2[z_2] \ast_{m-1} M^5) \ast_m (M^1 \ast_{m-1} \partial_m^+ \tilde{E}_1[z_1] \ast_{m-1} M^3 \ast_{m-1} \tilde{E}_2[z_2] \ast_{m-1} M^5) = (M^1 \ast_{m-1} \partial_m^+ (\tilde{E}_1[z_1]) \ast_{m-1} M^3 \ast_{m-1} \tilde{E}_2[z_2] \ast_{m-1} M^5) \ast_m (M^1 \ast_{m-1} \tilde{E}_1[z_1] \ast_{m-1} M^3 \ast_{m-1} \partial_m^+ (\tilde{E}_2[z_2]) \ast_{m-1} M^5)
\]
which is of the form \(E_1[z_1] \ast_m E_1'[z_1]\) as wanted.

Proof of Lemma 3.6.13. In order to show that \(z_i \preceq_U z_j\) implies \(i < j\), we only need to prove that \(z_i \preceq_U z_j\) implies \(i < j\), since \(\preceq_U\) is the transitive closure of \(\preceq_U\).

So suppose given \(i, j \in \mathbb{N}_p\) such that \(z_i \preceq_U z_j\), that is, \(z_i^+ \cap z_j^- \neq \emptyset\). By (G1), \(i \neq j\). Consider
\[
Y = E_1[z_1] \ast_n \cdots \ast_n E_{i-1}[z_{i-1}] \quad \text{and} \quad Z = E_1[z_1] \ast_n \cdots \ast_n E_p[z_p].
\]
Then, \(Y' := \text{eval}(Y)\) and \(Z' := \text{eval}(Z)\) are \(n\)-composable \((n+1)\)-cells. By Lemma 2.3.1(a), \(Y'_{n+1}^+ \cap (Z'_{n+1})^+ = \emptyset\). But, by Lemma 3.6.4, \(z_i \in Z_{n+1}'^+\) and \(Y_{n+1}' = \{z_1, \ldots, z_{i-1}\}\). Hence, since \(z_i^+ \cap z_j^- \neq \emptyset\), we have \(i < j\).

For the second part, note first that the first part implies that \(\tau: \mathbb{N}_p \rightarrow U\) defined by \(\tau(i) = z_i\) is a linear extension of \((U, \preceq_U)\). Let \(f = \sigma^{-1} \circ \tau\) be a morphism of linear extensions between \(\sigma\) and \(\tau\). By Lemma 3.6.9, we can suppose that \(f = (i \ i+1)\) for some \(i \in \mathbb{N}_{p-1}\). To conclude, we just need to show that \(x_i\) and \(x_{i+1}\) can be swapped in \(X = E_1[x_1] \ast_n \cdots \ast_n E_p[x_p]\). By contradiction, suppose that \(\langle x_i \rangle_{n+} \cap \langle x_{i+1} \rangle_{n-} \neq \emptyset\). Then, \(\tau(i) \preceq_U \tau(i+1)\). But \(\tau = \sigma \circ (i \ i+1)\), so \(\sigma(i+1) \preceq_U \sigma(i)\) and, since \(\sigma\) is a linear extension, \(i+1 < i\) which is a contradiction. So \(\langle x_i \rangle_{n+} \cap \langle x_{i+1} \rangle_{n-} = \emptyset\). By Lemma 3.6.12 (which is proved for the current value of \(n\)), there exist adapted \(n\)-contexts \(E_i'\) and \(E_{i+1}'\) such that
\[
X = E_1[x_1] \ast_n \cdots \ast_n E_{i-1}[x_{i-1}] \ast_n E_i'[x_{i+1}] \ast_n E_{i+2}[x_{i+2}] \ast_n \cdots \ast_n E_p[x_p]\n\]
which concludes the proof.

Proof of Lemma 3.6.14. By symmetry, we will only handle the case when \(\partial_m^- E_1[z] = \partial_m^+ E_2[z]\). We prove this property by an induction on \(m\). If \(m = 0\), the result is trivial. So suppose \(m > 0\). Consider the following subsets of \(P_m:\n\]
\[
S = (\pi^- E_1)_m \cup (\pi^+ E_1)_m,
\]
\[
S' = S \cup \langle z \rangle_{m-},
\]
\[
U = \{x \in S \mid x \not\in S'\},
\]
\[
V = S \setminus U.
\]
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By Lemma 3.6.7, \( S = (\pi^- E_2)_m \cup (\pi^+ E_2)_m \). By Lemma 3.6.11, there are \( m \)-contexts \( F_1, F_2 \) such that

\[
F_1[\bar{z}] = E_i[\bar{z}] \quad \text{and} \quad (\pi^- F_1)_m = U \quad \text{and} \quad (\pi^+ F_1)_m = V.
\]

For \( i \in \{1, 2\} \), we have that \( \partial^n_{m-1}(\pi^- F_1) = \partial^n_{m-1}E_i[\bar{z}] = \partial^n \partial^n_{m-1}E_i[\bar{z}], \) so

\[
\partial^n_{m-1}(\pi^- F_1) = \partial^n_{m-1}(\pi^- F_2).
\]

Since \( \pi^- F_1, \pi^- F_2 \in \text{Cell}(P)_m \), by Lemma 3.6.6, we have

\[
\pi^- F_1 = \pi^- F_2.
\]

Moreover, for \( i \in \{1, 2\} \), \( \partial^n_{m-1}(\pi^- F_i) = \partial^n_{m-1}(\tilde{F}_i[\bar{z}]), \) so

\[
\partial^n_{m-1}(\tilde{F}_1[\bar{z}]) = \partial^n_{m-1}(\tilde{F}_2[\bar{z}]).
\]

By induction on \( m \), we have that

\[
\tilde{F}_1[\bar{z}] = \tilde{F}_2[\bar{z}].
\]

But \( \partial^n_{m-1}(\tilde{F}_1[\bar{z}]) = \partial^n_{m-1}(\pi^+ F_i), \) so

\[
\partial^n_{m-1}(\pi^+ F_1) = \partial^n_{m-1}(\pi^+ F_2).
\]

Since \( \pi^+ F_1, \pi^+ F_2 \in \text{Cell}(P)_m \), by Lemma 3.6.6, we have

\[
\pi^+ F_1 = \pi^+ F_2.
\]

Hence,

\[
F_1[\bar{z}] = F_2[\bar{z}]
\]

which concludes the proof.

Proof of Lemma 3.6.15. By symmetry, we only give a proof for \( \epsilon = - \). We do an induction on \( N = |\text{eval}(X)_{n+1}| \). If \( N = 0 \), then by Lemma 3.4.4 and Lemma 3.6.4, there exist \( n \)-cells \( X', Y' \) such that \( X = \text{id}_{n+1}(X') \) and \( Y = \text{id}_{n+1}(Y') \)

so, since \( \partial^n X = \partial^n Y \), it holds that \( X' = Y' \). Hence, \( X = Y \).

Otherwise, if \( N = 1 \), then we can conclude by Lemma 3.6.14 and Lemma 3.6.4. Otherwise, \( N > 1 \). Then, by Lemma 3.4.4 and Lemma 3.6.3, \( X, Y \) can be written

\[
X = E_1[\bar{x}_1] *_n \cdots *_n E_p[\bar{x}_p] \quad \text{and} \quad Y = F_1[\bar{y}_1] *_n \cdots *_n F_j[\bar{y}_j]
\]

where \( p, q \in \mathbb{N}, \ x_i, y_j \in P_{n+1} \) and \( E_i, F_j \) \( n \)-contexts. By Lemma 3.6.4,

\[
\{x_1, \ldots, x_p\} = \text{eval}(X)_{n+1} = \text{eval}(Y)_{n+1} = \{y_1, \ldots, y_q\}
\]

and the \( x_i \)'s are all different, and the \( y_j \)'s too. So \( p = q = N \). Using Lemma 3.6.12 to reorder the \( y_j \)'s, we can suppose that \( x_i = y_i \) for \( 1 \leq i \leq N \). Since \( \partial^n X = \partial^n Y \), it holds that \( \partial^n E_1[\bar{x}_1] = \partial^n F_1[\bar{y}_1] \). So by Lemma 3.6.14,

\[
E_1[\bar{x}_1] = F_1[\bar{y}_1].
\]

Moreover,

\[
\partial^n (E_2[\bar{x}_2] *_n \cdots *_n E_p[\bar{x}_p]) = \partial^n E_1[\bar{x}_1] = \partial^n F_1[\bar{y}_1] = \partial^n (F_2[\bar{y}_2] *_n \cdots *_n F_p[\bar{y}_p]).
\]

So, by induction on \( N \),

\[
E_2[\bar{x}_2] *_n \cdots *_n E_p[\bar{x}_p] = F_2[\bar{y}_2] *_n \cdots *_n F_p[\bar{y}_p].
\]

Hence, \( X = Y \).
Proof of Lemma 3.6.16. Given \( X \in \text{Cell}(P)_{n+1} \), if \( X \) can be written
\[
X = \text{id}_{n+1}(X')
\]
with \( X' \in \text{Cell}(P)_n \), then \( X = \text{eval}(\text{id}_{n+1}(X')) \). Otherwise, if \( X \) can be written
\[
X = E_1[(x_1)] *_n \cdots *_n E_p[(x_p)]
\]
then \( X = \text{eval}(E_1[\widehat{x}_1] *_n \cdots E_p[\widehat{x}_p]) \). So, by Lemma 3.4.4,
\[
\text{eval}: (\text{Cell}(P)^+_{\leq n})_{n+1} \rightarrow \text{Cell}(P)_{n+1}
\]
is a surjective function. Let
\[
\text{reval}: \text{Cell}(P)_{n+1} \rightarrow (\text{Cell}(P)^+_{\leq n})_{n+1}
\]
be a section of \( \text{eval} \). For \( X \in \text{Cell}(P)_{n+1} \) and \( \epsilon \in \{-, +\} \), we have \( \partial^\epsilon_n(\text{reval}(X)) = \partial^\epsilon_n(\text{eval} \circ \text{reval}(X)) = \partial^\epsilon_n X \). So \( \text{reval} \) can be naturally extended to a morphism of cellular extensions
\[
\text{reval}: (\text{Cell}(P)^+_{\leq n}, \text{Cell}(P)_{n+1}) \rightarrow (\text{Cell}(P)^-_{\leq n}, (\text{Cell}(P)^+_{\leq n})_{n+1}).
\]
Moreover, if \( X = \text{id}_{n+1}(X') \) for some \( X' \in \text{Cell}(P)_n \), we have \( \text{eval}(\text{reval}(\text{id}_{n+1}(X'))) = \text{eval}(\text{id}_{n+1}(X')) = \text{id}_{n+1}(X') \). So, by Lemma 3.6.15,
\[
\text{reval}(\text{id}_{n+1}(X')) = \text{id}_{n+1}(X').
\]
Otherwise, if \( X = Y *_i Z \) for some \( i \leq n \) and \( Y, Z \in \text{Cell}(P)_{n+1} \), we have \( \text{eval}(\text{reval}(X *_i Y)) = \text{eval}(\text{reval}(X) *_i \text{reval}(Y)) = X \). So, by Lemma 3.6.15,
\[
\text{reval}(X *_i Y) = \text{reval}(X) *_i \text{reval}(Y).
\]
Thus, \( \text{reval} \) is an \((n+1)\)-functor and is an inverse to \( \text{eval} \). Hence, \( \text{Cell}(P)^+_{\leq n} \) is isomorphic to \( \text{Cell}(P)^{\leq n+1} \).

### 3.6.17 Freeness property

We are able to conclude the proof of freeness for the \( \omega \)-category of cells \( \text{Cell}(P) \).

**Theorem 3.6.18.** \( \text{Cell}(P) \) is a free \( \omega \)-category generated by the atoms \( \widehat{x} \) for \( x \in P \).

**Proof.** By Lemma 3.6.16, for \( n \in \mathbb{N} \), \( \text{Cell}(P)^{\leq n+1} \) is a free extension of \( \text{Cell}(P)^{\leq n} \) by \( P_{n+1} \). So \( \text{Cell}(P) \) is a free \( \omega \)-category. The generating property is given by Theorem 3.2.1.

### 4 Alternative notions of cells

Before being able to relate generalized parity complexes to the other formalisms, we first give alternative notions of cells to the one of Paragraph ?? that are still suited for describing the \( \omega \)-category of pasting diagrams.

