Unifying notions of pasting diagrams

Simon Forest

LIX, École Polytechnique
IRIF, Université Paris Diderot
simon.forest@normalesup.org

Abstract

In this work, we relate the three main formalisms for the notion of pasting diagram in strict ω-categories: Street's *parity complexes*, Johnson's *pasting schemes* and Steiner's *augmented directed complexes*. We first show that parity complexes and pasting schemes do not induce free ω-categories in general, contrarily to the claims made in their respective papers, by providing a counter-example. Then, we introduce a new formalism that is a strict generalization of augmented directed complexes, and corrected versions of parity complexes and pasting schemes, which moreover satisfies the aforementioned freeness property. Finally, we show that there are no other embeddings between these four formalisms.

Introduction

From an original idea of S. Mimram.

Pasting diagrams. Central to the theory of strict ω-categories is the notion of pasting diagram, which describes collections of morphisms for which a composite is expected to be defined and unambiguous. Reasonable definitions are easy to achieve in low dimensions, but the notion is far from being straightforward in general. The three main proposals are Johnsons' *pasting schemes* [8], Street's *parity complexes* [19, 20] and Steiner's *augmented directed complexes* [16, 17]. Even though the ideas underlying the definitions of those formalisms are quite similar, they differ on many points and comparing them precisely is uneasy, and actually, to the best of our knowledge, no formal account of the differences was ever made. In this article, we achieve the task of formally relating them. It turns out that the three notions are incomparable in terms of expressive power (each of the three allows a pasting diagram which is not allowed by others), and the way the comparison is performed here is by embedding them into a generalization of parity complexes which is able to encompass all the various flavors of pasting diagrams.

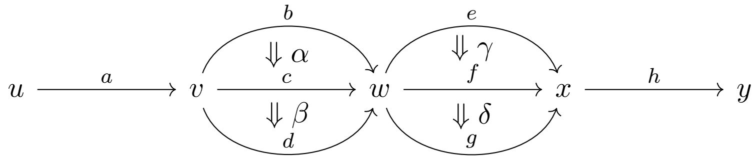
Originally, the motivation behind pasting diagrams was to give a simpler representation of formal composition of cells in (free) ω-categories. More precisely, given the data, for $i \geq 0$, of generating i-cells with their source and target boundaries (under the form of a *polygraph* [2], also called *computad* [18]), the cells of the associated free ω-category can be described as the formal composites of generators quotiented by the axioms of ω-categories. This representation is difficult to handle in practice, because the equivalence relation induced by the axioms is hard to describe. Instead, a graphical representation of the cells involved in the composite appeared to be sufficient to designate a cell. For instance, consider the two formal composites

$$a *_0 (\alpha *_1 \beta) *_0 ((\gamma *_0 h) *_1 (\delta *_0 h))$$

and

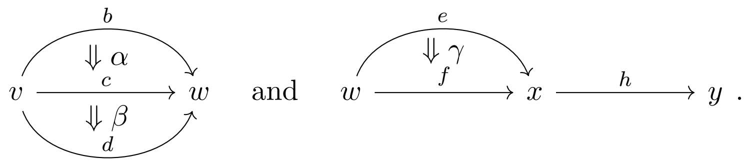
$$(a *_0 \alpha *_0 e *_0 h) *_1 (a *_0 c *_0 \gamma *_0 h) *_1 (a *_0 \beta *_0 \delta *_0 h).$$

Under the axioms of ω-categories, it can be checked, though it is not immediate, that both represent the same cell. However, both are formal composites of the elements of the following diagram



(1)

In fact, all formal composites involving all the generators of this diagram are equal and the data of the diagram enables to refer to the cell obtained by composing u, $v, \ldots, y$, a, $b, \ldots, h$, α, β, γ, δ together unambiguously without giving an explicit composite for them. We call *pasting diagrams* the diagrams satisfying this property. It can be observed that this pasting diagram is made of smaller pasting diagrams like



Moreover, the two can be composed along w by taking the union of the pasting diagrams. More generally, given a set of generators and their source-target borders satisfying sufficient properties, one can obtain a category of pasting diagrams on such a set, which is isomorphic to the free category mentioned earlier, justifying the use of pasting diagrams instead of formal composites to designate particular cells.

Hence, pasting diagram formalisms give effective descriptions of free ω-categories. In particular, they give a precise definition of the notion of commutative diagrams and model generic compositions. Moreover, they make it possible to study higher categories by probing them through pasting diagrams. For example, augmented directed complexes were used to give an effective description of the Gray tensor product in [17]. In a related manner, Kapranov and Voevodsky studied topological properties of pasting schemes in [10] and used them in an attempt to give a description of ω-groupoids in [11], but their results were shown paradoxical [15].

Several other works studied pasting diagrams. In [1], Buckley gives a mechanized Coq proof of the results of [19] but stops at the excision theorem [19, Theorem 4.1]. In particular, the proof of the freeness claim [19, Theorem 4.2] was not formally verified, and could not be, since this claim does not hold in general, as is shown in the present paper. In [3], Campbell isolates a common structure behind parity complexes and pasting schemes, called *parity structure*, and gives stronger axioms than the ones of parity complexes and pasting schemes, taking an opposite path from this work which seeks a more general formalism. In [13], Nguyen studies *pre-polytopes with labeled structures* and shows that they give a parity structure that satisfies a variant of Campbell's axioms that are enough to obtain another correct notion of pasting diagrams. In [6], Henry defines a theoretical notion of pasting diagrams, called polyplexes, to show that certain classes of polygraphs are presheaf categories, and uses them to prove a variant of the Simpson's conjecture in [7]. However, his pasting diagrams can involve some looping behaviors, and are then out of the scope of the formalisms studied in the present work. Using similar ideas, Hadzihasanovic [5] defines a class of pasting diagrams, called *regular polygraphs*, that is "big enough" to study semi-strict categories and which is well-behaved for several constructions (notably, their realizations as topological spaces are CW complexes).

Pasting diagrams in 1-categories. The most simple instance of a pasting diagram is in a 1-category: in this case, those are of the form

$$x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} x_n \tag{2}$$

and admit $a_n \circ \cdots \circ a_1$ as composite. On the contrary, diagrams such as

$$y \xleftarrow{a} x \xrightarrow{b} z \qquad\qquad \text{or} \qquad\qquad x \circlearrowleft a \tag{3}$$

are not expected to be pasting diagrams: in the first one, the two arrows are not even composable, and the second one is ambiguous in the sense that it might denote a, or $a \circ a$, etc. Also note that the diagram (2) can be freely obtained as the composite of generating diagrams of the form

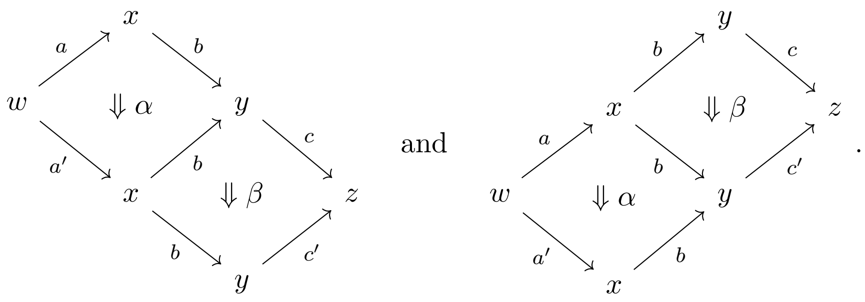
$$x_i \xrightarrow{a_i} x_{i+1}$$

(composition amounts here to identify the target object of a diagram with the source of the second), whereas this is not the case for the diagrams of (3). Note that the pasting diagrams of the form (2) can be characterized as the graphs which are acyclic, connected and non-branching (in the sense that no two arrows have the same source or the same target).

Pasting diagrams in higher categories. In order to extend this construction to higher dimensions, we first need to generalize the notion of graph: an ω*-hypergraph* is a sequence of sets $(P_i)_{i\geq 0}$ together with, for $i \geq 0$ and $x \in P_{i+1}$, two subsets $x^-, x^+ \subseteq P_i$ representing the source and target elements of x. Then, pasting diagrams can be generalized to higher dimensions inductively: an n-pasting diagram is given by source and target $(n-1)$-pasting diagrams and a (compatible) set of n-generators. Conditions need to be put on the ω-hypergraphs in order for the pasting diagrams to have the structure of an ω-category. But, contrarily to dimension one, giving such conditions is hard in higher dimensions, because guessing which formal composites are going to be identified by the axioms of ω-categories can be tricky. For example, the order in which we are supposed to compose the elements of (1) is ambiguous. Considering only the 2-generators, the orders of composition $\alpha, \beta, \gamma, \delta$ and $\alpha, \gamma, \delta, \beta$ are both possible. However, it can be proved that all possible orders of composition are equivalent by the axioms of strict ω-categories, so this ambiguity is not important. On the contrary, given the 2-cells α and β described by the diagrams

$$\begin{tikzcd} & x \arrow[dr, "b"] & \\ w \arrow[ur, "a"] \arrow[dr, "a'"'] & \Downarrow \alpha & y \\ & x \arrow[ur, "b"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "c"] & \\ x \arrow[ur, "b"] \arrow[dr, "b"'] & \Downarrow \beta & z \\ & y \arrow[ur, "c'"'] & \end{tikzcd}\ , \tag{4}$$

α and β can be composed together in two possible orders: α then β or β then α, which can be represented as



But here, these two composites are different. Even more subtle problems arise starting from dimension three, justifying the somewhat sophisticated axioms given for parity complexes and pasting schemes.

Pasting diagrams as cells. Given an ω-hypergraph P, the pasting diagrams on P can be described as *cells* on P, that is, as organized collections of generators of P. For good enough axioms on P, we expect these cells to be ω-categorical cells. There are different flavours for these cells, which reflects as different formalisms for pasting diagrams.

A first notion of cell is given by tuples of elements of P that are kept organized by dimension and by source/target status. This is the solution adopted by parity complexes. For example, the pasting diagram (1) is represented by five sets

$$X_2 = \{\alpha, \beta, \gamma, \delta\},$$
$$X_{1,-} = \{a, b, e, h\}, \qquad X_{1,+} = \{a, d, g, h\},$$
$$X_{0,-} = \{u\}, \qquad X_{0,+} = \{y\}$$

where $X_{i,-}$ represent the i-source, $X_{i,+}$ the i-target, and X_2 the 2-dimensional part of the diagram.

Another notion of cell is given by sets that gather all the elements appearing in the pasting diagram, regardless of their dimension or source/target status. This is the solution adopted by pasting schemes. For example, (1) will be represented by the set

$$X = \{u, v, w, x, y, a, b, c, d, e, f, g, h, \alpha, \beta, \gamma, \delta\}.$$

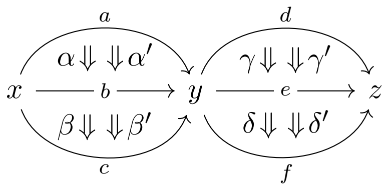
This notion of cell seems the most natural for our goals since it enables one to refer to the "cell obtained by composing together the generators of a set S" directly as the set S. However, it is arguably harder to work with.

A last notion of cells can be obtained by interpreting ω-hypergraph as directed complexes of abelian groups. Similarly to the first notion of cell, cells are then given by group elements for each dimension and source/target status. This is the notion adopted by augmented directed complexes. For example, (1) will be represented by the 5 group elements

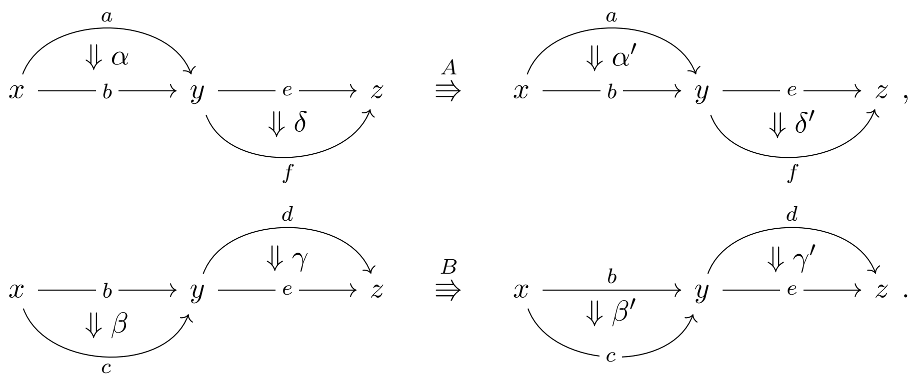
$$X_2 = \alpha + \beta + \gamma + \delta,$$
$$X_{1,-} = a + b + e + h, \qquad X_{1,+} = a + d + g + h,$$
$$X_{0,-} = u, \qquad X_{0,+} = y.$$

This has the advantage of allowing tools from group theory or linear algebra, and is currently the most widely used formalism.

Outline and results. In Section 1, we recall the definitions of each formalism: parity complexes (Subsection 1.3), pasting schemes (Subsection 1.4) and augmented directed complexes (Subsection 1.5). Then, we introduce *generalized parity complexes* (Subsection 1.6, with axioms given in Paragraph 1.6.3). We relate each definition to the unifying notion of ω-hypergraph (Paragraph 1.2.1): a formalism is then a class of ω-hypergraphs (given by axioms) together with a notion of cell and operations on these cells. In Paragraph 1.3.9, we discuss the counter-example to the freeness property of parity complexes, which involves the diagram made of



together with two 3-generators



In Paragraph 1.4.9, we explain that the counter-example above also contradicts the freeness property claimed for pasting schemes. In Paragraph 1.6.7, we give alternative axioms for generalized parity complexes that are simpler to check in practice.

In Section 2, we show that, given a generalized parity complex P, the set of cells $\mathrm{Cell}(P)$ on P has the structure of an ω-category. In Subsection 2.2, we prove an adapted version of [19, Lemma 3.2] which is the main tool to build new cells from known cells (Theorem 2.2.3). In Subsection 2.3, we use this property to show that cells on a generalized parity complex have the structure of an ω-category (Theorem 2.3.3).

In Section 3, we show the freeness result for generalized parity complexes. In Subsection 3.2, we prove that the atomic cells (i.e. cells induced by one generator) are generating $\mathrm{Cell}(P)$ (Theorem 3.2.2). In Subsection 3.4, we introduce *contexts* that are used to obtain canonical form for the cell of an ω-category (as given by Lemma 3.4.4). In Subsection 3.5, we formally define the notion of freeness we are using for ω-categories. In Subsection 3.6, we prove that $\mathrm{Cell}(P)$ is free (Theorem 3.6.18).

In Section 4, we define other notions of cells for generalized parity complexes, namely *maximal-well-formed* and *closed-well-formed sets*. Closed-well-formed sets should be understood as the equivalent of the notion of cell for pasting scheme in generalized parity complexes. Maximal-well-formed sets are then a convenient intermediate for proofs between the original notion of cell for parity complexes (as defined in Paragraph 1.3.3) and closed-well-formed sets. We show that the both new notions induce ω-categories of cells isomorphic to $\mathrm{Cell}(P)$ (Theorem 4.5.5 and Theorem 4.5.7).

In Section 5, we relate generalized parity complexes to the three other formalisms. In Subsection 5.1, we show that parity complexes are generalized parity complexes (Theorem 5.1.3). In Subsection 5.2, we show that loop-free pasting schemes are generalized parity complexes (Theorem 5.2.9) and that both formalisms induce isomorphic ω-categories (Theorem 5.2.10). In Subsection 5.3, we show that loop-free unital augmented directed complexes are generalized parity complexes (Theorem 5.3.23) and that both formalisms induce isomorphic ω-categories (Theorem 5.3.24). In Subsection 5.4, we give counter-examples to other embeddings between the formalisms.

Acknowledgements. I would like to deeply thank Samuel Mimram and Yves Guiraud for their supervision, help and useful feedback during this work. I would also like to thank Simon Henry, Ross Street and Léonard Guetta for the interesting exchanges on the subject. Finally, I would like to thank École Normale Supérieure de Paris for funding my PhD thesis.

1 Definitions

In this section, we recall the definition of strict ω-categories and then we present the three main formalisms for pasting diagrams studied in this article: *parity complexes* [19], *pasting schemes* [8] and *augmented directed complexes* [17]. They all roughly follow the same pattern: starting from what we call an ω-hypergraph, encoding the generating elements of the considered ω-category, they define a notion of *cell*, consisting of sub-hypergraphs satisfying some conditions. Then, they give conditions on these ω-hypergraph such that these cells can be composed and form an ω-category.

1.1 Higher categories

1.1.1 Graded sets. A graded set C is a set together with a partition

$$C = \bigsqcup_{i\in\mathbb{N}} C_i$$

the elements of C_i being *of dimension* i. For $n \in \mathbb{N}$ and $S \subseteq C$, we write $S_{\leq n}$ for $\cup_{i\leq n} S_i$.

1.1.2 Globular sets. A *globular set* C is a graded set, the elements of dimension n being called n*-cells*, together with functions $\partial_n^-, \partial_n^+ : C_{n+1} \to C_n$, respectively associating to an $(n{+}1)$-cell its n*-source* and n*-target*, in such a way that the globular identities are satisfied for every $n \in \mathbb{N}$:

$$\partial_n^- \circ \partial_{n+1}^- = \partial_n^- \circ \partial_{n+1}^+ \qquad \text{and} \qquad \partial_n^+ \circ \partial_{n+1}^- = \partial_n^+ \circ \partial_{n+1}^+.$$

Given $m, n \in \mathbb{N}$ with $n \leq m$, we write $\partial_n^- : C_m \to C_n$ for the function

$$\partial_n^- = \partial_n^- \circ \partial_{n+1}^- \circ \cdots \circ \partial_{m-1}^-$$

and similarly for ∂_n^+. Given $i \leq m$, we write $C_m \times_i C_m$ for the pullback

$$\begin{array}{ccccc} & & C_m \times_i C_m & & \\ & \swarrow & & \searrow & \\ C_m & & & & C_m. \\ & \underset{\partial_i^+}{\searrow} & & \underset{\partial_i^-}{\swarrow} & \\ & & C_i & & \end{array}$$

Moreover, given $x, y \in C_m$, we say that x and y are i*-composable* when $(x, y) \in C_m \times_i C_m$, that is, when $\partial_i^+ x = \partial_i^- y$. More generally, given $k \geq 0$ and $x_1, \ldots, x_k \in P_m$, we say that $x_1, \ldots, x_p$ are i*-composable* when, for $1 \leq j < k$, x_j and x_{j+1} are i-composable.

1.1.3 Strict higher categories. Given $n \in \mathbb{N} \cup \{\omega\}$, an n*-category* C is a globular set such that $C_i = \emptyset$ for $i > n$ and equipped with composition operations

$$*_j \colon C_i \times_j C_i \to C_i$$

for $j \leq i < n + 1$ and identity operations

$$\mathrm{id}_i \colon C_{i-1} \to C_i$$

for $0 < i < n + 1$, which satisfy the axioms (i) to (vi) below. For $j \leq i < n + 1$ and $x \in C_j$, we write $\mathrm{id}_i(x)$ for $\mathrm{id}_i \circ \cdots \circ \mathrm{id}_{j+1}(x)$. The axioms are the following:

(i) for $i < n+1$, $j,k \leq i$, $(x,y) \in C_i \times_j C_i$ and $\epsilon \in \{-,+\}$,

$$\partial_k^\epsilon(\alpha *_j \beta) = \begin{cases} \partial_k^\epsilon(\alpha) = \partial_k^\epsilon(\beta) & \text{if } k < j, \\ \partial_j^-(\alpha) & \text{if } k = j \text{ and } \epsilon = -, \\ \partial_j^+(\beta) & \text{if } k = j \text{ and } \epsilon = +, \\ \partial_k^\epsilon(\alpha) *_j \partial_k^\epsilon(\beta) & \text{if } k > j, \end{cases}$$

(ii) for $i < n$ and $x \in C_i$,

$$\partial_i^-(\mathrm{id}_{i+1}(x)) = \partial_i^+(\mathrm{id}_{i+1}(x)) = x,$$

(iii) for $j \leq i < n+1$, $x,y,z \in C_i$ with $(x,y),(y,z) \in C_i \times_j C_i$,

$$(x *_j y) *_j z = x *_j (y *_j z),$$

(iv) for $j \leq i < n+1$ and $x \in C_i$,

$$\mathrm{id}_i(\partial_j^- x) *_j x = x = x *_j \mathrm{id}_i(\partial_j^+ x),$$

(v) for $k < j \leq i < n+1$ and $x,x',y,y' \in C_i$ with $(x,y),(x',y') \in C_i \times_j C_i$ and $(x,x'),(y,y') \in C_i \times_k C_i$,

$$(x *_j y) *_k (x' *_j y') = (x *_k x') *_j (y *_k y'),$$

(vi) for $j \leq i < n$ and $(x,y) \in C_i \times_j C_i$,

$$\mathrm{id}_{i+1}(x *_j y) = \mathrm{id}_{i+1}(x) *_j \mathrm{id}_{i+1}(y).$$

Note that, given $n \in \mathbb{N}$, an n-category C can equivalently be defined as an ω-category such that, for $i > n$ and $x \in C_i$, $x = \mathrm{id}_i(x')$ for some $x' \in C_n$. Given $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\omega\}$ and an $(n+k)$-category C, we write $C_{\leq n}$ for the underlying n-category.

1.1.4 Strict higher functors. Given $n \in \mathbb{N} \cup \{\omega\}$ and two n-categories C and D, an *n-functor* $F\colon C \to D$ is given by a sequence of functions $F_i\colon C_i \to D_i$, for $0 \leq i < n+1$, such that

– for $j \leq i < n+1$, $x,y \in C_i$ such that $\partial_j^+ x = \partial_j^- y$,

$$F_i(x *_j y) = F_i(x) *_j F_i(y),$$

– for $i < n$, $x \in C_i$,

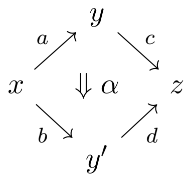
$$F_{i+1}(\mathrm{id}_{i+1}(x)) = \mathrm{id}_{i+1}(F_i(x)).$$

Given $x \in C_i$, we write $F(x)$ for $F_i(x)$.

1.2 Higher graphs

1.2.1 Hypergraphs. An ω*-hypergraph* P is a graded set, the elements of dimension i being called i*-generators*, together with, for $i \geq 0$ and for each generator $x \in P_{i+1}$, two finite subsets $x^-, x^+ \subseteq P_n$ called the *source* and *target* of x. Given a subset $U \subseteq P$ and $\epsilon \in \{-,+\}$, we write U^ϵ for $\cup_{x \in U} x^\epsilon$. Simple ω-hypergraphs can be represented graphically using *diagrams*, where 0-generators are represented by their names, and higher generators by arrows $\rightarrow$, $\Rightarrow$, $\Rrightarrow$ etc. . . that represent respectively 1-generators, 2-generators, 3-generators etc. . .

For example, the diagram



(5)

can be encoded as the ω-hypergraph P with

$$P_0 = \{x, y, y', z\}, \qquad P_1 = \{a, b, c, d\}, \qquad P_2 = \{\alpha\},$$

and $P_n = \emptyset$ for $n \geq 3$, source and target being

$$a^- = \{x\}, \qquad a^+ = \{y\}, \qquad \alpha^- = \{a, c\}, \qquad \alpha^+ = \{b, d\},$$

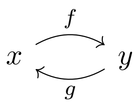
and so on.

1.2.2 Fork-freeness. Given an ω-hypergraph P and $n \in \mathbb{N}$, a subset $U \subseteq P_n$ is *fork-free* (also called *well-formed* in [20]) when:

- if $n = 0$ then $|U| = 1$,
- if $n > 0$ then for all $x, y \in U$ and $\epsilon \in \{-,+\}$, we have $x^\epsilon \cap y^\epsilon = \emptyset$.

Note that the definition of fork-freeness depends on the intended dimension n. This subtlety is important in the case of the empty set: $\emptyset$ is not well-formed as a subset of P_0 but it is as a subset of P_n when $n > 0$. For example, the subset $\{a, b\}$ of (5) is not fork-free since $a^- \cap b^- = \{x\}$, but $\{a, c\}$ is.

1.2.3 The relation $\triangleleft$. Given an ω-hypergraph P, $n > 0$ and $U \subseteq P_n$, for $x, y \in U$, we write $x \triangleleft^1_U y$ when $x^+ \cap y^- \neq \emptyset$ and we define the relation $\triangleleft_U$ on U as the transitive closure of $\triangleleft^1_U$. Given $V, W \subseteq U$, we write $V \triangleleft_U W$ when there exist $x \in V$ and $y \in W$ such that $x \triangleleft_U y$. We define the relation $\triangleleft$ on P as $\cup_{i>0} \triangleleft_{P_i}$. The ω-hypergraph P is then said *acyclic* when $\triangleleft$ is irreflexive. For example, the following ω-hypergraph



(6)

is not acyclic since $f \triangleleft g \triangleleft f$. The ω-hypergraph (1) is acyclic.

For $V \subseteq U$, we say that V is a *segment for* $\triangleleft_U$ when for all $x, y, z \in U$ with $x, z \in V$ and $x \triangleleft_U y \triangleleft_U z$, it holds that $y \in V$. For $V \subseteq U$, we say that V is *initial (resp. terminal) for* $\triangleleft_U$ when, for all $x \in U$, if there exists $y \in V$ such that $x \triangleleft_U y$ (resp. $y \triangleleft_U x$), then $x \in V$.

1.2.4 Remark. In [19], $\lhd$ is defined as a transitive and reflexive relation whereas in [8], it is only defined as a transitive relation. In this paper, a transitive (and not reflexive) definition is preferred, since it carries more information than a transitive and reflexive definition.

1.2.5 Other source/target operations. Given an ω-hypergraph P, for $n \geq 2$, $x \in P_n$ and $\epsilon, \eta \in \{-,+\}$, we write $x^{\epsilon\eta}$ for $(x^\epsilon)^\eta$. We extend the notation to subsets $S \subseteq P_n$ and write $S^{\epsilon\eta}$ for $(S^\epsilon)^\eta$. Moreover, we write $x^\mp$ for $x^- \setminus x^+$ and $x^\pm$ for $x^+ \setminus x^-$. We also extend the notation to subsets $S \subseteq P_n$ and write $S^\mp$ for $S^- \setminus S^+$ and $S^\pm$ for $S^+ \setminus S^-$. For example, in the ω-hypergraph (5), we have

$$\alpha^{--} = \{x, y\}, \qquad \alpha^{+-} = \{x, y'\}, \qquad \alpha^{-\mp} = \{x\}, \qquad \alpha^{+\pm} = \{z\}.$$

1.3 Parity complexes

In this subsection, we recall the formalism of parity complexes developed by Street in [19]. Most of the content will be reused when defining generalized parity complexes. The idea behind the formalism is to represent an $(n+1)$-cell as source and target n-cells together with a subset of P_{n+1} which "moves" the source n-cell to the target n-cell. Under the axioms of parity complexes, the set of cells will have a structure of ω-category.

1.3.1 Pre-cells. Let P be an ω-hypergraph. For $n \in \mathbb{N}$, an *n-pre-cell of P* is a tuple

$$X \quad = \quad (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n)$$

of finite subsets of P, such that $X_{i,\epsilon} \subseteq P_i$ for $0 \leq i < n$ and $\epsilon \in \{-,+\}$, and $X_n \subseteq P_n$. By convention, we sometimes write $X_{n,-}$ and $X_{n,+}$ for X_n. We write $\mathrm{PCell}(P)$ for the graded set of pre-cells of P.

Given $n \geq 0$, $\epsilon \in \{-,+\}$ and an $(n+1)$-pre-cell X of P, we define the n-pre-cell $\partial^\epsilon X$ as

$$\partial^\epsilon X = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_{n,\epsilon}).$$

The globular conditions $\partial^\epsilon \circ \partial^- = \partial^\epsilon \circ \partial^+$ are then trivially satisfied and ∂^-, ∂^+ equip $\mathrm{PCell}(P)$ with a structure of globular set.

1.3.2 Movement and orthogonality. Let P be an ω-hypergraph. Given $n \in \mathbb{N}$ and finite sets $M \subseteq P_{n+1}$, $U \subseteq P_n$ and $V \subseteq P_n$, we say that M *moves* U to V when

$$U = (V \cup M^-) \setminus M^+ \qquad \text{and} \qquad V = (U \cup M^+) \setminus M^-.$$

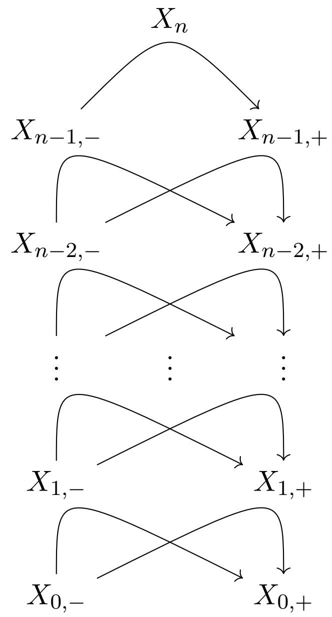
The idea here is that V is the subset obtained from U by replacing the source of M by its target.

Given $n \in \mathbb{N}$ and finite sets $S, T \subseteq P_n$, we say that S and T are *orthogonal*, written $S \perp T$, when $(S^- \cap T^-) \cup (S^+ \cap T^+) = \emptyset$. Orthogonality, as we will see in Lemma 2.1.4, is a condition that enables decomposition of movements.

1.3.3 Cells. Let P be an ω-hypergraph. Given $n \in \mathbb{N}$, an *n-cell of P* is an n-pre-cell of P, such that

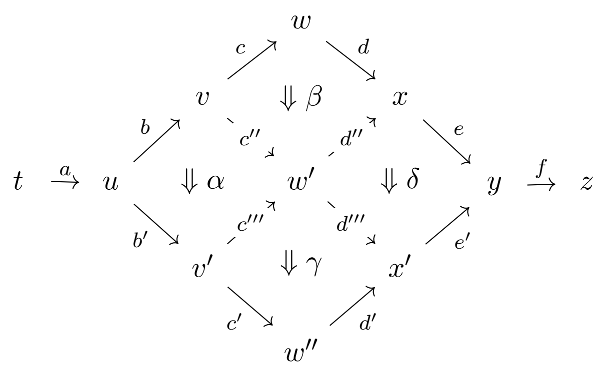
- $X_{i,\epsilon}$ is fork-free for $0 \leq i \leq n$ and $\epsilon \in \{-,+\}$,
- $X_{i+1,\epsilon}$ moves $X_{i,-}$ to $X_{i,+}$ for $0 \leq i < n$, $\epsilon \in \{-,+\}$.

We denote by Cell(P) the graded set of cells of P, which inherits the structure of globular set from PCell(P). Graphically, an n-cell X can be informally represented in the following way:



where an arrow $U \xrightarrow{M} V$ means that M moves U to V.

For example, consider the ω-hypergraph associated to the diagram



(7)

It contains (among other) the cells

$$(\{t\}),$$

$$(\{t\}, \{w'\}, \{a, b, c''\}, \{a, b, c'''\}, \{\alpha\}),$$

$$(\{t\}, \{z\}, \{a, b, c, d, e, f\}, \{a, b', c', d', e', f\}, \{\alpha, \beta, \gamma, \delta\}) \dots$$

1.3.4 Link with Street's definition. In [19], cells are defined as pairs (M, N) with $M, N \subseteq P$ satisfying conditions similar to the fork-freeness and movement conditions. This definition is equivalent to the one given above: given an n-cell (in the sense of Street) (M, N), one obtains an n-cell X (in our sense), by setting $X_n = M_n$ and, for $i < n$, $X_{i,-} = M_i$ and $X_{i,+} = N_i$, and an inverse translation is defined similarly.

1.3.5 Compositions and identities of cells. Let P be an ω-hypergraph. Given $0 \leq i \leq n \in \mathbb{N}$ and $X, Y \in \mathrm{Cell}(P)_n$ such that X and Y are i-composable, the *i-composite* $X *_i Y$ of X and Y is defined as the n-pre-cell Z such that

$$Z_{j,\epsilon} \quad = \quad \begin{cases} X_{j,\epsilon} & \text{if } j < i, \\ X_{i,-} & \text{if } j = i \text{ and } \epsilon = -, \\ Y_{i,+} & \text{if } j = i \text{ and } \epsilon = +, \\ X_{j,\epsilon} \cup Y_{j,\epsilon} & \text{if } j > i. \end{cases}$$

It will be shown in Section 2 that, under suitable assumptions, the composite of two n-cells is actually an n-cell. Given an n-cell X, the *identity of* X is the $(n{+}1)$-cell $\mathrm{id}_{n+1}(X)$ given by

$$\mathrm{id}(X) = (X_{0,-}, X_{0,+}, \ldots, X_{n-1,-}, X_{n-1,+}, X_n, X_n, \emptyset).$$

and for $m \geq n$, we define $\mathrm{id}_m(X)$ with an induction on m by

$$\mathrm{id}_n(X) = X \quad \text{and} \quad \mathrm{id}_{m+1}(X) = \mathrm{id}(\mathrm{id}_m(X)).$$

1.3.6 Atoms and relevance. Let P be an ω-hypergraph. Given $n \in \mathbb{N}$ and $x \in P_n$, we define $\langle x\rangle_{i,\epsilon} \subseteq P_i$ for $0 \leq i \leq n$ and $\epsilon \in \{-,+\}$ inductively by

$$\langle x\rangle_{n,-} = \langle x\rangle_{n,+} = \{x\}$$

and

$$\langle x\rangle_{j,-} = \langle x\rangle^{\mp}_{j+1,-} \quad \text{and} \quad \langle x\rangle_{j,+} = \langle x\rangle^{\pm}_{j+1,+}$$

for $0 \leq j < n$. For sake of symmetry, we write $\langle x\rangle_n$ for $\langle x\rangle_{n,\epsilon}$. The *atom associated to* x is then the n-pre-cell of P

$$\langle x\rangle = (\langle x\rangle_{0,-}, \langle x\rangle_{0,+}, \ldots, \langle x\rangle_{n-1,-}, \langle x\rangle_{n-1,+}, \langle x\rangle_n).$$

A generator x is said *relevant* when the atom $\langle x\rangle$ is a cell. For example, the atom associated to α in (5) is $\langle\alpha\rangle$ with

$$\begin{aligned} &\langle\alpha\rangle_{0,-} = \{u\}, && \langle\alpha\rangle_{1,-} = \{b, c''\}, && \langle\alpha\rangle_2 = \{\alpha\}, \\ &\langle\alpha\rangle_{0,+} = \{w'\}, && \langle\alpha\rangle_{1,+} = \{b', c'''\} \end{aligned}$$

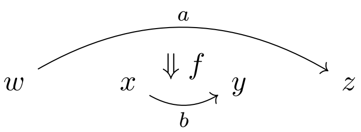
and, since it is a cell, α is relevant.