In this section, we suppose given an \( \omega \)-hypergraph \( P \).
4.1 Closed and maximal pre-cells

We write \( \text{Closed}(P) \) for the graded set of closed \( n \)-fgs of \( P \). Given an \( n \)-fgs \( X \) of \( P \), \( x \in X \) is said to be \textit{maximal in} \( X \) when for all \( y \in P \) such that \( x \not\subseteq y \) and \( x \neq y \), it holds that \( y \not\subseteq X \). We write \( \max(X) \) for the \( n \)-fgs of \( P \) made of the maximal elements of \( X \). The \( n \)-fgs \( X \) is then said to be \textit{maximal} when \( \max(X) = X \).

We write \( \text{Max}(P) \) for the graded set of maximal \( n \)-fgs. Given \( n \in \mathbb{N} \) and \( X \) an \( n \)-pre-cell of \( P \), we write \( \cup X \) for the \( n \)-fgs of \( P \) given by \( \cup_{0 \leq i \leq n}(X_{i,-} \cup X_{i,+}) \).

4.2 Maximality lemma

Here, we show that there is a simple criterion to know whether an element is maximal in a cell of \( \text{Cell}(P) \).

In this subsection, we suppose that \( P \) satisfies (G0), (G1), (G2) and (G3).

\textbf{Lemma 4.2.1} (Maximality lemma). Let \( m < n \in \mathbb{N} \) and \( X \) be an \( n \)-cell of \( P \). For \( x \in X_{m,-} \) (resp. \( x \in X_{m,+} \)) with \( x \) not maximal in \( \cup X \), we have \( x \in X_{m+1,-}^\pm \) (resp. \( x \in X_{m+1,+}^\pm \)).

\textit{Proof.} We prove this property by induction on \( p := n - m \). By symmetry, we only prove the case where \( x \in X_{m,-} \). Since \( x \) is not maximal, by definition of \( \partial \), there exist \( p > 0, \eta \in \{-,+,\}, x_0, x_1, \ldots, x_p \in P \) and \( \epsilon_1, \ldots, \epsilon_p \in \{-,+,\} \) such that \( x_0 = x, x_p \in X_{m+p,\eta} \) and \( x_i \in x_{i+1} \) for \( i < p \).

Suppose that \( p = 1 \). By Lemma 2.1.1, \( X_{m,-} \cap X_{m+1,\eta}^+ = \emptyset \). Since \( x \in x_1^1 \) and \( x_1 \in X_{m+1,\eta} \), we have \( \epsilon_1 = - \) and \( x \in X_{m+1,\eta}^- \). By Lemma 2.1.5, \( x \in X_{m+1,-}^- \). Otherwise, suppose that \( p > 1 \). Let \( y \in X_{m+p,\eta} \) be the smallest element of \( X_{m+p,\eta} \) for \( \leq_{X_{m+p,\eta}} \) such that \( y \not\subseteq x_{p-1} \). If \( x_{p-1} \not\subseteq y^- \), then, by minimality of \( y \), there is no \( y' \in X_{m+p,0} \) such that \( x_{p-1} \subseteq y'^+ \). Therefore,

\[ x_{p-1} \in X_{m+p,\eta}^\pm \subseteq X_{m+p,-}^- \]

Hence, \( x \) is not minimal in \( \partial_{m+p-1} X \) and we conclude by induction. Otherwise, \( x_{p-1} \subseteq y^+ \). Consider

\[ G = \{ z \in X_{m+p,\eta} \mid z \not\subseteq X_{m+p,\eta} \} \cup \{ y \} \] and \( Y = \text{Act}(\partial_{m+p-1} X, G) \).

We have \( x \in Y_{m,-} \) and \( x_{p-1} \in Y_{m+p-1} \subseteq \cup Y \) and, by Theorem 2.2.3, \( Y \) is a cell.

By induction, \( x \in Y_{m+1,-}^\pm \). Since \( X_{m+1,-} \) and \( Y_{m+1,-} \) both move \( X_{m,-} \) to \( X_{m,+} \), by Lemma 2.1.5, \( x \in X_{m+1,-} \) which concludes the proof.

This criterion gives a simple description of the set of maximal elements of a cell of \( \text{Cell}(P) \).

\textbf{Lemma 4.2.2.} Let \( m < n \in \mathbb{N}, \epsilon \in \{-,+,\} \) and \( X \) be an \( n \)-cell of \( P \). Then,

\[ \max(\cup X) \cap P_m = X_{m,-} \cap X_{m,+} \]

\textit{Proof.} By Lemma 4.2.1,

\[ \max(\cup X) \cap P_m = (X_{m,-} \setminus X_{m+1,-}^+) \cup (X_{m,+} \setminus X_{m+1,+}^+) \]

By Lemma 2.1.6, it can be simplified to

\[ \max(\cup X) \cap P_m = X_{m,-} \cap X_{m+1,-}^+ \]
4.3 Relating representations of pre-cells

In this subsection, we relate \( \text{Cell}(P) \), \( \text{Max}(P) \) and \( \text{Closed}(P) \) by giving translations functions and properties on these translations.

4.3.1 The translation functions. We define several functions between \( \text{PCell}(P) \), \( \text{Max}(P) \) and \( \text{Closed}(P) \), as represented by

\[
\begin{array}{ccc}
\text{PCell}(P) & \xrightarrow{T_M^{\text{PC}}} & \text{Max}(P) \\
& \xleftarrow{T_M^{\text{PC}}} & \xrightarrow{T_C^{\text{M}}} \text{Closed}(P) \\
& \xleftarrow{T_C^{\text{PC}}} & \xrightarrow{T_M^{\text{PC}}} \text{PCell}(P)
\end{array}
\]

where

- \( T_M^{\text{PC}} : \text{PCell}(P) \to \text{Max}(P) \) with, for \( X \) an \( n \)-pre-cell of \( P \),
  \[ T_M^{\text{PC}}(X) = \max(\cup X) \],
- \( T_M^{\text{PC}} : \text{Max}(P) \to \text{PCell}(P) \) with, for \( X \) an \( n \)-fgs of \( P \), \( T_M^{\text{PC}}(X) \) is the \( n \)-pre-cell \( Y \) of \( P \) such that
  \[
  \begin{align*}
  Y_n &= X_n, \\
  Y_i, - &= X_i \cup Y_{i+1}, - \quad &\text{for } i < n, \\
  Y_i, + &= X_i \cup Y_{i+1}, + \quad &\text{for } i < n,
  \end{align*}
  \]
- \( T_M^{\text{Cl}} : \text{Max}(P) \to \text{Closed}(P) \) with, for \( X \) a maximal \( n \)-fgs of \( P \),
  \[ T_M^{\text{Cl}}(X) = R(X) \],
- \( T_M^{\text{M}} : \text{Closed}(P) \to \text{Max}(P) \) with, for \( X \) a closed \( n \)-fgs,
  \[ T_M^{\text{Cl}}(X) = \max(X) \],
- \( T_M^{\text{PC}} : \text{PCell}(P) \to \text{Closed}(P) \) with, for \( X \) an \( n \)-pre-cell of \( P \),
  \[ T_M^{\text{PC}}(X) = R(\cup X) \],
- \( T_M^{\text{Cl}} : \text{Closed}(P) \to \text{PCell}(P) \) defined by
  \[ T_M^{\text{Cl}} = T_M^{\text{PC}} \circ T_M^{\text{Cl}} \].

These operations can be related to each other, as state the following lemmas.

**Lemma 4.3.2.** \( T_M^{\text{Cl}} \circ T_M^{\text{Cl}} = \text{id}_{\text{Closed}(P)} \) and \( T_M^{\text{Cl}} \circ T_M^{\text{Cl}} = \text{id}_{\text{Max}(P)} \).
Proof. Let $X$ be a closed $n$-fns of $P$ and $x \in X$. We have $T^C_M(X) \subseteq X$ so

$$T^M_C \circ T^C_M(X) \subseteq X.$$ 

Moreover, for $x \in X$, since $X$ is finite, there is $y \in \max(X)$ with $y \mathbin{R} x$. It implies that $y \in T^C_M(X)$ and $x \in T^M_C \circ T^C_M(X)$. Therefore,

$$X \subseteq T^M_C \circ T^C_M(X),$$

which shows that

$$T^M_C \circ T^C_M = \text{id}_{\text{Closed}(P)}.$$

For the other equality, note that, for all $n$-fns $X$ of $P$, $R(X)$ has the same maximal elements as $X$. It implies that

$$T^C_M \circ T^M_C = \text{id}_{\text{Max}(P)}.$$ 

Lemma 4.3.3. Suppose that $P$ satisfies axioms (G0), (G1), (G2) and (G3). Let $n \in \mathbb{N}$, $X \in \text{Cell}(P)_n$ and $Y = T^PC_M(X)$. Then,

$$Y_n = X_n \quad \text{and} \quad Y_i = X_{i,-} \cap X_{i,+} \quad \text{for } i < n.$$ 

Proof. This is a direct consequence of Lemma 4.2.2. \hfill \Box

Lemma 4.3.4. Suppose that $P$ satisfies axioms (G0), (G1), (G2) and (G3). Then, for $X \in \text{Cell}(P)$, $T^M_P \circ T^PC_M(X) = X$.

Proof. Let $n \in \mathbb{N}$, $X \in \text{Cell}(P)_n$, $Y = T^PC_M(X)$ and $Z = T^M_P(Y)$. We show that $X_{i,\epsilon} = Z_{i,\epsilon}$ by a decreasing induction on $i$. By Lemma 4.3.3, we have

$$Z_n = Y_n = X_n$$

and, for $i < n$, we have

$$Z_{i,-} = Y_i \cup Z^\mathbb{F}_{i+1,-}$$

$$= (X_{i,-} \cap X_{i,+}) \cup X^\mathbb{F}_{i+1,-}$$

$$= X_{i,-} \quad \text{(by Lemma 2.1.7)}.$$ 

Similarly, $Z_{i,+} = X_{i,+}$, so $X = Z$. Hence, $T^M_P \circ T^PC_M(X) = X$. \hfill \Box

Lemma 4.3.5. $T^M_P \circ T^PC_M = T^PC_C$

Proof. Let $n \in \mathbb{N}$ and $X \in \text{Cell}(P)_n$. Then,

$$T^M_P \circ T^PC_M(X) = R(\max(\cup X))$$

$$= R(\cup X)$$

$$= T^PC_C(X).$$

Hence, $T^M_P \circ T^PC_M = T^PC_C$. \hfill \Box
4.3.6 **Sources and targets.** Given $n > 0$ and $X$ a maximal $n$-fgs, define the *source* $	ilde{\partial}^- X$ (resp. *target* $\tilde{\partial}^+ X$) of $X$ as the maximal $(n-1)$-fgs $Y$ with

$$Y_{n-1} = X_{n-1} \cup X_n^- \quad \text{ (resp. } X_{n-1} \cup X_n^+)$$

$$Y_i = X_i \quad \text{ for } i < n - 1.$$  

Similarly, given $n > 0$ and $X$ a closed $n$-fgs, define the *source* $\bar{\partial}^- X$ (resp. *target* $\bar{\partial}^+ X$) of $X$ as the closed $(n-1)$-fgs $Y$ with

$$Y = R(X \setminus (X_n \cup R(X_n^+))) \quad \text{ (resp. } R(X \setminus (X_n \cup R(X_n^-)))).$$  

These sources and targets are compatible with the translation functions from Paragraph 4.3.1 as state the following lemmas.