1.3.7 Tightness. Given $n \in \mathbb{N}$, a subset $T \subseteq P_n$ is said to be *tight* when, for all $u, v \in P_n$ such that $u \triangleleft v$ and $v \in T$, we have $u^- \cap T^{\pm} = \emptyset$. For example, in (7), $X = \{\beta, \gamma\}$ is not tight since $\alpha \triangleleft \gamma$ and $c'' \in \alpha^- \cap X^{\pm}$. This notion appears in [20] to correct the original definition of parity complexes. However, it will not be used in the generalized formalism.

1.3.8 Parity complexes. A *parity complex* is an ω-hypergraph P satisfying the axioms (C0) to (C5) below:

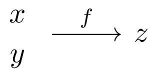
(C0) for $n > 0$ and $x \in P_n$, $x^- \neq \emptyset$ and $x^+ \neq \emptyset$;
(C1) for $n \geq 2$ and $x \in P_n$, $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$;
(C2) for $n \geq 1$ and $x \in P_n$, x^- and x^+ are fork-free;
(C3) P is acyclic;
(C4) for $n \geq 1$, $x, y \in P_n$, $z \in P_{n+1}$, if $x \triangleleft y$, $x \in z^\epsilon$ and $y \in z^\eta$ for some $\epsilon, \eta \in \{-,+\}$, then $\epsilon = \eta$;
(C5) for $0 \leq i < n$ and $x \in P_n$, $\langle x \rangle_{i,-}$ is tight.

(C0) ensures that each generator has defined source and target. (C1) enforces basic globular properties on generators. For example, it forbids the ω-hypergraph



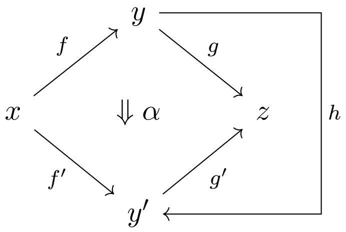
(8)

since $f^{--} \cup f^{++} = \{w, y\}$ and $f^{+-} \cup f^{-+} = \{x, z\}$. (C2) forbids generators with parallel elements in their sources or targets. For example, the ω-hypergraph



(9)

does not satisfy (C2) since $f^- = \{x, y\}$ is not fork-free. (C3) forbids ω-hypergraph like (6). (C4) can be informally described as forbidding "bridges": the ω-hypergraph



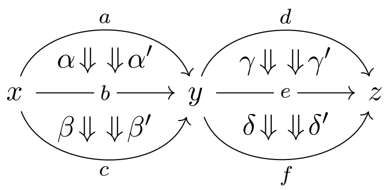
(10)

does not satisfy (C4). Indeed, $f \triangleleft g'$ and $f \in \alpha^-$ and $g' \in \alpha^+$.

The axioms (C0) to (C4) above were the only ones in [19]. But they appeared to be insufficient. The last axiom (C5) was then introduced in the corrigenda [20] which involves the mentioned notion of tightness. This axiom relates to segment Axiom (G3) of generalized parity complexes and prevent problematic ω-hypergraphs in the spirit of (16) discussed in Paragraph 1.6.5, even though (16) does not satisfy (C3) in the first place.

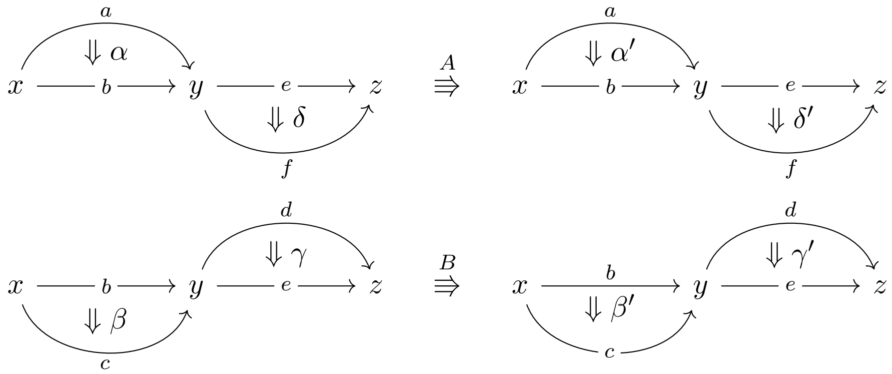
1.3.9 A counter-example to the freeness property. The main result claimed in [19] is that the globular set Cell(P) together with the source, target, identity and composition operations, has the structure of an ω-category, which is freely generated by the atoms $\langle x \rangle$ for $x \in P$ ([19, Theorem 4.2]). But, this property does not hold as we illustrate with a counter-example.

Consider the ω-hypergraph P defined by the diagram given by



(11)

together with two 3-generators



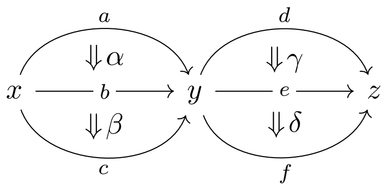
Then, it can be shown that P is a parity complex. Moreover, the diagram (11) defines a polygraph Q, which induces a free ω-category $\mathcal{A} = Q^*$ (see Paragraph 3.5.2 for the definitions). In $\mathcal{A}$, there are two ways of composing the 3-cells A and B, inducing two 3-cells H_1 and H_2 as follows:

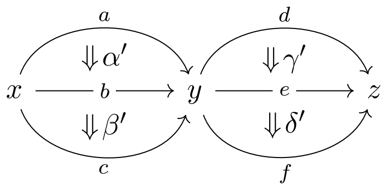
$$H_1 = ((a *_0 \gamma) *_1 A *_1 (\beta *_0 f)) *_2 ((\alpha' *_0 d) *_1 B *_1 (c *_0 \delta'))$$

and

$$H_2 = ((\alpha *_0 d) *_1 B *_1 (c *_0 \delta)) *_2 ((a *_0 \gamma') *_1 A *_1 (\beta' *_0 f)).$$

These two 3-cells are not equal with respect to the axioms of strict 3-categories (a proof is given in Appendix A), so they are different 3-cells of $\mathcal{A}$. But both have the same 2-source and 2-target:

$\partial_2^-(H_i) =$  ,

$\partial_2^+(H_i) =$  .

Moreover, the involved 3-generators in H_1 and H_2 are A and B. The information that makes H_1 and H_2 different is the order of composition of A and B, which is not included in the cell structure of a parity complex. So the universal morphism eval: $\mathcal{A} \to \mathrm{Cell}(P)$ maps H_1 and H_2 to the same cell X given by:

$$X_3 = \{A, B\},$$
$$X_{2,-} = \{\alpha, \beta, \gamma, \delta\}, \qquad X_{2,+} = \{\alpha', \beta', \gamma', \delta'\},$$
$$X_{1,-} = \{a, d\}, \qquad X_{1,+} = \{c, f\},$$
$$X_{0,-} = \{x\}, \qquad X_{0,-} = \{z\}.$$

Hence, H_1 and H_2 are identified in $\mathrm{Cell}(P)$. But [19, Theorem 4.2] was precisely stating that eval was an isomorphism. Thus, $\mathrm{Cell}(P)$ does not satisfy the freeness property claimed in [19].

1.4 Pasting schemes

Johnson's loop-free pasting schemes [8] is another proposed formalism for pasting diagrams. Like parity complexes, they are based on ω-hypergraphs, but the cells will now be represented as subsets of generators instead of tuples as for parity complexes. Moreover, pasting schemes will rely on relations, namely B and E, on the ω-hypergraph to define the globular operations on the cells. More precisely, B and E will encode which generators to remove to obtain respectively the target and the source of a cell.

1.4.1 Conventions for relations. First, we give some elementary definitions and notations for relations. A *binary relation between two sets X and Y* is a subset $\mathrm{L} \subseteq X \times Y$. For $(x, y) \in X \times Y$, we write $x \,\mathrm{L}\, y$ when $(x, y) \in \mathrm{L}$. The *identity relation on a set X* is the relation $\mathrm{L} \subseteq X \times X$ such that $x \,\mathrm{L}\, x'$ iff $x = x'$. Given a binary relation L between X and Y, for $x \in X$, we write $\mathrm{L}(x)$ for the set

$$\mathrm{L}(x) = \{y \in Y \mid x \,\mathrm{L}\, y\}.$$

Similarly, given $X' \subseteq X$, we denote by $\mathrm{L}(X')$ the set

$$\{y \in Y \mid \exists x \in X', x \,\mathrm{L}\, y\}.$$

The relation L is said *finitary* when, for all $x \in X$, $\mathrm{L}(x)$ is a finite set. If L is a relation on a graded set $P = \bigsqcup_{n \in \mathbb{N}} P_n$, we write L^i_j for the relation between P_i and P_j defined as $\mathrm{L} \cap (P_i \times P_j)$. Similarly, we write L^i for the relation between P_i and P defined as $\mathrm{L} \cap (P_i \times P)$.

Given relations L between X and Y and L' between Y and Z, we write $\mathrm{L}\mathrm{L}'$ for the relation between X and Z which is the composite relation defined as

$$\mathrm{L}\mathrm{L}' = \{(x, z) \in X \times Z \mid \exists y \in Y, x \,\mathrm{L}\, y \text{ and } y \,\mathrm{L}'\, z\}.$$

1.4.2 Pre-pasting schemes. A *pre-pasting scheme* $(P, \mathrm{B}, \mathrm{E})$ is given by a graded set P and two binary relations B, E (for "beginning" and "end") on P such that

- B and E are finitary,
- for $n < m$, $\mathrm{B}^n_m = \mathrm{E}^n_m = \emptyset$,
- B^n_n (resp. E^n_n) is the identity relation on P_n,
- for $m < n$, for $\mathrm{L} \in \{\mathrm{B}, \mathrm{E}\}$, $x \,\mathrm{L}^{n+1}_m\, y$ if and only if

$$x \,\mathrm{L}^{n+1}_n \mathrm{B}^n_m\, y \quad \text{and} \quad x \,\mathrm{L}^{n+1}_n \mathrm{E}^n_m\, y.$$

For example, the diagram (5) can be encoded as a pre-pasting scheme

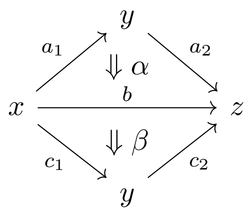
$$\begin{aligned} \mathrm{B}^2_1(f) &= \{a, c\}, & \mathrm{E}^2_1(f) &= \{b, d\}, \\ \mathrm{B}^2_0(f) &= \{y\}, & \mathrm{E}^2_0(f) &= \{y'\}, \\ \mathrm{B}^1_0(a) &= \{x\}, & \mathrm{E}^1_0(a) &= \{y\} \ldots \end{aligned}$$

1.4.3 Pre-pasting schemes as ω-hypergraphs. Note that the data of a pre-pasting scheme P is completely determined by the data of $\mathrm{B}^n_{n-1}(x)$ and $\mathrm{E}^n_{n-1}(x)$ for $n \geq 1$ and $x \in P_n$. In fact, the data of a pre-pasting scheme is equivalent to the data of an ω-hypergraph structure on P: the correspondence is given by $x^- = \mathrm{B}^n_{n-1}(x)$ and $x^+ = \mathrm{E}^n_{n-1}(x)$ for $n \geq 1$ and $x \in P_n$. In particular, the relation $\triangleleft$ on a pasting scheme is defined as the one on the associated ω-hypergraph.

1.4.4 Direct loops. Given an ω-hypergraph P, a *direct loop* is given by

- either $n > 0$ and $x, y \in P_n$ such that $x \triangleleft y$ and $\mathrm{E}(y) \cap \mathrm{B}(x) \neq \emptyset$,
- or $z \in P$ such that $\mathrm{E}(z) \cap \mathrm{B}(z) \neq \{z\}$.

For example, the ω-hypergraph



(12)

has a direct loop by the first criterion, because $\alpha \triangleleft \beta$ and $y \in \mathrm{B}(\alpha) \cap \mathrm{E}(\beta)$. An example of direct loop given by the second criterion is the ω-hypergraph (4) where $b \in \mathrm{B}(\alpha) \cap \mathrm{E}(\alpha)$.

1.4.5 Finite graded subsets. Let P be a pre-pasting scheme. We define the relation $\mathrm{R} \subseteq P \times P$ as the smallest reflexive transitive relation on P such that, for all $n \in \mathbb{N}$ and $x \in P_{n+1}$, $\mathrm{B}(x), \mathrm{E}(x) \subseteq \mathrm{R}(x)$. A *finite graded subset of dimension n of P* (abbreviated *n-fgs*) is an $(n+1)$-tuple $X = (X_0, \ldots, X_n)$ such that $X_i \subseteq P_i$ and X_i is finite for every i. We sometimes abuse notation and identify the n-fgs X with the set $\cup_{i \leq n} X_i$. We say that X is *closed* when $\mathrm{R}(X) = X$.

Given X an n-fgs of P, define the *source* and the *target* of X as the $(n-1)$-fgs's $\partial^- X$ and $\partial^+ X$ of P such that

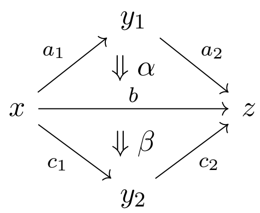
$$\partial^- X = X \setminus \mathrm{E}^n(X) \quad \text{and} \quad \partial^+ X = X \setminus \mathrm{B}^n(Y).$$

1.4.6 Well-formed sets. Given a pre-pasting scheme P, an n-fgs X of P is said to be *well-formed* when

- X is closed,
- X_n is fork-free,
- if $n > 0$, $\partial^- X$ and $\partial^+ X$ are well-formed $(n-1)$-fgs.

A well-formed n-fgs of P will be called a *well-formed set of dimension n*, abbreviated *n-wfs of P*. We denote by $\mathrm{WF}(P)$ the set of wfs's of P. By [8, Theorem 3], the operations ∂^- and ∂^+ equip $\mathrm{WF}(P)$ with a structure of globular set.

For example, the pre-pasting scheme



(13)

has among others the following wfs:

$$\{x\}, \{z\},$$

$$\{x, y_1, z, a_1, a_2\}, \{x, y_2, z, c_1, c_2\},$$

$$\{x, y_1, y_2, z, a_1, a_2, b, c_1, c_2, \alpha, \beta\} \ldots$$

1.4.7 Compositions and identities. Let P be a pre-pasting scheme. Given $i \leq n \in \mathbb{N}$ and X, Y two n-wfs such that $\partial_i^+ X = \partial_i^- Y$, the *$i$-composite* of X and Y is the n-fgs $X *_i Y$ such that

$$X *_i Y = X \cup Y.$$

Given $n \in \mathbb{N}$ and an n-wfs $X = (X_0, \ldots, X_n)$ of P, the *identity of* X is the $(n{+}1)$-wfs $\mathrm{id}_{n+1}(X)$ such that

$$\mathrm{id}_{n+1}(X) = (X_0, \ldots, X_n, \emptyset).$$

1.4.8 Loop-free pasting schemes. A *pasting scheme* is a pre-pasting scheme P satisfying the following two axioms:

(S0) for $n > 0$ and $x \in P_n$, $\mathrm{B}^n_{n-1}(x) \neq \emptyset$ and $\mathrm{E}^n_{n-1}(x) \neq \emptyset$;

(S1) for $\mathrm{L} \in \{\mathrm{B}, \mathrm{E}\}$, $k < n$ and $x \in P_n$ and $y \in P_k$,

- if $x \, \mathrm{E}^n_{n-1} \mathrm{L}^{n-1}_k \, y$ then $x \, \mathrm{E}^n_k \, y$ or $x \, \mathrm{B}^n_k \mathrm{L}^{n-1}_k \, y$,
- if $x \, \mathrm{B}^n_{n-1} \mathrm{L}^{n-1}_k \, y$ then $x \, \mathrm{B}^n_k \, y$ or $x \, \mathrm{E}^n_k \mathrm{L}^{n-1}_k \, y$.

The pasting scheme P is a *loop-free pasting scheme* when it moreover satisfies the following axioms:

(S2) P has no direct loops;

(S3) for $x \in P$, $\mathrm{R}(x) \in \mathrm{WF}(P)$;

(S4) for $k < n \in \mathbb{N}$, $X \in \mathrm{WF}(P)_k$ and $x \in P_n$,

- if $\partial_k^-(\mathrm{R}(x)) \subseteq X$, then $\langle x \rangle_{k,-}$ is a segment for $\triangleleft_{X_k}$,
- if $\partial_k^+(\mathrm{R}(x)) \subseteq X$, then $\langle x \rangle_{k,+}$ is a segment for $\triangleleft_{X_k}$;

(S5) for $n \in \mathbb{N}$, $X \in \mathrm{WF}(P)_n$ and $x \in P_{n+1}$, we have

(a) $X \cap \mathrm{E}(x) = \emptyset$,

(b) for $y \in X$, if $\mathrm{B}(x) \cap \mathrm{R}(y) \neq \emptyset$, then $y \in \mathrm{B}(x)$.

(S1) enforces basic globular properties on generators (for example, it forbids the ω-hypergraph (8)). (S3) enforces fork-freeness on the iterated sources and targets of a generator (for example, it forbids the ω-hypergraph (9)). (S4) relates to segment Axiom (G3) of generalized parity complexes and prevent problematic ω-hypergraphs in the spirit of (16) discussed in Paragraph 1.6.5, even though (16) does not satisfy (S2) in the first place. However, a more satisfying counter-example in dimension four exists (see [14, Example 3.11]). (S5) can be deduced from the other axioms (see [9, Theorem 3.7]) but it simplifies the proofs of [8].

1.4.9 A counter-example to the freeness property. The main result claimed in [8] is that, given a loop-free pasting scheme P, the globular set $\mathrm{WF}(P)$ together with the source, target, composition and identity operations has the structure of an ω-category, which is freely generated by the wfs's $\mathrm{R}(x)$ for $x \in P$ ([8, Theorem 13]). But the same flaw than in [19] is present, which makes the freeness result wrong. In fact, the same counter-example than for parity complexes (see Paragraph 1.3.9) can be used: the ω-hypergraph P is a loop-free pasting scheme and the universal map $\mathrm{eval}\colon \mathcal{A} \to \mathrm{WF}(P)$ sends H_1 and H_2 to the same wfs X where

$$X = \{x, y, z, \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta', A, B\}$$

contradicting the freeness property [8, Theorem 13].

1.5 Augmented directed complexes

Augmented directed complexes, designed by Steiner in [17], are not directly based on ω-hypergraphs but on chain complexes. Under the axioms required in [17], it happens that the data of a chain complex is equivalent to the data of an ω-hypergraph. The structure of cells used within this formalism strongly resembles the structure of cells used for parity complexes. The only difference is that the cells are tuples of group elements instead of subsets of an ω-hypergraph.

1.5.1 Augmented directed complex. A *pre-augmented directed complexes*, abbreviated *pre-adc*, $(K, \mathrm{d}, \mathrm{e})$ consists in

– for $n \geq 0$, an abelian group K_n together with a distinguished submonoid $K_n^* \subseteq K_n$,

– group morphisms called *boundary operators*, for $n \geq 0$,

$$\mathrm{d}_n \colon K_{n+1} \to K_n,$$

– an *augmentation*, that is, a group morphism

$$\mathrm{e}\colon K_0 \to \mathbb{Z}.$$

An *augmented directed complex*, abbreviated *adc*, is a pre-adc $(K, \mathrm{d}, \mathrm{e})$ such that $\mathrm{e} \circ \mathrm{d}_0 = 0$ and $\mathrm{d}_n \circ \mathrm{d}_{n+1}$ for $n \in \mathbb{N}$.

1.5.2 Bases for pre-adc's. Given a pre-adc $(K, \mathrm{d}, \mathrm{e})$, a *basis* of $(K, \mathrm{d}, \mathrm{e})$ is a graded set $B \subseteq \bigsqcup_{n \geq 0} K_n$ such that each K_n^* is the free commutative monoid on B_n and each K_n is the free abelian group on K_n^*.

Given a basis B of $(K, \mathrm{d}, \mathrm{e})$, note that every element $x \in K_n^*$ can be uniquely written as

$$x = \sum_{b \in B_n} x_b b,$$

with $x_b \in \mathbb{N}$ and $x_b \neq 0$ for a finite number of $b \in B_n$. This representation defines a partial order $\leq$ where for $x, y \in K_n^*$, $x \leq y$ when $x_b \leq y_b$ for all $b \in B_n$. Furthermore, a *greatest lower bound* $x \wedge y$ can be defined as

$$x \wedge y = \sum_{b \in B_n} \min(x_b, y_b) b.$$

Given $x \in K_{n+1}^*$, define $x^{\mp}, x^{\pm} \in K_n^*$ as the unique elements satisfying

$$\mathrm{d}_n\, x = x^{\pm} - x^{\mp} \quad \text{and} \quad x^{\mp} \wedge x^{\pm} = 0.$$

Also, if $x = \sum_{b \in B_{n+1}} x_b b$, we define x^-, x^+ as

$$x^- = \sum_{b \in B_{n+1}} x_b b^{\mp} \quad \text{and} \quad x^+ = \sum_{b \in B_{n+1}} x_b b^{\pm}$$

1.5.3 Remark. $x^{\mp}$ and $x^{\pm}$ are respectively denoted by $\partial^- x$ and $\partial^+ x$ in [17]. We adopt here this convention for consistency with those of Subsection 1.3.

1.5.4 From ω-hypergraphs to pre-adc's with basis. Given an ω-hypergraph P, we define the *pre-adc associated to* P as the pre-adc $(K, \mathrm{d}, \mathrm{e})$ defined as follows. For $n \in \mathbb{N}$, K_n^* is defined as the free commutative monoid on P_n and K_n as the free abelian on K_n^*. The augmentation $\mathrm{e}\colon K_0 \to \mathbb{Z}$ is defined as the unique morphism such that $\mathrm{e}(x) = 1$ for $x \in P_0$. Given $n \in \mathbb{N}$, for all finite subsets $S \subseteq P_n$, we write $\mathrm{M}_n(S)$ for $\sum_{x \in S} x \in K_n$. Then, $\mathrm{d}_n\colon K_{n+1} \to K_n$ is defined as the unique morphism such that $\mathrm{d}_n(x) = \mathrm{M}_n(x^+) - \mathrm{M}_n(x^-)$ for $x \in P_{n+1}$. Note that K admits canonically P as a basis. We say that P *is an adc* when K is an adc.

For example, consider the ω-hypergraph (13). We write S^* the free commutative monoid on the set S. Then, the pre-adc associated to (13) is defined by

$$K_0^* = \{x, y_1, y_2, z\}^*, \quad K_1^* = \{a_1, a_2, b, c_1, c_2\}^*, \quad K_2^* = \{\alpha, \beta\}^*$$

and $K_n^* = 0$ for $n \geq 3$. K_0, K_1, K_2 and K_n for $n \geq 3$ are then the induced free abelian groups on these monoids. d and e are defined by universal property to be the only morphisms such that

$$\mathrm{e}(x) = \mathrm{e}(y_1) = \mathrm{e}(y_2) = \mathrm{e}(z) = 1$$
$$\mathrm{d}_0(a_1) = y_1 - x, \quad \mathrm{d}_0(a_2) = z - y_1, \quad \mathrm{d}_0(b) = z - x,$$
$$\mathrm{d}_0(c_1) = y_2 - x, \qquad \mathrm{d}_0(c_2) = z - y_2,$$
$$\mathrm{d}_1(\alpha) = b - (a_1 + a_2), \qquad \mathrm{d}_1(\beta) = (c_1 + c_2) - b.$$

As an example for the $(-)^{\mp}$ and $(-)^{\pm}$ operations, we have

$$(a_1 + a_2)^{\mp} = x, \qquad (a_1 + a_2)^{\pm} = z,$$
$$(\alpha + \beta)^{\mp} = a_1 + a_2, \qquad (\alpha + \beta)^{\pm} = c_1 + c_2.$$

1.5.5 Cells. Given a pre-adc K, an *n-pre-cell of K* is given by an $(2n{+}1)$-tuple

$$X = (X_{0,-}, X_{0,+}, \dots, X_{n-1,-}, X_{n-1,+}, X_n)$$

with $X_n \in K_n^*$ and $X_{i,-}, X_{i,+} \in K_n^*$ for $0 \le i < n$. For simplicity, we often refer to X_n by $X_{n,-}$ or $X_{n,+}$. We write $\mathrm{PCell}(K)$ for the set of pre-cells of K.

Given $n \ge 0$, $\epsilon \in \{-,+\}$ and an $(n{+}1)$-pre-cell X of K, we define the n-pre-cell $\partial^\epsilon X$ as

$$\partial^\epsilon X = (X_{0,-}, X_{0,+}, \dots, X_{n-1,-}, X_{n-1,+}, X_{n,\epsilon}).$$

The globular conditions $\partial^\epsilon \circ \partial^- = \partial^\epsilon \circ \partial^+$ are then trivially satisfied and ∂^-, ∂^+ equip $\mathrm{PCell}(K)$ with a structure of globular set.

An *n-cell of K* is an n-pre-cell X of K such that

- for $0 \le i < n$, $\mathrm{d}_i(X_{i+1,-}) = \mathrm{d}_i(X_{i+1,+}) = X_{i,+} - X_{i,-}$,
- $\mathrm{e}(X_{0,-}) = \mathrm{e}(X_{0,+}) = 1$.

We denote by $\mathrm{Cell}(K)$ the set of cells of K, which inherits the globular structure from $\mathrm{PCell}(K)$.

1.5.6 Compositions and identities of cells. Let K be a pre-adc. For $i < n$, X, Y n-pre-cells such that $\partial_i^+ X = \partial_i^- Y$, we define the *$i$-composite* $X *_i Y$ as the n-pre-cell Z such that

$$Z_{j,\epsilon} = \begin{cases} X_{j,\epsilon} + Y_{j,\epsilon} & \text{when } j > i \\ X_{i,-} & \text{when } j = i \text{ and } \epsilon = - \\ Y_{i,+} & \text{when } j = i \text{ and } \epsilon = + \\ X_{j,\epsilon} \text{ (or equivalently } Y_{j,\epsilon}) & \text{when } j < i \end{cases}$$

Given an n-pre-cell X of K, we define the *identity of X* as the $(n{+}1)$-pre-cell $\mathrm{id}_{n+1}(X)$ of K such that

$$\mathrm{id}_{n+1}(X) = (X_{0,-}, X_{0,+}, \dots, X_{n-1,-}, X_{n-1,+}, X_n, X_n, 0)$$

1.5.7 Atoms. Let K be a pre-adc. Given $n \in \mathbb{N}$ and $x \in P_n$, we define $[x]_{i,\epsilon} \subseteq P_i$ for $0 \le i \le n$ and $\epsilon \in \{-,+\}$ inductively by

$$[x]_{n,-} = [x]_{n,+} = x$$

and

$$[x]_{j,-} = [x]_{j+1,-}^{\mp} \quad \text{and} \quad [x]_{j,+} = [x]_{j+1,+}^{\pm}$$

for $0 \le j < n$. For simplicity, we sometimes write $[x]_{n,\epsilon}$ for $[x]_n$. The *atom associated to x* is then the n-pre-cell of K

$$[x] = ([x]_{0,-}, [x]_{0,+}, \dots, [x]_{n-1,-}, [x]_{n-1,+}, [x]_n).$$

For example, in the pre-adc associated to the ω-hypergraph (13), the atom $[\alpha]$ associated to α is defined by

$$[\alpha]_2 = \alpha,$$
$$[\alpha]_{1,-} = a_1 + a_2, \qquad [\alpha]_{1,+} = b,$$
$$[\alpha]_{0,-} = x, \qquad [\alpha]_{0,+} = z.$$

1.5.8 The relations $<_i$. Let K be a pre-adc with a basis B. Given $i \in \mathbb{N}$, we define the relation $<_i$ on B as the smallest transitive relation such that, for $m, n > i$, $x \in B_m$ and $y \in B_n$ such that $[x]_{i,+} \wedge [y]_{i,-} \neq 0$, we have $x <_i y$.

1.5.9 Unital loop-free basis. Let K be a pre-adc with a basis B. The basis B is said

- *unital* when for all $x \in B$, $\mathrm{e}([x]_{0,-}) = \mathrm{e}([x]_{0,+}) = 1$,
- *loop-free* when, for all $i \in \mathbb{N}$, $<_i$ is irreflexive.

1.5.10 The freeness property. In [17], the author shows that, given an adc K with a loop-free unital basis B, the globular set $\mathrm{Cell}(K)$, together with composition and identity operations, has a structure of an ω-category which is freely generated by the atoms $[b]$ for $b \in B$. Contrarily to parity complexes and pasting schemes, the associated pre-adc to the ω-hypergraph (11) is not a loop-free adc.

1.6 Generalized parity complexes

In this subsection, we introduce *generalized parity complexes*. They are a new formalism for pasting diagrams and are based on parity complexes. More precisely, generalized parity complexes rely on the same notion of cell than parity complexes, but satisfy different axioms, namely the axioms (G0) to (G4) introduced in Paragraph 1.6.3. Whereas axioms (G0) to (G2) were already present in [19], (G3) generalizes (C4) and (C5) and can be thought as an equivalent of (S4). Axiom (G4) filters out the counter-example (given in Paragraph 1.3.9) to the freeness property provided for parity complexes. In fact, under these axioms, the category of cells (as defined in Subsection 1.3) is freely generated by the atoms (as proved in Section 3).

In the following, we suppose given an ω-hypergraph P.

1.6.1 The segment condition. For $m \geq 0$ and $x \in P_m$, we say that x *satisfies the segment condition* when, for all $n < m$ and every n-cell X such that $\langle x \rangle_{n,-} \subseteq X_n$, it holds that $\langle x \rangle_{n,-}$ is a segment for $\triangleleft_{X_n}$, and dually with $\langle x \rangle_{n,+}$.

1.6.2 Torsion. Given $0 < n < i, j$, $x \in P_i$, $y \in P_j$ and an n-cell Z, x and y are said to be *in torsion with respect to Z* when

$$\langle x \rangle_{n,+} \subseteq Z_n, \quad \langle y \rangle_{n,-} \subseteq Z_n,$$

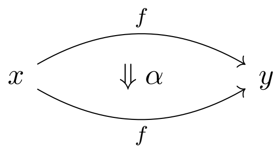
$$\langle x \rangle_{n,+} \cap \langle y \rangle_{n,-} = \emptyset \quad \text{and} \quad \langle x \rangle_{n,+} \underset{Z_n}{\triangleleft} \langle y \rangle_{n,-} \underset{Z_n}{\triangleleft} \langle x \rangle_{n,+}.$$

1.6.3 Generalized parity complexes. The ω-hypergraph P is said to be a *generalized parity complex* when it satisfies the following axioms:

- (G0) (non-emptiness) for all $x \in P$, $x^- \neq \emptyset$ and $x^+ \neq \emptyset$;
- (G1) (acyclicity) P is acyclic;
- (G2) (relevance) for all $x \in P$, x is relevant;
- (G3) (segment) for $x \in P$, x satisfies the segment condition;
- (G4) (torsion-freeness) for all $0 < n < i, j$, $x \in P_i$, $y \in P_j$ and every n-cell Z, x and y are not in torsion with respect to Z.

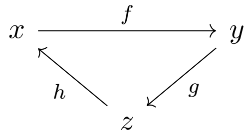
1.6.4 Axioms (G1) to (G2).

Axiom (G1) enforces the same notion of acyclicity than for parity complexes, forbidding cyclic n-cells and cycles of n-cells such as



(14)

and



. (15)

Axiom (G2) asks that the generators of the ω-hypergraph induce cells. For example, the ω-hypergraphs (8) and (9) are forbidden by this axiom. Note that (G2) entails Axioms (C1) and (C2) of parity complexes.

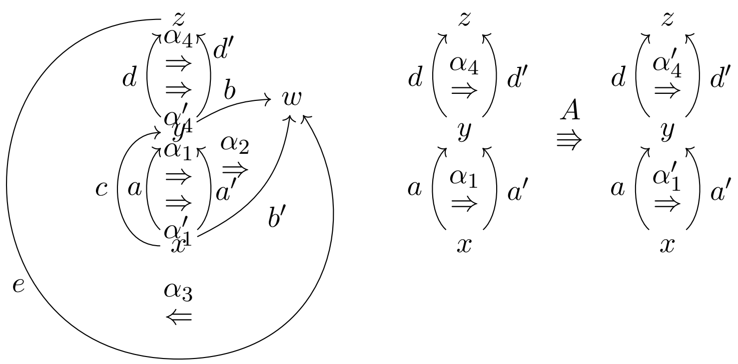
1.6.5 The segment Axiom (G3).

Recall that our goal is to obtain a category of cells which is freely generated by the atoms. A necessary condition for this is that all cells should be *decomposable*, that is, obtainable by identities and compositions of atoms. But the definition of cells does not require this property and, in fact, there are cells of ω-hypergraphs satisfying (G0) to (G2) that are not decomposable. The problem comes from an incompatibility between two concurrent phenomena:

(i) on the one side, the decomposition property that we want requires that some orders of compositions be allowed;

(ii) on the other side, $\lhd$ imposes restrictions on the orders in which the generators can be composed.

Axiom (G3) can then be understood as a condition to conciliate the two. For example, consider the ω-hypergraph P defined by the diagram



(16)

where, more precisely,

$$A^- = \{\alpha_1, \alpha_4\}, \qquad A^+ = \{\alpha'_1, \alpha'_4\},$$
$$\alpha_1^- = {\alpha'_1}^- = \{a\}, \qquad \alpha_1^+ = {\alpha'_1}^+ = \{a'\},$$
$$\alpha_4^- = {\alpha'_4}^- = \{d\}, \qquad \alpha_4^+ = {\alpha'_4}^+ = \{d'\} \quad \text{etc.}$$

Note that P satisfies (G0), (G1) and (G2). In this ω-hypergraph, there is a 3-cell Y given by

$$Y_3 = \{A\},$$
$$Y_{2,-} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \qquad Y_{2,+} = \{\alpha_1, \alpha'_2, \alpha'_3, \alpha_4\},$$
$$Y_{1,-} = \{a, b\}, \qquad Y_{1,+} = \{c, d', e\},$$
$$Y_{0,-} = \{x\}, \qquad Y_{0,+} = \{z\}.$$

and a 2-cell X given by

$$X_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$
$$X_{1,-} = \{a, b\}, \qquad X_{1,+} = \{c, d', e\},$$
$$X_{0,-} = \{x\}, \qquad X_{0,+} = \{z\}.$$

Suppose by contradiction that $\mathrm{Cell}(P)$ is an ω-category in which all cells are decomposable. Then, by a general property of ω-categories (Lemma 3.4.4), Y can be written

$$Y = \mathrm{id}_3(\phi) *_1 (\mathrm{id}_3(f) *_0 \langle A \rangle *_0 \mathrm{id}_3(g)) *_1 \mathrm{id}_3(\psi)$$

where $\phi, \psi \in \mathrm{Cell}(P)_2$ and $f, g \in \mathrm{Cell}(P)_1$. Since $X = \partial^- Y$, it implies that X can be written

$$X = \phi *_1 X' *_1 \psi$$

where $X' = (\mathrm{id}_2(f) *_0 \partial_2^-(\langle A \rangle) *_0 \mathrm{id}_2(g)) \in \mathrm{Cell}(P)_2$, illustrating (i). We have that $\mathrm{Cell}(P)_{\leq 2} \simeq \mathrm{Cell}(P \setminus \{A\})_{\leq 2}$ and it can be checked that $P \setminus \{A\}$ satisfies (G0), (G1), (G2) and (G3). So it can be shown (Lemma 2.3.1) that

$$\phi_2,\ X'_2 \text{ and } \psi_2 \text{ form a partition of } X_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \tag{17}$$

and that,

$$\text{for } (\beta, \gamma) \in (\phi_2 \times (X'_2 \cup \psi_2)) \cup ((\psi_2 \cup X'_2) \times \psi_2),\ \neg(\gamma \triangleleft^1_{X_2} \beta), \tag{18}$$

the last property illustrating (ii). Consider $\alpha_2 \in X_2$. Since we have

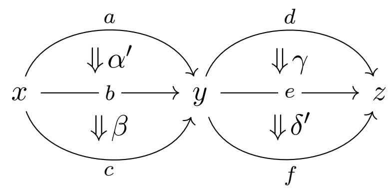
$$X'_2 = A^- = \{\alpha_1, \alpha_4\},$$

by (17), either $\alpha_2 \in \phi_2$ or $\alpha_2 \in \psi_2$. By (18), since $\alpha_1 \triangleleft^1_{X_2} \alpha_2$, we have $\alpha_2 \in \phi_2$. Now consider $\alpha_3 \in X_2$. By (17), either $\alpha_3 \in \phi_2$ or $\alpha_3 \in \psi_2$. By (18), since $\alpha_3 \triangleleft^1_{X_2} \alpha_4$, we have $\alpha_3 \in \phi_2$. But then, $\alpha_3 \in \phi_2$, $\alpha_2 \in \psi_2$ and $\alpha_2 \triangleleft^1_{X_2} \alpha_3$, contradicting (18). Hence, $\mathrm{Cell}(P)$ is not an ω-category in which all cells are decomposable.

Axiom (G3) prevents this kind of problems and, in particular, forbids the ω-hypergraph P. Indeed, A does not satisfy the segment condition: $\langle A \rangle_{2,-} = \{\alpha_1, \alpha_4\} \subseteq X_2$ but $\alpha_1 \triangleleft_{X_2} \alpha_2 \triangleleft_{X_2} \alpha_3 \triangleleft_{X_2} \alpha_4$ and $\alpha_2, \alpha_3 \notin \langle A \rangle_{2,-}$. Therefore, $\langle A \rangle_{2,-}$ is not a segment for $\triangleleft_{X_2}$.

1.6.6 The torsion-freeness Axiom (G4).

Note that the ω-hypergraph P defined by the diagram (11) satisfies (G0), (G1), (G2) and (G3). Thus, these axioms do not ensure the freeness property for $\mathrm{Cell}(P)$. However, P does not satisfy (G4). Indeed, there is the following 2-cell X associated to the diagram



that is,

$$X_2 = \{\alpha', \beta, \gamma, \delta'\},$$
$$X_{1,-} = \{a, d\}, \qquad X_{1,+} = \{c, f\},$$
$$X_{0,-} = \{x\}, \qquad X_{0,+} = \{z\}.$$

and A and B are in torsion with respect to X, breaking (G4).

The idea with the situations with torsion is that they are minimal cases where the freeness property fails for an ω-hypergraph P. When $x, y \in P$ are in torsion with respect to a cell Z of P, there are two possible order to compose x and y: x then y, or y then x. And both composites produce equal cells in $\mathrm{Cell}(P)$. However, this equality can not be deduced from an exchange law, since the torsion says basically that x and y cross each other, preventing to obtain the left-hand side of axiom (v) of ω-category (see Paragraph 1.1.3).

1.6.7 More computable axioms. Axioms (G3) and (G4) reveal to be hard to check in practice. Indeed, both involve a quantification on all the cells, and enumerating the cells can be tough since their number is exponential in the number of elements of the ω-hypergraph in the worst case. Here, we give stronger axioms that are simpler to verify, in the sense that they can be checked using an algorithm with polynomial complexity.

Given an ω-hypergraph P, for $n \geq 0$, $x, y \in P_n$, we write $x \curvearrowright y$ when there exists $z \in P_{n+1}$ such that $x \in z^-$ and $y \in z^+$. We write $\curvearrowright^*$ for the reflexive transitive closure of $\curvearrowright$. For $S, T \subseteq P_n$, we write $S \curvearrowright^* T$ when there exist $s \in S$ and $t \in T$ such that $s \curvearrowright^* t$.

Consider the following axiom on an ω-hypergraph P:

(G3') for $n > 0$, $x \in P_n$, $k < n$, we have $\neg(\langle x\rangle_{k,+} \curvearrowright^* \langle x\rangle_{k,-})$.

Then, (G3) can be safely replaced by (G3') in the axioms of generalized parity complexes, as stated by the following lemma.

Lemma 1.6.8. *Let P be an ω-hypergraph satisfying (G0), (G1) and (G2). If P satisfies (G3'), then it satisfies (G3).*

Proof. Suppose that P satisfies (G3'). Let $n < m$, $x \in P_m$ and X be an n-cell such that $\langle x\rangle_{n,-} \subseteq X_n$. If $n = 0$, there is nothing to prove, so we can assume $n > 0$. By contradiction, suppose that $\langle x\rangle_{n,-}$ is not a segment for $\triangleleft_{X_n}$. So there are $r > 2$, $x_1, \ldots, x_r \in P_n$ with $x_1, x_r \in \langle x\rangle_{n,-}$, $x_2, \ldots, x_{r-1} \notin \langle x\rangle_{n,-}$ and $x_i \triangleleft^1_{X_n} x_{i+1}$ for $1 \leq i < r$. Hence, there are $z_1, \ldots, z_{r-1} \in P_{n-1}$ such that $z_i \in x_i^+ \cap x_{i+1}^-$ for $1 \leq i < r$. For $u \in X_n$ such that $z_1 \in u^-$, since X_n is fork-free, $u = x_2 \notin X_n$. So, since x is relevant by (G2), $z_1 \in \langle x\rangle^{\pm}_{n,-} = \langle x\rangle_{n-1,+}$. Similarly, $z_{r-1} \in \langle x\rangle_{n-1,-}$. Thus, $\langle x\rangle_{n-1,+} \curvearrowright^* \langle x\rangle_{n-1,-}$, contradicting (G3'). Hence, P satisfies (G3). □

Now, consider the following axiom on an ω-hypergraph P:

(G4') for $n > 0$, $i > n$, $j > n$, $x \in P_i$, and $y \in P_j$, if $\langle x\rangle_{n,+} \cap \langle y\rangle_{n,-} = \emptyset$, then at most one of the following holds:

- $\langle x\rangle_{n-1,+} \curvearrowright^* \langle y\rangle_{n-1,-}$,
- $\langle y\rangle_{n-1,+} \curvearrowright^* \langle x\rangle_{n-1,-}$.

Then, (G4) can be safely replaced by (G4') in the axioms of generalized parity complexes, as stated by the following lemma.

Lemma 1.6.9. *Let P be an ω-hypergraph satisfying (G0), (G1) and (G2). If P satisfies (G4'), then it satisfies (G4).*

Proof. Suppose that P satisfies (G4'). By contradiction, assume that P does not satisfy (G4). So there are $n > 0$, $i > n$, $j > n$, $x \in P_i$, $y \in P_j$ and an n-cell Z such that x and y are in torsion with respect to Z. That is, $\langle x\rangle_{n,+} \subseteq Z_n$, $\langle y\rangle_{n,-} \subseteq Z_n$, $\langle x\rangle_{n,+} \cap \langle y\rangle_{n,-} = \emptyset$ and $\langle x\rangle_{n,+} \triangleleft_{Z_n} \langle y\rangle_{n,-} \triangleleft_{Z_n} \langle x\rangle_{n,+}$. So there are $r > 1$, $z_1, \ldots, z_r \in Z_n$ with $z_1 \in \langle x\rangle_{n,+}$, $z_r \in \langle y\rangle_{n,-}$, $z_2, \ldots, z_{r-1} \notin \langle x\rangle_{n,+} \cup \langle x\rangle_{n,-}$ and $z_k \triangleleft^1_{Z_n} z_{k+1}$ for $1 \leq k < r$. Hence, there are $w_1, \ldots, w_{r-1} \in P_{n-1}$ such that $w_k \in z_k^+ \cap z_{k+1}^-$ for $1 \leq k < r$. For all $u \in Z_n$ with $w_1 \in u^-$, $u = z_2 \notin \langle x\rangle_{n,+}$, since Z_n is fork-free. Thus, $w_1 \in \langle x\rangle^{\pm}_{n,+} = \langle x\rangle_{n-1,+}$. Similarly, $w_{r-1} \in \langle y\rangle_{n-1,-}$ so $\langle x\rangle_{n-1,+} \curvearrowright^* \langle y\rangle_{n-1,-}$. Similarly, using (G2), $\langle y\rangle_{n-1,+} \curvearrowright^* \langle x\rangle_{n-1,-}$, which contradicts (G4'). Hence, P satisfies (G4). ☐

2 The category of cells

In this section, we show that Cell(P) has a structure of an ω-category. For this purpose, we adapt the proofs of [19] and take the opportunity to simplify them.

2.1 Movement properties

Here, we state several useful properties of movement (as defined in Paragraph 1.3.2), some of which coming from [19].

In the following, we suppose given an ω-hypergraph P. The first property gives another criterion for movement.

Lemma 2.1.1 ([19, Proposition 2.1])**.** *For $n \in \mathbb{N}$, finite subsets $U \subseteq P_n$ and $S \subseteq P_{n+1}$, there exists $V \subseteq P_n$ such that S moves U to V if and only if $S^{\mp} \subseteq U$ and $U \cap S^+ = \emptyset$.*

Proof. If S moves U to V, then, by definition,

$$S^{\mp} \subseteq (V \cup S^-) \setminus S^+ = U$$

and

$$U \cap S^+ = ((V \cup S^-) \setminus S^+) \cap S^+ = \emptyset.$$

Conversely, if $S^{\mp} \subseteq U$ and $U \cap S^+ = \emptyset$, let $V = (U \cup S^+) \setminus S^-$. Then

$$\begin{aligned}(V \cup S^-) \setminus S^+ &= (U \cup S^+ \cup S^-) \setminus S^+ \\ &= (U \setminus S^+) \cup (S^- \setminus S^+) \\ &= U \cup S^{\mp} && \text{(since } U \cap S^+ = \emptyset) \\ &= U && \text{(since } S^{\mp} \subseteq U)\end{aligned}$$

and S moves U to V. ☐

The next property states that it is possible to modify a movement by adding or removing "independent" elements.

Lemma 2.1.2 ([19, Proposition 2.2])**.** *Let $n \in \mathbb{N}$, $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$ be finite subsets such that S moves U to V. Then, for all $X, Y \subseteq P_n$ with $X \subseteq U$, $X \cap S^{\mp} = \emptyset$ and $Y \cap (S^- \cup S^+) = \emptyset$, S moves $(U \cup Y) \setminus X$ to $(V \cup Y) \setminus X$.*

Proof. By Lemma 2.1.1, $S^{\mp} \subseteq U$ and $U \cap S^+ = \emptyset$. Using the hypothesis, we can refine both equalities to $S^{\mp} \subseteq (U \cup Y) \setminus X$ and $((U \cup Y) \setminus X) \cap S^+ = \emptyset$. Using Lemma 2.1.1 again, S moves $(U \cup Y) \setminus X$ to W where

$$\begin{aligned} W &= (((U \cup Y) \setminus X) \cup S^+) \setminus S^- \\ &= ((U \cup S^+ \cup Y) \setminus X) \setminus S^- && \text{(since } X \cap S^+ \subseteq U \cap S^+ = \emptyset) \\ &= (((U \cup S^+) \setminus S^-) \cup Y) \setminus X && \text{(since } Y \cap S^- = \emptyset) \\ &= (V \cup Y) \setminus X. \end{aligned}$$

□

The following property gives sufficient conditions for composing movements.

Lemma 2.1.3 ([19, Proposition 2.3])**.** *For $n \in \mathbb{N}$, finite subsets $U, V, W \subseteq P_n$ and $S, T \subseteq P_{n+1}$ such that S moves U to V and T moves V to W, if $S^- \cap T^+ = \emptyset$ then $S \cup T$ moves U to W.*

Proof. We compute $(U \cup (S \cup T)^+) \setminus (S \cup T)^-$:

$$\begin{aligned} (U \cup S^+ \cup T^+) \setminus (S^- \cup T^-) &= (((U \cup S^+) \setminus S^-) \cup T^+) \setminus T^- \\ &= (V \cup T^+) \setminus T^- \\ &= W. \end{aligned}$$

Similarly, $(W \cup (S \cup T)^-) \setminus (S \cup T)^+ = U$ and $S \cup T$ moves U to W. □

The next property gives sufficient conditions for decomposing movements.

Lemma 2.1.4 ([19, Proposition 2.4])**.** *For $n \in \mathbb{N}$, finite subsets $U, W \subseteq P_n$, $S, T \subseteq P_{n+1}$ such that $S \cup T$ moves U to W and $S^{\mp} \subseteq U$, if $S \perp T$ then there exists V such that S moves U to V and T moves V to W.*

Proof. Let $R = S \cup T$. By Lemma 2.1.1, $R^{\mp} \subseteq U$ and $U \cap S^+ \subseteq U \cap R^+ = \emptyset$. By Lemma 2.1.1 again, S moves U to $V = (U \cup S^+) \setminus S^-$. Moreover,

$$\begin{aligned} S^- \cap T^+ &= S^{\mp} \cap T^+ && \text{(since } S^+ \cap T^+ = \emptyset\text{, by } S \perp T) \\ &\subseteq U \cap T^+ && \text{(since } S^{\mp} \subseteq U\text{, by hypothesis)} \\ &\subseteq U \cap (S \cup T)^+ \\ &= \emptyset && \text{(by Lemma 2.1.1).} \end{aligned}$$

Therefore,

$$\begin{aligned} && R^{\mp} &\subseteq U \\ \Leftrightarrow && ((S^- \cup T^-) \setminus T^+) \setminus S^+ &\subseteq U \\ \Leftrightarrow && ((T^- \setminus T^+) \cup S^-) \setminus S^+ &\subseteq U && \text{(because } S^- \cap T^+ = \emptyset) \\ \Leftrightarrow && T^{\mp} \cup S^- &\subseteq U \cup S^+ \\ \Leftrightarrow && T^{\mp} &\subseteq (U \cup S^+) \setminus S^- && \text{(since } T^{\mp} \cap S^- = \emptyset\text{,} \\ &&&&& \text{by } S \perp T). \end{aligned}$$

Hence, $T^\mp \subseteq (U \cup S^+) \setminus S^- = V$ and

$$V \cap T^+ \subseteq (U \cup S^+) \cap T^+ \subseteq (U \cap R^+) \cup (S^+ \cap T^+) = \emptyset.$$

By Lemma 2.1.1, T moves V to $(V \cup T^+) \setminus T^-$. Moreover,

$$\begin{aligned}
S^- \cap T^+ &= S^\mp \cap T^+ && \text{(since } S \perp T\text{)} \\
&\subseteq U \cap R^+ && \text{(since } S^\mp \subseteq U \text{ by hypothesis)} \\
&= \emptyset.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(V \cup T^+) \setminus T^- &= (((U \cup S^+) \setminus S^-) \cup T^+) \setminus T^- \\
&= (U \cup S^+ \cup T^+) \setminus (S^- \cup T^-) && \text{(since } S^- \cap T^+ = \emptyset\text{)} \\
&= W.
\end{aligned}$$

Hence, T moves V to W. □

The last properties (not in [19]) describe which elements are touched or left untouched by movement.

Lemma 2.1.5. *For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if S moves U to V, then $S^\mp = U \setminus V$ and $S^\pm = V \setminus U$. In particular, if T moves U to V, then $S^\mp = T^\mp$ and $S^\pm = T^\pm$.*

Proof. By the definition of movement, we have

$$V = (U \cup S^+) \setminus S^- \quad \text{and} \quad U = (V \cup S^-) \setminus S^+$$

and therefore

$$\begin{aligned}
U \cap V &= U \cap ((U \setminus S^-) \cup S^\pm) \\
&= U \setminus S^\mp && \text{(since } U \cap S^+ = \emptyset\text{)}.
\end{aligned}$$

Similarly, $U \cap V = V \setminus S^\pm$. Hence, $S^\mp = U \setminus V$ and $S^\pm = V \setminus U$. □

Lemma 2.1.6. *For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if S moves U to V, then*

$$U \setminus S^- = U \setminus S^\mp = U \cap V = V \setminus S^\pm = V \setminus S^+.$$

Proof.

$$\begin{aligned}
U \setminus S^- &= U \setminus S^\mp && \text{(since } U \cap S^+ = \emptyset\text{, by definition of movement)} \\
&= U \cap V && \text{(by Lemma 2.1.5)} \\
&= V \setminus S^\pm \\
&= V \setminus S^+ && \text{(since } V \cap S^- = \emptyset\text{, by definition of movement)}
\end{aligned}$$
□

Lemma 2.1.7. *For $n \in \mathbb{N}$, finite subsets $U, V \subseteq P_n$ and $S \subseteq P_{n+1}$, if S moves U to V, then*

$$U = (U \cap V) \sqcup S^\mp \quad \textit{and} \quad V = (U \cap V) \sqcup S^\pm.$$

Proof. We have

$$\begin{aligned} U &= (V \cup S^-) \setminus S^+ \\ &= (V \setminus S^+) \cup (S^- \setminus S^+) \\ &= (U \cap V) \cup S^\mp && \text{(by Lemma 2.1.6)} \end{aligned}$$

and

$$\begin{aligned} (U \cap V) \cap S^\mp &\subseteq V \cap S^- \\ &= ((U \cup S^-) \setminus S^-) \cap S^- \\ &= \emptyset. \end{aligned}$$

Hence, $U = (U \cap V) \sqcup S^\mp$. Similarly $V = (U \cap V) \sqcup S^\pm$. □

2.2 An adapted proof of Street's Lemma 3.2

Here, we state and prove an a property similar to [19, Lemma 3.2] which enables to build new cells from other cells. We adapt the proof to the new set of axioms and simplify it (notably, we remove the need for the notion of receptivity and the apparent circularity of the proof). In the following, we suppose given an ω-hypergraph P.

2.2.1 Gluings and activations. Let $n \in \mathbb{N}$, X be an n-pre-cell of P and $G \subseteq P_{n+1}$ be a finite subset. We say that G is *glueable on* X if $G^\mp \subseteq X_n$. If so, we call *gluing of* X *on* C the $(n+1)$-pre-cell Y of P such that

$$\begin{aligned} Y_{n+1} &= G, \\ Y_{n,-} &= X_n, \\ Y_{n,+} &= (X_n \cup G^+) \setminus G^-, \\ Y_{i,\epsilon} &= X_{i,\epsilon}. \end{aligned}$$

We denote Y by $\mathrm{Glue}(X, G)$. Moreover, we call *activation of* G *on* X the n-pre-cell $\partial_n^+(\mathrm{Glue}(X, G))$ and we denote it by $\mathrm{Act}(X, G)$. We say that G is dually gluable on X when $G^\pm \subseteq X_n$ and we define the dual gluing $\overline{\mathrm{Glue}}(X, G)$ and the dual activation $\overline{\mathrm{Act}}(X, G)$ similarly.

For example, take the ω-hypergraph (16). Then $\{A\}$ is glueable of X and $\mathrm{Glue}(X, \{A\}) = Y$ and $\mathrm{Act}(X, \{A\})$ is the 2-pre-cell X' with

$$\begin{gathered} X'_2 = \{\alpha_1, \alpha'_2, \alpha'_3, \alpha_4\}, \\ X_{1,-} = \{a, b\}, \qquad X_{1,+} = \{c, d', e\}, \\ X_{0,-} = \{x\}, \qquad X_{0,+} = \{z\}. \end{gathered}$$

2.2.2 The gluing theorem. We prove the following theorem which is an adaptation of [19, Lemma 3.2].

Theorem 2.2.3. *Suppose that P satisfies axioms (G0), (G1), (G2) and (G3). Given $n \in \mathbb{N}$, an n-cell X of P and a finite fork-free set $G \subseteq P_{n+1}$ such that G is glueable on X, we have that*

(a) $\mathrm{Act}(X, G)$ *is a cell, and* $G^+ \cap X_n = \emptyset$*,*

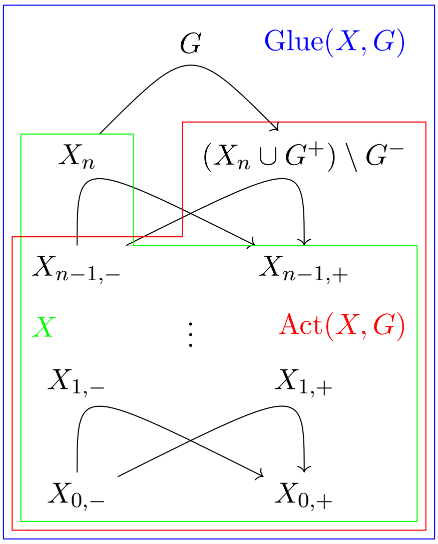


Figure 1: Cells involved and their movements in Theorem 2.2.3

(b) $\mathrm{Glue}(X, G)$ *is a cell,*

(c) for $G' \subseteq P_{n+1}$ *finite, fork-free and dually glueable on* X*,* $G'^{-} \cap G^{+} = \emptyset$*.*

This theorem naturally admits a dual statement, when G is dually glueable o X.

Proof. See Figure 1 for a representation of the cells in the statement of the theorem. The proof of this theorem (and its dual) is made with an induction on n. For a given n, there are four steps. Firstly, we show that (a) holds when $|G| = 1$. Secondly, we use the first step to show that (a) holds for all possible G. Thirdly, we prove that (b) holds. And fourthly, we prove that (c) holds. In the following, let S be $\mathrm{Act}(X, G)_n = (X_n \cup G^+) \setminus G^-$.

Step 1: (a) holds when $|G| = 1$*.* Let $x \in P_{n+1}$ be such that $\{x\} = G$. If $n = 0$, then there exists $y \in P_0$ such that $X_0 = \{y\}$. By axioms (G1) and (G2), there exists $z \in P_0$ with $y \neq z$ such that $x^- = \{y\}$ and $x^+ = \{z\}$. So $\mathrm{Act}(X, G) = \{z\}$ is a cell. Otherwise, $n > 0$. Then, $S = (X_n \cup x^+) \setminus x^-$ and in order to prove that $\mathrm{Act}(X, G)$ is a cell, we need to show that

– S moves $X_{n-1,-}$ to $X_{n-1,+}$;

– S is fork-free.

Using the segment Axiom (G3), we get that x^- is a segment in X_n for $\triangleleft_{X_n}$. By elementary properties of partial orders, we can decompose X_n as the partition

$$X_n = U \cup x^- \cup V$$

with U initial and V final for $\triangleleft_{X_n}$, which implies that $U^{\mp} \subseteq X_{n-1,-}$ and $V^{\pm} \subseteq X_{n-1,+}$. As subsets of the fork-free set X_n, U, x^- and V are fork-free and $U \perp x^-$, $x^- \perp V$, $U \perp V$. Using Lemma 2.1.4, we get $A, B \subseteq P_{n-1}$ such that

– U moves $X_{n-1,-}$ to A,

– x^- moves A to B,

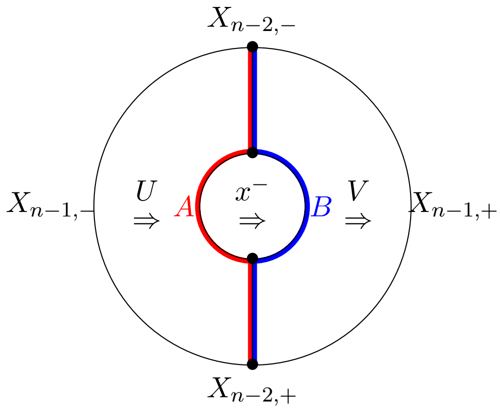


Figure 2: The decomposition of X_n

- V moves B to $X_{n-1,+}$

as pictured on Figure 2. In the following, for $Z \subseteq P_{n-1}$, we write $D(Z)$ for the $(n-1)$-pre-cell of P defined by

$$D(Z)_{n-1} = Z,$$
$$D(Z)_{i,\epsilon} = X_{i,\epsilon} \quad \text{for } i < n-1 \text{ and } \epsilon \in \{-,+\}.$$

Since $D(A) = \mathrm{Act}(D(X_{n-1,-}), U)$, $D(B) = \mathrm{Act}(D(A), x^-)$, $D(X_{n-1,-}) = \partial^- X$ is an $(n-1)$-cell and both U and x^- are fork-free, by using two times the induction hypothesis of Theorem 2.2.3, first on $D(X_{n-1,-})$, then on $D(A)$, we get that

$$D(A) \text{ and } D(B) \text{ are cells.} \tag{19}$$

By (G2), we have that

$$x^+ \text{ is fork-free.} \tag{20}$$

Since x^- moves A to B, by Lemma 2.1.1,

$$A \cap x^{-+} = \emptyset. \tag{21}$$

By (G2), it holds that $x^{+\mp} = x^{-\mp} \subseteq A$. By (19) and (20), using the induction hypothesis of Theorem 2.2.3 on $D(A)$, we get

$$A \cap x^{++} = \emptyset. \tag{22}$$

By Lemma 2.1.1, there exists B' such that x^+ moves A to B', and

$$\begin{aligned}
B' &= (A \cup x^{++}) \setminus x^{+-} \\
&= (A \setminus x^{+-}) \cup (x^{++} \setminus x^{+-}) \\
&= (A \setminus x^{+\mp}) \cup x^{+\pm} && \text{(by (22))} \\
&= (A \setminus x^{-\mp}) \cup x^{-\pm} && \text{(since } x^{+\mp} = x^{-\mp}\text{, by (G2))} \\
&= (A \setminus x^{--}) \cup (x^{-+} \setminus x^{--}) && \text{(by (21))} \\
&= (A \cup x^{-+}) \setminus x^{--} \\
&= B && \text{(since } x^- \text{ moves } A \text{ to } B\text{).}
\end{aligned}$$

Hence,

$$x^+ \text{ moves } A \text{ to } B. \tag{23}$$

Since $x^{+\mp} \subseteq D(A)_{n-1}$ and $U^\pm \subseteq D(A)_{n-1}$, using the induction hypothesis of Theorem 2.2.3, by (c) we get that

$$U^- \cap x^{++} = \emptyset. \tag{24}$$

Similarly with $D(B)$, we get that

$$x^{+-} \cap V^+ = \emptyset. \tag{25}$$

By definition, U moves $X_{n-1,-}$ to A and x^+ moves A to B. Moreover, by (24), $U^- \cap x^{++} = \emptyset$. Using Lemma 2.1.3, we deduce that

$$U \cup x^+ \text{ moves } X_{n-1,-} \text{ to } B. \tag{26}$$

Since U and V are disjoint and respectively initial and terminal for $\triangleleft_{X_n}$, we have that $U^- \cap V^+ = \emptyset$. Also, by (25), we have $(x^{+-} \cap V^+) = \emptyset$, therefore

$$\begin{aligned}(U \cup x^+)^- \cap V^+ &\subseteq (U^- \cap V^+) \cup (x^{+-} \cap V^+)\\ &= \emptyset.\end{aligned}$$

Using (26) and Lemma 2.1.3, knowing that $S = U \cup x^+ \cup V$, we deduce that

$$S \text{ moves } X_{n-1,-} \text{ to } X_{n-1,+}. \tag{27}$$

The set $U \cup V$ is fork-free as a subset of the fork-free X_n, and x^+ is fork-free since x is relevant by (G2). Moreover,

$$\begin{aligned}
U^- \cap x^{+-} &= U^- \cap x^{+\mp} && \text{(by (24))}\\
&\subseteq U^- \cap A && \text{(by (23) and Lemma 2.1.1)}\\
&= \emptyset && \text{(since } U \text{ moves } X_{n-1,-} \text{ to } A\text{)},\\
U^+ \cap x^{++} &= U^\pm \cap x^{++} && \text{(by (24))}\\
&\subseteq A \cap x^{++} && \text{(by Lemma 2.1.1 since } U \text{ moves } X_{n-1,-} \text{ to } A\text{)}\\
&= \emptyset && \text{(by (23) and Lemma 2.1.1).}
\end{aligned}$$

So $U \perp x^+$. Similarly, $x^+ \perp V$. Hence, since $S = U \cup x^+ \cup V$,

$$S \text{ is fork-free.} \tag{28}$$

Then, by (27) and (28),

$$\mathrm{Act}(X, G) \text{ is a cell.}$$

Finally, we prove the second part of (a). By Axiom (G1), $x^- \cap x^+ = \emptyset$. Since $U \perp x^+$ and $x^+ \perp V$ (by (28)), using (G0), we deduce that

$$U \cap x^+ = x^+ \cap V = \emptyset$$

Hence,

$$X_n \cap x^+ = (U \cup x^- \cup V) \cap x^+ = \emptyset$$

and it concludes the proof of the Step 1.

Step 2: (a) holds. We prove this by induction on $|G|$. If $|G| = 0$, then the result is trivial and the case $|G| = 1$ was proved in Step 1. So suppose that $|G| \geq 2$. Since the relation $\triangleleft$ is acyclic by (G1), we can consider a minimal $x \in G$ for $\triangleleft_G$. Let $\tilde{G}$ be $G \setminus \{x\}$ and recall that we defined S as $(X_n \cup G^+) \setminus G^-$. In order to show that $\mathrm{Act}(X, G)$ is a cell, we need to prove the following:

- S moves $X_{n-1,-}$ to $X_{n-1,+}$;
- S is fork-free.

For this purpose, we will first move X_n with $\{x\}$ to $U := (X_n \cup x^+) \setminus x^-$ and use Step 1, then move U by $\tilde{G}$ to $V := (U \cup \tilde{G}^+) \setminus \tilde{G}^-$ and use the induction of Step 2. Finally, we will prove that $V = S$. So, using Step 1 with X and $\{x\}$, we get that

- $\mathrm{Act}(X, \{x\})$ is a cell;
- in particular, U is fork-free and, if $n > 0$, U moves $X_{n-1,-}$ to $X_{n-1,+}$;
- $X_n \cap x^+ = \emptyset$.

By Lemma 2.1.1, we deduce that $\{x\}$ moves X_n to U. Moreover,

$$\begin{aligned}
\tilde{G}^{\mp} &= \tilde{G}^- \setminus \tilde{G}^+ \\
&= (G^- \setminus x^-) \setminus (G^+ \setminus x^+) && \text{(since fork-freeness implies that } G^\epsilon = \sqcup_{u \in G} u^\epsilon\text{)} \\
&\subseteq ((G^- \setminus x^-) \setminus G^+) \cup x^+ \\
&= ((G^- \setminus G^+) \setminus x^-) \cup x^+ \\
&\subseteq (X_n \setminus x^-) \cup x^+ && \text{(since } G^{\mp} \subseteq X_n \text{ by Lemma 2.1.1)} \\
&\subseteq (X_n \cup x^+) \setminus x^- && \text{(since } x^- \cap x^+ = \emptyset \text{ by (G1))} \\
&= U.
\end{aligned}$$

Also, $\tilde{G}$ is fork-free as a subset of the fork-free set G. Using the induction hypothesis of Step 2 for $\tilde{G}$, we get that

- $\mathrm{Act}(\mathrm{Act}(X, \{x\}), \tilde{G})$ is a cell;
- In particular, $V := (U \cup \tilde{G}^+) \setminus \tilde{G}^-$ is fork-free, and, if $n > 0$, V moves $X_{n-1,-}$ to $X_{n-1,+}$;
- $U \cap \tilde{G}^+ = \emptyset$.

By Lemma 2.1.1, we deduce that $\tilde{G}$ moves U to V. Also note that $x^- \cap \tilde{G}^+ = \emptyset$ since x was taken minimal in G. Using Lemma 2.1.3, we deduce that $G = \{x\} \cup \tilde{G}$ moves X_n to V. But $S = (X_n \cup G^+) \setminus G^-$. So $S = V$.

One still needs to show the second part of (a), that is, $X_n \cap G^+ = \emptyset$:

$$\begin{aligned}
X_n \cap G^+ &= (U \cup x^- \setminus x^+) \cap G^+ && \text{(by Lemma 2.1.1, since } \{x\} \text{ moves } X_n \text{ to } U\text{)} \\
&= ((U \cup x^-) \cap G^+) \setminus x^+ \\
&= (U \cap G^+) \setminus x^+ && \text{(since } x^- \cap G = x^- \cap (x^+ \cup \tilde{G}) = \emptyset\text{)} \\
&= (U \cap \tilde{G}^+) \\
&= \emptyset
\end{aligned}$$

which ends the proof of Step 2.

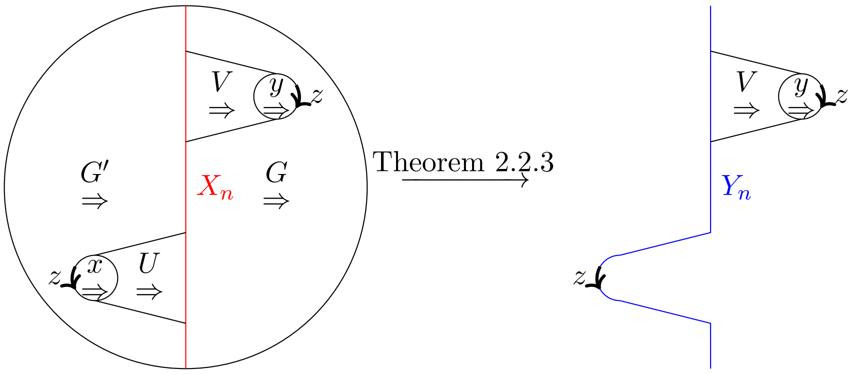


Figure 3: V, U and Y_n

Step 3: (b) holds. By (a), $\mathrm{Act}(X, G)$ is a cell. To conclude, we need to show that G moves X_n to S. By definition of S, we have that $S = (X_n \cup G^+) \setminus G^-$. Also:

$$
\begin{aligned}
(S \cup G^-) \setminus G^+ &= (((X_n \cup G^+) \setminus G^-) \cup G^-) \setminus G^+ \\
&= (X_n \cup G^+ \cup G^-) \setminus G^+ \\
&= (X_n \setminus G^+) \cup G^{\mp} \\
&= X_n \cup G^{\mp} && \text{(since } X_n \cap G^+ = \emptyset \text{ by (a))} \\
&= X_n && \text{(since } G \text{ is glueable on } X\text{)}.
\end{aligned}
$$

Hence, $\mathrm{Glue}(X, G)$ is a cell.