**Lemma 4.3.7.** Suppose that $P$ satisfies axioms $(G0)$, $(G1)$, $(G2)$ and $(G3)$. For $n > 0$, $\epsilon \in \{-, +\}$ and $X \in \text{Cell}(P)_n$, we have

$$T^M_{P^C}(\partial^\epsilon X) = \tilde{\partial}^\epsilon (T^M_P(X)).$$  

**Proof.** Let $Y = T^M_{P^C}(\partial^\epsilon X)$, $X' = T^M_P(X)$ and $Z = \tilde{\partial}^\epsilon X'$. By Lemma 4.3.3,

$$Y_{n-1} = X_{n-1,\epsilon},$$

$$Y_i = X_i^- \cap X_i^+ \quad \text{ for } i < n - 1,$$

and

$$X'_n = X_n,$$

$$X'_i = X_i^- \cap X_i^+ \quad \text{ for } i < n.$$  

If $\epsilon = -$, then

$$Z_{n-1} = (X_{n-1, -} \cap X_{n-1, +}) \cup X_n^- \quad \text{ (by Lemma 2.1.7)}$$

$$Z_i = X'_i \quad \text{ for } i < n - 1$$

so $Y = Z$. Similarly, when $\epsilon = +$, $Y = Z$. Which concludes the proof. \qed

**Lemma 4.3.8.** For $n > 0$, $\epsilon \in \{-, +\}$ and $X \in \text{Max}(P)_n$, we have

$$T^M_{P^C}(\partial^\epsilon X) = \bar{\partial}^\epsilon (T^M_{P^C}(X)).$$  

**Proof.** By symmetry, we only prove the case $\epsilon = -$. Let $Y = T^M_{P^C}(\partial^- X)$ and $Z = \bar{\partial}^-(T^M_{P^C}(X))$. By unfolding the definitions, we have

$$Y = R((X \setminus X_n) \cup X_n^-),$$

$$Z = R(R(X \setminus (X_n \cup R(X_n^+)))).$$

In order to show that $Y \subseteq Z$, we only need to prove that $Y' \subseteq Z$ where

$$Y' := (X \setminus X_n) \cup X_n^+.$$  

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First, we have that \( Y' \subseteq R(X) \). Moreover,
\[
Y' \cap (X_n \cup R(X_n^+)) \\
= ((X \setminus X_n) \cup X_n^\mp) \cap (X_n \cup R(X_n^+)) \\
= ((X \setminus X_n) \cup X_n^\mp) \cap R(X_n^+) \\
= (X \setminus X_n) \cap R(X_n^+) \\
= X \cap R(X_n^+) \\
= \emptyset \quad \text{(since } X \text{ is maximal}).
\]
So \( Y' \subseteq Z \), which implies that \( Y \subseteq Z \).
Similarly, to show that \( Z \subseteq Y \), we only need to prove that \( Z' \subseteq Y \) where
\[
Z' := R(X) \setminus (X_n \cup R(X_n^+)).
\]
But
\[
Z' \subseteq Y \Leftrightarrow R(X) \subseteq Y \cup X_n \cup R(X_n^+)
\]
and
\[
Y \cup X_n \cup R(X_n^+) = R((X \setminus X_n) \cup X_n^\mp) \cup X_n \cup R(X_n^+) \\
= R((X \setminus X_n) \cup X_n^\mp \cup X_n^+) \cup X_n \\
= R((X \setminus X_n) \cup X_n^\mp \cup X_n^+) \cup X_n \\
= R((X \setminus X_n) \cup X_n^- \cup X_n^+ \cup X_n) \\
= R((X \setminus X_n) \cup X_n^- \cup X_n^+ \cup X_n) \\
= R(X).
\]
So \( Z' \subseteq Y \), which implies that \( Z \subseteq Y \). Hence, \( Y = Z \), which concludes the proof.

**Lemma 4.3.9.** Suppose that \( P \) satisfies axioms \((G0)\), \((G1)\), \((G2)\) and \((G3)\). For \( n > 0 \), \( \epsilon \in \{-, +\} \) and \( X \in \text{Cell}(P)_n \),
\[
T_{\text{Cl}}^{\text{PC}}(\partial^\epsilon X) = \tilde{\delta}(T_{\text{Cl}}^{\text{PC}}(X)).
\]

**Proof.** We have
\[
T_{\text{Cl}}^{\text{PC}}(\partial^\epsilon X) = T_{\text{Cl}}^{\text{PC}} \circ T_{\text{M}}^{\text{PC}}(\partial^\epsilon X) \quad \text{(by Lemma 4.3.5)} \\
= T_{\text{Cl}}^{\text{PC}}(\tilde{\delta}(T_{\text{M}}^{\text{PC}}(X))) \quad \text{(by Lemma 4.3.7)} \\
= \tilde{\delta}(T_{\text{Cl}}^{\text{PC}} \circ T_{\text{M}}^{\text{PC}}(X)) \quad \text{(by Lemma 4.3.8)} \\
= \tilde{\delta}(T_{\text{Cl}}^{\text{PC}}(X))
\]
which concludes the proof.

### 4.4 Compositions and identities

Here, we define compositions and identities for the globular sets \( \text{Max}(P) \) and \( \text{Closed}(P) \), and prove some compatibility results with the translations functions.

Given \( i \leq n \in \mathbb{N} \) and \( X,Y \) two maximal \( n \)-fgs, we define the *maximal \( i \)-composition \( X \star^M_i Y \) of \( X \) and \( Y \) as a maximal \( n \)-fgs defined by
\[
X \star^M_i Y = \max(R(X) \cup R(Y)).
\]
Similarly, given $i \leq n \in \mathbb{N}$ and $X,Y$ two closed $n$-fgs, we define the closed composition $X \star_i^\text{cl} Y$ of $X$ and $Y$ as a closed $n$-fgs defined by

$$X \star_i^\text{cl} Y = X \cup Y.$$ 

For simplicity, we sometimes write $\star^\text{cl}$ (resp. $\star^\text{M}$) for $\star_i^\text{cl}$ (resp. $\star_i^\text{M}$). Given $n \in \mathbb{N}$ and a closed (resp. maximal) $n$-fgs $X$, we define the identity $\text{id}_{n+1}(X)$ of $X$ as the closed (resp. maximal) $(n+1)$-fgs 

$$(X_0, \ldots, X_n, \emptyset).$$

These compositions and identities are compatible with the translation functions from Paragraph 4.3.1 as state the following lemmas.

**Lemma 4.4.1.** For $k < n \in \mathbb{N}$ and $k$-composable $n$-cells $X$ and $Y$ of $P$,

$$T_{\text{Cl}}^{\text{PC}}(X \star_k Y) = T_{\text{Cl}}^{\text{PC}}(X) \star^\text{cl} T_{\text{Cl}}^{\text{PC}}(Y).$$

**Proof.** Let $Z = X \star_k Y$. We have

$$T_{\text{Cl}}^{\text{PC}}(X \star_k Y) = R(\cup Z)$$

and

$$T_{\text{Cl}}^{\text{PC}}(X) \star^\text{cl} T_{\text{Cl}}^{\text{PC}}(Y) = R(\cup X) \cup R(\cup Y) = R((\cup X) \cup (\cup Y)).$$

By definition of composition, $\cup Z \subseteq (\cup X) \cup (\cup Y)$, so

$$T_{\text{Cl}}^{\text{PC}}(X \star_k Y) \subseteq T_{\text{Cl}}^{\text{PC}}(X) \star^\text{cl} T_{\text{Cl}}^{\text{PC}}(Y).$$

For the other inclusion, note that $X_{i,\epsilon} \subseteq Z_{i,\epsilon}$ for $(i, \epsilon) \neq (k, +)$, and

$$X_{k,+} = (X_{k,-} \cup X_{k+1,-}^+) \setminus X_{k+1,-}^-$$

$$\subseteq Z_{k,-} \cup Z_{k+1,-}^+$$

$$\subseteq R(\cup Z)$$

so $\cup X \subseteq R(\cup Z)$. Similarly, $\cup Y \subseteq R(\cup Z)$, thus

$$(\cup X) \cup (\cup Y) \subseteq R(\cup Z),$$

which implies that

$$T_{\text{Cl}}^{\text{PC}}(X) \star^\text{cl} T_{\text{Cl}}^{\text{PC}}(Y) \subseteq T_{\text{Cl}}^{\text{PC}}(X \star_k Y).$$

Hence,

$$T_{\text{Cl}}^{\text{PC}}(X) \star^\text{cl} T_{\text{Cl}}^{\text{PC}}(Y) = T_{\text{Cl}}^{\text{PC}}(X \star_k Y).$$

**Lemma 4.4.2.** For $n \in \mathbb{N}$ and an $n$-cell $X \in \text{Cell}(P)$,

$$T_{\text{Cl}}^{\text{PC}}(\text{id}_{n+1}(X)) = \text{id}_{n+1}(T_{\text{Cl}}^{\text{PC}}(X)).$$

**Proof.** It readily follows from the definitions.

**Lemma 4.4.3.** For $n \in \mathbb{N}$ and $X,Y \in \text{Closed}(P)_n$,

$$T_{\text{M}}^{\text{Cl}}(X \star^\text{cl} Y) = T_{\text{M}}^{\text{Cl}}(X) \star^\text{M} T_{\text{M}}^{\text{Cl}}(Y).$$
Proof. We have
\[
T^\text{Cl}_M(X) \ast M T^\text{Cl}_M(Y) = \max(R(T^\text{Cl}_M(X)) \cup R(T^\text{Cl}_M(Y)))
\]
\[
= \max(X \cup Y) \quad \text{(by Lemma 4.3.2)}
\]
\[
= T^\text{Cl}_M(X \ast \text{Cl} Y)
\]
which concludes the proof.

Lemma 4.4.4. For \( k < n \in \mathbb{N} \) and \( k \)-composable \( n \)-cells \( X \) and \( Y \) of \( P \),
\[
T^\text{PC}_M(X \ast_k Y) = T^\text{PC}_M(X) \ast M T^\text{PC}_M(Y).
\]

Proof. We have
\[
T^\text{PC}_M(X \ast_k Y) = T^\text{Cl}_M(T^\text{PC}_M(X) \ast \text{Cl} T^\text{PC}_M(Y)) \quad \text{(by Lemma 4.3.2 and 4.3.5)}
\]
\[
= T^\text{Cl}_M \circ T^\text{PC}_M(X) \ast M T^\text{Cl}_M \circ T^\text{PC}_M(Y) \quad \text{(by Lemma 4.4.1)}
\]
\[
= T^\text{PC}_M(X) \ast M T^\text{PC}_M(Y) \quad \text{(by Lemma 4.4.3)}
\]
which concludes the proof.

Lemma 4.4.5. For \( n \in \mathbb{N} \) and an \( n \)-cell \( X \in \text{Cell}(P) \),
\[
T^\text{PC}_M(\text{id}_{n+1}(X)) = \text{id}_{n+1}(T^\text{PC}_M(X)).
\]

Proof. It readily follows from the definitions.

4.5 Alternative cells

In this subsection, we define notions of cells for \( \text{Max}(P) \) and \( \text{Closed}(P) \) and prove that the associated \( \omega \)-category of cells are isomorphic to \( \text{Cell}(P) \).

Given \( n \in \mathbb{N} \) and \( X \in \text{Max}(P)_n \), we say that \( X \) is \textit{maximal-well-formed} when
- \( X_n \) is fork-free,
- \( \text{\bar{\partial}}^- X \) and \( \text{\bar{\partial}}^+ X \) are maximal-well-formed,
- if \( n \geq 2 \), \( \text{\bar{\partial}}^- \circ \text{\bar{\partial}}^- (X) = \text{\bar{\partial}}^- \circ \text{\bar{\partial}}^+ (X) \) and \( \text{\bar{\partial}}^+ \circ \text{\bar{\partial}}^- (X) = \text{\bar{\partial}}^+ \circ \text{\bar{\partial}}^+ (X) \).

We write \( \text{Max}_{\text{WF}}(P) \) for the graded set of maximal-well-formed fgs of \( P \). Similarly, given \( n \in \mathbb{N} \) and \( X \in \text{Closed}(P)_n \), we say that \( X \) is \textit{closed-well-formed} when
- \( X_n \) is fork-free,
- \( \text{\bar{\partial}}^- X \) and \( \text{\bar{\partial}}^+ X \) are closed-well-formed,
- if \( n \geq 2 \), \( \text{\bar{\partial}}^- \circ \text{\bar{\partial}}^- (X) = \text{\bar{\partial}}^- \circ \text{\bar{\partial}}^+ (X) \) and \( \text{\bar{\partial}}^+ \circ \text{\bar{\partial}}^- (X) = \text{\bar{\partial}}^+ \circ \text{\bar{\partial}}^+ (X) \).

We write \( \text{Closed}_{\text{WF}}(P) \) for the graded set of closed-well-formed fgs of \( P \).

Lemma 4.5.1. \( T^\text{Cl}_M \) induces a bijection between \( \text{Max}_{\text{WF}}(P) \) and \( \text{Closed}_{\text{WF}}(P) \).
Proof. We already know that $T_{CM}^M$ is a bijection by Lemma 4.3.2. For $n \in \mathbb{N}$, we show that $T_{CM}^M$ sends a maximal-well-formed $n$-fgs $X$ to a closed-well-formed fgs by induction on $n$. If $n = 0$, the result is trivial. So suppose $n > 0$. Let $Y = T_{CM}^M(X)$. Then, $Y_n = X_n$ is fork-free and, for $\epsilon \in \{-, +\}$, $\partial Y = T_{CM}^M(\partial Y(X))$ by Lemma 4.3.8, and it is closed-well-formed by induction. Also, when $n \geq 2$,

$$\partial^\epsilon \circ \partial^\epsilon (Y) = T_{CM}^M(\partial^\epsilon \circ \partial^\epsilon (X))$$

(by Lemma 4.3.8)

$$= T_{CM}^M(\partial^\epsilon \circ \partial^\epsilon (X))$$

$$= \partial^\epsilon \circ \partial^\epsilon (Y)$$

So $Y$ is closed-well-formed. Similarly, $T_{CM}^M$ sends closed-well-formed fgs to maximal-well-formed fgs, which concludes the proof.