Step 4: (c) holds. By contradiction, suppose that $G'^- \cap G^+ \neq \emptyset$. By definition, there are $x \in G'$, $y \in G$ and $z \in x^- \cap y^+$. Consider

$$
\begin{aligned}
U &= \{x' \in G' \mid x \underset{G'}{\triangleleft} x'\} \cup \{x\}, \\
V &= \{y' \in G \mid y' \underset{G}{\triangleleft} y\} \cup \{y\}.
\end{aligned}
$$

By the acyclicity Axiom (G1), we have

$$U^+ \cap V^- = \emptyset.$$

Since U is a terminal set for $\triangleleft_{G'}$, we have in particular $U^+ \cap G'^- \subseteq U^-$. So,

$$U^+ = (U^+ \setminus G'^-) \cup (U^+ \cap G'^-) \subseteq G'^{\pm} \cup U^-.$$

Hence, $U^{\pm} \subseteq G'^{\pm} \subseteq X_n$ (since G' is dually glueable on X). Similarly, $V^{\mp} \subseteq X_n$. Using the dual version of (a), $Y := \overline{\mathrm{Act}}(X, U)$ is an n-cell with $Y_n = (X_n \cup U^-) \setminus U^+$(see Figure 3) and we have

$$
\begin{aligned}
V^{\mp} &= V^{\mp} \setminus U^+ && \text{(since } V^- \cap U^+ = \emptyset\text{)} \\
&\subseteq X_n \setminus U^+ && \text{(since } V^{\mp} \subseteq X_n\text{)} \\
&\subseteq (X_n \cup U^-) \setminus U^+ \\
&= Y_n.
\end{aligned}
$$

Using Theorem 2.2.3(a) with Y and V, we get

$$Y_n \cap V^+ = \emptyset.$$

But, since $z \in U^{\mp} \subseteq Y_n$ (by (G1)) and $U^{\mp} \subseteq Y_n$, $z \in Y_n \cap V^+$, which is a contradiction. Hence,

$$G'^{-} \cap G^{+} = \emptyset.$$

which ends the proof of (c). □

2.3 Cell(P) is an ω-category

Here, we finally prove that Cell(P) has a structure of an ω-category. In the following, we suppose given an ω-hypergraph P which satisfies (G0), (G1), (G2) and (G3).

Lemma 2.3.1. *Let $n > 0$ and X, Y be two n-cells of P that are $(n-1)$-composable. Then,*

(a) $X_n^- \cap Y_n^+ = \emptyset$,

(b) $X_n \cap Y_n = \emptyset$,

(c) $X *_{n-1} Y$ is an n-cell of P.

Proof. Using Theorem 2.2.3(c) with $\partial^+ X$, X_n and Y_n, we get

$$X_n^- \cap Y_n^+ = \emptyset.$$

Moreover,

$$\begin{aligned} X_n^+ \cap Y_n^+ &= X_n^{\pm} \cap Y_n^+ && (\text{since } X_n^- \cap Y_n^+ = \emptyset) \\ &\subseteq X_{n-1,+} \cap Y_n^+ \\ &= Y_{n-1,-} \cap Y_n^+ \\ &= \emptyset && (\text{by Theorem 2.2.3(a)}). \end{aligned}$$

By (G0), it implies that $X_n \cap Y_n = \emptyset$. Similarly,

$$X_n^- \cap Y_n^- = \emptyset$$

So $X_n \cup Y_n$ is fork-free. For $X *_{n-1} Y$ to be a cell, $X_n \cup Y_n$ must move $X_{n-1,-}$ to $Y_{n-1,+}$. But, since X and Y are cells and are $(n-1)$-composable, we know that X_n moves $X_{n-1,-}$ to $X_{n-1,+}$, Y_n moves $Y_{n-1,-}$ to $Y_{n-1,+}$ and $X_{n-1,+} = Y_{n-1,-}$. Since $X_n^- \cap Y_n^+$, using Lemma 2.1.3, we get that $X_n \cup Y_n$ moves $X_{n-1,-}$ to $Y_{n-1,+}$. Hence, $X *_{n-1} Y$ is a cell. □

Lemma 2.3.2. *Let $0 \le i < n$ and X, Y be two n-cells of P that are i-composable. Then,*

(a) for $i < j \le n$, $(X_{j,-}^- \cup X_{j,+}^-) \cap (Y_{j,-}^+ \cup Y_{j,+}^+) = \emptyset$,

(b) $X *_i Y$ is a cell.

Proof. By induction on $n - i$. If $n - i = 1$, the properties follow from Lemma 2.3.1. So suppose that $n - i > 1$. For $\epsilon, \eta \in \{-, +\}$, by induction with $\partial^\epsilon X$ and $\partial^\eta Y$, we get that

$$X_{n-1,\epsilon}^- \cap Y_{n-1,\eta}^+ = \emptyset.$$

Therefore,

$$(X_{n-1,-}^- \cup X_{n-1,+}^-) \cap (Y_{n-1,-}^+ \cup Y_{n-1,+}^+) = \emptyset.$$

We also get that

$$(X^-_{j,-} \cup X^-_{j,+}) \cap (Y^+_{j,-} \cup Y^+_{j,+}) = \emptyset \quad \text{for } i < j < n-1.$$

Let $Z = \partial^+ X *_i \partial^- Y$. Then, by induction, Z is a $(n-1)$-cell and

$$Z_{n-1} = X_{n-1,+} \cup Y_{n-1,-}.$$

Using Theorem 2.2.3(c), we get

$$X^-_n \cap Y^+_n = \emptyset$$

which concludes the proof of (a).

For (b), we already know that $\partial^- X *_i \partial^- Y$ and $\partial^+ X *_i \partial^+ Y$ are cells by induction. So, in order to prove that $X *_i Y$ is a cell, we just need to show that $X_n \cup Y_n$ is fork-free and moves $X_{n-1,-} \cup Y_{n-1,-}$ to $X_{n-1,+} \cup Y_{n-1,+}$. But

$$\begin{aligned} X^+_n \cap Y^+_n &= X^\pm_n \cap Y^+_n && \text{(by (a))} \\ &\subseteq Z_{n-1} \cap Y^+_n \\ &= \emptyset && \text{(by Theorem 2.2.3(a)).} \end{aligned}$$

Similarly,

$$X^-_n \cap Y^-_n = \emptyset$$

so $X_n \cup Y_n$ is fork-free. Using the dual of Theorem 2.2.3(a) with Z and X_n, we get that

$$X^-_n \cap (X_{n-1,+} \cup Y_{n-1,-}) = X^-_n \cap Y_{n-1,-} = \emptyset.$$

Similarly, if $Z' = \partial^- X *_i \partial^- Y$ then $Z'_{n-1} = X_{n-1,-} \cup Y_{n-1,-}$. Using Theorem 2.2.3(a) with Z' and X_n, we get that

$$X^+_n \cap (X_{n-1,-} \cup Y_{n-1,-}) = X^+_n \cap Y_{n-1,-} = \emptyset.$$

Since X_n moves $X_{n-1,-}$ to $X_{n-1,+}$, using Lemma 2.1.2, we deduce that

$$X_n \text{ moves } X_{n-1,-} \cup Y_{n-1,-} \text{ to } X_{n-1,+} \cup Y_{n-1,-}.$$

Similarly,

$$Y_n \text{ moves } X_{n-1,+} \cup Y_{n-1,-} \text{ to } X_{n-1,+} \cup Y_{n-1,+}.$$

Since $X^-_n \cap Y^+_n = \emptyset$, using Lemma 2.1.3, we get that

$$X_n \cup Y_n \text{ moves } X_{n-1,-} \cup Y_{n-1,-} \text{ to } X_{n-1,+} \cup Y_{n-1,+}.$$

Hence, $X *_i Y$ is a cell. □

Theorem 2.3.3. $(\mathrm{Cell}(P), \partial^-, \partial^+, *, \mathrm{id})$ *is an ω-category.*

Proof. We already know that $(\mathrm{Cell}(P), \partial^-, \partial^+)$ is a globular set. By Lemma 2.3.2, the composition operation $*$ is well-defined on composable cells. Moreover, all the axioms of ω-categories (given in Paragraph 1.1.3), follow readily from the definitions of $\partial^-, \partial^+, *, \mathrm{id}$. For example, consider the exchange law (vi). Given $j < i \le n \in N$, $X, X', Y, Y' \in \mathrm{Cell}(P)_n$ such that X, Y are i-composable, X', Y' are i-composable and X, X' are j-composable, let

$$Z = (X *_i X') *_j (Y *_i Y') \quad \text{and} \quad Z' = (X *_j Y) *_i (X' *_j Y').$$

For $k \leq n$ and $\epsilon \in \{-,+\}$, we have

$$Z_{k,\epsilon} = Z'_{k,\epsilon} = \begin{cases} X_{k,\epsilon} \cup Y_{k,\epsilon} \cup X'_{k,\epsilon} \cup Y'_{k,\epsilon} & \text{when } k > i, \\ X_{i,-} \cup X'_{i,-} & \text{when } k = i \text{ and } \epsilon = -, \\ Y_{i,+} \cup Y'_{i,+} & \text{when } k = i \text{ and } \epsilon = +, \\ X_{k,\epsilon} \cup X'_{k,\epsilon} & \text{when } j < k < i, \\ X_{j,-} & \text{when } k = j \text{ and } \epsilon = -, \\ X'_{j,+} & \text{when } k = j \text{ and } \epsilon = +, \\ X_{k,\epsilon} & \text{when } k < j, \end{cases}$$

so $Z = Z'$. Thus, $\mathrm{Cell}(P)$ satisfies axiom (vi) and the others as well. Hence, $(\mathrm{Cell}(P), \partial^-, \partial^+, *, \mathrm{id})$ is an ω-category. □

3 The freeness property

In this section, we give a complete proof of freeness for generalized parity complexes. We first define the freeness notion we are using and give some tools to manipulate the cells of an ω-category.

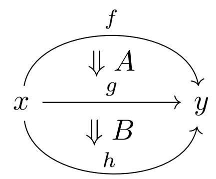
3.1 Generating sets

Suppose given $n \in \mathbb{N} \cup \{\omega\}$, $\mathcal{C}$ an n-category and S a subset of $\bigsqcup_{0 \leq i < n+1} \mathcal{C}_i$. The *set generated by S in $\mathcal{C}$*, denoted by S^*, is the smallest subset $T \subseteq \mathcal{C}$ such that

- $S \subseteq T$,
- if $X, Y \in T \cap \mathcal{C}_i$ and $\partial^+_j X = \partial^-_j Y$ for some $j < i$, then $X *_j Y \in T$,
- if $X \in T \cap \mathcal{C}_i$ and $i < n$ then $\mathrm{id}_{i+1}(X) \in T$,

and in this case, we say that *S generates T in $\mathcal{C}$*. Note that if S generates $\mathcal{C}$, every cell of $\mathcal{C}$ can be written by an expression involving only cells in S, compositions and identities.

For example, consider the following ω-hypergraph:



Then, the set $S_1 = \{A, B\}$ is *not* generating, whereas the set $S_2 = \{x, y, f, g, h, A, B\}$ is generating.

The notion of generating set for an ω-category can be reduced to the notion of generating set for an n-category, with $n \in \mathbb{N}$:

Lemma 3.1.1. *Let $\mathcal{C}$ be an ω-category and $S \subseteq \cup_{i \geq 0} \mathcal{C}_i$. Then S generates $\mathcal{C}$ in $\mathcal{C}$ if and only if for all $n \geq 0$, $S_{\leq n}$ generates $\mathcal{C}_{\leq n}$ in $\mathcal{C}_{\leq n}$.*

Proof. Let $n \geq 0$. Since $S_{\leq n} \subseteq (S^*)_{\leq n}$, we have $(S_{\leq n})^* \subseteq (S^*)_{\leq n}$. For the other side, since $S \subseteq (S_{\leq n})^* \cup \bigcup_{i>n} \mathcal{C}_i$, we have $S^* \subseteq (S_{\leq n})^* \cup \bigcup_{i>n} \mathcal{C}_i$, so $(S^*)_{\leq n} \subseteq (S_{\leq n})^*$. Hence, $(S_{\leq n})^* = (S^*)_{\leq n}$.

Suppose that S is generating $\mathcal{C}$. Then, for all $n \geq 0$, $(S_{\leq n})^* = (S^*)_{\leq n} = \mathcal{C}_{\leq n}$. Conversely, suppose that for all $n \geq 0$, $(S_{\leq n})^* = \mathcal{C}_{\leq n}$. For all n, we have $\mathcal{C}_{\leq n} = (S_{\leq n})^* = (S^*)_{\leq n} \subseteq S^*$. Hence, $\mathcal{C} = \cup_{n \geq 0} \mathcal{C}_{\leq n} = S^*$. □

3.2 Atoms are generating

Here, we show that the atoms are generating by adapting the results and proofs of [19].

In this subsection, we suppose given an ω-hypergraph P. We define the rank of an n-cell X of P as the n-tuple

$$\mathrm{Rank}(X) = (|X_{1,-} \cap X_{1,+}|, \ldots, |X_{n-1,-} \cap X_{n-1,+}|, |X_n|).$$

We order the ranks using a lexicographic ordering $<_{\mathrm{lex}}$: if $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are two n-tuples and there exists $1 \leq k \leq n$ such that $a_i = b_i$ for $i > k$ and $a_k < b_k$, then $(a_1, \ldots, a_n) <_{\mathrm{lex}} (b_1, \ldots, b_n)$. Note $<_{\mathrm{lex}}$ is well-founded. For example, in the ω-hypergraph (7), the 2-cell

$$X = (\{t\}, \{z\}, \{a, b, c, d, e, f\}, \{a, b', c', d', e', f\}, \{\alpha, \beta, \gamma, \delta\}) \ldots$$

has rank

$$\mathrm{Rank}(X) = (|\{a, f\}|, |\{\alpha, \beta, \gamma, \delta\}|) = (2, 4).$$

Theorem 3.2.1 (Excision of extremals)**.** *Suppose that P satisfies (G0), (G1), (G2) and (G3). Let $n \in \mathbb{N}$, X be an n-cell of P and $u \in X_n$ such that $X \neq \langle u \rangle$. Then there exist $i < n$ and n-cells Y, Z such that* $\mathrm{Rank}(Y) <_{\mathrm{lex}} \mathrm{Rank}(X)$, $\mathrm{Rank}(Z) <_{\mathrm{lex}} \mathrm{Rank}(X)$ *and $X = Y *_i Z$.*

Proof. Since $X \neq \langle u \rangle$, there is a least $i \geq -1$ such that $i < n$ and

$$(X_{j,-}, X_{j,+}) = (\langle u \rangle_{j,-}, \langle u \rangle_{j,+}) \quad \text{for } i + 1 < j \leq n.$$

In fact, $i \geq 0$. Indeed, if $i = -1$, then $X_{1,-} = \langle u \rangle_{1,-}$ and, since X and $\langle u \rangle$ are cells, $X_{0,-} = X_{1,-}^{\mp} = \langle u \rangle_{1,-}^{\mp} = \langle u \rangle_{0,-}$ and similarly $X_{0,+} = \langle u \rangle_{0,+}$, contradicting $X \neq \langle u \rangle$. If $i < n - 1$, since X is a cell, $X_{i+2,\epsilon} = \langle u \rangle_{i+2,\epsilon}$ moves $X_{i+1,-}$ to $X_{i+1,+}$, and, by Lemma 2.1.7, we have

$$X_{i+1,\epsilon} = \langle u \rangle_{i+1,\epsilon} \cup (X_{i+1,-} \cap X_{i+1,+}) \quad \text{for } \epsilon \in \{-, +\}.$$

This equality is still valid when $i = n - 1$ since $\langle u \rangle_{i+1,\epsilon} = \{x\} \subseteq X_n$. By definition of i, there exists $w \in (X_{i+1,-} \cap X_{i+1,+}) \setminus \langle u \rangle_{i+1,\epsilon}$. Let x be minimal in $X_{i+1,-}$ for $\triangleleft_{X_{i+1,-}}$ such that $x \triangleleft_{X_{i+1,-}} w$ or $x = w$ and let y be maximal for $\triangleleft_{X_{i+1,-}}$ in $X_{i+1,-}$ such that $w \triangleleft_{X_{i+1,-}} y$ or $w = y$. By Axiom (G3), either $x \notin \langle u \rangle_{i+1,-}$ or $y \notin \langle u \rangle_{i+1,-}$. By symmetry, we can suppose that $x \notin \langle u \rangle_{i+1,-}$. By minimality, we have $x^{\mp} \subseteq X_{i,-}$. Since $\partial_i^- X$ is a cell, by Theorem 2.2.3, $Y := \mathrm{id}_n(\mathrm{Glue}(\partial_i^- X, \{x\}))$ is an n-cell with

$$
\begin{aligned}
Y_{j,\epsilon} &= \emptyset && \text{for } i + 1 < j \leq n \text{ and } \epsilon \in \{-, +\} \\
Y_{i+1,\epsilon} &= \{x\} && \text{for } \epsilon \in \{-, +\} \\
Y_{i,-} &= X_{i,-} \\
Y_{i,+} &= (X_{i,-} \cup x^+) \setminus x^- \\
Y_{j,\epsilon} &= X_{j,\epsilon} && \text{for } j < i \text{ and } \epsilon \in \{-, +\}.
\end{aligned}
$$

For $\epsilon \in \{-,+\}$, since $X_{i+1,\epsilon} = \{x\} \sqcup (X_{i+1,\epsilon} \setminus \{x\})$ is fork-free and moves $X_{i,-}$ to $X_{i,+}$, by Lemma 2.1.4, we have that

$$\{x\} \text{ moves } X_{i,-} \text{ to } (X_{i,-} \cup x^+) \setminus x^-$$

and

$$X_{i+1,\epsilon} \setminus \{x\} \text{ moves } (X_{i,-} \cup x^+) \setminus x^- \text{ to } X_{i,+}.$$

If $i+2 \leq n$, for $\epsilon \in \{-,+\}$, since $X_{i+2,\epsilon} = \langle u \rangle_{i+2,\epsilon}$ moves $X_{i+1,-}$ to $X_{i+1,+}$ and $X^-_{i+2,\epsilon} \cap \{x\} = \emptyset$, using Lemma 2.1.2, we have that

$$X_{i+2,\epsilon} \text{ moves } X_{i+1,-} \setminus \{x\} \text{ to } X_{i+1,+} \setminus \{x\}.$$

So the following n-pre-cell Z is a cell:

$$\begin{aligned} Z_{j,\epsilon} &= X_{j,\epsilon} && \text{for } i+1 < j \leq n \text{ and } \epsilon \in \{-,+\} \\ Z_{i+1,\epsilon} &= X_{i+1,\epsilon} \setminus \{x\} && \text{for } \epsilon \in \{-,+\} \\ Z_{i,-} &= (X_{i,-} \cup x^+) \setminus x^- && \\ Z_{i,+} &= X_{i,+} && \\ Z_{j,\epsilon} &= X_{j,\epsilon} && \text{for } j < i \text{ and } \epsilon \in \{-,+\}. \end{aligned}$$

One readily checks that that $\mathrm{Rank}(Y) <_{\mathrm{lex}} \mathrm{Rank}(X)$, $\mathrm{Rank}(Z) <_{\mathrm{lex}} \mathrm{Rank}(X)$ and $X = Y *_i Z$. □

Theorem 3.2.2. *Suppose that P satisfies (G0), (G1), (G2) and (G3). Let $S = \{\langle x \rangle \mid x \in P\}$. Then S is generating* $\mathrm{Cell}(P)$*.*

Proof. Let T be the generated set by S in $\mathrm{Cell}(P)$. We show that $T = \mathrm{Cell}(P)$ by induction on the dimension of the cells. By the constraints on cells, the 0-cells of P are necessarily atoms. For $n > 0$, we do a second induction on the rank of the n-cell. So let X be an n-cell of P such that all n-cells with lower ranks are in T. If $X_n = \emptyset$, then $X = \mathrm{id}_n(X')$ for some X'. By induction, $X' \in T$, so $X \in T$. Otherwise, if $X_n \neq \emptyset$, then either X is an atom, in which case $X \in T$, or by the excision Theorem 3.2.1, there are $i < n$ and Y, Z n-cells of P with lower ranks than X such that $X = Y *_i Z$. By induction, $Y, Z \in T$ so $X \in T$. So $\mathrm{Cell}(P)_n \subseteq T$ for all n. Hence, $T = \mathrm{Cell}(P)$. □

3.3 Properties of the exchange law

Here, we show some useful properties related to the exchange law.

In this subsection, we suppose given an m-category $\mathcal{C}$ with $m \in \mathbb{N} \cup \{\omega\}$. The first lemma states that low level compositions commute with sequences of high level compositions.

Lemma 3.3.1. *Let $j < i \leq n < m+1$, $p \geq 0$ and $x_1, \ldots, x_p, y_1, \ldots, y_p$ n-cells of $\mathcal{C}$ such that $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_p)$ are i-composable, and $\partial^+_j x_k = \partial^-_j y_k$ for $1 \leq k \leq p$. Then*

$$(x_1 *_i \cdots *_i x_p) *_j (y_1 *_i \cdots *_i y_p) = (x_1 *_j y_1) *_i \cdots *_i (x_p *_j y_p).$$

Proof. We do an induction on p. If $p = 1$, the result is trivial. So suppose $p > 1$. Then

$$\begin{aligned}
&(x_1 *_i \cdots *_i x_p) *_j (y_1 *_i \cdots *_i y_p) \\
&= ((x_1 *_i \cdots *_i x_{p-1}) *_i x_p) *_j ((y_1 *_i \cdots *_i y_{p-1}) *_i y_p) \\
&= ((x_1 *_i \cdots *_i x_{p-1}) *_j (y_1 *_i \cdots *_i y_{p-1})) *_i (x_p *_j y_p) \\
&= ((x_1 *_j y_1) *_i \cdots *_i (x_{p-1} *_j y_{p-1})) *_i (x_p *_i y_p) \qquad \text{(by induction)} \\
&= (x_1 *_j y_1) *_i \cdots *_i (x_{p-1} *_j y_{p-1}) *_i (x_p *_i y_p).
\end{aligned}$$

□

The next lemma states that compositions with identities distribute over sequences of compositions.

Lemma 3.3.2. *Let $j < i \leq n < m+1$, $p \geq 0$ and $(x_1, \ldots, x_p)$ n-cells of $\mathcal{C}$ that are i-composable and y an i-cell of $\mathcal{C}$.*

– *If $\partial_j^+ x_k = \partial_j^- y$ for $1 \leq k \leq p$, then*

$$(x_1 *_i \cdots *_i x_p) *_j \mathrm{id}_n(y) = (x_1 *_j \mathrm{id}_n(y)) *_i \cdots *_i (x_p *_j \mathrm{id}_n(y)).$$

– *If $\partial_j^- x_k = \partial_j^+ y$ for $1 \leq k \leq p$, then*

$$\mathrm{id}_n(y) *_j (x_1 *_i \cdots *_i x_p) = (\mathrm{id}_n(y) *_j x_1) *_i \cdots *_i (\mathrm{id}_n(y) *_j x_p).$$

Proof. Note that $\mathrm{id}_n(y) = \underbrace{\mathrm{id}_n(y) *_i \cdots *_i \mathrm{id}_n(y)}_{p}$. Using Lemma 3.3.1, we conclude the proof. □

The next lemma states that composing two sequences of cells in low dimension is equivalent to composing the first one and then the second one in high dimension.

Lemma 3.3.3. *Let $j < i \leq n < m+1$, $p, q \geq 0$ and n-cells $x_1 \ldots, x_p$ and $y_1, \ldots, y_q$ such that $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_q)$ are i-composable and $\partial_j^+ x_k = \partial_j^- y_{k'}$ for $1 \leq k \leq p$ and $1 \leq k' \leq q$. Then,*

$$\begin{aligned}
&(x_1 *_i \cdots *_i x_p) *_j (y_1 *_i \cdots *_i y_q) \\
&= (x_1 *_j \mathrm{id}_n(\partial_i^- y_1)) *_i \cdots *_i (x_p *_j \mathrm{id}_n(\partial_i^- y_1)) \\
&*_i (\mathrm{id}_n(\partial_i^+ x_p) *_j y_1) *_i \cdots *_i (\mathrm{id}_n(\partial_i^+ x_p) *_j y_q) \\
&= (\mathrm{id}_n(\partial_i^- x_1) *_j y_1) *_i \cdots *_i (\mathrm{id}_n(\partial_i^- x_1) *_j y_q) \\
&*_i (x_1 *_j \mathrm{id}_n(\partial_i^+ y_q)) *_i \cdots *_i (x_p *_j \mathrm{id}_n(\partial_i^+ y_q)).
\end{aligned}$$

Proof. We have that

$$(x_1 *_i \cdots *_i x_p) = x_1 *_i \cdots *_i x_p *_i \underbrace{\mathrm{id}_n(\partial_i^+ x_p) *_i \cdots *_i \mathrm{id}_n(\partial_i^+ x_p)}_{q}$$

and

$$(y_1 *_i \cdots *_i y_q) = \underbrace{\mathrm{id}_n(\partial_i^- y_1) *_i \cdots *_i \mathrm{id}_n(\partial_i^- y_1)}_{p} *_i y_1 *_i \cdots *_i y_p.$$

So, using Lemma 3.3.1, we get that

$$\begin{aligned}
&(x_1 *_i \cdots *_i x_p) *_j (y_1 *_i \cdots *_i y_q) \\
&= (x_1 *_j \mathrm{id}_n(\partial_i^- y_1)) *_i \cdots *_i (x_p *_j \mathrm{id}_n(\partial_i^- y_1)) \\
&*_i (\mathrm{id}_n(\partial_i^+ x_p) *_j y_1) *_i \cdots *_i (\mathrm{id}_n(\partial_i^+ x_p) *_j y_q).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(x_1 *_i \cdots *_i x_p) *_j (y_1 *_i \cdots *_i y_q) \\
&= (\mathrm{id}_n(\partial_i^- x_1) *_j y_1) *_i \cdots *_i (\mathrm{id}_n(\partial_i^- x_1) *_j y_q) \\
&*_i (x_1 *_j \mathrm{id}_n(\partial_i^+ y_q)) *_i \cdots *_i (x_p *_j \mathrm{id}_n(\partial_i^+ y_q)).
\end{aligned}$$

□

The next lemma is a special case of Lemma 3.3.3.

Lemma 3.3.4. *Let $j < i \leq n < m+1$ and n-cells x, y such that $\partial_j^+ x = \partial_j^- y$. Then,*

$$\begin{aligned}
x *_j y &= (x *_j \mathrm{id}_n(\partial_i^- y)) *_i (\mathrm{id}_n(\partial_i^+ x) *_j y) \\
&= (\mathrm{id}_n(\partial_i^- x) *_j y) *_i (x *_j \mathrm{id}_n(\partial_i^+ y)).
\end{aligned}$$

Proof. By Lemma 3.3.3. □

3.4 Contexts

Here, we define contexts, a tool to manipulate the cells of an ω-category. They can be thought as a composite of cells with one "hole" that can be filled by another cell. Using contexts, we are able to define a canonical form for cells.

In the following, we suppose given an m-category $\mathcal{C}$ with $m \in \mathbb{N} \cup \{\omega\}$.

3.4.1 Definition. Given $n < m+1$, two n-cells $y, z \in C$ are said *parallel* when $n = 0$ or $\partial^- y = \partial^- z$ and $\partial^+ y = \partial^+ z$. Given p with $n \leq p < m+1$ and a p-cell x, we say that x *is adapted to* (y, z) when $\partial_n^- x = y$ and $\partial_n^+ x = z$. Given x,y,z as above, we define an *n-context E of type* (y, z) and the *evaluation $E[x]$ of E on x* by induction on the dimension n of y and z as follows.

- For 0-cells y and z, there is a unique context of type (y, z), noted $[-]$. Given a p-cell x as above, the induced cell is $[x] = x$.
- For parallel $(n+1)$-cells y and z, a context of type (y, z) is given by a context $\tilde{E}$ of type $(\partial^- y, \partial^+ z)$, together with a pair of $(n+1)$-cells x_{n+1} and x'_{n+1} such that

  $$\partial^+(x_{n+1}) = \tilde{E}[\partial^- y] \quad \text{and} \quad \partial^-(x'_{n+1}) = \tilde{E}[\partial^+ z].$$

  Given a p-cell x as above, the evaluation of E on x is

  $$E[x] = x_{n+1} *_n \tilde{E}[x] *_p x'_{n+1}$$

  so that we sometimes write

  $$E = x_{n+1} *_n \tilde{E} *_n x'_{n+1}$$

  for the context E itself.

Given parallel n-cells y and z, a context of type (y, z) thus consists of pairs of suitably typed k-cells (x_k, x'_k), for $0 \leq k \leq n$, and is of the form

$$E = x_n *_{n-1} (\cdots *_1 (x_1 *_0 [-] *_0 x'_1) *_1 \cdots) *_{n-1} x'_p$$

We write $\pi_k^- E = x_k$ and $\pi_k^+ E = x'_k$ for the k-cells of the context E. Moreover, an m-context E will be said *adapted to* x when its type is $(\partial_m^- x, \partial_m^+ x)$.

3.4.2 Canonical forms. Here, we show that contexts can be used to give a canonical form to the cells of an ω-category. First, we prove that the composition of an evaluated context with a cell results in an evaluated context.

Lemma 3.4.3. *Let $n \leq p < m+1$, a p-cell x and an n-context E adapted to x. Given $i \leq q \leq n$ and a q-cell y such that $\partial_i^- y = \partial_i^+ E[x]$, there exists an n-context E' adapted to x such that $E[x] *_i y = E'[x]$.*

Proof. Let $z = E[x] *_i y$. Then

$$z = (\pi_n^- E *_{n-1} (\cdots (\pi_1^- E *_0 x *_0 \pi_1^+ E) \cdots) *_{n-2} \pi_{n-1}^+ E) *_{n-1} \pi_n^+ E) *_i y.$$

We prove this lemma by an induction on (n, q) (ordered lexicographically). If $i = n$, then, since y is of dimension $q \leq n$,

$$z = E[x].$$

Otherwise, if $i = n-1$, then

$$z = (\pi_n^- E) *_{n-1} (\cdots (\pi_1^- E *_0 x *_0 \pi_1^+ E) \cdots) *_{n-2} (\pi_{n-1}^+ E)) *_{n-1} (\pi_n^+ E *_{n-1} y)$$

which is of the form $E'[x]$. Otherwise, $i < n-1$. Then,

- if $q < n$, using Lemma 3.3.2, we get

  $$z = (\pi_n^- E *_i y) *_{n-1} (\tilde{E}[x] *_i y) *_{n-1} (\pi_n^+ E *_i y)$$

  By induction hypothesis on the middle part, we get an $(n-1)$-context E' such that

  $$z = (\pi_n^- E *_i y) *_{n-1} E'[x] *_{n-1} (\pi_n^+ E *_i y).$$

  From this, one easily deduces an n-context E'' such that $z = E''[x]$.

- Otherwise, if $q = n$, using Lemma 3.3.3, we get

  $$z = (E[x] *_i \partial_{n-1}^- y) *_{n-1} (\partial_{n-1}^+ E[x] *_i y).$$

  Using the induction hypothesis on the left-hand side, we get an n-context E' such that

  $$z = E'[x] *_{n-1} (\partial_{n-1}^+ E[x] *_i y).$$

  Thus,

  $$\begin{aligned} z &= (\pi_n^- E' *_{n-1} (\cdots x \cdots) *_{n-1} \pi_n^+ E') *_{n-1} (\partial_{n-1}^+ (\pi_n^+ E) *_i y) \\ &= \pi_n^- E' *_{n-1} (\cdots x \cdots) *_{n-1} (\pi_n^+ E' *_{n-1} (\partial_{n-1}^+ (\pi_n^+ E) *_i y)) \end{aligned}$$

  which is the evaluation of an n-context E'' on x. □

Contexts can be used to obtain the canonical form for the cells of an ω-category, as in the following lemma.

Lemma 3.4.4. *Suppose that $P \subseteq \mathcal{C}$ generates $\mathcal{C}$. Then, for $n \geq 0$, every n-cell x of $\mathcal{C}$ can be written in one of the following form:*

- $\mathrm{id}_n(x')$ *with x' an $(n-1)$-cell.*

– $E_1[x_1] *_{n-1} E_2[x_2] *_{n-1} \cdots *_{n-1} E_p[x_p]$ *with* $p > 0$, $x_1, \ldots, x_p \in P_n$ *and* $E_1, \ldots, E_p$ $(n-1)$-*contexts.*

Proof. Let x be an n-cell of $\mathcal{C}$. Since P is generating, there is a formal expression involving only compositions, identities and elements of P that is equal to x. We make an induction on the structure of such an expression.

If $x \in P$, then

$$x = \partial^-_{n-1}(x) *_{n-1} (\cdots(\partial^-_0 x *_0 x *_0 \partial^+_0 x)\cdots) *_{n-1} \partial^+_{n-1}(x)$$

which has the form of an evaluated context. Otherwise, if $x = \mathrm{id}_n(x')$ with x' of dimension $(n-1)$, which is one of the desired forms. Otherwise, $x = y *_i z$. By induction, we can write y and z in one of the desired form. There are four cases:

– $y = \mathrm{id}_n(y')$ and $z = \mathrm{id}_n(z')$, with y', z' $(n-1)$-cells. Then $x = \mathrm{id}_n(y' *_i z')$ which is a desired form.

– $y = E_1[y_1] *_{n-1} \cdots *_{n-1} E_p[y_p]$ and $z = \mathrm{id}_n(z')$. If $i = n-1$, then $y *_i z = y$ and it is a desired form. Otherwise, $i < n-1$ and, using Lemma 3.3.2,

$$y *_i z = (E_1[y_1] *_i z) *_{n-1} \cdots *_{n-1} (E_p[y_p] *_i z).$$

By applying Lemma 3.4.3 to all $E_k[y_k] *_i z$ for $1 \leq k \leq p$, we get the desired form.

– $y = \mathrm{id}_n(y')$ and $z = E_1[z_1] *_{n-1} \cdots *_{n-1} E_p[z_p]$. This case is similar to the previous one.

– $y = E_1[y_1] *_{n-1} \cdots *_{n-1} E_p[y_p]$ and $z = E'_1[z_1] *_{n-1} \cdots *_{n-1} E'_q[z_q]$. If $i = n-1$, we can concatenate the two decompositions directly. Otherwise $i < n-1$ and, by Lemma 3.3.4,

$$y *_i z = (y *_i \partial^-_{n-1}(z)) *_{n-1} (\partial^+_{n-1}(y) *_{n-1} z).$$

Using Lemma 3.3.2,

$$\begin{aligned} y *_i z = {} & (E_1[y_1] *_i \partial^-_{n-1}(z)) *_{n-1} \cdots *_{n-1} (E_p[y_p] *_i \partial^-_{n-1}(z)) \\ & *_{n-1} (\partial^+_{n-1}(y) *_i E'_1[z_1]) *_{n-1} \cdots *_{n-1} (\partial^+_{n-1}(y) *_i E'_q[z_q]). \end{aligned}$$

By applying Lemma 3.4.3 to all $E_k[y_k] *_i \partial^-_{n-1}(z)$ for $1 \leq k \leq p$, and to all $\partial^+_{n-1}(y) *_i E'_k[z_k]$ for $1 \leq k \leq q$, we get a desired form. □

3.5 Free ω-categories

Here, we define briefly the notion of freeness we use for the ω-category of cells. We refer to [12, 2] for a more complete presentation.