**Lemma 4.5.2.** Suppose that $P$ satisfies (G0), (G1), (G2) and (G3). For $n \in \mathbb{N}$ and $X \in \text{Cell}(P)_n$, $T_{CM}^PC(X) \in \text{Max}_{WF}(P)_n$.

Proof. We proceed by induction on $n$. If $n = 0$, the result is trivial. So suppose that $n > 0$ and let $Y = T_{CM}^PC(X)$. Since $Y_n = X_n$, $Y_n$ is fork-free. Moreover, by Lemma 4.3.7, $\partial Y = T_{CM}^PC(\partial Y) X$ for $\epsilon \in \{-, +\}$. By the induction hypothesis, $\partial Y$ is maximal-well-formed. And, when $n \geq 2$, for $\eta \in \{-, +\}$,

$$\partial^\eta \circ \partial^\eta (Y) = T_{CM}^PC(\partial^\eta \circ \partial^\eta (X))$$

(by Lemma 4.3.7)

$$= T_{CM}^PC(\partial^\eta \circ \partial^\eta (X))$$

$$= \partial^\eta \circ \partial^\eta (Y).$$

Hence, $Y$ is maximal-well-formed.

**Lemma 4.5.3.** Suppose that $P$ satisfies (G0), (G1), (G2) and (G3). For $n \in \mathbb{N}$ and $X \in \text{Max}_{WF}(P)_n$, there exists an $n$-cell $Y$ such that $T_{CM}^PC(Y) = X$.

Proof. We proceed by induction on $n$. If $n = 0$, the result is trivial. So suppose that $n > 0$. By induction, let $S, T \in \text{Cell}(P)$ be such that $T_{CM}^PC(S) = \partial^- X$ and $T_{CM}^PC(T) = \partial^+ X$. When $n \geq 2$, for $\epsilon \in \{-, +\}$, we have

$$\partial^\epsilon S = T_{PC}^M \circ T_{CM}^PC(\partial^\epsilon S)$$

(by Lemma 4.3.4)

$$= T_{PC}^M(\partial^\epsilon (T_{CM}^PC(S)))$$

(by Lemma 4.3.7)

$$= T_{CM}^M(\partial^\epsilon \circ \partial^- (X))$$

$$= T_{CM}^M(\partial^\epsilon \circ \partial^- (X))$$

(because $X$ is maximal-well-formed)

$$= T_{CM}^M(\partial^\epsilon \circ \partial^- (X))$$

$$= T_{CM}^M(\partial^\epsilon \circ \partial^- (X))$$

$$= T_{CM}^M(\partial^\epsilon \circ \partial^- (X))$$

$$= T_{CM}^M(\partial^\epsilon T)$$

$$= \partial^\epsilon T.$$

Also,

$$(S_{n-1} \cup X_{n+}^+) \setminus X_n^- = (X_{n-1} \cup X_n^+ \cup X_n^+) \setminus X_n^-$$

$$= X_{n-1} \cup X_n^+$$

$$= T_{n-1}.$$
Similarly, \((T_{n-1} \cup X_{n}^-) \setminus X_{n}^+ = S_{n-1}\) so \(X_n\) moves \(S_{n-1}\) to \(T_{n-1}\). Hence, the \(n\)-pre-cell \(Y\) defined below is a cell:

\[
\begin{align*}
Y_n &= X_n \\
Y_{n-1,-} &= S_{n-1} \\
Y_{n-1,+} &= T_{n-1} \\
Y_i,\delta &= S_{i,\delta} \quad \text{for } i < n - 1 \text{ and } \delta \in \{-, +\}.
\end{align*}
\]

Let \(Z = T_{M}^{PC}(Y)\). We have \(Z_n = X_n\) and

\[
\begin{align*}
\partial^- Z &= \partial^- (T_{M}^{PC}(Y)) \\
&= T_{M}^{PC} (\partial^- Y) \quad \text{(by Lemma 4.3.7)} \\
&= T_{M}^{PC} (S) \\
&= \partial^- X.
\end{align*}
\]

So, by definition of \(\partial^-\), \(Z_i = X_i\) for \(i < n - 1\) and \(Z_{n-1} \cup X_{n}^- = X_{n-1} \cup X_{n}^-\).

Since \(X\) and \(Z\) are maximal, we have

\[
X_{n-1} \cap X_{n}^- = Z_{n-1} \cap X_{n}^- = \emptyset.
\]

Hence, \(X_{n-1} = Z_{n-1}\) and \(X = Z = T_{M}^{PC}(Y)\) which concludes the proof.

**Lemma 4.5.4.** Suppose that \(P\) satisfies (G0), (G1), (G2) and (G3). Then, \(T_{M}^{PC}\) induces a bijection between the \(Cell(P)\) and \(Max_{WF}(P)\).

**Proof.** By Lemma 4.5.3, \(T_{M}^{PC}: Cell(P) \to Max_{WF}(P)\) is onto, and by Lemma 4.3.4, it is one-to-one, so it is bijective.

**Theorem 4.5.5.** Suppose that \(P\) satisfies (G0), (G1), (G2) and (G3). Then, \(Max_{WF}(P)\) is an \(\omega\)-category and \(T_{M}^{PC}\) induces an isomorphism between \(Cell(P)\) and \(Max_{WF}(P)\).

**Proof.** We first prove that composition is well-defined. Let \(i \leq n \in \mathbb{N}\) and \(X, Y \in Max_{WF}(P)_n\) be such that \(\partial^+_i X = \partial^-_i Y\). By Lemma 4.5.4, there exist \(X', Y' \in Cell(P)_n\) such that \(T_{M}^{PC}(X') = X\) and \(T_{M}^{PC}(Y') = Y\). By Lemma 4.3.7, we have

\[
T_{M}^{PC}(\partial^+_i X') = \partial^+_i X = \partial^-_i Y = T_{M}^{PC}(\partial^-_i Y'),
\]

and, by Lemma 4.5.4, \(\partial^+_i X' = \partial^-_i Y'\) so \(X'\) and \(Y'\) are \(i\)-composable.

By Lemma 4.5.4, \(T_{M}^{PC}(X' \ast_i Y') \in Max_{WF}(P)\) and, by Lemma 4.4.4, \(X \ast M Y \in Max_{WF}(P)\).

Now, we prove that \(Max_{WF}(P)\) satisfies the axioms ?? to ?? of \(\omega\)-categories. But it readily follows from Lemmas 4.5.4, 4.4.4 and 4.4.5. Indeed, for example, for axiom ??, given \(i \leq n \in \mathbb{N}\) and \(i\)-composable \(X, Y, Z \in Max_{WF}(P)_n\), by Lemma 4.5.4, there exist \(X', Y', Z' \in Cell(P)_n\) such that \(X = T_{M}^{PC}(X')\), \(Y = T_{M}^{PC}(Y')\) and \(Z = T_{M}^{PC}(Z')\).

By a similar argument as above, \(X', Y', Z'\) are \(i\)-composable and, by Lemma 4.4.4,

\[
(X \ast M Y) \ast_i M Z = T_{M}^{PC}((X' \ast_i Y') \ast_i Z') = T_{M}^{PC}(X' \ast_i (Y' \ast_i Z')) = X \ast M (Y \ast M Z),
\]

so ?? is satisfied.

Hence, \(Max_{WF}(P)\) is an \(\omega\)-category, and \(T_{M}^{PC}\) is an isomorphism by Lemmas 4.5.4, 4.4.4, 4.4.5.
Lemma 4.5.6. Suppose that $P$ satisfies (G0), (G1), (G2) and (G3). Then, $T^{PC}_{\text{Cl}}$ induces a bijection between $\text{Cell}(P)$ and $\text{Closed}_{\text{WF}}(P)$.

Proof. The result is a consequence of Lemmas 4.3.5 and 4.5.1 and 4.5.4. 

Theorem 4.5.7. Suppose that $P$ satisfies (G0), (G1), (G2) and (G3). Then, $\text{Closed}_{\text{WF}}(P)$ is an $\omega$-category and $T^{PC}_{\text{Cl}}$ induces an isomorphism between $\text{Cell}(P)$ and $\text{Closed}_{\text{WF}}(P)$.

Proof. By a similar proof than for Theorem 4.5.5, using Lemmas 4.3.9, 4.4.1, 4.4.2 and 4.5.6.

5 Unifying formalisms of pasting diagrams

In this section, we show that parity complexes, pasting schemes and augmented directed complexes are generalized parity complexes.

5.1 Encoding parity complexes

In this subsection, we show that parity complexes are generalized parity complexes, after applying two reasonable restrictions. Firstly, parity complexes do not require all the generators to be relevant even though generalized parity complexes do. But, by [19, Theorem 4.2], irrelevant generators do not play any role in the generated $\omega$-category $\text{Cell}(P)$ for $P$ an $\omega$-hypergraph, and $\text{Cell}(P) \simeq \text{Cell}(P')$ where $P'$ is the set of relevant elements of $P$. Secondly, as discussed in Section ??, parity complexes do not ensure torsion-freeness, which can break the freeness property. A natural fix is to require parity complexes to moreover satisfy (G4). Then, in the following, we will suppose given an $\omega$-hypergraph $P$ satisfying axioms (C0) to (C5) and (G2) and (G4).

Lemma 5.1.1 ([20, Proposition 1.4]). For $n > 0$, $U,V \subseteq P_n$ with $U$ tight, $V$ fork-free and $U \subseteq V$, we have that $U$ is a segment for $\varsigma_V$.

Proof. Suppose given $x,y,z \in V$ such that $x,z \in U$ and $x \varsigma_U y \varsigma_V z$. Then, there is $w \in \{+\} \cap \{\rangle\}$. By definition of tightness, since $y \varsigma_V z$, we have $y^\rho \cap U^\rho = \emptyset$. So there is $y^\rho' \in U$ such that $w \in y'^\rho$. Since $V$ is fork-free, $y = y^\rho$. Hence, $U$ is a segment for $\varsigma_V$.

Lemma 5.1.2. Let $n \in \mathbb{N}$. For $x \in P_n$, $x$ satisfies the segment condition.

Proof. Let $m < n \in \mathbb{N}$, $x \in P_m$ and $X$ be an $m$-cell. Suppose that $\langle x \rangle_{m,-} \subseteq X_m$. By (C5), $\langle x \rangle_{m,-}$ is tight. Then, by [20, Proposition 1.4], $\langle x \rangle_{m,-}$ is a segment for $\varsigma_{X_m}$.

Now suppose that $\langle x \rangle_{m,+} \subseteq X_m$. By contradiction, assume that $\langle x \rangle_{m,+}$ is not a segment for $\varsigma_{X_m}$. Thus, by definition of $\varsigma_{X_m}$, there exist $p > 1$ and $u_0, \ldots, u_p \in X_m$ with $u_0,u_p \in \langle x \rangle_{m,+}$, $u_1, \ldots, u_{p-1} \notin \langle x \rangle_{m,+}$ and $u_1 \varsigma_{X_m} u_{p+1}$. By definition of $\varsigma_{X_m}$, there exist $z_0, \ldots, z_{p-1}$ such that $z_i \in u_i \cap u_{i+1}$. Note that $z_0 \in \langle x \rangle_{m,+}$. Indeed, if $z_0 \in v^\rho$ for some $v \in X_m$, then, since $X_m$ is fork-free, $v = u_1$, so $v \notin \langle x \rangle_{m,+}$. Similarly, $z_{p-1} \in \langle x \rangle_{m,+}$. Since $x$ is relevant by (G2), $\langle x \rangle_{m,+} = \langle x \rangle_{m,+} \subseteq X_m$. By [19, Lemma 3.2] (which is essentially Theorem 2.2.3, but for the axioms of parity complexes), $\overline{\Xi}(X, \langle x \rangle_{m,+})$
is a cell with $Y_m = (X_m \setminus \langle (x)_{m,-} \rangle) \cup \langle (x)_{m,-} \rangle$. So $\langle (x)_{m,-} \rangle \subseteq Y_m$ and, as previously shown, $\langle (x)_{m,-} \rangle$ is a segment for $\triangleleft Y_m$. But, since $\langle (x)_{m,-} \rangle$ and $\langle (x)_{m,+} \rangle$, there exist $u_0, u'_0 \in \langle (x)_{m,-} \rangle$ such that $z_0 \in u'_0$ and $z_{p-1} \in u'_0$. So $u'_0 \lessdot_{X_m} u_1 \lessdot_{X_m} \ldots \lessdot_{X_m} u_{p-1} \lessdot_{X_m} u'_0$ with $u_1, \ldots, u_{p-1} \notin \langle (x)_{m,-} \rangle$, contradicting the fact that $\langle (x)_{m,-} \rangle$ is a segment for $\triangledown Y_m$. Thus, $\langle (x)_{m,-} \rangle$ is a segment for $X_m$. Hence, $x$ satisfies the segment condition.