3.5.1 Cellular extensions.

An *n-cellular extension*

$$\mathcal{C} \underset{\mathrm{t}}{\overset{\mathrm{s}}{\leftleftarrows}} S$$

is given by an n-category $\mathcal{C}$, a set S and functions $\mathrm{s}, \mathrm{t}\colon S \to \mathcal{C}_n$ such that, if $n > 0$, $\partial^- \circ \mathrm{s} = \partial^- \circ \mathrm{t}$ and $\partial^+ \circ \mathrm{s} = \partial^+ \circ \mathrm{t}$. When there is no ambiguity, we denote by $(\mathcal{C}, S)$ such an extension. A *morphism* (F, f) *between two n-cellular extensions*

$$\mathcal{C} \underset{\mathrm{t}}{\overset{\mathrm{s}}{\leftleftarrows}} S \quad \text{and} \quad \mathcal{D} \underset{\mathrm{t}'}{\overset{\mathrm{s}'}{\leftleftarrows}} T$$

is given by an n-functor $F\colon \mathcal{C} \to \mathcal{D}$ and a function $f\colon S \to T$ such that the following diagrams commute:

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle \mathrm{s}}\downarrow & & \downarrow{\scriptstyle \mathrm{s}'} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle \mathrm{t}}\downarrow & & \downarrow{\scriptstyle \mathrm{t}'} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}.$$

Following [2], we denote by n-Cat$^+$ the category of n-cellular extensions. Note that there is a forgetful functor $U\colon (n{+}1)\text{-Cat} \to n\text{-Cat}^+$ where

$$U(\mathcal{C}) = \mathcal{C}_{\leq n} \underset{\partial^+}{\overset{\partial^-}{\leftleftarrows}} \mathcal{C}_{n+1}.$$

By generic categorical arguments, it can be shown that this functor has a left adjoint

$$-[-]\colon n\text{-Cat}^+ \to (n{+}1)\text{-Cat}$$

sending an n-cellular extension $(\mathcal{C}, S)$ to an $(n{+}1)$-category $\mathcal{C}[S]$. A *free extension of* $\mathcal{C}$ *by* S is a categorical extension of $\mathcal{C}$ isomorphic to $\mathcal{C}[S]$. Note that $\mathcal{C}[S]_{\leq n}$ is isomorphic to $\mathcal{C}$ and $\mathcal{C}[S]_{n+1}$ is the set of all the formal composites made with elements of S (considered as $(n{+}1)$-cells with sources and targets given by s and t) and cells of $\mathcal{C}$.

Given an n-cellular extension $(\mathcal{C}, S)$, the unit of the adjunction induces a morphism

$$\eta_{(\mathcal{C},S)}\colon (\mathcal{C}, S) \to (\mathcal{C}, \mathcal{C}[S]_{n+1})$$

satisfying the following universal property: for all $(n{+}1)$-category $\mathcal{D}$ and all morphism of cellular extension

$$(F, f)\colon (\mathcal{C}, S) \to (\mathcal{D}_{\leq n}, \mathcal{D}_{n+1})$$

there is a unique $(n{+}1)$-functor $G\colon \mathcal{C}[S] \to \mathcal{D}$ making the following diagram commute:

$$\begin{array}{ccc} (\mathcal{C}, S) & \xrightarrow{(F,f)} & (\mathcal{D}_{\leq n}, \mathcal{D}_{n+1}) \\ {\scriptstyle \eta_{(\mathcal{C},S)}}\downarrow & \nearrow_{UG} & \\ (\mathcal{C}, \mathcal{C}[S]_{n+1}) & & \end{array}.$$

3.5.2 Polygraphs. For $n \in \mathbb{N}$, we define inductively on n the notion of an *n-polygraph* P together with the *n-category* P^* *generated by* P:

- A 0-polygraph P is a set P_0. The generated 0-category P^* by P is P_0 (seen as a 0-category).
- An $(n{+}1)$-polygraph P is given by an n-polygraph $P_{\leq n}$ together with an n-cellular extension

  $$(P_{\leq n})^* \underset{\mathrm{t}_n}{\overset{\mathrm{s}_n}{\leftleftarrows}} P_{n+1}$$

  and the $(n{+}1)$-category P^* generated by P is the free extension

  $$(P_{\leq n})^*[P_{n+1}].$$

An *ω-polygraph* P is then a sequence $(P^i)_{i\geq 0}$ where P^i is an i-polygraph such that $(P^{i+1})_{\leq i} = P^i$ and the *ω-category generated by* P is the colimit $\cup_{i\geq 0}(P^i)^*$.

3.5.3 Freeness. For $n \in \mathbb{N} \cup \{\omega\}$, a *free n-category* is an n-category $\mathcal{C}$ such that there exists an n-polygraph P with $\mathcal{C} \simeq P^*$. By unfolding the definition, it means that, for $k < n$, $\mathcal{C}_{\leq k+1} \simeq \mathcal{C}_{\leq k}[P_{k+1}]$ for some k-cellular extension

$$\mathcal{C}_{\leq k} \underset{\mathrm{t}_k}{\overset{\mathrm{s}_k}{\leftleftarrows}} P_{k+1}.$$

3.6 Cell(P) is free

Here, we show that generalized parity complexes induce ω-categories as defined in Subsection 3.5.

In the following, we suppose given a generalized parity complex P.

3.6.1 A cellular extension for Cell(P). For $n \in \mathbb{N}$, there is an n-cellular extension

$$\mathrm{Cell}(P)_{\leq n} \underset{\partial^+\langle - \rangle}{\overset{\partial^-\langle - \rangle}{\leftleftarrows}} P_{n+1}$$

where, for $x \in P_{n+1}$ and $\epsilon \in \{-,+\}$, $\partial^\epsilon\langle - \rangle(x) = \partial^\epsilon\langle x \rangle$ (which is an n-cell by (G2)). In the following, we denote by $\mathrm{Cell}(P)^+_{\leq n}$ the (n+1)-category $\mathrm{Cell}(P)_{\leq n}[P_{n+1}]$. We have a morphism of cellular extensions

$$(\mathrm{Cell}(P)_{\leq n} \underset{\partial^+\langle - \rangle}{\overset{\partial^-\langle - \rangle}{\leftleftarrows}} P_{n+1}) \xrightarrow{(\mathrm{id}_{\mathrm{Cell}(P)_{\leq n}}, \langle - \rangle)} (\mathrm{Cell}(P)_{\leq n} \underset{\partial^+}{\overset{\partial^-}{\leftleftarrows}} \mathrm{Cell}(P)_{n+1})$$

where, for all $x \in P_{n+1}$, $\langle - \rangle(x) = \langle x \rangle$. By the universal property discussed in Subsection 3.5, there is an unique (n+1)-functor

$$\mathrm{eval}_{n+1} \colon \mathrm{Cell}(P)^+_{\leq n} \to \mathrm{Cell}(P)_{\leq n+1}$$

such that the following diagram commutes:

$$\begin{array}{ccc} P_{n+1} & \xrightarrow{\langle - \rangle} & \mathrm{Cell}(P)_{n+1} \\ {\scriptstyle \eta_{(\mathrm{Cell}(P)_{\leq n}, P_{n+1})}} \downarrow & \nearrow {\scriptstyle \mathrm{eval}_{n+1}} & \\ \mathrm{Cell}(P)^+_{\leq n} & & \end{array} .$$

For $x \in P_{n+1}$, we write $\widehat{x}$ for $\eta_{(\mathrm{Cell}(P)^{(n)}, P_{n+1})}(x) \in (\mathrm{Cell}(P)^+_{\leq n})_{n+1}$. For $m \leq n$ and $y \in P_m$, we write $\widehat{y}$ for $\langle y \rangle \in (\mathrm{Cell}(P)^+_{\leq n})_m = \mathrm{Cell}(P)_m$. For conciseness, we will sometimes write eval for eval_{n+1}.

3.6.2 The generators are preserved by eval. In order to show that eval is an isomorphism and in particular a monomorphism, we state lemmas relating the generators involved in the images of eval with the generators involved in the arguments of eval. The first lemma states that the evaluation of identities gives cells with no top-level generators.

Lemma 3.6.3. *For $n \in \mathbb{N}$ and $X \in (\mathrm{Cell}(P)^+_{\leq n})_n$, $\mathrm{eval}(\mathrm{id}_{n+1}(X))_{n+1} = \emptyset$.*

Proof. Since eval is an (n+1)-functor, we have

$$\mathrm{eval}(\mathrm{id}_{n+1}(X)) = \mathrm{id}_{n+1}(\mathrm{eval}(X)) = \mathrm{id}_{n+1}(X).$$

Hence,

$$\mathrm{eval}(\mathrm{id}_{n+1}(X))_{n+1} = (\mathrm{id}_{n+1}(X))_{n+1} = \emptyset. \qquad \square$$

The next lemma states that the generators involved in the decomposition of Lemma 3.4.4 are the top-level generators of the evaluation.

Lemma 3.6.4. *Given $n \in \mathbb{N}$, $k \geq 1$, $x_1, \dots, x_k \in P_{n+1}$, adapted n-contexts $E_1, \dots, E_k$ of* $\mathrm{Cell}(P)^+_{\leq n}$ *and* $X = E_1[\widehat{x_1}] *_n \cdots *_n E_k[\widehat{x_k}] \in \mathrm{Cell}(P)^+_{\leq n}$, *we have*

(a) $\mathrm{eval}(X)_{n+1}$ *is equal to* $\{x_1, \dots, x_k\}$,

(b) for $i \neq j$, $x_i \neq x_j$.

In particular, if

$$E_1[\widehat{x_1}] *_n \cdots *_n E_k[\widehat{x_k}] = E'_1[\widehat{y_1}] *_n \cdots *_n E'_l[\widehat{y_l}]$$

with $l \geq 1$, $y_1, \dots, y_l \in P_{n+1}$ *and* $E'_1, \dots, E'_l$ *being adapted n-contexts, then* $k = l$ *and*

$$\{x_1, \dots, x_k\} = \{y_1, \dots, y_l\}.$$

Proof. By definition, for all $1 \leq i \leq k$, $\mathrm{eval}(\widehat{x_i}) = \langle x_i \rangle$, thus $\mathrm{eval}(\widehat{x_i})_{n+1} = \{x_i\}$. With an induction on the structure of E_i, one can show that

$$\mathrm{eval}(E_i[\widehat{x_i}])_{n+1} = \mathrm{eval}(\widehat{x_i}) = \{x_i\}.$$

Moreover,

$$\begin{aligned}
\mathrm{eval}(X) &= \mathrm{eval}(E_1[\widehat{x_1}] *_n \cdots *_n E_k[\widehat{x_k}])_{n+1} \\
&= (\mathrm{eval}(E_1[\widehat{x_1}]) *_n \cdots *_n \mathrm{eval}(E_k[\widehat{x_k}]))_{n+1} \\
&= \mathrm{eval}(E_1[\widehat{x_1}])_{n+1} \cup \cdots \cup \mathrm{eval}(E_k[\widehat{x_k}])_{n+1} \\
&= \{x_1, \dots, x_k\}
\end{aligned}$$

so (a) holds. Now let $1 \leq i < j \leq k$ and Y, Z be $(n{+}1)$-cells defined by

$$Y = \mathrm{eval}(E_1[\widehat{x_1}] *_n \cdots *_n E_i[\widehat{x_i}]) \quad \text{and} \quad Z = \mathrm{eval}(E_{i+1}[\widehat{x_{i+1}}] *_n \cdots *_n E_k[\widehat{x_k}]).$$

Then $x_i \in Y_{n+1}$, $x_j \in Z_{n+1}$ and Y,Z are n-composable. Using Lemma 2.3.1, we have that $Y_{n+1} \cap Z_{n+1} = \emptyset$. Hence, $x_i \neq x_j$. □

3.6.5 Properties on $\mathrm{Cell}(P)$**.** We first state a simple criterion for the equality of two cells in $\mathrm{Cell}(P)$.

Lemma 3.6.6. *Given* $n \in \mathbb{N}$, $\epsilon \in \{-, +\}$ *and* $X, Y \in \mathrm{Cell}(P)_n$ *such that* $\partial^\epsilon X = \partial^\epsilon Y$ *and* $X_n = Y_n$, *we have* $X = Y$.

Proof. If $n = 0$, the result is trivial. So suppose $n > 0$. By symmetry, we can moreover suppose that $\epsilon = -$. By the hypothesis, we only need to prove that $X_{n-1,+} = Y_{n-1,+}$. But

$$X_{n-1,+} = (X_{n-1,-} \cup X^+_n) \setminus X^-_n = (Y_{n-1,-} \cup Y^+_n) \setminus Y^-_n = Y_{n-1,+}.$$ □

The next states that, given $m \in \mathbb{N}$, an evaluated m-context induces a partition of the m-generators involved in the associated cell.

Lemma 3.6.7. *Given $n \in \mathbb{N}$, $1 \le m \le n$, $x \in P_{n+1}$, $\epsilon \in \{-,+\}$, an adapted m-context E of* $\mathrm{Cell}(P)^+_{\le n}$, *we have that*

$$(\pi^-_m E)_m, \langle x\rangle_{m,\epsilon}, (\pi^+_m E)_m \text{ is a partition of } (\partial^\epsilon_m E[\widehat{x}])_m.$$

Moreover, given another m-context E' adapted to $\widehat{x}$ such that $E[\widehat{x}] = E'[\widehat{x}]$, we have

$$(\pi^-_m E)_m \cup (\pi^+_m E)_m = (\pi^-_m E')_m \cup (\pi^+_m E')_m.$$

Proof. For the first part, note that

$$\partial^\epsilon_m(E[\widehat{x}]) = \pi^-_m E *_{m-1} \partial^\epsilon_m\langle x\rangle *_{m-1} \pi^+_m E$$

By the definition of composition,

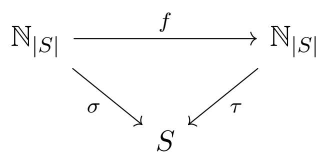
$$(\partial^\epsilon_m(E[\widehat{x}]))_m = (\pi^-_m E)_m \cup (\partial^\epsilon_m\langle x\rangle)_m \cup (\pi^+_m E)_m \tag{29}$$

and by the definition of atoms, $(\partial^\epsilon_m\langle x\rangle)_m = \langle x\rangle_{m,\epsilon}$. By Lemma 2.3.1(b), (29) is moreover a partition. The second part is a consequence of the first since we have

$$(\pi^-_m E)_m \cup (\partial^-_m\langle x\rangle)_m \cup (\pi^+_m E)_m = (\pi^-_m E')_m \cup (\partial^-_m\langle x\rangle)_m \cup (\pi^+_m E')_m$$

and both sides are partitions. □

3.6.8 Linear extensions. For $n \in \mathbb{N}$, we write $\mathbb{N}_n$ for $\{1,\dots,n\}$. Given a finite poset $(S,<)$, a *linear extension of* $(S,<)$ is given by a bijection $\sigma : \mathbb{N}_{|S|} \to S$ such that, for $1 \le i,j \le k$, if $\sigma(i) < \sigma(j)$, then $i < j$. The linear extensions of $(S,<)$ have a structure of an 1-category $\mathrm{LinExt}(S)$ where the objects are the linear extensions $\sigma : \mathbb{N}_{|S|} \to S$ and the morphisms between $\sigma,\tau \colon \mathbb{N}_{|S|} \to S$ are the functions $f \colon \mathbb{N}_{|S|} \to \mathbb{N}_{|S|}$ such that the triangle



is commutative.

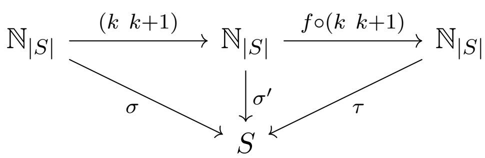
The next lemma states that the morphisms of linear extensions are generated by transpositions.

Lemma 3.6.9. *Let T be the set of consecutive transpositions in* $\mathrm{LinExt}(S)$*, that is,*

$$T = \{(i\ i{+}1) \colon \sigma \to \tau \in \mathrm{LinExt}(S)_1 \mid \sigma,\tau \in \mathrm{LinExt}(S), i \in \mathbb{N}_{|S|-1}\}.$$

Then, $\mathrm{LinExt}(S)$ *is generated by* $\mathrm{LinExt}(S)_0 \cup T$.

Proof. Let $\sigma,\tau \in \mathrm{LinExt}(S)_0$ and $f \colon \sigma \to \tau \in \mathrm{LinExt}(S)_1$. Since $\tau \circ f = \sigma$, f is a bijection. We prove the result by induction on the number N of inversions of f. If $N = 0$, then $f = \mathrm{id}_{\mathbb{N}_{|S|}} = \mathrm{id}_1(\sigma)$. Otherwise, $N > 0$. Thus, there exists $k \in \mathbb{N}_{|S|-1}$ such that $f(k) > f(k+1)$. Let $\sigma' = \sigma \circ (k\ k{+}1)$. σ' is then a linear extension of $(S,<)$ as in



.

Indeed, for $i \neq j \in \mathbb{N}_{|S|}$ such that $\sigma'(i) < \sigma'(j)$,

– if $\{i,j\} \cap \{k,k+1\} = \emptyset$, then $\sigma(i) < \sigma(j)$ and $i < j$;

– if $i = k$ and $j \neq k+1$, then $\sigma(i+1) < \sigma(j)$ and $i+1 < j$, so $i < j$;

– if $i = k$ and $j = k+1$, then $i < j$;

– if $i = k+1$ and $j \neq k$, then $\sigma(k) < \sigma(j)$, so $k < j$, and, since $j \neq i = k+1$, $i = k+1 < j$;

– if $i = k+1$ and $j = k$, then $\sigma(k) < \sigma(k+1)$, so $\tau(f(k)) < \tau(f(k+1))$, hence $f(k) < f(k+1)$, contradicting the hypothesis;

– if $i \notin \{k,k+1\}$ and $j \in \{k,k+1\}$, then $i < j$ similarly as when $i \in \{k,k+1\}$ and $j \notin \{k,k+1\}$.

Moreover, the number of inversions of $f \circ (k\ k{+}1)$ is $N-1$. By induction, it can be written using elements of T. Hence, f can be written as a composite of elements of T. □

3.6.10 The technical lemmas. We now state the technical lemmas that we use to prove the freeness property. They give properties satisfied by $\mathrm{Cell}(P)^+_{\leq n}$ and they should be thought as one big lemma since they are mutually dependent. However, we preferred to split them, for clarity.

The first lemma enables to modify a context E adapted a generator x without changing the evaluation.

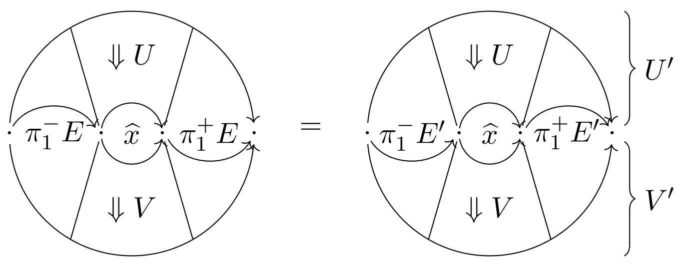
Lemma 3.6.11. *Let $n \in \mathbb{N}$ and $m \leq n$. Let $x \in P_{n+1}$, E be an adapted m-context of $\mathrm{Cell}(P)^+_{\leq n}$ with $m \leq n$. Consider the following subsets of P_m:*

$$S = (\pi^-_m E)_m \cup (\pi^+_m E)_m, \qquad U = \{y \in S \mid y \underset{S'}{\lhd} \langle x\rangle_{m,-}\},$$
$$S' = S \cup \langle x\rangle_{m,-}, \qquad V = \{y \in S \mid \langle x\rangle_{m,-} \underset{S'}{\lhd} y\}.$$

Then, for every partition $U' \cup V'$ of S such that $U \subseteq U'$, and $V \subseteq V'$, U' initial and V' final for $\lhd_S$, there exists an m-context E' adapted to $\widehat{x}$ such that

$$(\pi^-_m E')_m = U', \qquad (\pi^+_m E')_m = V', \qquad E[\widehat{x}] = E'[\widehat{x}].$$

Graphically, with $m = 2$, this can be illustrated as



Next, we show that if two $(n{+}1)$-generators in context do not have common n-generators in their source and target then we can apply the exchange rule.

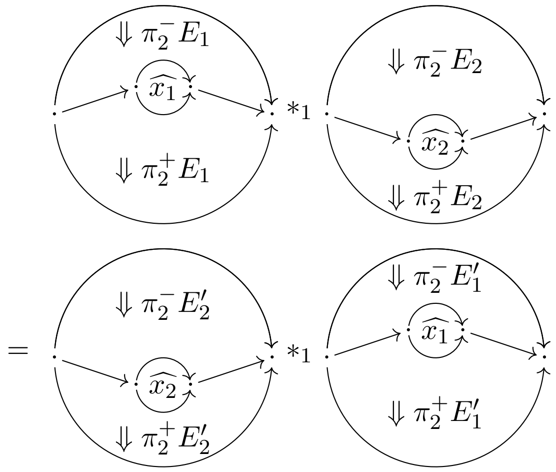
Lemma 3.6.12. *Let $n \in \mathbb{N}$ and $m \leq n$. Let $k_1, k_2 \geq 0$ be such that $\max(k_1,k_2) = n+1$, $x_1 \in P_{k_1}$, $x_2 \in P_{k_2}$, E_1, E_2 be adapted m-contexts of $\mathrm{Cell}(P)^+_{\leq n}$ with $0 \leq m < \min(k_1,k_2)$ such that $E_1[\widehat{x_1}]$ and $E_2[\widehat{x_2}]$ are m-composable. Then*

$$\langle x_1\rangle_{m,-} \cap \langle x_2\rangle_{m,+} = \emptyset.$$

Moreover, if $\langle x_1\rangle_{m,+} \cap \langle x_2\rangle_{m,-} = \emptyset$*, then there exist* m*-contexts* E'_1, E'_2 *such that*

$$E_1[\widehat{x_1}] *_m E_2[\widehat{x_2}] = E'_2[\widehat{x_2}] *_m E'_1[\widehat{x_1}].$$

An instance of the above lemma with $m = 2$ can be pictured as



We have seen in Lemma 3.4.4 that every n-cell can be expressed as a composition of n-generators in context. The following lemma states that there is such a decomposition for every linearization of the poset of such n-generators under the "dependency order" $\triangleleft$.

Lemma 3.6.13. *Let* $n \in \mathbb{N}$*. Let* $U = \{x_1, \ldots, x_k\} \subseteq P_{n+1}$ *be a set of generators and* $E_1, \ldots, E_k$ *be adapted* n*-contexts such that the cell*

$$X = E_1[\widehat{x_1}] *_n \cdots *_n E_k[\widehat{x_k}]$$

exists in $\mathrm{Cell}(P)^+_{\leq n}$*. Then*

$$x_i \underset{D}{\triangleleft} x_j \qquad \textit{implies} \qquad i < j$$

for all indices i, j *such that* $1 \leq i, j \leq k$*. Moreover, if* σ *is a linear extension of* $(U, \triangleleft_U)$*, then there exist* n*-contexts* $E'_1, \ldots, E'_k$ *such that*

$$X = E'_1[\widehat{\sigma(1)}] *_n \cdots *_n E'_k[\widehat{\sigma(k)}].$$

The following lemma states that, in order for two contexts applied to a generator to evaluate to the same cell, it is enough for them to have same source or target.

Lemma 3.6.14. *Let* $n \in \mathbb{N}$*. Let* $x \in P_{n+1}$ *and* E_1, E_2 *be adapted* m*-contexts with* $m \leq n$ *such that* $\partial^-_m E_1[\widehat{x}] = \partial^-_m E_2[\widehat{x}]$ *or* $\partial^+_m E_1[\widehat{x}] = \partial^+_m E_2[\widehat{x}]$*. Then*

$$E_1[\widehat{x}] = E_2[\widehat{x}].$$

The following lemma generalizes Lemma 3.6.14 to other cells.

Lemma 3.6.15. *Let* $n \in \mathbb{N}$*. Let* $X, Y \in \mathrm{Cell}(P)^+_{\leq n}$ *and* $\epsilon \in \{-,+\}$ *be such that* $(\mathrm{eval}(X))_{n+1} = (\mathrm{eval}(Y))_{n+1}$ *and* $\partial^\epsilon_m X = \partial^\epsilon_m Y$*. Then,* $X = Y$*.*

Finally, we can conclude that $\mathrm{Cell}(P)_{\leq n+1}$ is a free extension of $\mathrm{Cell}(P)^{(n)}$ by P_{n+1}.

Lemma 3.6.16. $\mathrm{Cell}(P)^+_{\leq n}$ *is isomorphic to* $\mathrm{Cell}(P)_{\leq n+1}$*.*

Proof. We will prove the lemmas above using an induction on n. For a fixed n we will prove Lemma 3.6.11 and Lemma 3.6.12 together by induction on m.

Proof of Lemma 3.6.11. If $m = 0$, the property is trivial. So suppose $m > 0$. By Lemma 3.4.4, $\pi_m^- E$ can be written

$$\pi_m^- E = E_1[\widehat{y_1}] *_{m-1} \cdots *_{m-1} E_p[\widehat{y_p}]$$

with $p \in \mathbb{N}$, $y_1, \ldots, y_p \in P_m$ and $E_1, \ldots, E_p$ $(m-1)$-contexts. Let Y be $\{y_1, \ldots, y_p\}$. Since U' is initial for $\lhd_S$, $U' \cap Y$ is initial for $\lhd_Y$, so there exists a linear extension of $(Y, \lhd_Y)$

$$\sigma \colon \mathbb{N}_p \to Y$$

such that $\{i \in \mathbb{N}_p \mid \sigma(i) \in U'\} = \{1, \ldots, i_0\}$ for some $i_0 \in \{0\} \cup \mathbb{N}_p$. Since we can use Lemma 3.6.12 to permute the y_i's in $\pi_m^- E$ according to σ, we can suppose that $\pi_m^- E$ is such that

$$\{i \in \mathbb{N}_p \mid y_i \in U'\} = \{1, \ldots, i_0\}.$$

Remember that

$$E[\widehat{x}] = \pi_m^- E *_{m-1} \tilde{E}[\widehat{x}] *_{m-1} \pi_m^+ E$$

If $i_0 < p$, we want to swap all the $E_i[y_i]$ for $i > i_0$ with $\tilde{E}[\widehat{x}]$ using Lemma 3.6.12. We just need to show how to do it for $i = p$ when $i_0 < p$, and then iterate this procedure for $i_0 < i < p$. So, suppose that $i_0 < p$. Let T be $E_p[\widehat{y_p}] *_{m-1} \tilde{E}[\widehat{x}]$. Since $y_p \notin U'$, we have $y_p \notin U$. In particular, $\langle y_p \rangle_{m-1,+} \cap \langle x \rangle_{m-1,-} = \emptyset$. So, using Lemma 3.6.12, we get adapted $(m-1)$-contexts F and F' such that

$$T = F[\widehat{x}] *_{m-1} F'[\widehat{y_p}].$$

Thus, we obtain an m-context E' adapted to $\widehat{x}$ with $E'[\widehat{x}] = E[\widehat{x}]$ and

$$\begin{aligned} \pi_m^- E' &= E_1[\widehat{y_1}] *_{m-1} \cdots *_{m-1} E_{p-1}[\widehat{y_{p-1}}] \\ \tilde{E}' &= F[\widehat{x}] \\ \pi_m^+ E' &= F'[\widehat{y_p}] *_{m-1} \pi_m^+ E \end{aligned}$$

After iterating this procedure for all $i > i_0$, we get an adapted m-context E' for $\widehat{x}$ such that $E'[\widehat{x}] = E[\widehat{x}]$ and $(\pi_m^- E')_m = (\pi_m^- E)_m \cap U'$.

Using a similar procedure to transfer elements from $\pi_m^+ E'$ to $\pi_m^- E'$, we get an adapted m-context E'' for $\widehat{x}$ such that $E''[\widehat{x}] = E[\widehat{x}]$ and $(\pi_m^+ E'')_m = (\pi_m^+ E') \cap V'$. By Lemma 3.6.7, $(\pi_m^- E')_m \cup (\pi_m^+ E')_m$ and $(\pi_m^- E'')_m \cup (\pi_m^+ E'')_m$ are partitions of S, as $U' \cup V'$ (by hypothesis), thus

$$\begin{aligned} (\pi_m^- E'')_m &= S \setminus (\pi_m^+ E'')_m \\ &= S \setminus ((\pi_m^+ E')_m \cap V') \\ &= (\pi_m^- E')_m \cup U' \\ &= ((\pi_m^- E)_m \cap U') \cup U' \\ &= U'. \end{aligned}$$

By partition, we have $(\pi_m^+ E'')_m = V'$. Hence, E'' satisfies the wanted properties.

Proof of Lemma 3.6.12. We have

$$\operatorname{eval}(E_1[\widehat{z_1}] *_m E_2[\widehat{z_2}]) = \operatorname{eval}(E_1[\widehat{z_1}]) *_m \operatorname{eval}(E_2[\widehat{z_2}]) = E_1[\langle z_1 \rangle] *_m E_2[\langle z_2 \rangle].$$

By Lemma 2.3.2, $(E_1[\langle z_1 \rangle]_{m+1,-})^- \cap (E_2[\langle z_2 \rangle]_{m+1,+})^+ = \emptyset$. But

$$E_1[\langle z_1 \rangle]_{m+1,-} = \langle z_1 \rangle_{m+1,-} \qquad \text{and} \qquad E_2[\langle z_2 \rangle]_{m+1,+} = \langle z_2 \rangle_{m+1,+}.$$

Therefore,

$$\langle z_1\rangle_{m,-} \cap \langle z_2\rangle_{m,+} \subseteq \langle z_1\rangle^-_{m+1,-} \cap \langle z_2\rangle^+_{m+1,+} = \emptyset.$$

For the second part, suppose that $\langle z_1\rangle_{m,+} \cap \langle z_2\rangle_{m,-} = \emptyset$. If $m = 0$, then since $\widehat{z_1} *_0 \widehat{z_2}$ exists, we have $\langle z_1\rangle_{0,+} = \langle z_2\rangle_{0,-}$ and they are non-empty by Axiom (G2), which contradicts $\langle z_1\rangle_{0,+} \cap \langle z_2\rangle_{0,-} = \emptyset$. Hence, $m > 0$. Consider the following:

$$\begin{aligned}
M &= \partial^+_m E_1[\widehat{z_1}] && \text{(or equivalently } \partial^-_m E_2[\widehat{z_2}]),\\
S_i &= (\pi^-_m E_i)_m \cup (\pi^+_m E_i)_m && \text{for } i \in \{1,2\},\\
S' &= M_m,\\
U_i &= \{x \in S_i \mid x \underset{S'}{\lhd} \langle z_i\rangle_{m,+}\} && \text{for } i \in \{1,2\},\\
V_i &= \{x \in S_i \mid \langle z_i\rangle_{m,+} \underset{S'}{\lhd} x\} && \text{for } i \in \{1,2\}.
\end{aligned}$$

We have $\langle z_1\rangle_{m,+} \subseteq S'$ and $\langle z_2\rangle_{m,-} \subseteq S'$. By (G4), we do not have both $\langle z_1\rangle_{m,+} \lhd_{S'} \langle z_2\rangle_{m,-}$ and $\langle z_2\rangle_{m,-} \lhd_{S'} \langle z_1\rangle_{m,+}$. By symmetry, we can suppose that $\neg(\langle z_2\rangle_{m,-} \lhd_{S'} \langle z_1\rangle_{m,+})$. Since we can use Lemma 3.6.11 (which is proved for the current values of m and n) to change E_1 and E_2, we can suppose that

$$\begin{aligned}
(\pi^-_m E_1)_m &= U_1, & (\pi^+_m E_1)_m &= S_1 \setminus U_1,\\
(\pi^-_m E_2)_m &= S_2 \setminus V_2, & (\pi^+_m E_2)_m &= V_2.
\end{aligned}$$

Then

$$(U_1 \cup \langle z_1\rangle_{m,+}) \cap (\langle z_2\rangle_{m,-} \cup V_2) = \emptyset$$

since, otherwise, it would contradict the condition $\neg(\langle z_2\rangle_{m,-} \lhd_{S'} \langle z_1\rangle_{m,+})$. Consider the following sets:

$$\begin{aligned}
Q_1 &= U_1, & Q_2 &= \langle z_1\rangle_{m,+},\\
Q_3 &= S' \setminus (U_1 \cup \langle z_1\rangle_{m,+} \cup \langle z_2\rangle_{m,-} \cup V_2),\\
Q_4 &= \langle z_2\rangle_{m,-}, & Q_5 &= V_2.
\end{aligned}$$

Then Q_1, Q_2, Q_3, Q_4, Q_5 form a partition of S'. Let $R_1, \dots, R_5$ be such that $R_5 = Q_5$ and $R_i = R_{i+1} \cup Q_i$. Note that

- R_5 is final for $\lhd_{S'}$, by segment Axiom (G3) used for z_2 and M,
- R_4 is final for $\lhd_{S'}$, by definition of V_2,
- R_3 is final for $\lhd_{S'}$, because $S' \setminus R_3 = Q_1 \cup Q_2$ is initial for $\lhd_{S'}$,
- R_2 is final for $\lhd_{S'}$, by segment Axiom (G3) used for z_1 and M.

This implies that there exists a linear extension for $(S', \lhd_{S'})$

$$\sigma \colon \mathbb{N}_{S'} \to S'$$

such that for $i, j \in \mathbb{N}_{S'}$ and $1 \le k, l \le 5$, if $\sigma(i) \in Q_k$ and $\sigma(j) \in Q_l$ with $k < l$, then $i < j$. Since $S' = M_m$, using Lemma 3.6.12 inductively, M can be written

$$M = \prod_{i=1}^{|S'|} F_i[\widehat{\sigma(i)}]$$

with $F_1, \ldots, F_{|S'|}$ adapted $(m-1)$-contexts. By regrouping the terms corresponding to $Q_1, \ldots, Q_5$ respectively, we obtain five m-cells $M^1, M^2, M^3, M^4, M^5 \in \mathrm{Cell}(P)_m$ where

$$M^j = \prod_{i \in \sigma^{-1}(Q_j)} F_i[\widehat{\sigma(i)}]$$

and such that

$$M = M^1 *_{m-1} M^2 *_{m-1} M^3 *_{m-1} M^4 *_{m-1} M^5.$$

Since

$$\partial^-_{m-1}(\pi^-_m E_1) = \partial^-_{m-1} M = \partial^-_{m-1} M^1$$

and

$$(\pi^-_m E_1)_m = U_1 = M^1_m,$$

by Lemma 3.6.6, it implies that $\pi^-_m E_1 = M^1$. Moreover, since

$$\partial^-_{m-1} \tilde{E}_1[\partial^+_m \hat{z}_1] = \partial^+_{m-1}(\pi^-_m E_1) = \partial^+_{m-1} M^1 = \partial^-_{m-1} M^2$$

and

$$(\tilde{E}_1[\partial^+_m \hat{z}_1])_m = \langle z_1 \rangle_{m,+} = M^2_m,$$

by Lemma 3.6.6, it implies that $\tilde{E}_1[\partial^+_m \hat{z}_1] = M^2$. Similarly, we can show that

$$\tilde{E}_2[\partial^-_m \hat{z}_2] = M^4 \quad \text{and} \quad \pi^+_m E_2 = M^5.$$

Moreover, since

$$\begin{aligned}(\pi^-_m E_2)_m &= S_2 \setminus V_2 \\ &= S' \setminus (\langle z_2 \rangle_{m,-} \cup V_2) \\ &= Q_1 \cup Q_2 \cup Q_3\end{aligned}$$

and

$$\partial^-_{m-1}(\pi^-_m E_2) = \partial^-_{m-1} M = \partial^-_{m-1}(M^1 *_{m-1} M^2 *_{m-1} M^3),$$

by Lemma 3.6.6, we have that

$$\pi^-_m E_2 = M^1 *_{m-1} M^2 *_{m-1} M^3.$$

Similarly, we can show that

$$\pi^+_m E_1 = M^3 *_{m-1} M^4 *_{m-1} M^5.$$

Hence,

$$\begin{aligned} E_1[\hat{z}_1] *_m E_2[\hat{z}_2] = &\ (M^1 *_{m-1} \tilde{E}_1[\hat{z}_1] *_{m-1} M^3 *_{m-1} \partial^-_m \tilde{E}_2[\hat{z}_2] *_{m-1} M^5) \\ &*_m (M^1 *_{m-1} \partial^+_m \tilde{E}_1[\hat{z}_1] *_{m-1} M^3 *_{m-1} \tilde{E}_2[\hat{z}_2] *_{m-1} M^5) \\ = &\ M^1 *_{m-1} \tilde{E}_1[\hat{z}_1] *_{m-1} M^3 *_{m-1} \tilde{E}_2[\hat{z}_2] *_{m-1} M^5 \\ = &\ (M^1 *_{m-1} \partial^-_m(\tilde{E}_1[\hat{z}_1]) *_{m-1} M^3 *_{m-1} \tilde{E}_2[\hat{z}_2] *_{m-1} M^5) \\ &*_m (M^1 *_{m-1} \tilde{E}_1[\hat{z}_1] *_{m-1} M^3 *_{m-1} \partial^+_m(\tilde{E}_2[\hat{z}_2]) *_{m-1} M^5) \end{aligned}$$

which is of the form $E'_2[\hat{z}_2] *_m E'_1[\hat{z}_1]$ as wanted.

Proof of Lemma 3.6.13. In order to show that $z_i \lhd_U z_j$ implies $i < j$, we only need to prove that $z_i \lhd_U^1 z_j$ implies $i < j$, since $\lhd_U$ is the transitive closure of $\lhd_U^1$. So suppose given $i, j \in \mathbb{N}_p$ such that $z_i \lhd_U^1 z_j$, that is, $z_i^+ \cap z_j^- \neq \emptyset$. By (G1), $i \neq j$. Consider

$$Y = E_1[\widehat{z_1}] *_n \cdots *_n E_{i-1}[\widehat{z_{i-1}}] \quad \text{and} \quad Z = E_i[\widehat{z_i}] *_n \cdots *_n E_p[\widehat{z_p}].$$

Then, $Y' := \operatorname{eval}(Y)$ and $Z' := \operatorname{eval}(Z)$ are n-composable $(n{+}1)$-cells. By Lemma 2.3.1(a), $(Y'_{n+1})^- \cap (Z'_{n+1})^+ = \emptyset$. But, by Lemma 3.6.4, $z_i \in Z'_{n+1}$ and $Y'_{n+1} = \{z_1, \ldots, z_{i-1}\}$. Hence, since $z_i^+ \cap z_j^- \neq \emptyset$, we have $i < j$.

For the second part, note first that the first part implies that $\tau \colon \mathbb{N}_p \to U$ defined by $\tau(i) = z_i$ is a linear extension of $(U, \lhd_U)$. Let $f = \sigma^{-1} \circ \tau$ be a morphism of linear extensions between σ and τ. By Lemma 3.6.9, we can suppose that $f = (i\ i{+}1)$ for some $i \in \mathbb{N}_{p-1}$. To conclude, we just need to show that $\widehat{x_i}$ and $\widehat{x_{i+1}}$ can be swapped in $X = E_1[\widehat{x_1}] *_n \cdots *_n E_p[\widehat{x_p}]$. By contradiction, suppose that $\langle x_i \rangle_{n,+} \cap \langle x_{i+1} \rangle_{n,-} \neq \emptyset$. Then, $\tau(i) \lhd_U \tau(i+1)$. But $\tau = \sigma \circ (i\ i{+}1)$, so $\sigma(i+1) \lhd_U \sigma(i)$ and, since σ is a linear extension, $i + 1 < i$ which is a contradiction. So $\langle x_i \rangle_{n,+} \cap \langle x_{i+1} \rangle_{n,-} = \emptyset$. By Lemma 3.6.12 (which is proved for the current value of n), there exist adapted n-contexts E'_i and E'_{i+1} such that

$$X = E_1[\widehat{x_1}] *_n \cdots *_n E_{i-1}[\widehat{x_{i-1}}] *_n E'_i[\widehat{x_{i+1}}] *_n E'_{i+1}[\widehat{x_i}] *_n E_{i+2}[\widehat{x_{i+2}}] *_n \cdots *_n E_p[\widehat{x_p}]$$

which concludes the proof.

Proof of Lemma 3.6.14. By symmetry, we will only handle the case when $\partial_m^- E_1[\widehat{z}] = \partial_m^- E_2[\widehat{z}]$. We prove this property by an induction on m. If $m = 0$, the result is trivial. So suppose $m > 0$. Consider the following subsets of P_m:

$$\begin{aligned} S &= (\pi^- E_1)_m \cup (\pi^+ E_1)_m, \\ S' &= S \cup \langle z \rangle_{m,-}, \\ U &= \{x \in S \mid x \underset{S'}{\lhd} \langle z \rangle_{m,-}\}, \\ V &= S \setminus U. \end{aligned}$$

By Lemma 3.6.7, $S = (\pi^- E_2)_m \cup (\pi^+ E_2)_m$. By Lemma 3.6.11, there are m-contexts F_1, F_2 such that

$$F_i[\widehat{z}] = E_i[\widehat{z}] \quad \text{and} \quad (\pi_m^- F_i)_m = U \quad \text{and} \quad (\pi_m^+ F_i)_m = V.$$

For $i \in \{1, 2\}$, we have that $\partial_{m-1}^-(\pi_m^- F_i) = \partial_{m-1}^- E_i[\widehat{z}] = \partial^- \partial_m^- E_i[\widehat{z}]$, so

$$\partial_{m-1}^-(\pi_m^- F_1) = \partial_{m-1}^-(\pi_m^- F_2).$$

Since $\pi_m^- F_1, \pi_m^- F_2 \in \operatorname{Cell}(P)_m$, by Lemma 3.6.6, we have

$$\pi_m^- F_1 = \pi_m^- F_2.$$

Moreover, for $i \in \{1, 2\}$, $\partial_{m-1}^+(\pi_m^- F_i) = \partial_{m-1}^-(\tilde{F}_i[\widehat{z}])$, so

$$\partial_{m-1}^-(\tilde{F}_1[\widehat{z}]) = \partial_{m-1}^-(\tilde{F}_2[\widehat{z}]).$$

By induction on m, we have that

$$\tilde{F}_1[\widehat{z}] = \tilde{F}_2[\widehat{z}].$$

But $\partial^+_{m-1}(\tilde{F}_i[\widehat{z}]) = \partial^-_{m-1}(\pi^+_m F_i)$, so

$$\partial^-_{m-1}(\pi^+_m F_1) = \partial^-_{m-1}(\pi^+_m F_2).$$

Since $\pi^+_m F_1, \pi^+_m F_2 \in \mathrm{Cell}(P)_m$, by Lemma 3.6.6, we have

$$\pi^+_m F_1 = \pi^+_m F_2.$$

Hence,

$$F_1[\widehat{z}] = F_2[\widehat{z}]$$

which concludes the proof.

Proof of Lemma 3.6.15. By symmetry, we only give a proof for $\epsilon = -$. We do an induction on $N = |\mathrm{eval}(X)_{n+1}|$. If $N = 0$, then by Lemma 3.4.4 and Lemma 3.6.4, there exist n-cells X', Y' such that

$$X = \mathrm{id}_{n+1}(X') \quad \text{and} \quad Y = \mathrm{id}_{n+1}(Y')$$

so, since $\partial^-_n X = \partial^-_n Y$, it holds that $X' = Y'$. Hence, $X = Y$.

Otherwise, if $N = 1$, then we can conclude by Lemma 3.6.14 and Lemma 3.6.4. Otherwise, $N > 1$. Then, by Lemma 3.4.4 and Lemma 3.6.3, X, Y can be written

$$X = E_1[\widehat{x_1}] *_n \cdots *_n E_p[\widehat{x_p}] \quad \text{and} \quad Y = F_1[\widehat{y_1}] *_n \cdots *_n F_q[\widehat{y_q}]$$

with $p, q \in \mathbb{N}$, $x_i, y_j \in P_{n+1}$ and E_i, F_j n-contexts. By Lemma 3.6.4,

$$\{x_1, \ldots, x_p\} = \mathrm{eval}(X)_{n+1} = \mathrm{eval}(Y)_{n+1} = \{y_1, \ldots, y_q\}$$

and the x_i's are all different, and the y_j's too. So $p = q = N$. Using Lemma 3.6.12 to reorder the y_i's, we can suppose that $x_i = y_i$ for $1 \le i \le N$. Since $\partial^-_n X = \partial^-_n Y$, it holds that $\partial^-_n E_1[\widehat{x_1}] = \partial^-_n F_1[\widehat{y_1}]$. So, by Lemma 3.6.14,

$$E_1[\widehat{x_1}] = F_1[\widehat{y_1}].$$

Moreover,

$$\partial^-_n(E_2[\widehat{x_2}] *_n \cdots *_n E_p[\widehat{x_p}]) = \partial^+_n E_1[\widehat{x_1}] = \partial^+_n F_1[\widehat{y_1}] = \partial^-_n(F_2[\widehat{y_2}] *_n \cdots *_n F_p[\widehat{y_p}]).$$

So, by induction on N,

$$E_2[\widehat{x_2}] *_n \cdots *_n E_p[\widehat{x_p}] = F_2[\widehat{y_2}] *_n \cdots *_n F_p[\widehat{y_p}].$$

Hence, $X = Y$.

Proof of Lemma 3.6.16. Given $X \in \mathrm{Cell}(P)_{n+1}$, if X can be written

$$X = \mathrm{id}_{n+1}(X')$$

with $X' \in \mathrm{Cell}(P)_n$, then $X = \mathrm{eval}(\mathrm{id}_{n+1}(X'))$. Otherwise, if X can be written

$$X = E_1[\langle x_1 \rangle] *_n \cdots *_n E_p[\langle x_p \rangle]$$

then $X = \mathrm{eval}(E_1[\widehat{x_1}] *_n \cdots E_p[\widehat{x_p}])$. So, by Lemma 3.4.4,

$$\mathrm{eval}\colon (\mathrm{Cell}(P)^+_{\le n})_{n+1} \to \mathrm{Cell}(P)_{n+1}$$

is a surjective function. Let

$$\text{reval}\colon \text{Cell}(P)_{n+1} \to (\text{Cell}(P)^+_{\le n})_{n+1}$$

be a section of eval. For $X \in \text{Cell}(P)_{n+1}$ and $\epsilon \in \{-,+\}$, we have $\partial^\epsilon_n(\text{reval}(X)) = \partial^\epsilon_n(\text{eval} \circ \text{reval}(X)) = \partial^\epsilon_n X$. So reval can be naturally extended to a morphism of cellular extensions

$$\text{reval}\colon (\text{Cell}(P)_{\le n}, \text{Cell}(P)_{n+1}) \to (\text{Cell}(P)_{\le n}, (\text{Cell}(P)^+_{\le n})_{n+1}).$$

Moreover, if $X = \text{id}_{n+1}(X')$ for some $X' \in \text{Cell}(P)_n$, we have $\text{eval}(\text{reval}(\text{id}_{n+1}(X'))) = \text{eval}(\text{id}_{n+1}(X')) = \text{id}_{n+1}(X')$. So, by Lemma 3.6.15,

$$\text{reval}(\text{id}_{n+1}(X')) = \text{id}_{n+1}(X').$$

Otherwise, if $X = Y *_i Z$ for some $i \le n$ and $Y, Z \in \text{Cell}(P)_{n+1}$, we have $\text{eval}(\text{reval}(X *_i Y)) = \text{eval}(\text{reval}(X) *_i \text{reval}(Y)) = X$. So, by Lemma 3.6.15,

$$\text{reval}(X *_i Y) = \text{reval}(X) *_i \text{reval}(Y).$$

Thus, reval is an $(n+1)$-functor and is an inverse to eval. Hence, $\text{Cell}(P)^+_{\le n}$ is isomorphic to $\text{Cell}(P)_{\le n+1}$. □

3.6.17 Freeness property. We are able to conclude the proof of freeness for the ω-category of cells $\text{Cell}(P)$.

Theorem 3.6.18. $\text{Cell}(P)$ *is a free ω-category generated by the atoms $\widehat{x}$ for $x \in P$.*

Proof. By Lemma 3.6.16, for $n \in \mathbb{N}$, $\text{Cell}(P)_{\le n+1}$ is a free extension of $\text{Cell}(P)_{\le n}$ by P_{n+1}. So $\text{Cell}(P)$ is a free ω-category. The generating property is given by Theorem 3.2.1. □

4 Alternative notions of cells

Before being able to relate generalized parity complexes to the other formalisms, we first give alternative notions of cells to the one of Paragraph 1.3.3 that are still suited for describing the ω-category of pasting diagrams.

In this section, we suppose given an ω-hypergraph P.

4.1 Closed and maximal pre-cells

We write $\text{Closed}(P)$ for the graded set of closed fgs of P. Given an n-fgs X of P, $x \in X$ is said to be *maximal in* X when for all $y \in P$ such that $x \mathrel{\mathrm{R}} y$ and $x \neq y$, it holds that $y \notin X$. We write $\max(X)$ for the n-fgs of P made of the maximal elements of X. The n-fgs X is then said to be *maximal* when $\max(X) = X$. We write $\text{Max}(P)$ for the gradet set of maximal fgs. Given $n \in \mathbb{N}$ and X an n-pre-cell of P, we write $\cup X$ for the n-fgs of P given by $\cup_{0 \le i \le n}(X_{i,-} \cup X_{i,+})$.

4.2 Maximality lemma

Here, we show that there is a simple criterion to know whether an element is maximal in a cell of Cell(P).

In this subsection, we suppose that P satisfies (G0), (G1), (G2) and (G3).

Lemma 4.2.1 (Maximality lemma)**.** *Let $m < n \in \mathbb{N}$ and X be an n-cell of P. For $x \in X_{m,-}$ (resp. $x \in X_{m,+}$) with x not maximal in $\cup X$, we have $x \in X^{\mp}_{m+1,-}$ (resp. $x \in X^{\pm}_{m+1,+}$).*

Proof. We prove this property by induction on $p := n - m$. By symmetry, we only prove the case where $x \in X_{m,-}$. Since x is not maximal, by definition of R, there exist $p > 0$, $\eta \in \{-,+\}$, $x_0, x_1, \ldots, x_p \in P$ and $\epsilon_1, \ldots, \epsilon_p \in \{-,+\}$ such that $x_0 = x$, $x_p \in X_{m+p,\eta}$ and $x_i \in x_{i+1}^{\epsilon_{i+1}}$ for $i < p$.

Suppose that $p = 1$. By Lemma 2.1.1, $X_{m,-} \cap X^{+}_{m+1,\eta} = \emptyset$. Since $x \in x_1^{\epsilon}$ and $x_1 \in X_{m+1,\eta}$, we have $\epsilon_1 = -$ and $x \in X^{\mp}_{m+1,\eta}$. By Lemma 2.1.5, $x \in X^{\mp}_{m+1,-}$. Otherwise, suppose that $p > 1$. Let $y \in X_{m+p,\eta}$ be the smallest element of $X_{m+p,\eta}$ for $\lhd_{X_{m+p,\eta}}$ such that $y \,\mathrm{R}\, x_{p-1}$. If $x_{p-1} \in y^{-}$, then, by minimality of y, there is no $y' \in X_{m+p,\eta}$ such that $x_{p-1} \in y'^{+}$. Therefore,

$$x_{p-1} \in X^{\mp}_{m+p,\eta} \subseteq X_{m+p-1,-}.$$

Hence, x is not minimal in $\partial^{-}_{m+p-1} X$ and we conclude by induction. Otherwise, $x_{p-1} \in y^{+}$. Consider

$$G = \{z \in X_{m+p,\eta} \mid z \underset{X_{m+p,\eta}}{\lhd} y\} \cup \{y\} \quad \text{and} \quad Y = \mathrm{Act}(\partial^{-}_{m+p-1} X, G).$$

We have $x \in Y_{m,-}$ and $x_{p-1} \in Y_{m+p-1} \subseteq \cup Y$ and, by Theorem 2.2.3, Y is a cell. By induction, $x \in Y^{\mp}_{m+1,-}$. Since $X_{m+1,-}$ and $Y_{m+1,-}$ both move $X_{m,-}$ to $X_{m,+}$, by Lemma 2.1.5, $x \in X^{\mp}_{m+1,-}$ which concludes the proof. □

This criterion gives a simple description of the set of maximal elements of a cell of Cell(P).

Lemma 4.2.2. *Let $m < n \in \mathbb{N}$, $\epsilon \in \{-,+\}$ and X be an n-cell of P. Then,*

$$\max(\cup X) \cap P_m = X_{m,-} \cap X_{m,+}.$$

Proof. By Lemma 4.2.1,

$$\max(\cup X) \cap P_m = (X_{m,-} \setminus X^{\mp}_{m+1,-}) \cup (X_{m,+} \setminus X^{\pm}_{m+1,+}).$$

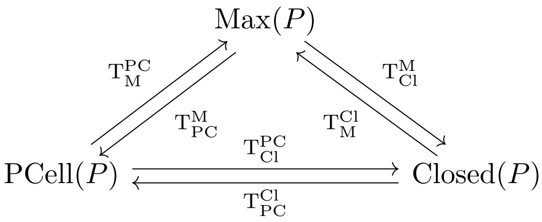
By Lemma 2.1.6, it can be simplified to

$$\max(\cup X) \cap P_m = X_{m,-} \cap X^{\mp}_{m+1,-}.$$ □

4.3 Relating representations of pre-cells

In this subsection, we relate Cell(P), Max(P) and Closed(P) by giving translations functions and properties on these translations.

4.3.1 The translation functions. We define several functions between PCell(P), Max(P) and Closed(P), as represented by



where

– $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}\colon \mathrm{PCell}(P) \to \mathrm{Max}(P)$ with, for X an n-pre-cell of P,

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) = \max(\cup X),$$

– $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}\colon \mathrm{Max}(P) \to \mathrm{PCell}(P)$ with, for X an n-fgs of P, $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(X)$ is the n-pre-cell Y of P such that

$$\begin{aligned} Y_n &= X_n, & \\ Y_{i,-} &= X_i \cup Y^{\mp}_{i+1,-} & \text{for } i < n, \\ Y_{i,+} &= X_i \cup Y^{\pm}_{i+1,+} & \text{for } i < n, \end{aligned}$$

– $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}\colon \mathrm{Max}(P) \to \mathrm{Closed}(P)$ with, for X a maximal n-fgs of P,

$$\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X) = \mathrm{R}(X),$$

– $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}\colon \mathrm{Closed}(P) \to \mathrm{Max}(P)$ with, for X a closed n-fgs,

$$\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) = \max(X),$$

– $\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}\colon \mathrm{PCell}(P) \to \mathrm{Closed}(P)$ with, for X an n-pre-cell of P,

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) = \mathrm{R}(\cup X),$$

– $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{PC}}\colon \mathrm{Closed}(P) \to \mathrm{PCell}(P)$ defined by

$$\mathrm{T}^{\mathrm{Cl}}_{\mathrm{PC}} = \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}.$$

These operations can be related to each other, as state the following lemmas.

Lemma 4.3.2. $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} = \mathrm{id}_{\mathrm{Closed}(P)}$ *and* $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} = \mathrm{id}_{\mathrm{Max}(P)}$.

Proof. Let X be a closed n-fgs of P and $x \in X$. We have $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) \subseteq X$ so

$$\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) \subseteq X.$$

Moreover, for $x \in X$, since X is finite, there is $y \in \max(X)$ with $y \,\mathrm{R}\, x$. It implies that $y \in \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X)$ and $x \in \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X)$. Therefore,

$$X \subseteq \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X),$$

which shows that

$$\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} = \mathrm{id}_{\mathrm{Closed}(P)}.$$

For the other equality, note that, for all n-fgs X of P, $\mathrm{R}(X)$ has the same maximal elements as X. It implies that

$$\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} = \mathrm{id}_{\mathrm{Max}(P)}. \qquad \square$$

Lemma 4.3.3. *Suppose that P satisfies axioms (G0), (G1), (G2) and (G3). Let $n \in \mathbb{N}$, $X \in \mathrm{Cell}(P)_n$ and $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)$. Then,*

$$Y_n = X_n \quad \textit{and} \quad Y_i = X_{i,-} \cap X_{i,+} \quad \textit{for } i < n.$$

Proof. This is a direct consequence of Lemma 4.2.2. □

Lemma 4.3.4. *Suppose that P satisfies axioms (G0), (G1), (G2) and (G3). Then, for $X \in \mathrm{Cell}(P)$, $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) = X$*

Proof. Let $n \in \mathbb{N}$, $X \in \mathrm{Cell}(P)_n$, $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)$ and $Z = \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(Y)$. We show that $X_{i,\epsilon} = Z_{i,\epsilon}$ by a decreasing induction on i. By Lemma 4.3.3, we have

$$Z_n = Y_n = X_n$$

and, for $i < n$, we have

$$\begin{aligned} Z_{i,-} &= Y_i \cup Z^{\mp}_{i+1,-} \\ &= (X_{i,-} \cap X_{i,+}) \cup X^{\mp}_{i+1,-} \\ &= X_{i,-} && \text{(by Lemma 2.1.7).} \end{aligned}$$

Similarly,

$$Z_{i,+} = X_{i,+},$$

so $X = Z$. Hence, $\mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) = X$. □

Lemma 4.3.5. $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}} = \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$

Proof. Let $n \in \mathbb{N}$ and $X \in \mathrm{Cell}(P)_n$. Then,

$$\begin{aligned} \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) &= \mathrm{R}(\max(\cup X)) \\ &= \mathrm{R}(\cup X) \\ &= \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X). \end{aligned}$$

Hence, $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}} = \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}$. □

4.3.6 Sources and targets. Given $n > 0$ and X a maximal n-fgs, define the *source* $\tilde{\partial}^- X$ (resp. *target* $\tilde{\partial}^+ X$) *of* X as the maximal $(n-1)$-fgs Y with

$$\begin{aligned} Y_{n-1} &= X_{n-1} \cup X^{\mp}_n && \text{(resp. } X_{n-1} \cup X^{\pm}_n\text{),} \\ Y_i &= X_i && \text{for } i < n-1. \end{aligned}$$

Similarly, given $n > 0$ and X a closed n-fgs, define the *source* $\bar{\partial}^- X$ (resp. *target* $\bar{\partial}^+ X$) of X as the closed $(n-1)$-fgs Y with

$$Y = \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^+_n))) \quad \text{(resp. } \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^-_n)))\text{).}$$

These sources and targets are compatible with the translation functions from Paragraph 4.3.1 as state the following lemmas.

Lemma 4.3.7. *Suppose that P satisfies axioms (G0), (G1), (G2) and (G3). For $n > 0$, $\epsilon \in \{-,+\}$ and $X \in \mathrm{Cell}(P)_n$, we have*

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^\epsilon X) = \tilde{\partial}^\epsilon(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)).$$

Proof. Let $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^\epsilon X)$, $X' = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)$ and $Z = \tilde{\partial}^\epsilon X'$. By Lemma 4.3.3,

$$\begin{aligned} Y_{n-1} &= X_{n-1,\epsilon}, \\ Y_i &= X_{i,-} \cap X_{i,+} && \text{for } i < n-1, \end{aligned}$$

and

$$\begin{aligned} X'_n &= X_n, \\ X'_i &= X_{i,-} \cap X_{i,+} && \text{for } i < n. \end{aligned}$$

If $\epsilon = -$, then

$$\begin{aligned} Z_{n-1} &= (X_{n-1,-} \cap X_{n-1,+}) \cup X_n^{\mp} \\ &= X_{n-1,-} && \text{(by Lemma 2.1.7)} \\ Z_i &= X'_i \\ &= X_{i,-} \cap X_{i,+} && \text{for } i < n-1 \end{aligned}$$

so $Y = Z$. Similarly, when $\epsilon = +$, $Y = Z$. Which concludes the proof. □

Lemma 4.3.8. *For $n > 0$, $\epsilon \in \{-,+\}$ and $X \in \mathrm{Max}(P)_n$, we have*

$$\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon X) = \bar{\partial}^\epsilon(\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X)).$$

Proof. By symmetry, we only prove the case $\epsilon = -$. Let $Y = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^- X)$ and $Z = \bar{\partial}^-(\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X))$. By unfolding the definitions, we have

$$\begin{aligned} Y &= \mathrm{R}((X \setminus X_n) \cup X_n^{\mp}), \\ Z &= \mathrm{R}(\mathrm{R}(X) \setminus (X_n \cup \mathrm{R}(X_n^{+}))). \end{aligned}$$

In order to show that $Y \subseteq Z$, we only need to prove that $Y' \subseteq Z$ where

$$Y' := (X \setminus X_n) \cup X_n^{\mp}.$$

First, we have that $Y' \subseteq \mathrm{R}(X)$. Moreover,

$$\begin{aligned} & Y' \cap (X_n \cup \mathrm{R}(X_n^{+})) \\ & = ((X \setminus X_n) \cup X_n^{\mp}) \cap (X_n \cup \mathrm{R}(X_n^{+})) \\ & = ((X \setminus X_n) \cup X_n^{\mp}) \cap \mathrm{R}(X_n^{+}) \\ & = (X \setminus X_n) \cap \mathrm{R}(X_n^{+}) \\ & = X \cap \mathrm{R}(X_n^{+}) \\ & = \emptyset && \text{(since } X \text{ is maximal).} \end{aligned}$$

So $Y' \subseteq Z$, which implies that $Y \subseteq Z$.

Similarly, to show that $Z \subseteq Y$, we only need to prove that $Z' \subseteq Y$ where

$$Z' := \mathrm{R}(X) \setminus (X_n \cup \mathrm{R}(X_n^+)).$$

But

$$Z' \subseteq Y \Leftrightarrow \mathrm{R}(X) \subseteq Y \cup X_n \cup \mathrm{R}(X_n^+)$$

and

$$\begin{aligned} Y \cup X_n \cup \mathrm{R}(X_n^+) &= \mathrm{R}((X \setminus X_n) \cup X_n^{\mp}) \cup X_n \cup \mathrm{R}(X_n^+) \\ &= \mathrm{R}((X \setminus X_n) \cup X_n^{\mp} \cup X_n^+) \cup X_n \\ &= \mathrm{R}((X \setminus X_n) \cup X_n^- \cup X_n^+) \cup X_n \\ &= \mathrm{R}((X \setminus X_n) \cup X_n^- \cup X_n^+ \cup X_n) \\ &= \mathrm{R}(X). \end{aligned}$$

So $Z' \subseteq Y$, which implies that $Z \subseteq Y$. Hence, $Y = Z$, which concludes the proof. $\square$

Lemma 4.3.9. *Suppose that P satisfies axioms (G0), (G1), (G2) and (G3). For $n > 0$, $\epsilon \in \{-,+\}$ and $X \in \mathrm{Cell}(P)_n$,*

$$\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(\partial^\epsilon X) = \bar{\partial}^\epsilon(\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X)).$$

Proof. We have

$$\begin{aligned} \mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(\partial^\epsilon X) &= \mathrm{T}_{\mathrm{Cl}}^{\mathrm{M}} \circ \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial^\epsilon X) && \text{(by Lemma 4.3.5)} \\ &= \mathrm{T}_{\mathrm{Cl}}^{\mathrm{M}}(\tilde{\partial}^\epsilon(\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(X))) && \text{(by Lemma 4.3.7)} \\ &= \bar{\partial}^\epsilon(\mathrm{T}_{\mathrm{Cl}}^{\mathrm{M}} \circ \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(X)) && \text{(by Lemma 4.3.8)} \\ &= \bar{\partial}^\epsilon(\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X)) \end{aligned}$$

which concludes the proof. $\square$

4.4 Compositions and identities

Here, we define compositions and identities for the globular sets $\mathrm{Max}(P)$ and $\mathrm{Closed}(P)$, and prove some compatibility results with the translations functions.

Given $i \leq n \in \mathbb{N}$ and X, Y two maximal n-fgs, we define the *maximal i-composition $X *^{\mathrm{M}} Y$ of X and Y* as a maximal n-fgs defined by

$$X *_i^{\mathrm{M}} Y = \max(\mathrm{R}(X) \cup \mathrm{R}(Y)).$$

Similarly, given $i \leq n \in \mathbb{N}$ and X, Y two closed n-fgs, we define the *closed composition $X *_i^{\mathrm{Cl}} Y$ of X and Y* as a closed n-fgs defined by

$$X *_i^{\mathrm{Cl}} Y = X \cup Y.$$

For simplicity, we sometimes write $*^{\mathrm{Cl}}$ (resp. $*^{\mathrm{M}}$) for $*_i^{\mathrm{Cl}}$ (resp. $*_i^{\mathrm{M}}$). Given $n \in \mathbb{N}$ and a closed (resp. maximal) n-fgs X, we define the *identity* $\mathrm{id}_{n+1}(X)$ *of X* as the closed (resp. maximal) $(n{+}1)$-fgs

$$(X_0, \ldots, X_n, \emptyset).$$

These compositions and identities are compatible with the translation functions from Paragraph 4.3.1 as state the following lemmas.

Lemma 4.4.1. *For $k < n \in \mathbb{N}$ and k-composable n-cells X and Y of P,*

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_k Y) = \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y).$$

Proof. Let $Z = X *_k Y$. We have

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_k Y) = \mathrm{R}(\cup Z)$$

and

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y) = \mathrm{R}(\cup X) \cup \mathrm{R}(\cup Y) = \mathrm{R}((\cup X) \cup (\cup Y)).$$

By definition of composition, $\cup Z \subseteq (\cup X) \cup (\cup Y)$, so

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_k Y) \subseteq \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y).$$

For the other inclusion, note that $X_{i,\epsilon} \subseteq Z_{i,\epsilon}$ for $(i,\epsilon) \neq (k,+)$, and

$$\begin{aligned} X_{k,+} &= (X_{k,-} \cup X^{+}_{k+1,-}) \setminus X^{-}_{k+1,-} \\ &\subseteq Z_{k,-} \cup Z^{+}_{k+1,-} \\ &\subseteq \mathrm{R}(\cup Z) \end{aligned}$$

so $\cup X \subseteq \mathrm{R}(\cup Z)$. Similarly, $\cup Y \subseteq \mathrm{R}(\cup Z)$, thus

$$(\cup X) \cup (\cup Y) \subseteq \mathrm{R}(\cup Z),$$

which implies that

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y) \subseteq \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_k Y).$$

Hence,

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y) = \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_k Y). \qquad \square$$

Lemma 4.4.2. *For $n \in \mathbb{N}$ and an n-cell $X \in \mathrm{Cell}(P)$,*

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(\mathrm{id}_{n+1}(X)) = \mathrm{id}_{n+1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X)).$$

Proof. It readily follows from the definitions. □

Lemma 4.4.3. *For $n \in \mathbb{N}$ and $X, Y \in \mathrm{Closed}(P)_n$,*

$$\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X *^{\mathrm{Cl}} Y) = \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) *^{\mathrm{M}} \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(Y).$$

Proof. We have

$$\begin{aligned} \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X) *^{\mathrm{M}} \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(Y) &= \max(\mathrm{R}(\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X)) \cup \mathrm{R}(\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(Y))) \\ &= \max(X \cup Y) && \text{(by Lemma 4.3.2)} \\ &= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(X *^{\mathrm{Cl}} Y) \end{aligned}$$

which concludes the proof. □

Lemma 4.4.4. *For $k < n \in \mathbb{N}$ and k-composable n-cells X and Y of P,*

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X *_k Y) = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) *^{\mathrm{M}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y).$$

Proof. We have

$$\begin{aligned}
\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X *_k Y) &= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X *_k Y) && \text{(by Lemmas 4.3.2 and 4.3.5)} \\
&= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{Cl}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y)) && \text{(by Lemma 4.4.1)} \\
&= \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(X) *^{\mathrm{M}} \mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{Cl}}(Y) && \text{(by Lemma 4.4.3)} \\
&= \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) *^{\mathrm{M}} \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y) && \text{(by Lemmas 4.3.2 and 4.3.5)}
\end{aligned}$$

which concludes the proof. □

Lemma 4.4.5. *For $n \in \mathbb{N}$ and an n-cell $X \in \mathrm{Cell}(P)$,*

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\mathrm{id}_{n+1}(X)) = \mathrm{id}_{n+1}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)).$$

Proof. It readily follows from the definitions. □

4.5 Alternative cells

In this subsection, we define notions of cells for $\mathrm{Max}(P)$ and $\mathrm{Closed}(P)$ and prove that the associated ω-category of cells are isomorphic to $\mathrm{Cell}(P)$.

Given $n \in \mathbb{N}$ and $X \in \mathrm{Max}(P)_n$, we say that X is *maximal-well-formed* when

- X_n is fork-free,
- $\tilde{\partial}^- X$ and $\tilde{\partial}^+ X$ are maximal-well-formed,
- if $n \geq 2$, $\tilde{\partial}^- \circ \tilde{\partial}^-(X) = \tilde{\partial}^- \circ \tilde{\partial}^+(X)$ and $\tilde{\partial}^+ \circ \tilde{\partial}^-(X) = \tilde{\partial}^+ \circ \tilde{\partial}^+(X)$.