**Theorem 5.1.3.** $P$ is a generalized parity complex.

**Proof.** (G0) is a consequence of (C0). (G1) is a consequence of (C3). And (G3) is a consequence of Lemma 5.1.2.

The category of cells of the parity complex is, of course, isomorphic to the category of cells of the associated generalized parity complex.

### 5.2 Encoding pasting schemes

In this subsection, we embed loop-free pasting schemes in generalized parity complexes. More precisely, we will only embed the loop-free pasting schemes that are torsion-free (that is, whose $\omega$-hypergraph satisfies (G4)) since, like for parity complexes, the ones that are not torsion-free can not be expected to induce free $\omega$-categories. So, in the following, we suppose given an $\omega$-hypergraph $P$ satisfying (S0), (S1), (S2), (S3), (S4), (S5) and (G4).

**Lemma 5.2.1.** Let $k < n \in \mathbb{N}$, $x \in P_n$ and $y \in P_k$. If $x B_{n-1}^n B_{k}^{n-1} y$ then $y \in B_k^n(x)$ or $x E_{n-1}^n B_{k}^{n-1} y$. Dually, if $x E_{n-1}^n B_{k}^{n-1} y$ then $y \in E_k^n(x)$ or $x E_{n-1}^n B_{k}^{n-1} y$.

**Proof.** We do an induction on $n-k$. If $k = n-1$, the result is trivial. If $k = n-2$, the result is a consequence of (S1). Otherwise, suppose that $k < n-2$. We will only prove the first part, since the second is dual. So suppose that $y \notin B_k^n(x)$.

By the definition of $B$, we have

$$\neg(x B_{n-1}^n B_{k}^{n-1} y ) \quad \text{or} \quad \neg(x E_{n-1}^n E_{k}^{n-1} y).$$

By symmetry, we can suppose that $\neg(x B_{n-1}^n E_{k}^{n-1} y)$. Let $u \in P_{n-1}$ be minimal for $\triangledown$ such that $x B_{n-1}^n u B_{k}^{n-1} y$. Then, there are two possible cases: either $u B_{n-2}^n R_{k}^{n-2} y$ or $u E_{n-2}^n B_{k}^{n-2} y$.

In the first case, let $v \in P_{n-2}$ be such that $u B_{n-2}^n v R_{k}^{n-2} y$. By the minimality of $u$, we have $\neg(x B_{n-1}^n E_{n-2}^n v)$, so $\neg(x B_{n-2}^n v)$ by definition of $B$. By Axiom (S1), we have $x E_{n-1}^n E_{n-2}^n v$. So $x E_{n-1}^n B_{k}^{n-1} y$.

In the second case, since we supposed $\neg(x B_{n-1}^n E_{k}^{n-1} y)$, we have $\neg(u E_{n}^{n-1} y)$. By induction, $u B_{n-2}^n R_{k}^{n-2} y$ and we can conclude using the first case.

**Lemma 5.2.2.** Let $n > 0$ and $X$ be an $n$-wfs of $P$. Then $\partial^+ X = \partial X$.

**Proof.** We only prove the case $\epsilon = -$. So let $n > 0$ and $X$ be an $n$-wfs of $P$. Recall that $\partial^- X = X \setminus E(X)$ and $\partial^+ X = R(X \setminus (X_n \cup R(X^+_n)))$.

**Step 1:** $\partial^- X \subseteq \partial X$. We have to show that

$$R(X \setminus (X_n \cup R(X^+_n))) \subseteq X \setminus E(X).$$

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Since $X \setminus E(X)$ is closed (by [8, Theorem 12]), it is equivalent to

$$X \setminus (X_n \cup R(X_n^+)) \subseteq X \setminus E(X)$$

which is itself equivalent to

$$E(X) \subseteq (X_n \cup R(X_n^+))$$

which holds.

Step 2: $\partial^- X \subseteq \bar{\partial}^- X$. We have to show that

$$X \setminus E(X) \subseteq R(X \setminus (X_n \cup R(X_n^+))) = \bar{\partial}^- (X).$$

Let $m < n \in \mathbb{N}$ and $x \in (X \setminus E(X))_m$. If $x \not\in R(X_n^+)$ then $x \in \bar{\partial}^- (X)$. So suppose that $x \in R(X_n^+)$. Since $(X_n)_n = X_n^+$, it implies that $m < n - 1$. By definition of $R(X_n^+)$, there exists $y \in X_n$ such that $yE_n R_{m-1}^n x$ and, by Axiom (S2), we can take $y$ minimal for $\epsilon$ satisfying this property. By Lemma 5.2.1, it holds that $yB_n x R_{m-1}^n$. Then, there is no $y' \in X_n$ such that $y' E_n z$: otherwise, $y' E_{n-1}^n x$ and $y' \epsilon y$, contradicting the minimality of $y$. So $z \not\in R(X_n^+)$ and $z R x$. It implies that $z \in X \setminus (X_n \cup R(X_n^+))$ and $x \in \bar{\partial}^- X$.

Lemma 5.2.3. Let $n \in \mathbb{N}$ and $X \in WF(P)_n$. Then $X \in \text{Closed}_{WF(P)}(P)_n$.

Proof. We prove this lemma by induction on $n$. If $n = 0$, the result is trivial. So suppose $n > 0$. Since $X$ is well-formed, $X_n$ is fork-free. Moreover, using Lemma 5.2.2, for $\epsilon \in \{-, +\}$, $\bar{\partial}^\epsilon (X) = \partial^\epsilon (X)$ which is well-formed. By induction, $\bar{\partial}^\epsilon (X) \subseteq \text{Closed}_{WF(P)}(P)_{n-1}$. Also, when $n \geq 2$, since $\partial^\epsilon \circ \partial^- (X) = \partial^\epsilon \circ \partial^+ (X)$, by Lemma 5.2.2, $\partial^\epsilon \circ \partial^- (X) = \partial^\epsilon \circ \partial^+ (X)$. Hence, $X \in \text{Closed}_{WF(P)}(P)_n$.

Lemma 5.2.4. Let $n \in \mathbb{N}$, $X$ be an $n$-wfs, $S \subseteq P_{n+1}$ be a finite subset with $S$ fork-free, $S^\top \subseteq X$ and let $Y = X \cup R(S)$. Then $Y$ is an $(n+1)$-wfs of $P$ and $\partial^- Y = X$.

Proof. Let $k = |S|$. We show this lemma by induction on $k$. If $k = 0$, the result is trivial. If $k = 1$, the result is a consequence of [8, Proposition 8]. So suppose $k > 1$. By Axiom (S2), take $x \in S$ minimal for $\epsilon$. By minimality, we have

$$x^- \subseteq S^\top \subseteq X.$$

Using [8, Proposition 8], $X \cup R(x)$ is well-formed. By (S5), $X \cap E(x) = \emptyset$, so $\partial^- (X \cup R(x)) = X$.

Let $X' = \partial^+ (X \cup R(x))$ and $S' = S \setminus \{x\}$. We have

$$S'^\top \subseteq X' \iff S'^\top \subseteq X' \cup S'^\top$$

$$\iff S^- \subseteq X' \cup S'^\top \cup x^-$$

$$\iff S^- \subseteq (X' \setminus x^-) \cup x^+ \cup S'^\top \cup x^-$$

$$\iff S^- \subseteq X_n \cup S^+$$

$$\iff S^\top \subseteq X_n$$

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so \(S^{\circ} \subseteq X'\). By induction, \(X' \cup R(S')\) is well-formed and \(\partial^- (X' \cup R(S')) = X'\). Since \(WF(P)\) has the structure of an \(\omega\)-category by [8, Theorem 12], we can compose \(X \cup R(x)\) and \(X' \cup R(S')\). So

\[X \cup R(S) = X \cup R(x) \cup X' \cup R(S')\]

is well-formed and \(\partial^- (X \cup R(S)) = X\).

\(\Box\)

**Lemma 5.2.5.** For \(X \in \text{Closed}_{WF}(P)\), we have \(X \in WF(P)\).

**Proof.** Let \(n \in \mathbb{N}\) and \(X \in \text{Closed}_{WF}(P)_n\). We prove this lemma by induction on \(n\). If \(n = 0\), the result is trivial. So suppose \(n > 0\). Let \(Y = \partial^- X\). By definition, \(Y \in \text{Closed}_{WF}(P)\) and, by induction, \(Y \in WF(P)\). By definition of \(\partial^-\), we have \(X^{\circ} \subseteq Y\). By Lemma 5.2.4, \(Y \cup R(X_n)\) is well-formed. But \(Y = R(X \setminus (X_n \cup R(X_n)))\), hence \(X \cup R(X_n)\) is well-formed.

\(\Box\)

**Lemma 5.2.6.** Let \(n \in \mathbb{N}\) and \(x \in P_n\). Then, for \(i < n\) and \(\epsilon \in \{-, +\}\),

\[\partial^-_i R(x) = R((x)_{i, \epsilon}).\]

**Proof.** Let \(n \in \mathbb{N}\), \(x \in P_n\) and \(i < n\). By symmetry, we will only prove that \(\partial^-_i (R(x)) = R((x)_{i, -})\). We have

\[
\partial^-_i (R(x)) = \partial^-_i (T^M_C_i (\{x\})) = T^M_C_i (\partial^-_i (\{x\})) = T^M_C_i ((x)_{i, -}) = R((x)_{i, -}).
\]

Hence, \(\partial^-_i R(x) = R((x)_{i, -})\).

\(\Box\)

**Lemma 5.2.7.** For all \(n \in \mathbb{N}\) and \(x \in P_n\), \(x\) is relevant.

**Proof.** Let \(n \in \mathbb{N}\) and \(x \in P_n\). By axiom (S3), \(R(x)\) is well-formed. So, for \(i \leq n\) and \(\epsilon \in \{-, +\}\), \(\partial_\epsilon^x R(x)\) is well-formed. Then, by Lemma 5.2.6, \((x)_{i, -}\) and \((x)_{i, +}\) are fork-free. We show that \((x)_{i, -}^\pm = (x)_{i, -}\) and \((x)_{i, +}^\pm = (x)_{i, -}\).

We have \((x)_{n, -}^\pm = (x)_{n, +} = (x)_{n-1, +}\) and, similarly, \((x)_{n, +}^\pm = (x)_{n-1, -}\). For \(i < n-1\) and for \(\epsilon \in \{-, +\}\), we have

\[
(x)_{i+1, \pm} = \partial^x_\epsilon (x)_{i+1} = \partial^x_\epsilon \partial^x_\epsilon (\{x\}) = \partial^x_\epsilon (\{x\}) = (x)_{i, \epsilon}^\pm
\]

and similarly, \((x)_{i, +}^\pm = (x)_{i, -}\). By definition of \(\partial^x\), it gives \((x)_{i, -} = (x)_{i, +}^\pm\) and \((x)_{i, +} = (x)_{i, -}^\pm\). From these equalities, it readily follows that, for \(0 \leq i < n\) and \(\epsilon \in \{-, +\}\), \((x)_{i+1, \epsilon}\) moves \((x)_{i, -}\) to \((x)_{i, +}\). Hence, \((x)\) is a cell.

\(\Box\)

**Lemma 5.2.8.** Let \(n \geq 0\). Then,

(a) for \(x \in P_n\), \(x\) satisfies the segment condition,

(b) for \(X\) an \(n\)-cell, \(T^X_C(X) \in WF(P)\).
5.3.1 Adc’s as ω-categories. In the following, given \( n \in \mathbb{N} \) and \( x \in P_n \), we will write \( \bar{x} \) to refer to \( x \) as an element of the graded set \( P \) whereas \( x \) alone refer to \( x \) as an element of the monoid \( K_n^\ast \). Given \( n \in \mathbb{N} \),

- for \( s \in K_n^\ast \), we write \( S_n(s) \) for \( \{ \bar{x} \in P_n \mid x \leq s \} \),
- for \( S \subseteq P_n \) finite, we write \( M_n(S) \) for \( \sum_{x \in S} x \).

The \( \omega \)-hypergraph associated to \( K \) is the \( \omega \)-hypergraph structure on \( P \) defined as follows. Given \( n \geq 0 \) and \( \bar{x} \in P_{n+1} \), we define \( \bar{x}^-, \bar{x}^+ \subseteq P_n \) as

\[
\bar{x}^- = S_n(x^-) \quad \bar{x}^+ = S_n(x^+).
\]

where \( x^- \) and \( x^+ \) were defined in Paragraph ??.
5.3.2 Fork-freeness and radicality. For \( n > 0 \), \( s \in K_n^* \) is said to be fork-free when for all \( \bar{x}, \bar{y} \in P_n \) such that \( x + y \leq s \), it holds that \( \bar{x}^\epsilon \cap \bar{y}^\epsilon = \emptyset \) for \( \epsilon \in \{-, +\} \). Given \( s \in K_n^* \), \( s \) is said to be fork-free when \( e(s) = 1 \). Given \( X \) an \( n \)-cell of \( K \), \( X \) is said fork-free when, for \( i \leq n \) and \( \epsilon \in \{-, +\} \), \( X_{i,\epsilon} \) is fork-free. For \( n \geq 0 \) and \( s \in K_n^* \), \( s \) is said radical when for all \( z \in K_n^* \) such that \( 2z \leq s \), \( z = 0 \). We then have the following properties.

Lemma 5.3.3. For \( n > 0 \) and \( \bar{x} \in P_n \), \( \bar{x}^- \neq \emptyset \) and \( \bar{x}^+ \neq \emptyset \). That is, \( P \) satisfies \((G0)\).

Proof. By contradiction, if \( \bar{x}^- = \emptyset \), it implies that \([x]_{n-1,-} = 0\). Hence, \([x]_{i,-} = 0\) for \( i < n \). In particular, \( e([x]_{0,-}) = 0 \), contradicting the fact that the basis is unital. Hence, \( \bar{x}^- \neq \emptyset \) and similarly \( \bar{x}^+ \neq \emptyset \). \(\square\)

Lemma 5.3.4. For \( n \geq 0 \) and \( s \in K_n^* \), if \( s \) is fork-free, then \( s \) is radical.

Proof. If \( n = 0 \), \( s \in K_n^* \) can be written \( s = \sum_{1 \leq i \leq k} x_i \) with \( x_i \in P_0 \). So \( e(s) = k \), and, by fork-freeness, \( k = 1 \). Hence, \( s \) is radical.

Otherwise, assume that \( n > 0 \). By contradiction, suppose that there is \( \bar{x} \in P_n \) such that \( 2x \leq s \). By Lemma 5.3.3, it means that \( \bar{x}^- \cap \bar{x}^- = \emptyset \), contradicting the fact that \( s \) is fork-free. Hence, \( s \) is radical. \(\square\)

Lemma 5.3.5. For \( n \geq 0 \) and an \( n \)-cell \( X \) of \( K \), \( X \) is fork-free.

Proof. We prove this lemma using an induction on \( n \). If \( n = 0 \), since \( e(X_0) = 1 \), \( X \) is fork-free by definition.

Otherwise, suppose that \( n > 0 \). By induction, \( \partial^- X \) and \( \partial^+ X \) are fork-free, so \( X_{i,\epsilon} \) is fork-free for \( i < n \) and \( \epsilon \in \{-, +\} \). Let \( \bar{x}, \bar{y} \in P_n \) be such that \( x + y \leq X_n \). By contradiction, suppose that there is \( \bar{z} \in P_{n-1} \) such that \( \bar{z} \in \bar{x}^- \cap \bar{y}^- \). By [17, Proposition 5.4], there are \( k \geq 1 \), \( \bar{x}_1, \ldots, \bar{x}_k \in P_n \) and \( n \)-cells \( X_1, \ldots, X_k \) of \( K \) with \( X_i = \bar{x}_i \) such that

\[
X = X^1 \ast_{n-1} \cdots \ast_{n-1} X^k
\]

so \( X_n = x_1 + \cdots + x_k \). Hence, there are \( 1 \leq i_1, i_2 \leq k \) with \( i_1 \neq i_2 \) such that \( x_{i_1} = x \) and \( x_{i_2} = y \). By symmetry, we can suppose that \( i_1 < i_2 \). If there is some \( i \) such that \( \bar{z} \notin \bar{x}_{i}^+ \) by [17, Proposition 5.4], \( i < i_1 \). So, for \( i_1 \leq i \leq i_2 \), \( \bar{z} \notin \bar{x}_{i}^+ \). Let \( Y = X^{i_1} \ast_{n-1} X^{i_1+1} \ast_{n-1} \cdots \ast_{n-1} X^{i_2} \) which is a cell of \( K \). We have

\[
Y_{n-1,-} = \sum_{i_1 \leq i \leq i_2} [x_{i}]_{n-1,-} - \sum_{i_1 \leq i \leq i_2} [x_{i}]_{n-1,+} + Y_{n-1,+}
\]

with

\[
2z \leq \sum_{i_1 \leq i \leq i_2} [x_{i}]_{n-1,-} \quad \text{and} \quad 2\epsilon(z \leq \sum_{i_1 \leq i \leq i_2} [x_{i}]_{n-1,+}) \quad \text{and} \quad Y_{n-1,+} \geq 0
\]

so \( 2z \leq Y_{n-1,-} \), contradicting the fact that \( \partial^- Y \) is fork-free by induction. Thus \( \bar{x}^- \cap \bar{y}^- = \emptyset \) and, similarly, \( \bar{x}^+ \cap \bar{y}^+ = \emptyset \). Hence, \( X \) is fork-free. \(\square\)

Lemma 5.3.6. For all \( n \geq 0 \), \( S_n \circ M_n = id_{P_n} \).

Proof. The result is a direct consequence of the definitions. \(\square\)

Lemma 5.3.7. For all \( n \geq 0 \) and \( s \in K_n^* \) radical, \( M_n \circ S_n(s) = s \)

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Proof. The result is a direct consequence of the definitions. \qed

Lemma 5.3.8. Let \( n \geq 0, U, V \subseteq P_n \) be finite sets and \( x \in P_n \). We have the following properties:

(a) if \( U \cap V = \emptyset \), then \( M_n(U) \cap M_n(V) = 0 \) and \( M_n(U \cup V) = M_n(U) + M_n(V) \),
(b) if \( U \subseteq V \), then \( M_n(U) \leq M_n(V) \) and \( M_n(V \setminus U) = M_n(V) - M_n(U) \),
(c) if \( n > 0 \), then \( M_{n-1}(\bar{x}^\epsilon) = x^\epsilon \),
(d) Suppose that \( U \) is fork-free. Then \( M_n(U) \) is fork-free. Moreover, when \( n > 0 \), \( M_{n-1}(U^\epsilon) = (M_n(U))^\epsilon \).

Proof. (a) and (b) are direct consequences of the definitions. For (c), note that \( \bar{x}^\epsilon = S_{n-1}(x^\epsilon) \). By Lemma 5.3.5, \([x]_{n-1, \epsilon}\) is fork-free and, by Lemma 5.3.4, it is radical. So, by Lemma 5.3.7, \( M_{n-1}(\bar{x}^\epsilon) = x^\epsilon \).

For (d), suppose that \( U \subseteq P_n \) is fork-free. If \( n = 0 \), the result is trivial, so we can suppose \( n > 0 \). For \( x, y \in P_n \) with \( x \leq M_n(U) \) and \( y \leq M_n(U) \) such that there exist \( z \in P_{n-1} \) and \( \epsilon \in \{-, +\} \) with \( z \leq x^\epsilon \) and \( z \leq y^\epsilon \), we have \( z \in \bar{x}^\epsilon \) and \( z \in \bar{y}^\epsilon \). Since \( U \) is fork-free, \( x = y \). Also, since \( M_n(U) \) is radical, \(-(x + y \leq M_n(U))\). So \( M_n(U) \) is fork-free. For the second part, note that for \( \bar{x}, \bar{y} \in U \) with \( x \neq y \), we have \( \bar{x}^\epsilon \cap \bar{y}^\epsilon = \emptyset \). Hence,

\[
M_{n-1}(U^\epsilon) = M_{n-1}(\bigcup_{\bar{x} \in U} \bar{x}^\epsilon) = \sum_{\bar{x} \in U} M_{n-1}(\bar{x}^\epsilon) \quad \text{(by (a))}
\]

\[
= \sum_{\bar{x} \in U} x^\epsilon \quad \text{(by (c))}
\]

\[
= s^\epsilon.
\]

Lemma 5.3.9. Let \( n \geq 0, u, v \in K_n^* \) be such that \( u, v \) are radical and \( z \in P_n \). We have the following properties:

(a) if \( u \wedge v = 0 \), then \( S_n(u) \cap S_n(v) = \emptyset \) and \( S_n(u + v) = S_n(u) \cup S_n(v) \),
(b) if \( u \leq v \), then \( S_n(u) \subseteq S_n(v) \) and \( S_n(v - u) = (S_n(v)) \setminus (S_n(u)) \),
(c) if \( n > 0 \), then \( S_{n-1}(z^\epsilon) = z^\epsilon \),
(d) if \( u \) is fork-free, then \( S_n(u) \) is fork-free. Moreover, when \( n > 0 \), \( S_{n-1}(u^\epsilon) = (S_n(u))^\epsilon \).

Proof. (a), (b) and (c) are direct consequences of the definitions. For (d), suppose that \( u \) is fork-free. If \( n = 0 \), the result is trivial, so suppose that \( n > 0 \). For \( x, y \in S_n(u) \) such that there exist \( \epsilon \in \{-, +\} \) and \( \bar{z} \in \bar{x}^\epsilon \cap \bar{y}^\epsilon \), we have \( z \leq x^\epsilon \) and \( z \leq y^\epsilon \). By fork-freeness, \( \neg(x + y \leq u) \). But \( x \leq u \) and \( y \leq u \). So \( x = y \) and \( S_n(u) \) is fork-free. For the second part, note that for \( x, y \in P_n \) with
Lemma 5.3.11. Let $x \neq y$, $x \leq u$ and $y \leq u$, we have $x^e \wedge y^e = 0$. Hence,

$$S_{n-1}(u^e) = S_{n-1}\left( \sum_{x \in P_n, x \leq u} x^e \right)$$

$$= \bigcup_{x \in P_n, x \leq u} S_{n-1}(x^e) \quad \text{(by (a))}$$

$$= \bigcup_{x \in P_n, x \leq u} x^e \quad \text{(by (c))}$$

$$= (S_n(u))^e. \qed$$

5.3.10 Movement properties. Here, we prove several lemmas relating movement properties on $P$ with properties on $K$.

Lemma 5.3.11. Let $n > 0$, $u \in K_n^*$ fork-free and $U = S_n(u)$. Then,

$$u^\perp = M_{n-1}(U^\perp) \quad \text{and} \quad u^\pm = M_{n-1}(U^\pm).$$

Proof. We have

$$d u = u^\pm - u^\perp$$

$$= u^+ - u^-$$

$$= M_{n-1}(U^+) - M_{n-1}(U^-) \quad \text{(by Lemma 5.3.8)}$$

$$= (M_{n-1}(U^\pm) + M_{n-1}(U^+ \cap U^-))$$

$$- (M_{n-1}(U^\perp) + M_{n-1}(U^+ \cap U^-)) \quad \text{(by Lemma 5.3.8)}$$

$$= M_{n-1}(U^\pm) - M_{n-1}(U^\perp).$$

Since $U^\pm \cap U^\perp = \emptyset$, we have $M_{n-1}(U^\pm) \cap M_{n-1}(U^\perp) = \emptyset$. By uniqueness of the decomposition,

$$u^\perp = M_{n-1}(U^\perp) \quad \text{and} \quad u^\pm = M_{n-1}(U^\pm). \qed$$

Lemma 5.3.12. Let $n \geq 0$, $S \subseteq P_{n+1}$ be a finite and fork-free set, $U, V \subseteq P_n$ be finite sets, such that $S$ moves $U$ to $V$. Then, $d(M_{n+1}(S)) = M_n(V) - M_n(U)$.

Proof. By definition of movement, $V = (U \cup S^+) \setminus S^-$. Hence,

$$M_n(V) = M_n((U \cup S^+) \setminus S^-)$$

$$= M_n(U \cup S^+) - M_n(S^-) \quad \text{(by Lemma 5.3.8, since $S^- \subseteq U \cup S^+$)}$$

$$= M_n(U) + M_n(S^+) - M_n(S^-) \quad \text{(since $U \cap S^+ = \emptyset$ by Lemma 2.1.1)}$$

$$= M_n(U) + (M_{n+1}(S))^+ - (M_{n+1}(S))^- \quad \text{(by Lemma 5.3.8)}$$

$$= M_n(U) + d(M_{n+1}(S)). \qed$$

Lemma 5.3.13. Let $n \geq 0$, $s \in K_{n-1}^*$ fork-free, $u, v \in K_n^*$ with $u, v$ radical, such that $d s = v-u$, $u \wedge s^+ = 0$ and $s^- \wedge v = 0$. Then, $S_{n+1}(s)$ moves $S_n(u)$ to $S_n(v)$.