We write $\mathrm{Max}_{\mathrm{WF}}(P)$ for the graded set of maximal-well-formed fgs of P. Similarly, given $n \in \mathbb{N}$ and $X \in \mathrm{Closed}(P)_n$, we say that X is *closed-well-formed* when

- X_n is fork-free,
- $\bar{\partial}^- X$ and $\bar{\partial}^+ X$ are closed-well-formed,
- if $n \geq 2$, $\bar{\partial}^- \circ \bar{\partial}^-(X) = \bar{\partial}^- \circ \bar{\partial}^+(X)$ and $\bar{\partial}^+ \circ \bar{\partial}^-(X) = \bar{\partial}^+ \circ \bar{\partial}^+(X)$.

We write $\mathrm{Closed}_{\mathrm{WF}}(P)$ for the graded set of closed-well-formed fgs of P.

Lemma 4.5.1. $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}$ *induces a bijection between* $\mathrm{Max}_{WF}(P)$ *and* $\mathrm{Closed}_{WF}(P)$.

Proof. We already know that $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}$ is a bijection by Lemma 4.3.2. For $n \in \mathbb{N}$, we show that $\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}$ sends a maximal-well-formed n-fgs X to a closed-well-formed fgs by induction on n. If $n = 0$, the result is trivial. So suppose $n > 0$. Let $Y = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(X)$. Then, $Y_n = X_n$ is fork-free and, for $\epsilon \in \{-,+\}$, $\bar{\partial}^\epsilon(Y) = \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon(X))$ by Lemma 4.3.8, and it is closed-well-formed by induction. Also, when $n \geq 2$,

$$\begin{aligned}
\bar{\partial}^\epsilon \circ \bar{\partial}^-(Y) &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon \circ \tilde{\partial}^-(X)) && \text{(by Lemma 4.3.8)} \\
&= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}^\epsilon \circ \tilde{\partial}^+(X)) \\
&= \bar{\partial}^\epsilon \circ \bar{\partial}^+(Y)
\end{aligned}$$

So Y is closed-well-formed. Similarly, $\mathrm{T}^{\mathrm{Cl}}_{\mathrm{M}}$ sends closed-well-formed fgs to maximal-well-formed fgs, which concludes the proof. □

Lemma 4.5.2. *Suppose that P satisfies (G0), (G1), (G2) and (G3). For $n \in \mathbb{N}$ and $X \in \mathrm{Cell}(P)_n$, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X) \in \mathrm{Max}_{WF}(P)_n$.*

Proof. We proceed by induction on n. If $n = 0$, the result is trivial. So suppose that $n > 0$ and let $Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(X)$. Since $Y_n = X_n$, Y_n is fork-free. Moreover, by Lemma 4.3.7, $\tilde{\partial}^{\epsilon} Y = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^{\epsilon} X)$ for $\epsilon \in \{-,+\}$. By the induction hypothesis, $\tilde{\partial}^{\epsilon} Y$ is maximal-well-formed. And, when $n \geq 2$, for $\eta \in \{-,+\}$,

$$\begin{aligned}
\tilde{\partial}^{\eta} \circ \tilde{\partial}^{-}(Y) &= \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^{\eta} \circ \partial^{-}(X)) && \text{(by Lemma 4.3.7)} \\
&= \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^{\eta} \circ \partial^{+}(X)) \\
&= \tilde{\partial}^{\eta} \circ \tilde{\partial}^{+}(Y).
\end{aligned}$$

Hence, Y is maximal-well-formed. □

Lemma 4.5.3. *Suppose that P satisfies (G0), (G1), (G2) and (G3). For $n \in \mathbb{N}$ and $X \in \mathrm{Max}_{WF}(P)_n$, there exists an n-cell Y such that $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y) = X$.*

Proof. We proceed by induction on n. If $n = 0$, the result is trivial. So suppose that $n > 0$. By induction, let $S, T \in \mathrm{Cell}(P)_{n-1}$ be such that $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(S) = \tilde{\partial}^{-} X$ and $\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(T) = \tilde{\partial}^{+} X$. When $n \geq 2$, for $\epsilon \in \{-,+\}$, we have

$$\begin{aligned}
\partial^{\epsilon} S &= \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^{\epsilon} S) && \text{(by Lemma 4.3.4)} \\
&= \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(\tilde{\partial}^{\epsilon}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(S))) && \text{(by Lemma 4.3.7)} \\
&= \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(\tilde{\partial}^{\epsilon} \circ \tilde{\partial}^{-}(X)) \\
&= \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(\tilde{\partial}^{\epsilon} \circ \tilde{\partial}^{+}(X)) && \text{(because } X \text{ is maximal-well-formed)} \\
&= \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}}(\tilde{\partial}^{\epsilon}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(T))) \\
&= \mathrm{T}^{\mathrm{M}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^{\epsilon} T) \\
&= \partial^{\epsilon} T.
\end{aligned}$$

Also,

$$\begin{aligned}
(S_{n-1} \cup X_n^{+}) \setminus X_n^{-} &= (X_{n-1} \cup X_n^{\mp} \cup X_n^{+}) \setminus X_n^{-} \\
&= X_{n-1} \cup X_n^{\pm} \\
&= T_{n-1}.
\end{aligned}$$

Similarly, $(T_{n-1} \cup X_n^{-}) \setminus X_n^{+} = S_{n-1}$ so X_n moves S_{n-1} to T_{n-1}. Hence, the n-pre-cell Y defined below is a cell:

$$\begin{aligned}
Y_n &= X_n \\
Y_{n-1,-} &= S_{n-1} \\
Y_{n-1,+} &= T_{n-1} \\
Y_{i,\delta} &= S_{i,\delta} && \text{for } i < n-1 \text{ and } \delta \in \{-,+\}.
\end{aligned}$$

Let $Z = \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y)$. We have $Z_n = X_n$ and

$$\begin{aligned}
\tilde{\partial}^{-} Z &= \tilde{\partial}^{-}(\mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(Y)) \\
&= \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(\partial^{-} Y) && \text{(by Lemma 4.3.7)} \\
&= \mathrm{T}^{\mathrm{PC}}_{\mathrm{M}}(S) \\
&= \tilde{\partial}^{-} X.
\end{aligned}$$

So, by definition of $\tilde{\partial}^-$, $Z_i = X_i$ for $i < n-1$ and $Z_{n-1} \cup X_n^\mp = X_{n-1} \cup X_n^\mp$. Since X and Z are maximal, we have

$$X_{n-1} \cap X_n^\mp = Z_{n-1} \cap X_n^\mp = \emptyset.$$

Hence, $X_{n-1} = Z_{n-1}$ and $X = Z = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Y)$ which concludes the proof. □

Lemma 4.5.4. *Suppose that P satisfies (G0), (G1), (G2) and (G3). Then, $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}$ induces a bijection between the* $\mathrm{Cell}(P)$ *and* $\mathrm{Max}_{WF}(P)$*.*

Proof. By Lemma 4.5.3, $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}} \colon \mathrm{Cell}(P) \to \mathrm{Max}_{\mathrm{WF}}(P)$ is onto, and by Lemma 4.3.4, it is one-to-one, so it is bijective. □

Theorem 4.5.5. *Suppose that P satisfies (G0), (G1), (G2) and (G3). Then,* $\mathrm{Max}_{WF}(P)$ *is an ω-category and $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}$ induces an isomorphism between* $\mathrm{Cell}(P)$ *and* $\mathrm{Max}_{WF}(P)$*.*

Proof. We first prove that composition is well-defined. Let $i \le n \in \mathbb{N}$ and $X, Y \in \mathrm{Max}_{\mathrm{WF}}(P)_n$ be such that $\tilde{\partial}_i^+ X = \tilde{\partial}_i^- Y$. By Lemma 4.5.4, there exist $X', Y' \in \mathrm{Cell}(P)_n$ such that $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(X') = X$ and $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Y') = Y$. By Lemma 4.3.7, we have

$$\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial_i^+ X') = \tilde{\partial}_i^+ X = \tilde{\partial}_i^- Y = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\partial_i^- Y'),$$

and, by Lemma 4.5.4, $\partial_i^+ X' = \partial_i^- Y'$ so X' and Y' are i-composable. By Lemma 4.5.4, $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(X' *_i Y') \in \mathrm{Max}_{\mathrm{WF}}(P)$ and, by Lemma 4.4.4, $X *^{\mathrm{M}} Y \in \mathrm{Max}_{\mathrm{WF}}(P)$.

Now, we prove that $\mathrm{Max}_{\mathrm{WF}}(P)$ satisfies the axioms (i) to (vi) of ω-categories. But it readily follows from Lemmas 4.5.4, 4.4.4 and 4.4.5. Indeed, for example, for axiom (iii), given $i \le n \in \mathbb{N}$ and i-composable $X, Y, Z \in \mathrm{Max}_{\mathrm{WF}}(P)_n$, by Lemma 4.5.4, there exist $X', Y', Z' \in \mathrm{Cell}(P)_n$ such that $X = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(X')$, $Y = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Y')$ and $Z = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(Z')$. By a similar argument as above, X', Y', Z' are i-composable and, by Lemma 4.4.4,

$$(X *_i^{\mathrm{M}} Y) *_i^{\mathrm{M}} Z = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}((X' *_i Y') *_i Z') = \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(X' *_i (Y' *_i Z')) = X *_i^{\mathrm{M}} (Y *_i^{\mathrm{M}} Z),$$

so (iii) is satisfied.

Hence, $\mathrm{Max}_{\mathrm{WF}}(P)$ is an ω-category, and $\mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}$ is an isomorphism by Lemmas 4.5.4, 4.4.4, 4.4.5. □

Lemma 4.5.6. *Suppose that P satisfies (G0), (G1), (G2) and (G3). Then, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}$ induces a bijection between* $\mathrm{Cell}(P)$ *and* $\mathrm{Closed}_{WF}(P)$*.*

Proof. The result is a consequence of Lemmas 4.3.5 and 4.5.1 and 4.5.4. □

Theorem 4.5.7. *Suppose that P satisfies (G0), (G1), (G2) and (G3). Then,* $\mathrm{Closed}_{WF}(P)$ *is an ω-category and $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}$ induces an isomorphism between* $\mathrm{Cell}(P)$ *and* $\mathrm{Closed}_{WF}(P)$*.*

Proof. By a similar proof than for Theorem 4.5.5, using Lemmas 4.3.9, 4.4.1, 4.4.2 and 4.5.6. □

5 Unifying formalisms of pasting diagrams

In this section, we show that parity complexes, pasting schemes and augmented directed complexes are generalized parity complexes.

5.1 Encoding parity complexes

In this subsection, we show that parity complexes are generalized parity complexes, after applying two reasonable restrictions. Firstly, parity complexes do not require all the generators to be relevant even though generalized parity complexes do. But, by [19, Theorem 4.2], irrelevant generators do not play any role in the generated ω-category $\mathrm{Cell}(P)$ for P an ω-hypergraph, and $\mathrm{Cell}(P) \simeq \mathrm{Cell}(P')$ where P' is the set of relevant elements of P. Secondly, as discussed in Section 1.6.6, parity complexes do not ensure torsion-freeness, which can break the freeness property. A natural fix is to require parity complexes to moreover satisfy (G4). Then, in the following, we will suppose given an ω-hypergraph P satisfying axioms (C0) to (C5) and (G2) and (G4).

Lemma 5.1.1 ([20, Proposition 1.4]). *For $n > 0$, $U, V \subseteq P_n$ with U tight, V fork-free and $U \subseteq V$, we have that U is a segment for $\triangleleft_V$.*

Proof. Suppose given $x, y, z \in V$ such that $x, z \in U$ and $x \triangleleft^1_V y \triangleleft_V z$. Then, there is $w \in x^+ \cap y^-$. By definition of tightness, since $y \triangleleft_V z$, we have $y^- \cap U^\pm = \emptyset$. So there is $y' \in U$ such that $w \in y'^-$. Since V is fork-free, $y = y'$. Hence, U is a segment for $\triangleleft_V$. □

Lemma 5.1.2. *Let $n \in \mathbb{N}$. For $x \in P_n$, x satisfies the segment condition.*

Proof. Let $m < n \in \mathbb{N}$, $x \in P_n$ and X be an m-cell. Suppose that $\langle x \rangle_{m,-} \subseteq X_m$. By (C5), $\langle x \rangle_{m,-}$ is tight. Then, by [20, Proposition 1.4], $\langle x \rangle_{m,-}$ is a segment for $\triangleleft_{X_m}$.

Now suppose that $\langle x \rangle_{m,+} \subseteq X_m$. By contradiction, assume that $\langle x \rangle_{m,+}$ is not a segment for $\triangleleft_{X_m}$. Thus, by definition of $\triangleleft_{X_m}$, there exist $p > 1$ and $u_0, \ldots, u_p \in X_m$ with $u_0, u_p \in \langle x \rangle_{m,+}$, $u_1, \ldots, u_{p-1} \notin \langle x \rangle_{m,+}$ and $u_i \triangleleft^1_{X_m} u_{i+1}$. By definition of $\triangleleft^1_{X_m}$, there exist $z_0, \ldots, z_{p-1}$ such that $z_i \in u_i^+ \cap u_{i+1}^-$. Note that $z_0 \in \langle x \rangle^\pm_{m,+}$. Indeed, if $z_0 \in v^-$ for some $v \in X_m$, then, since X_m is fork-free, $v = u_1$, so $v \notin \langle x \rangle_{m,+}$. Similarly, $z_{p-1} \in \langle x \rangle^\mp_{m,+}$. Since x is relevant by (G2), $\langle x \rangle^\pm_{m+1,+} = \langle x \rangle_{m,+} \subseteq X_m$. By [19, Lemma 3.2] (which is essentially Theorem 2.2.3, but for the axioms of parity complexes), $Y = \overline{\mathrm{Act}}(X, \langle x \rangle_{m+1,+})$ is a cell with $Y_m = (X_m \setminus \langle x \rangle_{m,+}) \cup \langle x \rangle_{m,-}$. So $\langle x \rangle_{m,-} \subseteq Y_m$ and, as previously shown, $\langle x \rangle_{m,-}$ is a segment for $\triangleleft_{Y_m}$. But, since $\langle x \rangle^\mp_{m,-} = \langle x \rangle^\mp_{m,+}$ and $\langle x \rangle^\pm_{m,-} = \langle x \rangle^\pm_{m,+}$, there exist $u'_0, u'_p \in \langle x \rangle_{m,-}$ such that $z_0 \in u_0'^+$ and $z_{p-1} \in u_p'^-$. So $u'_0 \triangleleft^1_{X_m} u_1 \triangleleft^1_{X_m} \cdots \triangleleft^1_{X_m} u_{p-1} \triangleleft^1_{X_m} u'_p$ with $u_1, \ldots, u_{p-1} \notin \langle x \rangle_{m,-}$, contradicting the fact that $\langle x \rangle_{m,-}$ is a segment for $\triangleleft_{Y_m}$. Thus, $\langle x \rangle_{m,+}$ is a segment for X_m. Hence, x satisfies the segment condition. □

Theorem 5.1.3. *P is a generalized parity complex.*

Proof. (G0) is a consequence of (C0). (G1) is a consequence of (C3). And (G3) is a consequence of Lemma 5.1.2. □

The category of cells of the parity complex is, of course, isomorphic to the category of cells of the associated generalized parity complex.

5.2 Encoding pasting schemes

In this subsection, we embed loop-free pasting schemes in generalized parity complexes. More precisely, we will only embed the loop-free pasting schemes that are torsion-free (that is, whose ω-hypergraph satisfies (G4)) since, like for parity complexes, the ones that are not torsion-free can not be expected to induce free ω-categories. So, in the following, we suppose given an ω-hypergraph P satisfying (S0), (S1), (S2), (S3), (S4), (S5) and (G4).

Lemma 5.2.1. *Let $k < n \in \mathbb{N}$, $x \in P_n$ and $y \in P_k$. If $x\mathrm{B}^n_{n-1}\mathrm{R}^{n-1}_k y$ then $y \in \mathrm{B}^n_k(x)$ or $x\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k y$. Dually, if $x\,\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k\,y$ then $y \in \mathrm{E}^n_k(x)$ or $x\,\mathrm{B}^n_{n-1}\mathrm{R}^{n-1}_k\,y$.*

Proof. We do an induction on $n-k$. If $k = n-1$, the result is trivial. If $k = n-2$, the result is a consequence of (S1). Otherwise, suppose that $k < n-2$. We will only prove the first part, since the second is dual. So suppose that $y \notin \mathrm{B}^n_k(x)$. By the definition of B, we have

$$\neg(x\,\mathrm{B}^n_{n-1}\mathrm{B}^{n-1}_k\,y) \quad \text{or} \quad \neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_k\,y).$$

By symmetry, we can suppose that $\neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_k\,y)$. Let $u \in P_{n-1}$ be minimal for $\lhd$ such that $x\,\mathrm{B}^n_{n-1}\,u\,\mathrm{R}^{n-1}_k\,y$. Then, there are two possible cases: either $u\,\mathrm{B}^{n-1}_{n-2}\mathrm{R}^{n-2}_k\,y$ or $u\,\mathrm{E}^{n-1}_{n-2}\mathrm{R}^{n-2}_k\,y$.

In the first case, let $v \in P_{n-2}$ be such that $u\,\mathrm{B}^{n-1}_{n-2}\,v\,\mathrm{R}^{n-2}_k\,y$. By the minimality of u, we have $\neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_{n-2}\,v)$, so $\neg(x\,\mathrm{B}^n_{n-2}\,v)$ by definition of B. By Axiom (S1), we have $x\,\mathrm{E}^n_{n-1}\mathrm{E}^{n-1}_{n-2}\,v$. So $x\,\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_k\,y$.

In the second case, since we supposed $\neg(x\,\mathrm{B}^n_{n-1}\mathrm{E}^{n-1}_k\,y)$, we have $\neg(u\,\mathrm{E}^{n-1}_k\,y)$. By induction, $u\,\mathrm{B}^{n-1}_{n-2}\mathrm{R}^{n-2}_k\,y$ and we can conclude using the first case. ☐

Lemma 5.2.2. *Let $n > 0$ and X be an n-wfs of P. Then $\partial^\epsilon X = \bar{\partial}^\epsilon X$.*

Proof. We only prove the case $\epsilon = -$. So let $n > 0$ and X be an n-wfs of P. Recall that $\partial^- X = X \setminus \mathrm{E}(X)$ and $\bar{\partial}^- X = \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^+_n)))$.

Step 1: $\bar{\partial}^- X \subseteq \partial^- X$. We have to show that

$$\mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^+_n))) \subseteq X \setminus \mathrm{E}(X).$$

Since $X \setminus \mathrm{E}(X)$ is closed (by [8, Theorem 12]), it is equivalent to

$$X \setminus (X_n \cup \mathrm{R}(X^+_n)) \subseteq X \setminus \mathrm{E}(X)$$

which is itself equivalent to

$$\mathrm{E}(X) \subseteq (X_n \cup \mathrm{R}(X^+_n))$$

which holds.

Step 2: $\partial^- X \subseteq \bar{\partial}^- X$. We have to show that

$$X \setminus \mathrm{E}(X) \subseteq \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X^+_n))) = \bar{\partial}^-(X).$$

Let $m < n \in \mathbb{N}$ and $x \in (X \setminus \mathrm{E}(X))_m$. If $x \notin \mathrm{R}(X^+_n)$ then $x \in \bar{\partial}^-(X)$. So suppose that $x \in \mathrm{R}(X^+_n)$. Since $\mathrm{E}(X)_{n-1} = X^+_n$, it implies that $m < n-1$. By definition of $\mathrm{R}(X^+_n)$, there exists $y \in X_n$ such that $y\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_m x$ and, by Axiom (S2), we can take y minimal for $\lhd$ satisfying this property. By Lemma 5.2.1, it holds that $y\,\mathrm{B}^n_{n-1}\mathrm{R}^{n-1}_m\,x$. Let $z \in P_{n-1}$ be such that $y\,\mathrm{B}^n_{n-1}\,z\,\mathrm{R}^{n-1}_m\,x$. Then, there is no $y' \in X_n$ such that $y'\,\mathrm{E}^n_{n-1}\,z$: otherwise, $y'\,\mathrm{E}^n_{n-1}\mathrm{R}^{n-1}_m\,x$ and $y' \lhd y$, contradicting the minimality of y. So $z \notin \mathrm{R}(X^+_n)$ and $z\,\mathrm{R}\,x$. It implies that $z \in X \setminus (X_n \cup \mathrm{R}(X^+_n))$ and $x \in \bar{\partial}^- X$. ☐

Lemma 5.2.3. *Let $n \in \mathbb{N}$ and $X \in \mathrm{WF}(P)_n$. Then $X \in \mathrm{Closed}_{WF}(P)_n$.*

Proof. We prove this lemma by induction on n. If $n = 0$, the result is trivial. So suppose $n > 0$. Since X is well-formed, X_n is fork-free. Moreover, using Lemma 5.2.2, for $\epsilon \in \{-,+\}$, $\bar{\partial}^\epsilon(X) = \partial^\epsilon(X)$ which is well-formed. By induction, $\bar{\partial}^\epsilon(X) \in \mathrm{Closed}_{\mathrm{WF}}(P)_{n-1}$. Also, when $n \geq 2$, since $\partial^\epsilon \circ \partial^-(X) = \partial^\epsilon \circ \partial^+(X)$, by Lemma 5.2.2, $\bar{\partial}^\epsilon \circ \bar{\partial}^-(X) = \bar{\partial}^\epsilon \circ \bar{\partial}^+(X)$. Hence, $X \in \mathrm{Closed}_{\mathrm{WF}}(P)_n$. ☐

Lemma 5.2.4. *Let $n \in \mathbb{N}$, X be an n-wfs, $S \subseteq P_{n+1}$ be a finite subset with S fork-free, $S^{\mp} \subseteq X$ and let $Y = X \cup \mathrm{R}(S)$. Then Y is an $(n{+}1)$-wfs of P and $\partial^- Y = X$.*

Proof. Let $k = |S|$. We show this lemma by induction on k. If $k = 0$, the result is trivial. If $k = 1$, the result is a consequence of [8, Proposition 8]. So suppose $k > 1$. By Axiom (S2), take $x \in S$ minimal for $\lhd$. By minimality, we have

$$x^- \subseteq S^{\mp} \subseteq X.$$

Using [8, Proposition 8], $X \cup \mathrm{R}(x)$ is well-formed. By (S5), $X \cap \mathrm{E}(x) = \emptyset$, so $\partial^-(X \cup \mathrm{R}(x)) = X$. Let $X' = \partial^+(X \cup \mathrm{R}(x))$ and $S' = S \setminus \{x\}$. We have

$$\begin{aligned} S'^{\mp} \subseteq X'_n &\Leftrightarrow S'^- \subseteq X'_n \cup S'^+ \\ &\Leftrightarrow S^- \subseteq X'_n \cup S'^+ \cup x^- \\ &\Leftrightarrow S^- \subseteq (X_n \setminus x^-) \cup x^+ \cup S'^+ \cup x^- \\ &\Leftrightarrow S^- \subseteq X_n \cup S^+ \\ &\Leftrightarrow S^{\mp} \subseteq X_n \end{aligned}$$

so $S'^{\mp} \subseteq X'$. By induction, $X' \cup \mathrm{R}(S')$ is well-formed and $\partial^-(X' \cup \mathrm{R}(S')) = X'$. Since $\mathrm{WF}(P)$ has the structure of an ω-category by [8, Theorem 12], we can compose $X \cup \mathrm{R}(x)$ and $X' \cup \mathrm{R}(S')$. So

$$X \cup \mathrm{R}(S) = X \cup \mathrm{R}(x) \cup X' \cup \mathrm{R}(S')$$

is well-formed and $\partial^-(X \cup \mathrm{R}(S)) = X$. ☐

Lemma 5.2.5. *For $X \in \mathrm{Closed}_{WF}(P)$, we have $X \in \mathrm{WF}(P)$.*

Proof. Let $n \in \mathbb{N}$ and $X \in \mathrm{Closed}_{\mathrm{WF}}(P)_n$. We prove this lemma by induction on n. If $n = 0$, the result is trivial. So suppose $n > 0$. Let $Y = \bar{\partial}^- X$. By definition, $Y \in \mathrm{Closed}_{\mathrm{WF}}(P)$ and, by induction, $Y \in \mathrm{WF}(P)$. By definition of $\bar{\partial}^-$, we have $X_n^{\mp} \subseteq Y$. By Lemma 5.2.4, $Y \cup \mathrm{R}(X_n)$ is well-formed. But $Y = \mathrm{R}(X \setminus (X_n \cup \mathrm{R}(X_n^+)))$, hence $X = Y \cup \mathrm{R}(X_n)$ is well-formed. ☐

Lemma 5.2.6. *Let $n \in \mathbb{N}$ and $x \in P_n$. Then, for $i < n$ and $\epsilon \in \{-,+\}$,*

$$\partial_i^\epsilon \mathrm{R}(x) = \mathrm{R}(\langle x \rangle_{i,\epsilon}).$$

Proof. Let $n \in \mathbb{N}$, $x \in P_n$ and $i < n$. By symmetry, we will only prove that $\partial_i^-(\mathrm{R}(x)) = \mathrm{R}(\langle x \rangle_{i,-})$. We have

$$\begin{aligned} \partial_i^-(\mathrm{R}(x)) &= \partial_i^-(\mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\{x\})) \\ &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\tilde{\partial}_i^-\{x\}) && \text{(by Lemmas 4.3.8 and 5.2.2)} \\ &= \mathrm{T}^{\mathrm{M}}_{\mathrm{Cl}}(\langle x \rangle_{i,-}) \\ &= \mathrm{R}(\langle x \rangle_{i,-}). \end{aligned}$$

Hence, $\partial_i^- \mathrm{R}(x) = \mathrm{R}(\langle x \rangle_{i,-})$. ☐

Lemma 5.2.7. *For all $n \in \mathbb{N}$ and $x \in P_n$, x is relevant.*

Proof. Let $n \in \mathbb{N}$ and $x \in P_n$. By axiom (S3), $\mathrm{R}(x)$ is well-formed. So, for $i \leq n$ and $\epsilon \in \{-,+\}$, $\partial_i^\epsilon(\mathrm{R}(x))$ is well-formed. Then, by Lemma 5.2.6, $\langle x\rangle_{i,-}$ and $\langle x\rangle_{i,+}$ are fork-free. We show that $\langle x\rangle_{i+1,-}^{\pm} = \langle x\rangle_{i,+}$ and $\langle x\rangle_{i+1,+}^{\mp} = \langle x\rangle_{i,-}$. We have $\langle x\rangle_{n,-}^{\pm} = \langle x\rangle^{\pm} = x^+ = \langle x\rangle_{n-1,+}$ and, similarly, $\langle x\rangle_{n,+}^{\mp} = \langle x\rangle_{n-1,-}$. For $i < n-1$ and for $\epsilon \in \{-,+\}$, we have

$$\begin{aligned}\langle x\rangle_{i+1,-}^{\pm} &= \tilde{\partial}^+ \circ \tilde{\partial}_{i+1}^-(\{x\}) \\ &= \tilde{\partial}_i^+(\{x\}) && \text{(by globularity on maximal-well-formed fgs)} \\ &= \langle x\rangle_{i,+}\end{aligned}$$

and similarly, $\langle x\rangle_{i+1,+}^{\mp} = \langle x\rangle_{i,-}$. By definition of $\tilde{\partial}^\epsilon$, it gives $\langle x\rangle_{i,-} = \langle x\rangle_{i+1,+}^{\mp}$ and $\langle x\rangle_{i+1,-}^{\pm} = \langle x\rangle_{i,+}$. From these equalities, it readily follows that, for $0 \leq i < n$ and $\epsilon \in \{-,+\}$, $\langle x\rangle_{i+1,\epsilon}$ moves $\langle x\rangle_{i,-}$ to $\langle x\rangle_{i,+}$. Hence, $\langle x\rangle$ is a cell. □

Lemma 5.2.8. *Let $n \geq 0$. Then,*

(a) for $x \in P_n$, x satisfies the segment condition,

(b) for X an n-cell, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X) \in \mathrm{WF}(P)$.

Proof. We prove this lemma by an induction on n. If $n = 0$, the result is trivial. So suppose that $n > 0$.

Step 1: (a) holds. Let $x \in P_n$ and X be an m-cell with $m < n$ such that $\langle x\rangle_{m,-} \subseteq X_m$. Let $Y = \mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X)$. By induction, $Y \in \mathrm{WF}(P)$. Also, by Lemma 5.2.6, $\partial_m^-(\mathrm{R}(x)) = \mathrm{R}(\langle x\rangle_{m,-}) \subseteq Y$. So, by (S4), $\langle x\rangle_{m,-}$ is a segment for $\triangleleft_{Y_m} = \triangleleft_{X_m}$. Hence, x satisfies the segment condition.

Step 2: (b) holds. Let X be an n-cell. By Lemma 5.2.5, we just have to show that $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X)$ is closed-well-formed and this fact is a consequence of Theorem 4.5.7 which requires the full segment axiom. But we can restrain our ω-hypergraph P to an ω-hypergraph P' where $P'_i = P_i$ for $i \leq n$ and $P'_i = \emptyset$ for $i > n$. By (a), P' satisfies (G3). Then, using Theorem 4.5.7, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X)$ is closed-well-formed and is still closed-well-formed in P. Hence, by Lemma 5.2.5, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X) \in \mathrm{WF}(P)$. □

Theorem 5.2.9. *P is a generalized parity complex.*

Proof. (G0) is a consequence of (S0). (G1) is a consequence of (S2). (G2) is a consequence of Lemma 5.2.7. (G3) is a consequence of Lemma 5.2.8. □

Theorem 5.2.10. *$\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}$ is an isomorphism between the ω-categories $\mathrm{Cell}(P)$ and $\mathrm{WF}(P)$. Moreover, for all $x \in P$, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(\langle x\rangle) = \mathrm{R}(x)$.*

Proof. By Theorem 4.5.7 and Lemmas 5.2.3 and 5.2.5, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}$ induces a bijection between $\mathrm{Cell}(P)$ and $\mathrm{WF}(P)$. By Lemmas 4.3.5, 4.3.7 and 4.3.8, for $n > 0$ and X an n-cell and $\epsilon \in \{-,+\}$, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(\partial^\epsilon X) = \partial^\epsilon(\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X))$. By Lemma 4.4.1, for $i < n \in \mathbb{N}$, n-cells X, Y that are i-composable, $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X *_i Y) = \mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X) *^{\mathrm{Cl}} \mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(Y)$. Also, for $n \geq 0$ and X an n-cell, by unfolding the definitions, we have $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(\mathrm{id}_{n+1}(X)) = \mathrm{id}_{n+1}(\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(X))$. Lastly, for $x \in P$, by Lemma 4.3.5, we have $\mathrm{T}_{\mathrm{Cl}}^{\mathrm{PC}}(\langle x\rangle) = \mathrm{T}_{\mathrm{Cl}}^{\mathrm{M}} \circ \mathrm{T}_{\mathrm{M}}^{\mathrm{PC}}(\langle x\rangle) = \mathrm{R}(x)$. □

5.3 Encoding augmented directed complexes

In this subsection, we embed augmented directed complexes with loop-free unital basis in generalized parity complexes. In the following, we suppose given an adc $(K, \mathrm{d}, \mathrm{e})$ with a loop-free unital basis P.

5.3.1 Adc's as ω-hypergraphs. In the following, given $n \in \mathbb{N}$ and $x \in P_n$, we will write $\bar{x}$ to refer to x *as an element of the graded set* P whereas x alone refer to x *as an element of the monoid* K_n^*. Given $n \in \mathbb{N}$,

– for $s \in K_n^*$, we write $\mathrm{S}_n(s)$ for $\{\bar{x} \in P_n \mid x \leq s\}$,

– for $S \subseteq P_n$ finite, we write $\mathrm{M}_n(S)$ for $\sum_{x \in S} x$.

The *ω-hypergraph associated to* K is the ω-hypergraph structure on P defined as follows. Given $n \geq 0$ and $\bar{x} \in P_{n+1}$, we define $\bar{x}^-, \bar{x}^+ \subseteq P_n$ as

$$\bar{x}^- = \mathrm{S}_n(x^-) \quad \bar{x}^+ = \mathrm{S}_n(x^+).$$

where x^- and x^+ were defined in Paragraph 1.5.2.

5.3.2 Fork-freeness and radicality. For $n > 0$, $s \in K_n^*$ is said to be *fork-free* when for all $\bar{x}, \bar{y} \in P_n$ such that $x + y \leq s$, it holds that $\bar{x}^\epsilon \cap \bar{y}^\epsilon = \emptyset$ for $\epsilon \in \{-,+\}$. Given $s \in K_0^*$, s is said to be fork-free when $\mathrm{e}(s) = 1$. Given X an n-cell of K, X is said *fork-free* when, for $i \leq n$ and $\epsilon \in \{-,+\}$, $X_{i,\epsilon}$ is fork-free. For $n \geq 0$ and $s \in K_n^*$, s is said *radical* when for all $z \in K_n^*$ such that $2z \leq s$, $z = 0$. We then have the following properties.

Lemma 5.3.3. *For $n > 0$ and $\bar{x} \in P_n$, $\bar{x}^- \neq \emptyset$ and $\bar{x}^+ \neq \emptyset$. That is, P satisfies (G0).*

Proof. By contradiction, if $\bar{x}^- = \emptyset$, it implies that $[x]_{n-1,-} = 0$. Hence, $[x]_{i,-} = 0$ for $i < n$. In particular, $\mathrm{e}([x]_{0,-}) = 0$, contradicting the fact that the basis is unital. Hence, $\bar{x}^- \neq \emptyset$ and similarly $\bar{x}^+ \neq \emptyset$. □

Lemma 5.3.4. *For $n \geq 0$ and $s \in K_n^*$, if s is fork-free, then s is radical.*

Proof. If $n = 0$, $s \in K_n^*$ can be written $s = \sum_{1 \leq i \leq k} x_i$ with $x_i \in P_0$. So $\mathrm{e}(s) = k$, and, by fork-freeness, $k = 1$. Hence, s is radical.

Otherwise, assume that $n > 0$. By contradiction, suppose that there is $\bar{x} \in P_n$ such that $2x \leq s$. By Lemma 5.3.3, it means that $\bar{x}^- \cap \bar{x}^- \neq \emptyset$, contradicting the fact that s is fork-free. Hence, s is radical. □

Lemma 5.3.5. *For $n \geq 0$ and an n-cell X of K, X is fork-free.*

Proof. We prove this lemma using an induction on n. If $n = 0$, since $\mathrm{e}(X_0) = 1$, X is fork-free by definition.