Proof. Let $S = S_{n+1}(s)$, $U = S_n(u)$ and $V = S_n(v)$. Since $d s = v-u$, we have

$$s^- \leq s^- + v = u + s^+$$

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Thus, \[ S^- = S_n(s^-) \subseteq S_n(u + s^+) = U \cup S^+. \]

Thus,
\[
M_n((U \cup S^+) \setminus S^-) = M_n(U \cup S^+) - M_n(S^-) \\
= M_n \circ S_n(u + s^+) - s^- \\
= u + s^+ - s^- \\
= u + d s \\
= v \\
= M_n(V)
\]

so, by Lemma 5.3.6, \( V = (U \cup S^+) \setminus S^- \). Similarly, \( U = (V \cup S^-) \setminus S^+ \). Hence, \( S \) moves \( U \) to \( V \).

\[ \square \]

**Lemma 5.3.14.** Let \( n > 0 \) and \( X \) be an \( n \)-cell of \( K \). Then, for \( i < n \) and \( \epsilon \in \{-, +\} \),
\[
X_{i, -} \wedge X^+_{i+1, \epsilon} = 0 \quad \text{and} \quad X^-_{i+1, \epsilon} \wedge X^+_{i, +} = 0.
\]

**Proof.** By contradiction, suppose given \( n > 0, X \) an \( n \)-cell, \( i < n \) and \( \epsilon \in \{-, +\} \) that give a counter-example for this property. By applying \( \partial^-, \partial^+ \) sufficiently, we can suppose that \( i = n - 1 \). Also, by symmetry, we only need to handle the first case, that is, when there is \( z \in P_{n-1} \) such that \( z \leq X_{n-1, -} \wedge X^+_{n-1, +} \). So there is \( x \in P_n \) such that \( x \leq X_n \) and \( z \leq x^+ \). By the definition of a cell, we have \( dX_n = X_{n-1, +} - X_{n-1, -} \), thus
\[
X_{n-1, +} + \sum_{u \in P_n, u \leq X_n} u^- = X_{n-1, -} + \sum_{u \in P_n, u \leq X_n} u^+ \\
\geq 2z.
\]

and, since \( X_{n-1, +} \) is radical, there is \( y \in P_n \) with \( y \leq X_n \) such that \( z \leq y^- \). By [17, Proposition 5.1], there are \( k \geq 1, x_1, \ldots, x_k \in P_n \) with \( x_1 + \cdots + x_k = X_n \) and \( i_1 < i_2 \) with \( x_{i_1} = x \) and \( x_{i_2} = y \) and \( n \)-cells \( X^1, \ldots, X^k \) with \( X^i_n = x_i \) such that \( X = X^1 \cdot \ldots \cdot X^k \). Let \( Y = X^1 \cdot \ldots \cdot X^k \). Since \( Y \) is a cell, we have
\[
Y_{n-1, +} + \sum_{1 \leq i \leq k} x^-_i = Y_{n-1, -} + \sum_{1 \leq i \leq k} x^+_i \\
= X_{n-1, -} + \sum_{1 \leq i \leq k} x^+_i \\
\geq 2z.
\]

Moreover, since \( X \) is fork-free and \( z \leq x^-_{i_1} \), we have \( \neg(z \leq x^-_i) \) for \( i \leq i_1 \). So \( 2z \leq Y_{n-1, +} \), contradicting the fact that \( Y_{n-1, +} \) is radical by Lemmas 5.3.5 and 5.3.4. Hence, \( X_{i, -} \wedge X^+_{n} = 0 \).

\[ \square \]
5.3.15 The translation operations. Given an $n$-pre-cell $X$ of $P$, we define $T_{\text{ADC}}^{PC}(X)$ as the $n$-pre-cell $Y$ of $K$ such that $Y_{i,\epsilon} = M_i(X_{i,\epsilon})$ for $i \leq n$ and $\epsilon \in \{-,+,\}$. Similarly, given an $n$-pre-cell $X$ of $K$, we define $T_{\text{PC}}^{ADC}(X)$ as the $n$-pre-cell $Y$ of $P$ such that $Y_{i,\epsilon} = S_i(X_{i,\epsilon})$ for $i \leq n$ and $\epsilon \in \{-,+,\}$. We then have the following properties.

**Lemma 5.3.16.** $T_{\text{PC}}^{PC}$ is a bijection with inverse $T_{\text{PC}}^{ADC}$ from $\text{Cell}(P)$ to $\text{Cell}(K)$.

**Proof.** Let $n \in \mathbb{N}$ and $X \in \text{Cell}(P)$. Then, by Lemma 5.3.8, $M_i(X_{i,\epsilon})$ is fork-free for $i \leq n$ and $\epsilon \in \{-,+,\}$. Moreover, by Lemma 5.3.12, for $i < n$ and $\epsilon \in \{-,+,\}$,

$$d(M_{i+1}(X_{i+1,\epsilon})) = M_i(X_{i,\epsilon}) - M_i(X_{i,-})$$

so $T_{\text{PC}}^{ADC}(X) \in \text{Cell}(K)$. Conversely, given $X \in \text{Cell}(K)$, by Lemma 5.3.9, $S_i(X_{i,\epsilon})$ is fork-free for $i \leq n$ and $\epsilon \in \{-,+,\}$. By Lemmas 5.3.13 and 5.3.14, for $i < n$ and $\epsilon \in \{-,+,\}$,

$$S_{i+1}(X_{i+1,\epsilon}) \text{ moves } S_i(X_{i,\epsilon}) \text{ to } S_i(X_{i,+})$$

so $T_{\text{PC}}^{ADC}(X) \in \text{Cell}(P)$. By Lemma 5.3.6, for $X \in \text{Cell}(P)$,

$$T_{\text{PC}}^{ADC} \circ T_{\text{PC}}^{PC}(X) = X,$$

and, by Lemmas 5.3.5, 5.3.4 and 5.3.7, for $X \in \text{Cell}(K)$,

$$T_{\text{PC}}^{ADC} \circ T_{\text{PC}}^{PC}(X) = X.$$

Hence, $T_{\text{PC}}^{PC}$ and $T_{\text{PC}}^{ADC}$ induce bijections between $\text{Cell}(P)$ and $\text{Cell}(K)$ and are inverse of each other. \hfill $\Box$

**Lemma 5.3.17.** For $x \in P$, we have $T_{\text{PC}}^{ADC}([x]) = \langle \bar{x} \rangle$.

**Proof.** Let $X = T_{\text{PC}}^{ADC}([x])$. We have $X_n = S_n([x]_n) = \{x\}$. We show by induction on $i$ that $X_{i,\epsilon} = \langle x \rangle_{i,\epsilon}$ for $i < n$ and $\epsilon \in \{-,+,\}$. We have $[x]_{i,-} = [x]_{i+1,-}$ so $X_{i,-} = S_i([x]_{i+1,-}) = X_{i+1,-}$ by Lemmas 5.3.5 and 5.3.11. So $X_{i,-} = \langle x \rangle_{i,-}$. Similarly, $X_{i,+} = \langle x \rangle_{i,+}$. Hence, $T_{\text{PC}}^{ADC}([x]) = \langle \bar{x} \rangle$. \hfill $\Box$

5.3.18 Adc’s are generalized parity complexes.

**Lemma 5.3.19.** $P$ satisfies (G1).

**Proof.** Note that, for $n > 0$ and $\bar{x}, \bar{y} \in P_n$, $\bar{x} \triangleleft_{P_n} \bar{y}$ implies $\bar{x} <_{n-1} \bar{y}$. So, by transitivity, we have $\triangleleft_{P_n} \subseteq <_{n-1}$. Since the basis $P$ is loop-free, $<_{n-1}$ is irreflexive and so is $\triangleleft_{P_n}$. Hence, $\triangleleft$ is irreflexive. \hfill $\Box$

**Lemma 5.3.20.** $P$ satisfies (G2).

**Proof.** Let $\bar{x} \in P$. By Lemma 5.3.17, $T_{\text{PC}}^{ADC}([x]) = \langle \bar{x} \rangle$. And, by Lemma 5.3.16, $T_{\text{PC}}^{ADC}([x]) \in \text{Cell}(P)$. Hence, $\bar{x}$ is relevant. \hfill $\Box$

**Lemma 5.3.21.** $P$ satisfies (G3').
Proof. By contradiction, suppose that there are \( n > 0, \ i < n, \ \bar{x} \in P_n \) with \( \langle \bar{x} \rangle_{i,+} \cap^* \langle \bar{x} \rangle_{i,-} \). So there are \( k \geq 1, \ \bar{y}_1, \ldots, \bar{y}_k \in P_i \) with \( \bar{y}_1 \in \langle \bar{x} \rangle_{i,+}, \ \bar{y}_k \in \langle \bar{x} \rangle_{i,-} \) and \( \bar{y}_j \cap \bar{y}_{j+1} \) for \( 1 \leq j < k \). By definition of \( \cap \), it gives \( \bar{z}_1, \ldots, \bar{z}_{k-1} \in P_{i+1} \) with \( \bar{y}_j \in \bar{z}_j^* \) and \( \bar{y}_{j+1} \in \bar{z}_{j+1}^* \) for \( 1 \leq j < k \). So we have

\[
\bar{x} \prec_i \bar{z}_1 \cdots \prec_i \bar{z}_{k-1} \prec_i \bar{x},
\]

contradicting the loop-freeness of the basis \( P \). Hence, \( P \) satisfies (G3').

Lemma 5.3.22. \( P \) satisfies (G4').

Proof. By contradiction, suppose that there are \( i > 0, \ m > i, \ n > i, \ \bar{x} \in P_m \) and \( \bar{y} \in P_n \) with \( \langle \bar{x} \rangle_{i,+} \cap \langle \bar{y} \rangle_{i,-} = \emptyset \), \( \langle \bar{x} \rangle_{i-1,+} \cap^* \langle \bar{y} \rangle_{i-1,-} \) and \( \langle \bar{y} \rangle_{i-1,+} \cap^* \langle \bar{x} \rangle_{i-1,-} \). By the same method than for Lemma 5.3.21, we get \( r, s \in \mathbb{N}, \ u_1, \ldots, u_r \in P_i, \ v_1, \ldots, v_s \in P_j \) such that

\[
\bar{x} \prec_i \bar{u}_1 \cdots \prec_i \bar{u}_r \prec_i \bar{y} \prec_i \bar{v}_1 \cdots \prec_i \bar{v}_s \prec_i \bar{x},
\]

contradicting the loop-freeness of the basis \( P \). Hence, \( P \) satisfies (G4').

Theorem 5.3.23. \( P \) is a generalized parity complex.

Proof. The result is a consequence of Lemmas 5.3.3, 5.3.19, 5.3.20, 5.3.21, \( \text{??} \), 5.3.22 and \( \text{??} \).

Theorem 5.3.24. \( T_{ADC}^{PC} \) is an isomorphism of \( \omega \)-categories. Moreover, for \( \bar{x} \in P \), \( T_{ADC}^{PC}(\langle \bar{x} \rangle) = [x] \).

Proof. The fact that \( T_{ADC}^{PC} \) is bijective is given by Lemma 5.3.16. The fact that \( T_{ADC}^{PC} \) commutes with source, target and identities is trivial.

Given \( i < n \in \mathbb{N}, \ i \)-composable cells \( X,Y \in \text{Cell}(P)_n \), we have that \( X_{j,\epsilon} \cap Y_{j,\epsilon} = \emptyset \) for \( i < j \leq n \) and \( \epsilon \in \{-,+,\} \). Indeed, by applying \( \partial^\epsilon \) sufficiently, we can suppose that \( j = n \). Then, by Lemma 3.3.4, \( X \ast_{i-1} Y = X' \ast_{i-1} Y' \) where \( X' = X \ast_{n-1} \text{id}_n(\partial^\epsilon_{n-1} Y) \) and \( Y' = \text{id}_n(\partial^\epsilon_{n-1} X') \ast_{i} Y \). Note that \( X'' = X_n \) and \( Y'' = Y_n \). Hence, by Lemma 2.3.1, \( X_n \cap Y_n = \emptyset \). Then, by Lemma 5.3.8, it follows readily that \( M_n(X \ast_{i} Y) = M_n(X) \ast_{i} M_n(Y) \). Thus, \( T_{ADC}^{PC} \) is an isomorphism of \( \omega \)-categories.

Lastly, given \( \bar{x} \in P \), by Lemmas 5.3.17 and 5.3.7, we have \( T_{ADC}^{PC}(\langle \bar{x} \rangle) = [x] \).

5.4 Absence of other embeddings

In this subsection, we show that there are no embeddings between the four formalisms except the ones already proved, that is, that parity complexes, pasting scheme and augmented directed complexes are generalized parity complexes. For the comparison with adc's, we use the translation from \( \omega \)-hypergraphs to pre-adc's defined in Paragraph \text{??} and the translation from adc's to \( \omega \)-hypergraphs defined in Paragraph 5.3.1.
5.4.1 No embedding in parity complexes. Axiom (C4) is relatively strong, so it can be used for building counter-examples to inclusion. The \( \omega \)-hypergraph (??) is a pasting scheme satisfying (G4) (and thus is a generalized parity complex) and is an adc with loop-free unital basis. But it is not a parity complex as we have seen in Paragraph ??, because it does not satisfy (C4). So there is no embedding from pasting schemes, augmented directed complexes or generalized parity complexes in parity complexes (fixed version).

5.4.2 No embedding in pasting schemes. For pasting schemes, we use the relatively strong Axiom (S2) for building counter-examples to inclusion. The following \( \omega \)-hypergraph is a parity complex satisfying (G4) (and thus it is a generalized parity complex) and is an adc with loop-free unital basis but it not as pasting scheme:

\[
\begin{align*}
\alpha_2 &\Rightarrow \alpha_3 \\
\text{Indeed, (S2) is not satisfied because } &\alpha_2 \searrow \alpha_3 \text{ but } y \in B(\alpha_2) \cap E(\alpha_3) \neq \emptyset. \text{ Note that (16) is essentially the } \omega \text{-hypergraph (??) without the 3-generator } A \text{ and the 2-generators } \alpha_1 \text{ and } \alpha_4.
\end{align*}
\]

5.4.3 No embedding in augmented directed complexes. For augmented directed complexes, the loop-free basis axiom is used for building counter-examples to inclusion. It enforces a strong version of the torsion-freeness (G4). So, counter-examples of inclusion can be found with fancy torsion-free situations. The following \( \omega \)-hypergraph is a parity complex and a pasting scheme, and moreover satisfies (G4), so it is a generalized parity com-
plex:

\[
\begin{array}{c}
\text{w} \\
\alpha \downarrow b \\
\beta \downarrow c \\
\gamma \downarrow e \\
\delta \downarrow f \\
\epsilon \downarrow h \\
\zeta \downarrow i \\
\text{z} \\
\end{array}
\]

\[
\begin{array}{c}
\text{w} \\
\alpha \downarrow b \\
\beta \downarrow c \\
\gamma' \downarrow e \\
\delta \downarrow f \\
\epsilon \downarrow h \\
\zeta \downarrow i \\
\text{z} \\
\end{array}
\]

\[
\begin{array}{c}
\text{w} \\
\alpha \downarrow b \\
\beta' \downarrow c \\
\gamma' \downarrow e \\
\delta' \downarrow f \\
\epsilon \downarrow h \\
\zeta \downarrow i \\
\text{z} \\
\end{array}
\]

\[
\begin{array}{c}
\text{w} \\
\alpha' \downarrow b \\
\beta' \downarrow c \\
\gamma'' \downarrow e \\
\delta' \downarrow f \\
\epsilon \downarrow h \\
\zeta' \downarrow i \\
\text{z} \\
\end{array}
\]

where

\[
A^- = \{\beta, \gamma\}, \quad A^+ = \{\beta', \gamma'\}, \\
B^- = \{\delta, \epsilon\}, \quad B^+ = \{\delta', \epsilon'\}, \\
C^- = \{\alpha, \gamma', \delta', \zeta\}, \quad C^+ = \{\alpha', \gamma'', \zeta'\}.
\]

But its associated pre-ade is an ade with a basis which is not loop-free unital. Indeed, we have \(e < [A]_{1,+} \wedge [B]_{1,-}\), \(h < [B]_{1,+} \wedge [C]_{1,-}\) and \(b < [C]_{1,-} \wedge [A]_{1,+}\), so

\[
A <_1 B <_1 C <_1 A.
\]

Hence, the basis of the associated augmented directed complex is not loop-free.

**Conclusion**

We hope that this work brought some understanding on the formalisms of pasting diagrams in several ways. First, by gathering all the three existing formalisms in one paper in the perspective of a unified treatment. Second, by giving some intuition on the axioms behind each of them. Third, by providing a generalization that encompasses the three other ones, with complete proofs. Last, by answering negatively to the questions of inclusions between formalisms. Moreover, this work was the opportunity to carry out a deep verification of the existing theories, allowing to discover a flaw that affects the freeness properties claimed for parity complexes and pasting schemes.
The generalized parity complexes presented in this paper seem to leave room for even more generalization, through at least two directions. First, it can be observed in several situations that Axiom (G1) is too strong. For example, the $\omega$-hypergraph

\[
\begin{array}{c}
  x \\
  a \\
  w \\
  \downarrow \alpha \\
  a' \\
  b \\
  y \\
  \downarrow \beta \\
  b \\
  z \\
\end{array}
\]

(18)

should be considered as a pasting diagram. However, $\alpha \cdot \alpha$, so (18) does not satisfy (G1), and is therefore not a generalized parity complex. Second, by using multisets instead of sets, it seems possible to authorize some looping behaviors. This could enable to represent unambiguously “the morphism with one copy of $\alpha$ and two copies of $\beta$” in the diagram

\[
\begin{array}{c}
  x \\
  a \\
  \downarrow \alpha \\
  a' \\
  b \\
  y \\
  \downarrow \beta \\
  b \\
  z \\
\end{array}
\]

(19)

These improvements would allow a bigger class of free $\omega$-categories to be described explicitly. In particular, related objects, such as opetopes or the pasting diagrams defined by Henry [6] in order to study the Simpson’s conjecture [7], could benefit from an effective description. Hence, future work on pasting diagrams might prove valuable and, among others, could help better understand the difficult world of weak $\omega$-categories.

References


A Details about the counter-example to parity complexes and pasting schemes

It was earlier claimed that the $\omega$-hypergraph $\omega$ was a counter-example to the freeness properties [19, Theorem 4.2] and [8, Theorem 13]. This claim assumes that $F_1$ and $F_2$ are two different 3-cells of the induced free $\omega$-category $\mathcal{A}$. In this section, we give two methods to show this fact.

A.1 A mechanized counter-example

A proof that $F_1$ and $F_2$ are different has been formalized in Agda and is discussed in [4]. The proof relies on a verified definition of a 3-category $\mathcal{A}'$ such that there is a 3-functor $I: \mathcal{A} \to \mathcal{A}'$ for which $I(F_1) \neq I(F_2)$ by definition. The source code is accessible through a GitLab repository.

A.2 A categorical model

A more theoretical proof is given by an interpretation of $\mathcal{A}$ in the 3-category of 2-categories, functors, pseudo-natural transformations, and modifications, that is, a 3-functor $K: \mathcal{A} \to 2\text{-Cat}$, for which $K(F_1) \neq K(F_2)$. After recalling some definitions, we define $K$ and show that $K(F_1) \neq K(F_2)$.

A.2.1 Pseudo-natural transformations. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and $G, H: \mathcal{C} \to \mathcal{D}$ two 2-functors. A pseudo-natural transformation $\alpha: G \Rightarrow H$ is given by

- for all $x \in \mathcal{C}_0$, an 1-cell $\alpha_x: G(x) \to H(x) \in \mathcal{D}_1$, and
- for all $x, y \in \mathcal{C}_0$, $f: x \to y \in \mathcal{C}_1$, a 2-cell $\alpha_f$ as in

\[
\begin{CD}
F(x) @>{F(f)}>> F(y) \\
\alpha_x \downarrow \alpha_f \downarrow \alpha_y \\
G(x) @>{G(f)}>> G(y)
\end{CD}
\]

such that some natural conditions hold.

A.2.2 Modifications. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories, $G, H: \mathcal{C} \to \mathcal{D}$ be two 2-functors, $\alpha, \beta: G \Rightarrow H$ be two pseudo-natural transformations. A modification $M: \alpha \Rightarrow \beta$ is given by 2-cells $M_x: \alpha_x \Rightarrow \beta_x$ for all $x \in \mathcal{C}_0$ such that some natural conditions hold.

\footnote{See https://gitlab.inria.fr/sforest/3-pasting-example: the definition of the 3-category can be found in \texttt{ex.agda} and the main result in \texttt{ex-is-cat.agda}.}
A.2.3 The interpretation. Let $\mathcal{C}$ be the free 2-category induced by the 2-polygraph consisting in one 0-generator $\star$. Let $\mathcal{D}$ be the free 2-category induced by the 1-polygraph

$$y_1 \xrightarrow{\alpha''} y_2 \xrightarrow{\beta''} y_3.$$  \hspace{1cm} (20)

Let $\mathcal{E}$ be the free 2-category induced by the 2-polygraph

$$
\begin{array}{c}
\xymatrix{
A' \ar[r]_{\tau} \ar[d]_{\iota} & e \ar[d]_{\tau'} \ar[dl]^i \\
B' &}
\end{array}
$$ \hspace{1cm} (21)

Let $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$ be the 2-functors defined by $F_i(\star) = y_i$. Let $G_1, G_2, G_3 : \mathcal{D} \to \mathcal{E}$ be the 2-functors defined by

- $G_1(y_1) = G_1(y_2) = i$, $G_1(y_3) = e$, $G_1(\alpha'') = \text{id}_1(i)$ and $G_1(\beta'') = \tau$,
- $G_2(y_1) = i$, $G_2(y_2) = G_2(y_3) = e$, $G_2(\alpha'') = \tau$ and $G_2(\beta'') = \text{id}_1(e)$,
- $G_3(y_1) = G_3(y_2) = G_3(y_3) = e$ and $G_3(\alpha'') = G_3(\beta'') = \text{id}_1(e)$.

Consider the following pseudo-natural transformations:

- $\bar{\alpha} : F_1 \Rightarrow F_2$ defined by $\bar{\alpha}_* = \alpha''$, 
- $\bar{\beta} : F_2 \Rightarrow F_3$ defined by $\bar{\beta}_* = \beta''$, 
- $\bar{\gamma} : G_1 \Rightarrow G_2$ defined by $\bar{\gamma}_{y_1} = \text{id}_i, \bar{\gamma}_{y_2} = \tau, \bar{\gamma}_{y_3} = \text{id}_e$, 
- $\bar{\delta} : G_2 \Rightarrow G_3$ defined by $\bar{\delta}_{y_1} = \tau, \bar{\delta}_{y_2} = \text{id}_e, \bar{\delta}_{y_3} = \text{id}_e$.

Note that $\bar{\alpha}_*\bar{\delta}$ is given by $(\bar{\alpha}_*\bar{\delta})_* = \delta_{F_2(\star)}G_2(\alpha_*) = \tau$. Similarly, $\bar{\beta}_*\bar{\gamma}$ is given by $(\bar{\beta}_*\bar{\gamma})_* = \tau$. So a modification $M : \bar{\alpha}_*\bar{\delta} \Rightarrow \bar{\alpha}_*\bar{\delta}$ (resp. $M : \bar{\beta}_*\bar{\gamma} \Rightarrow \bar{\beta}_*\bar{\gamma}$) is given by a 2-cell $M: \tau \Rightarrow \tau$ in $\mathcal{E}$. The interpretation $K : \mathcal{C} \to 2\text{-Cat}$ is then defined by

- $K(x) = \mathcal{C}, K(y) = \mathcal{D}, K(z) = \mathcal{E}$,
- $K(a) = F_1, K(b) = F_2, K(c) = F_3, K(d) = G_1, K(e) = G_2, K(f) = G_3$,
- $K(\alpha) = K(\alpha') = \bar{\alpha}, K(\beta) = K(\beta') = \bar{\beta}, K(\gamma) = K(\gamma') = \bar{\gamma}, K(\delta) = K(\delta') = \bar{\delta}$,
- $K(A) = A'$ and $K(B) = B'$.

Under this interpretation, we have

$$(K(F_1))_* = A' *_1 B' \quad \text{and} \quad (K(F_2))_* = B' *_1 A'$$

Since $A' *_1 B' \neq B' *_1 A'$ in $\mathcal{E}$, we have $F_1 \neq F_2$ in $\mathcal{A}$.