Otherwise, suppose that $n > 0$. By induction, $\partial^- X$ and $\partial^+ X$ are fork-free, so $X_{i,\epsilon}$ is fork-free for $i < n$ and $\epsilon \in \{-,+\}$. Let $\bar{x}, \bar{y} \in P_n$ be such that $x + y \leq X_n$. By contradiction, suppose that there is $\bar{z} \in P_{n-1}$ such that $\bar{z} \in \bar{x}^- \cap \bar{y}^-$. By [17, Proposition 5.4], there are $k \geq 1$, $\bar{x}_1, \ldots, \bar{x}_k \in P_n$ and n-cells $X^1, \ldots, X^k$ of K with $X_n^i = \bar{x}_i$ such that

$$X = X^1 *_{n-1} \cdots *_{n-1} X^k$$

so $X_n = x_1 + \cdots + x_k$. Hence, there are $1 \leq i_1, i_2 \leq k$ with $i_1 \neq i_2$ such that $x_{i_1} = x$ and $x_{i_2} = y$. By symmetry, we can suppose that $i_1 < i_2$. If there is some i such that $\bar{z} \in \bar{x}_i^+$, by [17, Proposition 5.4], $i < i_1$. So, for $i_1 \leq i \leq i_2$, $\bar{z} \notin \bar{x}_i^+$. Let $Y = X^{i_1} *_{n-1} X^{i_1+1} *_{n-1} \cdots *_{n-1} X^{i_2}$ which is a cell of K. We have

$$Y_{n-1,-} = \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,-} - \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,+} + Y_{n-1,+}$$

with

$$2z \leq \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,-} \quad \text{and} \quad \neg(z \leq \sum_{i_1 \leq i \leq i_2} [x_i]_{n-1,+}) \quad \text{and} \quad Y_{n-1,+} \geq 0$$

so $2z \leq Y_{n-1,-}$, contradicting the fact that $\partial^- Y$ is fork-free by induction. Thus $\bar{x}^- \cap \bar{y}^- = \emptyset$ and, similarly, $\bar{x}^+ \cap \bar{y}^+ = \emptyset$. Hence, X is fork-free. ☐

Lemma 5.3.6. *For all* $n \geq 0$, $\mathrm{S}_n \circ \mathrm{M}_n = \mathrm{id}_{P_n}$.

Proof. The result is a direct consequence of the definitions. ☐

Lemma 5.3.7. *For all* $n \geq 0$ *and* $s \in K_n^*$ *radical,* $\mathrm{M}_n \circ \mathrm{S}_n(s) = s$

Proof. The result is a direct consequence of the definitions. ☐

Lemma 5.3.8. *Let* $n \geq 0$, $U, V \subseteq P_n$ *be finite sets and* $x \in P_n$. *We have the following properties:*

(a) if $U \cap V = \emptyset$, *then* $\mathrm{M}_n(U) \wedge \mathrm{M}_n(V) = 0$ *and* $\mathrm{M}_n(U \cup V) = \mathrm{M}_n(U) + \mathrm{M}_n(V)$,

(b) if $U \subseteq V$, *then* $\mathrm{M}_n(U) \leq \mathrm{M}_n(V)$ *and* $\mathrm{M}_n(V \setminus U) = \mathrm{M}_n(V) - \mathrm{M}_n(U)$,

(c) if $n > 0$, *then* $\mathrm{M}_{n-1}(\bar{x}^\epsilon) = x^\epsilon$,

(d) Suppose that U *is fork-free. Then* $\mathrm{M}_n(U)$ *is fork-free. Moreover, when* $n > 0$, $\mathrm{M}_{n-1}(U^\epsilon) = (\mathrm{M}_n(U))^\epsilon$.

Proof. (a) and (b) are direct consequences of the definitions. For (c), note that $\bar{x}^\epsilon = \mathrm{S}_{n-1}(x^\epsilon)$. By Lemma 5.3.5, $[x]_{n-1,\epsilon}$ is fork-free and, by Lemma 5.3.4, it is radical. So, by Lemma 5.3.7, $\mathrm{M}_{n-1}(\bar{x}^\epsilon) = x^\epsilon$.

For (d), suppose that $U \subseteq P_n$ is fork-free. If $n = 0$, the result is trivial, so we can suppose $n > 0$. For $\bar{x}, \bar{y} \in P_n$ with $x \leq \mathrm{M}_n(U)$ and $y \leq \mathrm{M}_n(U)$ such that there exist $\bar{z} \in P_{n-1}$ and $\epsilon \in \{-,+\}$ with $z \leq x^\epsilon$ and $z \leq y^\epsilon$, we have $\bar{z} \in \bar{x}^\epsilon$ and $\bar{z} \in \bar{y}^\epsilon$. Since U is fork-free, $x = y$. Also, since $\mathrm{M}_n(U)$ is radical, $\neg(x + y \leq \mathrm{M}_n(U))$. So $\mathrm{M}_n(U)$ is fork-free. For the second part, note that for $\bar{x}, \bar{y} \in U$ with $x \neq y$, we have $\bar{x}^\epsilon \cap \bar{y}^\epsilon = \emptyset$. Hence,

$$\begin{aligned}
\mathrm{M}_{n-1}(U^\epsilon) &= \mathrm{M}_{n-1}(\cup_{\bar{x} \in U} \bar{x}^\epsilon) \\
&= \sum_{\bar{x} \in U} \mathrm{M}_{n-1}(\bar{x}^\epsilon) && \text{(by (a))} \\
&= \sum_{\bar{x} \in U} x^\epsilon && \text{(by (c))} \\
&= s^\epsilon.
\end{aligned}$$

☐

Lemma 5.3.9. *Let* $n \geq 0$, $u, v \in K_n^*$ *be such that* u, v *are radical and* $z \in P_n$. *We have the following properties:*

(a) if $u \wedge v = 0$, *then* $\mathrm{S}_n(u) \cap \mathrm{S}_n(v) = \emptyset$ *and* $\mathrm{S}_n(u + v) = \mathrm{S}_n(u) \cup \mathrm{S}_n(v)$,

(b) if $u \leq v$, *then* $\mathrm{S}_n(u) \subseteq \mathrm{S}_n(v)$ *and* $\mathrm{S}_n(v - u) = (\mathrm{S}_n(v)) \setminus (\mathrm{S}_n(u))$,

(c) if $n > 0$, *then* $\mathrm{S}_{n-1}(z^\epsilon) = \bar{z}^\epsilon$,

(d) if u *is fork-free, then* $\mathrm{S}_n(u)$ *is fork-free. Moreover, when* $n > 0$, $\mathrm{S}_{n-1}(u^\epsilon) = (\mathrm{S}_n(u))^\epsilon$.

Proof. (a), (b) and (c) are direct consequences of the definitions. For (d), suppose that u is fork-free. If $n = 0$, the result is trivial, so suppose that $n > 0$. For $\bar{x}, \bar{y} \in \mathrm{S}_n(u)$ such that there exist $\epsilon \in \{-,+\}$ and $\bar{z} \in \bar{x}^\epsilon \cap \bar{y}^\epsilon$, we have $z \le x^\epsilon$ and $z \le y^\epsilon$. By fork-freeness, $\neg(x+y \le u)$. But $x \le u$ and $y \le u$. So $x = y$ and $\mathrm{S}_n(u)$ is fork-free. For the second part, note that for $x, y \in P_n$ with $x \neq y$, $x \le u$ and $y \le u$, we have $x^\epsilon \wedge y^\epsilon = 0$. Hence,

$$\begin{aligned}
\mathrm{S}_{n-1}(u^\epsilon) &= \mathrm{S}_{n-1}\Big(\sum_{x \in P_n, x \le u} x^\epsilon\Big) \\
&= \bigcup_{x \in P_n, x \le u} \mathrm{S}_{n-1}(x^\epsilon) && \text{(by (a))} \\
&= \bigcup_{x \in P_n, x \le u} \bar{x}^\epsilon && \text{(by (c))} \\
&= (\mathrm{S}_n(u))^\epsilon.
\end{aligned}$$

□

5.3.10 Movement properties. Here, we prove several lemmas relating movement properties on P with properties on K.

Lemma 5.3.11. *Let $n > 0$, $u \in K_n^*$ fork-free and $U = \mathrm{S}_n(u)$. Then,*

$$u^\mp = \mathrm{M}_{n-1}(U^\mp) \quad \textit{and} \quad u^\pm = \mathrm{M}_{n-1}(U^\pm).$$

Proof. We have

$$\begin{aligned}
\mathrm{d}\, u &= u^\pm - u^\mp \\
&= u^+ - u^- \\
&= \mathrm{M}_{n-1}(U^+) - \mathrm{M}_{n-1}(U^-) && \text{(by Lemma 5.3.8)} \\
&= (\mathrm{M}_{n-1}(U^\pm) + \mathrm{M}_{n-1}(U^+ \cap U^-)) \\
&\quad - (\mathrm{M}_{n-1}(U^\mp) + \mathrm{M}_{n-1}(U^+ \cap U^-)) && \text{(by Lemma 5.3.8)} \\
&= \mathrm{M}_{n-1}(U^\pm) - \mathrm{M}_{n-1}(U^\mp).
\end{aligned}$$

Since $U^\pm \cap U^\mp = \emptyset$, we have $\mathrm{M}_{n-1}(U^\pm) \wedge \mathrm{M}_{n-1}(U^\mp) = \emptyset$. By uniqueness of the decomposition,

$$u^\mp = \mathrm{M}_{n-1}(U^\mp) \quad \text{and} \quad u^\pm = \mathrm{M}_{n-1}(U^\pm).$$

□

Lemma 5.3.12. *Let $n \ge 0$, $S \subseteq P_{n+1}$ be a finite and fork-free set, $U, V \subseteq P_n$ be finite sets, such that S moves U to V. Then, $\mathrm{d}(\mathrm{M}_{n+1}(S)) = \mathrm{M}_n(V) - \mathrm{M}_n(U)$.*

Proof. By definition of movement, $V = (U \cup S^+) \setminus S^-$. Hence,

$$\begin{aligned}
\mathrm{M}_n(V) &= \mathrm{M}_n((U \cup S^+) \setminus S^-) \\
&= \mathrm{M}_n(U \cup S^+) - \mathrm{M}_n(S^-) && \text{(by Lemma 5.3.8, since } S^- \subseteq U \cup S^+\text{)} \\
&= \mathrm{M}_n(U) + \mathrm{M}_n(S^+) - \mathrm{M}_n(S^-) && \text{(since } U \cap S^+ = \emptyset \text{ by Lemma 2.1.1)} \\
&= \mathrm{M}_n(U) + (\mathrm{M}_{n+1}(S))^+ - (\mathrm{M}_{n+1}(S))^- && \text{(by Lemma 5.3.8)} \\
&= \mathrm{M}_n(U) + \mathrm{d}(\mathrm{M}_{n+1}(S)).
\end{aligned}$$

□

Lemma 5.3.13. *Let $n \ge 0$, $s \in K_{n-1}^*$ fork-free, $u, v \in K_n^*$ with u, v radical, such that $\mathrm{d}\, s = v - u$, $u \wedge s^+ = 0$ and $s^- \wedge v = 0$. Then, $\mathrm{S}_{n+1}(s)$ moves $\mathrm{S}_n(u)$ to $\mathrm{S}_n(v)$.*

Proof. Let $S = \mathrm{S}_{n+1}(s)$, $U = \mathrm{S}_n(u)$ and $V = \mathrm{S}_n(v)$. Since $\mathrm{d}\, s = v - u$, we have

$$s^- \le s^- + v = u + s^+$$

so

$$S^- = \mathrm{S}_n(s^-) \subseteq \mathrm{S}_n(u + s^+) = U \cup S^+.$$

Thus,

$$\begin{aligned}
\mathrm{M}_n((U \cup S^+) \setminus S^-) &= \mathrm{M}_n(U \cup S^+) - \mathrm{M}_n(S^-) \\
&= \mathrm{M}_n \circ \mathrm{S}_n(u + s^+) - s^- && \text{(by Lemma 5.3.8)} \\
&= u + s^+ - s^- \\
&= u + \mathrm{d}\, s \\
&= v \\
&= \mathrm{M}_n(V)
\end{aligned}$$

so, by Lemma 5.3.6, $V = (U \cup S^+) \setminus S^-$. Similarly, $U = (V \cup S^-) \setminus S^+$. Hence, S moves U to V. ☐

Lemma 5.3.14. *Let $n > 0$ and X be an n-cell of K. Then, for $i < n$ and $\epsilon \in \{-, +\}$,*

$$X_{i,-} \wedge X^+_{i+1,\epsilon} = 0 \quad \textit{and} \quad X^-_{i+1,\epsilon} \wedge X_{i,+} = 0.$$

Proof. By contradiction, suppose given $n > 0$, X an n-cell, $i < n$ and $\epsilon \in \{-, +\}$ that give a counter-example for this property. By applying ∂^-, ∂^+ sufficiently, we can suppose that $i = n-1$. Also, by symmetry, we only need to handle the first case, that is, when there is $z \in P_{n-1}$ such that $z \le X_{n-1,-} \wedge X^+_n$. So there is $x \in P_n$ such that $x \le X_n$ and $z \le x^+$. By the definition of a cell, we have $\mathrm{d}\, X_n = X_{n-1,+} - X_{n-1,-}$, thus

$$\begin{aligned}
X_{n-1,+} + \sum_{u \in P_n, u \le X_n} u^- &= X_{n-1,-} + \sum_{u \in P_n, u \le X_n} u^+ \\
&\ge 2z
\end{aligned}$$

and, since $X_{n-1,+}$ is radical, there is $y \in P_n$ with $y \le X_n$ such that $z \le y^-$. By [17, Proposition 5.1], there are $k \ge 1$, $x_1, \ldots, x_k \in P_n$ with $x_1 + \cdots + x_k = X_n$ and $i_1 < i_2$ with $x_{i_1} = x$ and $x_{i_2} = y$ and n-cells $X^1, \ldots, X^k$ with $X^i_n = x_i$ such that $X = X^1 *_{n-1} \cdots *_{n-1} X^k$. Let $Y = X^1 *_{n-1} \cdots *_{n-1} X^{i_1}$. Since Y is a cell, we have

$$\begin{aligned}
Y_{n-1,+} + \sum_{1 \le i \le k} x_i^- &= Y_{n-1,-} + \sum_{1 \le i \le k} x_i^+ \\
&= X_{n-1,-} + \sum_{1 \le i \le k} x_i^+ \\
&\ge 2z.
\end{aligned}$$

Moreover, since X is fork-free and $z \le x^-_{i_2}$, we have $\neg(z \le x^-_i)$ for $i \le i_1$. So $2z \le Y_{n-1,+}$, contradicting the fact that $Y_{n-1,+}$ is radical by Lemmas 5.3.5 and 5.3.4. Hence, $X_{i,-} \wedge X^+_n = 0$. ☐

5.3.15 The translation operations. Given an n-pre-cell X of P, we define $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}(X)$ as the n-pre-cell Y of K such that $Y_{i,\epsilon} = \mathrm{M}_i(X_{i,\epsilon})$ for $i \le n$ and $\epsilon \in \{-,+\}$. Similarly, given an n-pre-cell X of K, we define $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}(X)$ as the n-pre-cell Y of P such that $Y_{i,\epsilon} = \mathrm{S}_i(X_{i,\epsilon})$ for $i \le n$ and $\epsilon \in \{-,+\}$. We then have the following properties.

Lemma 5.3.16. $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}$ *is a bijection with inverse* $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}$ *from* $\mathrm{Cell}(P)$ *to* $\mathrm{Cell}(K)$.

Proof. Let $n \in \mathbb{N}$ and $X \in \mathrm{Cell}(P)$. Then, by Lemma 5.3.8, $\mathrm{M}_i(X_{i,\epsilon})$ is fork-free for $i \le n$ and $\epsilon \in \{-,+\}$. Moreover, by Lemma 5.3.12, for $i < n$ and $\epsilon \in \{-,+\}$,

$$\mathrm{d}(\mathrm{M}_{i+1}(X_{i+1,\epsilon})) = \mathrm{M}_i(X_{i,+}) - \mathrm{M}_i(X_{i,-})$$

so $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}(X) \in \mathrm{Cell}(K)$. Conversely, given $X \in \mathrm{Cell}(K)$, by Lemma 5.3.9, $\mathrm{S}_i(X_{i,\epsilon})$ is fork-free for $i \le n$ and $\epsilon \in \{-,+\}$. By Lemmas 5.3.13 and 5.3.14, for $i < n$ and $\epsilon \in \{-,+\}$,

$$\mathrm{S}_{i+1}(X_{i+1,\epsilon}) \text{ moves } \mathrm{S}_i(X_{i,-}) \text{ to } \mathrm{S}_i(X_{i,+})$$

so $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}(X) \in \mathrm{Cell}(P)$. By Lemma 5.3.6, for $X \in \mathrm{Cell}(P)$,

$$\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}} \circ \mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}(X) = X,$$

and, by Lemmas 5.3.5, 5.3.4 and 5.3.7, for $X \in \mathrm{Cell}(K)$,

$$\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}} \circ \mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}(X) = X.$$

Hence, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}$ and $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}$ induce bijections between $\mathrm{Cell}(P)$ and $\mathrm{Cell}(K)$ and are inverse of each other. □

Lemma 5.3.17. *For* $x \in P$*, we have* $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}([x]) = \langle \bar{x} \rangle$.

Proof. Let $X = \mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}([x])$. We have $X_n = \mathrm{S}_n([x]_n) = \{x\}$. We show by induction on i that $X_{i,\epsilon} = \langle x \rangle_{i,\epsilon}$ for $i < n$ and $\epsilon \in \{-,+\}$. We have $[x]_{i,-} = [x]^{\mp}_{i+1,-}$ so $X_{i,-} = \mathrm{S}_i([x]^{\mp}_{i+1,-}) = X^{\mp}_{i+1,-}$ by Lemmas 5.3.5 and 5.3.11. So $X_{i,-} = \langle x \rangle_{i,-}$. Similarly, $X_{i,+} = \langle x \rangle_{i,+}$. Hence, $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}([x]) = \langle \bar{x} \rangle$. □

5.3.18 Adc's are generalized parity complexes.

Lemma 5.3.19. *P satisfies (G1).*

Proof. Note that, for $n > 0$ and $\bar{x}, \bar{y} \in P_n$, $\bar{x} \triangleleft^1_{P_n} \bar{y}$ implies $\bar{x} <_{n-1} \bar{y}$. So, by transitivity, we have $\triangleleft_{P_n} \subseteq <_{n-1}$. Since the basis P is loop-free, $<_{n-1}$ is irreflexive and so is $\triangleleft_{P_n}$. Hence, $\triangleleft$ is irreflexive. □

Lemma 5.3.20. *P satisfies (G2).*

Proof. Let $\bar{x} \in P$. By Lemma 5.3.17, $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}([x]) = \langle \bar{x} \rangle$. And, by Lemma 5.3.16, $\mathrm{T}^{\mathrm{ADC}}_{\mathrm{PC}}([x]) \in \mathrm{Cell}(P)$. Hence, $\bar{x}$ is relevant. □

Lemma 5.3.21. *P satisfies (G3').*

Proof. By contradiction, suppose that there are $n > 0$, $i < n$, $\bar{x} \in P_n$ with $\langle \bar{x} \rangle_{i,+} \curvearrowright^* \langle \bar{x} \rangle_{i,-}$. So there are $k \geq 1$, $\bar{y}_1, \ldots, \bar{y}_k \in P_i$ with $\bar{y}_1 \in \langle \bar{x} \rangle_{i,+}$, $\bar{y}_k \in \langle \bar{x} \rangle_{i,-}$ and $\bar{y}_j \curvearrowright \bar{y}_{j+1}$ for $1 \leq j < k$. By definition of $\curvearrowright$, it gives $\bar{z}_1, \ldots, \bar{z}_{k-1} \in P_{i+1}$ with $\bar{y}_j \in \bar{z}_j^-$ and $\bar{y}_{j+1} \in \bar{z}_j^+$ for $1 \leq j < k$. So we have

$$\bar{x} <_i \bar{z}_1 <_i \cdots <_i \bar{z}_{k-1} <_i \bar{x},$$

contradicting the loop-freeness of the basis P. Hence, P satisfies (G3'). ☐

Lemma 5.3.22. *P satisfies (G4').*

Proof. By contradiction, suppose that there are $i > 0$, $m > i$, $n > i$, $\bar{x} \in P_m$ and $\bar{y} \in P_n$ with $\langle \bar{x} \rangle_{i,+} \cap \langle \bar{y} \rangle_{i,-} = \emptyset$, $\langle \bar{x} \rangle_{i-1,+} \curvearrowright^* \langle \bar{y} \rangle_{i-1,-}$ and $\langle \bar{y} \rangle_{i-1,+} \curvearrowright^* \langle \bar{x} \rangle_{i-1,-}$. By the same method than for Lemma 5.3.21, we get $r, s \in \mathbb{N}$, $\bar{u}_1, \ldots, \bar{u}_r \in P_i$, $\bar{v}_1, \ldots, \bar{v}_s \in P_i$ such that

$$\bar{x} <_i \bar{u}_1 <_i \cdots <_i \bar{u}_r <_i \bar{y} <_i \bar{v}_1 <_i \cdots <_i \bar{v}_s <_i \bar{x},$$

contradicting the loop-freeness of the basis P. Hence, P satisfies (G4'). ☐

Theorem 5.3.23. *P is a generalized parity complex.*

Proof. The result is a consequence of Lemmas 5.3.3, 5.3.19, 5.3.20, 5.3.21, 1.6.8, 5.3.22 and 1.6.9. ☐

Theorem 5.3.24. $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}$ *is an isomorphism of ω-categories. Moreover, for $\bar{x} \in P$, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}(\langle \bar{x} \rangle) = [x]$.*

Proof. The fact that $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}$ is bijective is given by Lemma 5.3.16. The fact that $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}$ commutes with source, target and identities is trivial.

Given $i < n \in \mathbb{N}$, i-composable cells $X, Y \in \mathrm{Cell}(P)_n$, we have that $X_{j,\epsilon} \cap Y_{j,\epsilon} = \emptyset$ for $i < j \leq n$ and $\epsilon \in \{-,+\}$. Indeed, by applying ∂^ϵ sufficiently, we can suppose that $j = n$. Then, by Lemma 3.3.4, $X *_i Y = X' *_{n-1} Y'$ where $X' = X *_{n-1} \mathrm{id}_n(\partial^-_{n-1} Y)$ and $Y' = \mathrm{id}_n(\partial^+_{n-1} X) *_i Y$. Note that $X'_n = X_n$ and $Y'_n = Y_n$. Hence, by Lemma 2.3.1, $X_n \cap Y_n = \emptyset$. Then, by Lemma 5.3.8, it follows readily that $\mathrm{M}_n(X *_i Y) = \mathrm{M}_n(X) *_i \mathrm{M}_n(Y)$. Thus, $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}$ is an isomorphism of ω-categories.

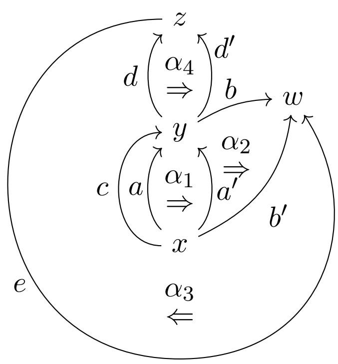
Lastly, given $\bar{x} \in P$, by Lemmas 5.3.17 and 5.3.7, we have $\mathrm{T}^{\mathrm{PC}}_{\mathrm{ADC}}(\langle \bar{x} \rangle) = [x]$. ☐

5.4 Absence of other embeddings

In this subsection, we show that there are no embeddings between the four formalisms except the ones already proved, that is, that parity complexes, pasting scheme and augmented directed complexes are generalized parity complexes. For the comparison with adc's, we use the translation from ω-hypergraphs to pre-adc's defined in Paragraph 1.5.4 and the translation from adc's to ω-hypergraphs defined in Paragraph 5.3.1.

5.4.1 No embedding in parity complexes. Axiom (C4) is relatively strong, so it can be used for building counter-examples to inclusion. The ω-hypergraph (10) is a pasting scheme satisfying (G4) (and thus is a generalized parity complex) and is an adc with loop-free unital basis. But it is not a parity complex as we have seen in Paragraph 1.3.8, because it does not satisfy (C4). So there is no embedding from pasting schemes, augmented directed complexes or generalized parity complexes in parity complexes (fixed version).

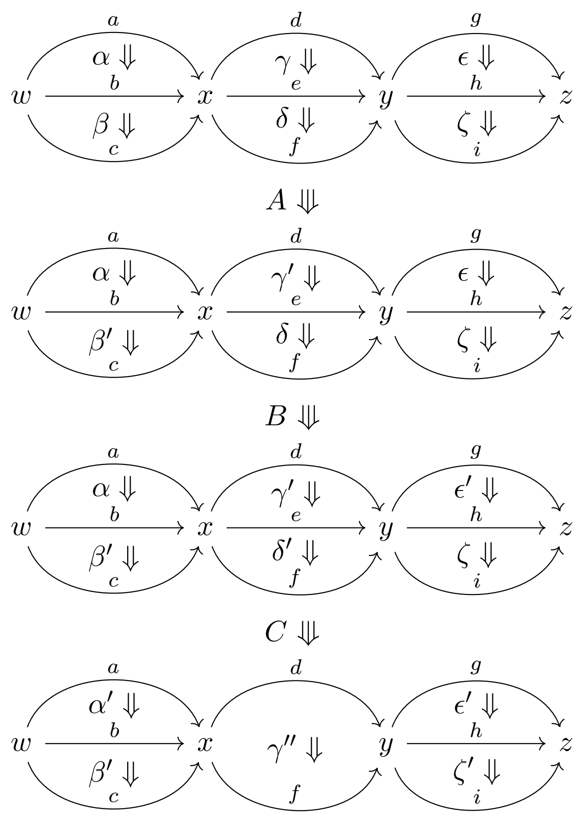
5.4.2 No embedding in pasting schemes. For pasting schemes, we use the relatively strong Axiom (S2) for building counter-examples to inclusion. The following ω-hypergraph is a parity complex satisfying (G4) (and thus it is a generalized parity complex) and is an adc with loop-free unital basis but it not as pasting scheme:



. (30)

Indeed, (S2) is not satisfied because $\alpha_2 \triangleleft \alpha_3$ but $y \in B(\alpha_2) \cap E(\alpha_3) \neq \emptyset$. Note that (30) is essentially the ω-hypergraph (16) without the 3-generator A and the 2-generators α'_1 and α'_4.

5.4.3 No embedding in augmented directed complexes. For augmented directed complexes, the loop-free basis axiom is used for building counter-examples to inclusion. It enforces a strong version of the torsion-freeness (G4). So, counter-examples of inclusion can be found with fancy torsion-free situations. The following ω-hypergraph is a parity complex and a pasting scheme, and moreover satisfies (G4), so it is a generalized parity complex:



(31)

where

$$\begin{aligned} A^- &= \{\beta, \gamma\}, & A^+ &= \{\beta', \gamma'\}, \\ B^- &= \{\delta, \epsilon\}, & B^+ &= \{\delta', \epsilon'\}, \\ C^- &= \{\alpha, \gamma', \delta', \zeta\}, & C^+ &= \{\alpha', \gamma'', \zeta'\}. \end{aligned}$$

But its associated pre-adc is an adc with a basis which is not loop-free unital. Indeed, we have $e < [A]_{1,+} \wedge [B]_{1,-}$, $h < [B]_{1,+} \wedge [C]_{1,-}$ and $b < [C]_{1,-} \wedge [A]_{1,+}$, so

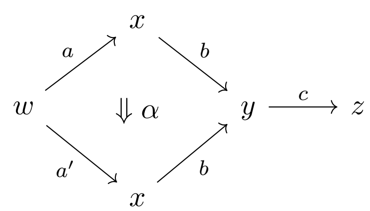
$$A <_1 B <_1 C <_1 A.$$

Hence, the basis of the associated augmented directed complex is not loop-free.

Conclusion

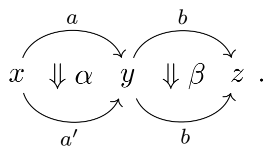
We hope that this work brought some understanding on the formalisms of pasting diagrams in several ways. First, by gathering all the three existing formalisms in one paper in the perspective of a unified treatment. Second, by giving some intuition on the axioms behind each of them. Third, by providing a generalization that encompasses the three other ones, with complete proofs. Last, by answering negatively to the questions of inclusions between formalisms. Moreover, this work was the opportunity to carry out a deep verification of the existing theories, allowing to discover a flaw that affects the freeness properties claimed for parity complexes and pasting schemes.

The generalized parity complexes presented in this paper seem to leave room for even more generalization, through at least two directions. First, it can be observed in several situations that Axiom (G1) is too strong. For example, the ω-hypergraph



(32)

should be considered as a pasting diagram. However, $\alpha \triangleleft \alpha$, so (32) does not satisfy (G1), and is therefore not a generalized parity complex. Second, by using multisets instead of sets, it seems possible to authorize some looping behaviors. This could enable to represent unambiguously "the morphism with one copy of α and two copies of β" in the diagram



(33)

These improvements would allow a bigger class of free ω-categories to be described explicitly. In particular, related objects, such as opetopes or the pasting diagrams defined by Henry [6] in order to study the Simpson's conjecture [7], could benefit from an effective description. Hence, future work on pasting diagrams might prove valuable and, among others, could help better understand the difficult world of weak ω-categories.

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A Details about the counter-example to parity complexes and pasting schemes

It was earlier claimed that the ω-hypergraph (11) was a counter-example to the freeness properties [19, Theorem 4.2] and [8, Theorem 13]. This claim assumes that F_1 and F_2 are two different 3-cells of the induced free ω-category $\mathcal{A}$. In this section, we give two methods to show this fact.

A.1 A mechanized counter-example

A proof that F_1 and F_2 are different has been formalized in Agda and is discussed in [4]. The proof relies on a verified definition of a 3-category $\mathcal{A}'$ such that there is a 3-functor $I\colon \mathcal{A} \to \mathcal{A}'$ for which $I(F_1) \neq I(F_2)$ by definition. The source code is accessible through a GitLab repository[1].

A.2 A categorical model

A more theoretical proof is given by an interpretation of $\mathcal{A}$ in the 3-category of 2-categories, functors, pseudo-natural transformations, and modifications, that is, a 3-functor $K\colon \mathcal{A} \to$ 2-Cat, for which $K(F_1) \neq K(F_2)$. After recalling some definitions, we define K and show that $K(F_1) \neq K(F_2)$.

A.2.1 Pseudo-natural transformations. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and $G, H\colon \mathcal{C} \to \mathcal{D}$ two 2-functors. A *pseudo-natural transformation* $\alpha\colon G \Rightarrow H$ is given by

- for all $x \in \mathcal{C}_0$, an 1-cell $\alpha_x\colon G(x) \to H(x) \in \mathcal{D}_1$,
- for all $x, y \in \mathcal{C}_0$, $f\colon x \to y \in \mathcal{C}_1$, a 2-cell α_f as in

$$\begin{array}{ccc} F(x) & \xrightarrow{\quad F(f) \quad} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & \Downarrow \alpha_f & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[\quad G(f) \quad]{} & G(y) \end{array}$$

such that some natural conditions hold.

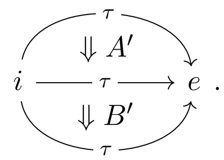
A.2.2 Modifications. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories, $G, H\colon \mathcal{C} \to \mathcal{D}$ be two 2-functors, $\alpha, \beta\colon G \Rightarrow H$ be two pseudo-natural transformations. A *modification* $M\colon \alpha \Rrightarrow \beta$ is given by 2-cells $M_x\colon \alpha_x \Rightarrow \beta_x$ for all $x \in \mathcal{C}_0$ such that some natural conditions hold.

A.2.3 The interpretation. Let $\mathcal{C}$ be the free 2-category induced by the 2-polygraph consisting in one 0-generator $\star$. Let $\mathcal{D}$ be the free 2-category induced by the 1-polygraph

$$y_1 \xrightarrow{\alpha''} y_2 \xrightarrow{\beta''} y_3 \ . \tag{34}$$

[1]See `https://gitlab.inria.fr/sforest/3-pasting-example`: the definition of the 3-category can be found in `ex.agda` and the main result in `ex-is-cat.agda`.

Let $\mathcal{E}$ be the free 2-category induced by the 2-polygraph



(35)

Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ be the 2-functors defined by $F_i(\star) = y_i$. Let $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be the 2-functors defined by

- $G_1(y_1) = G_1(y_2) = i$, $G_1(y_3) = e$, $G_1(\alpha'') = \mathrm{id}_1(i)$ and $G_1(\beta'') = \tau$,
- $G_2(y_1) = i$, $G_2(y_2) = G_2(y_3) = e$, $G_2(\alpha'') = \tau$ and $G_2(\beta'') = \mathrm{id}_1(e)$,
- $G_3(y_1) = G_3(y_2) = G_3(y_3) = e$ and $G_3(\alpha'') = G_3(\beta'') = \mathrm{id}_1(e)$.

Consider the following pseudo-natural transformations:

- $\bar{\alpha} \colon F_1 \Rightarrow F_2$ defined by $\bar{\alpha}_\star = \alpha''$,
- $\bar{\beta} \colon F_2 \Rightarrow F_3$ defined by $\bar{\beta}_\star = \beta''$,
- $\bar{\gamma} \colon G_1 \Rightarrow G_2$ defined by $\bar{\gamma}_{y_1} = \mathrm{id}_i$, $\bar{\gamma}_{y_2} = \tau$, $\bar{\gamma}_{y_3} = \mathrm{id}_e$,
- $\bar{\delta} \colon G_2 \Rightarrow G_3$ defined by $\bar{\delta}_{y_1} = \tau$, $\bar{\delta}_{y_2} = \mathrm{id}_e$, $\bar{\delta}_{y_3} = \mathrm{id}_e$.

Note that $\bar{\alpha} *_0 \bar{\delta}$ is given by $(\bar{\alpha} *_0 \bar{\delta})_\star = \delta_{F_2(\star)} \circ G_2(\alpha_\star) = \tau$. Similarly, $\bar{\beta} *_0 \bar{\gamma}$ is given by $(\bar{\beta} *_0 \bar{\gamma})_\star = \tau$. So a modification $M \colon \bar{\alpha} *_0 \bar{\delta} \Rrightarrow \bar{\alpha} *_0 \bar{\delta}$ (resp. $M \colon \bar{\beta} *_0 \bar{\gamma} \Rrightarrow \bar{\beta} *_0 \bar{\gamma}$) is given by a 2-cell $M_\star \colon \tau \Rightarrow \tau$ in $\mathcal{E}$. The interpretation $K \colon \mathcal{C} \to$ 2-Cat is then defined by

- $K(x) = \mathcal{C}$, $K(y) = \mathcal{D}$, $K(z) = \mathcal{E}$,
- $K(a) = F_1$, $K(b) = F_2$, $K(c) = F_3$, $K(d) = G_1$, $K(e) = G_2$, $K(f) = G_3$,
- $K(\alpha) = K(\alpha') = \bar{\alpha}$, $K(\beta) = K(\beta') = \bar{\beta}$, $K(\gamma) = K(\gamma') = \bar{\gamma}$, $K(\delta) = K(\delta') = \bar{\delta}$,
- $K(A) = A'$ and $K(B) = B'$.

Under this interpretation, we have

$$(K(F_1))_\star = A' *_1 B' \quad \text{and} \quad (K(F_2))_\star = B' *_1 A'$$

Since $A' *_1 B' \neq B' *_1 A'$ in $\mathcal{E}$, we have $F_1 \neq F_2$ in $\mathcal{A}$.